THE TOPOLOGY OF RANDOM REAