Shear Induced Rigidity in Athermal Materials: A Unified Statistical Framework

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Recent studies of athermal systems such as dry grains and dense, non-Brownian suspensions have shown that shear can lead to solidification through the process of shear jamming in grains and discontinuous shear thickening in suspensions. The similarities observed between these two distinct phenomena suggest that the physical processes leading to shear-induced rigidity in athermal materials are universal. We present a unified statistical mechanics model for these shear-driven transitions, which exhibits the phenomenology of shear jamming and discontinuous shear thickening in different regions of the predicted phase diagram. Our analysis identifies the crucial physical processes underlying shear-driven rigidity transitions, and clarifies the distinct roles played by shearing forces and the density of grains.

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Athermal materials such as dry grains and dense non-Brownian suspensions can respond to shear by organizing into structures that support the imposed load[1]: a process that has been termed shear-jamming (SJ) in grains[2–5], and discontinuous shear thickening (DST) in suspensions[6–13]. The nature of this self-organization process has been intensely investigated in recent months, and striking similarities have been observed between the two transitions even though SJ occurs through a quasi static process involving static states and DST occurs through a dynamical process in non-equilibrium steady states. Foremost amongst these is the proliferation of frictional contacts as the shear strain or the strain rate is increased. For a range of packing fractions, \( \phi_s < \phi < \phi_J \), below the isotropic jamming density, \( \phi_J \), quasi static shearing causes grains to come into contact leading to the SJ transition[2, 4, 5]. In a similar range of \( \phi \), athermal suspensions exhibit DST as increasing shearing rate leads to a loss of lubrication forces and increasing number of frictional contacts[7, 9]. There are other features that are shared by the two transitions: the anisotropy of the stress tensor decreases following both the SJ and DST transitions, and both dry grains and suspensions exhibit hysteresis in cyclic shearing processes[4, 9, 14].

In this work, we propose a statistical model that captures the universal features of shear-driven rigidity, and the nonlinear response of these self-organized states in both dry grains and athermal suspensions. The model incorporates a driving field that (i) creates frictional contacts, and (ii) increases the stress anisotropy of grains in a low-contact environment such as a force-chain. A particularly important aspect of disordered granular assemblies is the non-affine displacements of grains induced by the imposed strain: displacements that are inhomogeneous, and cannot be described by any type of homogeneous deformation of the unstrained state. Experiments indicate that the distribution of the non-affine strains depends on \( \phi \) but evolves little during the shear-jamming process[5, 14]. Based on this observation, we represent the positional disorder by a quenched random field whose strength depends only on \( \phi \). We show that a model with these ingredients captures the properties of shear-induced transitions in both SJ and DST if we associate the field with the strain in the former and the strain rate in the latter.

Model Lattice models have a venerable history of identifying universal features of equilibrium and non-equilibrium phase transitions. Our objective is to achieve this goal for transitions induced by shearing of athermal systems. The stress state of a grain can be rigorously mapped to a continuous spin variable[15]. Illustrating in 2D, the stress tensor of a grain can be written as:

\[
\sigma = P \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{(\sigma_{xx} - \sigma_{yy})}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sigma_{xy} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

where \( P \) is the hydrostatic pressure. The deviatoric part of the stress, which excludes this hydrostatic part, can therefore be mapped on to a 2D vector whose components are the normal stress, \( \frac{(\sigma_{xx} - \sigma_{yy})}{2} \), and the shear stress, \( \sigma_{xy} \), in a vector space spanned by two \( 2 \times 2 \) matrices. We simplify the picture further by dividing the granular assembly into two species[5, 14], and mapping to a spin-1 Ising model: grains with two contacts that form force chains and have a large deviatoric stress are represented by \( S_i = \pm 1 \), whereas grains with \( > 2 \) contacts, which have small deviatoric stress, are represented by \( S_i = 0 \). The external strain then maps naturally to a uniform magnetic field, \( H \), and the non-affine strain at grain level maps on to a random magnetic field. As we show in this paper, this minimal mode captures the essential features of SJ and DST and connects the myriad observations to a few core physical principles.

The Hamiltonian of the minimal model is a variant of the random-field-Blume-Emery-Griffiths (RFBEG) model[16–18]:

\[
H = -J \sum_{<i,j>} S_i S_j - h_i S_i - H \sum_i S_i + \Delta(H) \sum_i S_i^2.
\]

Spins, \( S_i = 0, 1, -1 \), are situated on a hyper cubic lattice, \( H \) is the external magnetic field, \( J \) is the strength of
nearest-neighbor ferromagnetic interactions, $\Delta$ is a chemical potential that controls the fraction of $S_i = 0$, and $h_i$ are random fields chosen from a Gaussian distribution with zero mean and standard deviation $R$, which measures the strength of the disorder. The ferromagnetic interaction arises from the constraints imposed by force and torque balance: grains in close proximity have similar stress anisotropy (Eq. 1)[5]. The $H$ dependence of $\Delta$ in Eq. 2 captures the proliferation of contacts with driving, which is an essential feature of SJ and DST. Removing this dependence reduces the model to RFBEG, or a diluted random-field-Ising model. Since external driving is necessary to change the configuration of athermal systems, we are interested in the zero-temperature dynamics[18].

For $N$ spins, we define two global order parameters:

$$X = 1 - \frac{1}{N} \sum_i S_i^2$$
$$M = \frac{1}{N} \sum_i S_i$$

that correspond to the fraction of grains with $> 2$ contact ($f_{\geq 2}$), and the stress anisotropy, respectively. We emphasize that $\Delta$ controls the fraction of grains with nearly-isotropic stresses: it does not control $\phi$, which remains constant during the shearing process. Since both SJ and DST are associated with an increase of $X$ with $H$, we focus solely on the $\Delta > 0$ regime. In general, any function, $f = f(H)$, defines a trajectory in the $H - \Delta$ phase space (a protocol). For the current study, we adopt the protocol $\Delta = \alpha |H| + \Delta_0$, which is the simplest that admits an increase in $X$ with the magnitude of the driving field. This protocol is parametrized by $\Delta_0$ and $\alpha$, with $\Delta_0$ determining $X$ at zero field.

**Meanfield Phase Diagram:** Meanfield solutions[19] demonstrate the essential properties of the order parameters of this non-equilibrium model. These solutions (Eqs. S7, S8 in [19]) can be used to construct a phase diagram in the $R - \Delta_0$ space. The phase diagrams are qualitatively different for $\alpha \leq 1$ and $\alpha > 1$. As shown in Fig. S2 of [19], only the $\alpha > 1$ protocols lead to a monotonic increase of $X(H)$, which is a feature shared by both SJ and DST, and we focus our study on these. Unless otherwise stated, the results in this paper are for $\alpha = 4$. As seen in Fig. 1(a), there is a critical point, $(\Delta_c, R_c)$, which marks the end of three transition lines: (i) $R = R_t(\Delta_0)$ identifies the transition from a single solution to multiple solutions for $M(H = 0)$, (ii) system-size avalanches in which spins flip from $\pm 1$ to 0 appear for $R < R_{DST}(\Delta_0)$, and (ii) $R = R_{sc}(\Delta_0)$ marks a line of discontinuous decrease (increase) of $M(H = 0)$ ($X(H = 0)$). Additional insight into the different regimes is provided by the spin configurations shown in Figs. S5 - S7 of [19].

For $\Delta_0 > \Delta_c$, $M$ is a non-monotonic function of $X$, exhibiting a peak at a characteristic value. In the $\Delta_0 < \Delta_c$ regime, $M$ decreases monotonically with $X$. The peak anisotropy, $M_{\text{peak}}(\Delta_0, R)$, undergoes a scaling collapse (Fig. 1b) as a function of the distance between $R$ and $R_c(\Delta_0) \approx \Delta_0/6$, showing a dramatic decrease as $R \rightarrow R_c(\Delta_0)$ from above. Meanfield calculations predict hysteresis only in the region of parameter space where there are system-spanning avalanches[20, 21]. However, simulations exhibit hysteresis over most of the region in Fig. 1(a). We have, therefore, used both simulations and meanfield calculations to study hysteresis loops appearing in cyclic trajectories.

**SJ transition:** In the $\Delta_0 > \Delta_c$ regime of the phase diagram, $X$ is significantly above zero at $H = 0$. Since the corresponding situation for $f_{\geq 2}$ holds for the SJ experiments[14], we analyze our model in this regime to understand and predict features of the SJ transition in dry grains. As discussed earlier, the mapping from the spin model to sheared grains is: (i) magnetic field $H \rightarrow$ strain, (ii) $M \rightarrow$ stress-anisotropy, (iii) $X \rightarrow f_{\geq 2}$, and (iv) $R$ in a one-to-one relationship with $\phi$.

Fig. 2 illustrates the salient features of our model in the SJ regime. Both order parameters are observed to obey a scaling form: $X_{\text{sc}}(R, H) = g_X(H/H_{\text{peak}}(R))$ and $M_{\text{sc}}(R, H) = g_M(H/H_{\text{peak}}(R))$. Numerically, $H_{\text{peak}}(R) \propto (R/R_c - 1)^{1/2}$, and $X_{\text{sc}}$ and $M_{\text{sc}}$ are scaled variables: $x_{\text{sc}} \equiv x_{\text{sc},\Delta_0} = x_{\text{sc},\Delta_0}$. The characteristic peak in

![Fig. 1: (Color Online) (a) Meanfield phase diagram for $\alpha = 4$. The colorbar indicates $M_{\text{peak}}$, the peak value of $M(H)$. The critical point $(\Delta_c, R_c)$ (yellow circle) marks the end point of three transition lines (see text): $R_t(\Delta_0)$ (black), $R_{DST}(\Delta_0)$ (light blue dotted) and $R_{sc}(\Delta_0)$ (white dashed). (b) Scaling collapse of $M_{\text{peak}}(R, \Delta_0)$ for $0.2 (dark blue) \leq \Delta_0 \leq 2$ (brown) with $R_{DST}(\Delta_0)$. (c) $M$ vs $X$ for $\Delta_0 = 0.9$, and $R = 0.4$ (blue), 0.6 (green), 0.8 (orange), 1 (red), and 1.2 (brown). (d) $M$ vs $X$ for $\Delta_0 = 0.2$, $R = 0.3$ (blue), 0.5 (green), 0.6 (orange), 0.8 (red), and 0.9 (brown). Curves with $R > R_c$ are denoted by dashed lines.](image-url)
\( M \) at any \( R \) is consequence of two competing effects of \( H \): increase in alignment of \( S_i = \pm 1 \) and increase in \( X \), the fraction of \( S_i = 0 \). The maximum anisotropy depends on the fraction of \( S_i = 0 \) at \( H = 0 \). Since \( H_{\text{peak}}(R) \rightarrow 0 \) as \( R \rightarrow R_J \), the scaling shown in Fig. 2 (a) and (b) indicates \( X(R, H) \) asymptotes to unity for \( H \approx 0 \) and the spin system reaches the analog of an isotropic jammed state \( (M \approx 0) \) as the disorder is reduced towards \( R_J \): a feature that is consistent with the sharp decrease of \( M_{\text{peak}} \) shown in Fig. 1 b. This suggests identification of \( R_J \) with \( \phi_J \). Remarkably, this mapping leads to scaling forms for experimentally measured stress anisotropy and \( f_{>2} \) during a forward shear run that are identical to those of \( M \) and \( X \), respectively[14].

Another aspect of the zero-temperature dynamics is the appearance of hysteresis loops. In the mean-field approximation, there is no hysteresis for \( \Delta_0 > \Delta_c \). To study hysteresis associated with SJ, we performed zero temperature Monte Carlo dynamics on a \( 64 \times 64 \) square lattice with periodic boundary conditions. The order parameters were averaged over two hundred different realizations of the disorder field. The results, shown in Fig. 2 (d), demonstrate the evolution of hysteresis loops with \( R \). SJ experiments in a range of \( \phi < \phi_J \), exhibit a similar trend of hysteresis loop shapes for symmetric shear cycles[4, 14]. The model predicts that the area of the hysteresis loops increases with \( R \) for \( \phi \) closer to \( \phi_J \), and it would be interesting to test this prediction in experiments.

The SJ experiments exhibit the phenomenon of Reynolds pressure[4]: pressure increasing quadratically with shear strain at small strains, with a Reynolds coefficient that depends only on \( \phi \) and appears to diverge at \( \phi_J \). Very general arguments lead to the quadratic dependence of the pressure on shear strain[22]. If we make the logical assumption that the pressure increase is tied to the increase of \( f_{>2} \), and hence \( X \), then our model provides a natural explanation for the \( \phi \) dependence. The scaling form of \( X(R, H) \) implies that the pressure scales as: \( P(R, H) \propto g_p(H/H_{\text{peak}}(R)) \). If we assume that \( g_p(x) \) increases quadratically or some other power of \( x \) for \( x << 1 \), the divergence of the Reynolds coefficient follows from \( 1/H_{\text{peak}} \propto (R/R_J - 1)^{-1.2} \). The source of the divergence observed in experiments is, therefore, directly related to the rapid rise in the number of contacts with shear strain as \( \phi \) increases towards \( \phi_J \).

Our analysis shows that the diversity of phenomena associated with SJ are unified by the zero-temperature dynamics of a quenched disorder model, and that all features of the SJ transition arise from a competition between the two distinct effects of shear: increase of \( f_{>2} \) and increasing stress anisotropy, with \( \phi \) controlling the maximum anisotropy.

\[ \Delta_0 = 0.9 \] (a) \( X(H) \) and (b) \( M(H) \) for \( 0.2 \) (blue) \( \leq R \leq 1.5 \) (brown) with increments chosen such that \( \Delta_0 \rightarrow 1 \) increases logarithmically between 1 and 10. (c) Plots (see text) of \( g_X(H/H_{\text{peak}}(R)) \) (main ) and \( g_M(H/H_{\text{peak}}(R)) \) (inset). (d) Hysteresis loops obtained from numerical simulations \( (\Delta_0 = 2) \), in the same range of values of \( \Delta_0 \) as the disorder is reduced towards \( \Delta_c \). Remarkably, this mapping leads to scaling form of \( \phi \) provides a natural explanation for the \( \phi \) dependence.

\[ |R - R_DST| \] (a) (b) (c)

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**FIG. 2:** (Color Online) SJ regime \( (\Delta_0 = 0.9) \): (a) \( X(H) \) and (b) \( M(H) \) for \( 0.2 \) (blue) \( \leq R \leq 1.5 \) (brown) with increments chosen such that \( \Delta_0 \rightarrow 1 \) increases logarithmically between 1 and 10. (c) Plots (see text) of \( g_X(H/H_{\text{peak}}(R)) \) (main ) and \( g_M(H/H_{\text{peak}}(R)) \) (inset). (d) Hysteresis loops obtained from numerical simulations \( (\Delta_0 = 2) \), in the same range of values of \( \Delta_0 \) as the disorder is reduced towards \( \Delta_c \).

**FIG. 3:** (Color online) DST regime: Meanfield results at \( \Delta_0 = 0.2 \) for (a) \( M \) vs \( H \) and (b) \( X(H) \) hysteresis loops from cyclic runs for \( 0.09 \) (blue)< \( R \leq 0.6 \) (brown). Dotted (solid) lines are below (above) \( R_{DST} = 0.4 \) (red dashed lines). The blue line is below \( R_m \). (c) Extent of \( X(H) \) hysteresis loops, \( H_+ - H_- \), as a function of \( |R - R_{DST}| \) for \( \Delta_0 = 0.2 \) (square), and \( \Delta_0 = 0.4 \) (circle). The solid black line is a power law with exponent 0.5.

**DST transition** Experiment, theory and simulations of the discontinuous shear thickening transition in dense, athermal suspensions are converging on a scenario where increasing strain rate leads to a proliferation of frictional contacts[6, 7, 9, 10], which ultimately results in a robust force bearing network. For DST, the relevant mapping to our model is, therefore, \( H \rightarrow \) strain rate, and \( M \rightarrow \) stress anisotropy arising from the contact forces.

In sheared suspensions, the fraction of frictional contacts at zero strain rate (\( H \)) is insignificant, and therefore the regime of our model relevant for DST is \( \Delta_0 \ll \Delta_c \). Within the dome bounded by \( R_m(\Delta_0) \) and \( R_t(\Delta_0) \) in Fig. 1 (a), \( M \) decreases and \( X \) increases, monotonically
with $H$ (Fig. 3 (a),(b)). The behavior of $M(H)$ is reminiscent of the sharp decrease in the stress anisotropy at the DST transition[7]. As $R$ is reduced towards $R_m$, the curves sharpen up, as shown in Fig. 3(a),(b). For $R < R_m(\Delta_0)$, $X(H = 0) \approx 1$, and $M(H)$ stays close to zero for all $H$ indicating a nearly isotropic system. In the regime, $R_m \leq R < R_{DST}(\Delta_0)$, the meanfield solutions for $M(H)$ and $X(H)$ exhibit vertical slopes at a characteristic $H(R)$, indicating the appearance of flowcurve hysteresis. Mean field solutions predict hysteresis loops that span a range of $H$ from $H_-$ to $H_+$. Below $R_{DST}$, $H_+ - H_-$ grows as $|R - R_{DST}|^2$ (Fig.3(c)), and $H_-$ approaches zero at $R_m$ with a power law that depends on $\Delta_0$. These observations are in accord with the scaling results of DST[9], based on the premise that proliferation of frictional contacts is the necessary for DST. Our model provides a unifying framework for the scaling relations observed in both DST and SJ transitions. Similar to the SJ transition, our analysis shows that the DST transition is triggered by a rapid rise in $X$. In contrast to the SJ regime, $X(0) \approx 0$ in the DST regime, which leads to sharper transitions and more pronounced hysteresis.

Discussion

We have constructed a driven, disordered, zero-temperature statistical mechanics model, which captures all essential features of shear-induced rigidity transitions in granular materials and dense athermal suspensions. It emphasizes that shear-induced proliferation of contacts is essential for the SJ and DST transitions: simulation or experimental protocols that do not achieve this cannot induce the transitions. Shearing at constant volume, which frustrates dilation[23] is one mechanism for introducing new contacts. Our analysis highlights the distinct roles played by density and driving: density controls the strength of quenched disorder, whereas rigidity emerges as the driving field introduces new contacts. Our model is not capable of describing states where compression of grains is a significant component of the stress tensor, and therefore is applicable only for $\phi < \phi_1$.

The phase diagram of the model unifies the seemingly distinct phenomena of SJ and DST. In the range of $\phi$ where experiments on SJ transitions have been performed[14], $X(0) >> 0$, and the phenomenology corresponds to the $\Delta_0 > \Delta_1$ part of the phase diagram. In contrast, protocols leading to DST have vanishing number of frictional contacts at zero strain rate ($X(0) \approx 0$), and explore the $\Delta_0 < \Delta_1$ region. The phenomenology of SJ and DST differ because the dynamics is qualitatively different in these two regimes of the phase diagram. The physical processes underlying them, however, are the same and encapsulated in the Hamiltonian (Eq. 2). Shearing dense granular materials exhibits intermittent response[24]. In contrast to existing models, our approach focuses on changes in forces and force networks while grain positions establish a disordered, quenched background. We plan to compare the predictions of our model regarding avalanche distributions to experimental and numerical results in the near future.

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[19] See Supplemental material at [url will be inserted by publisher] for meanfield equations, additional figures and movies.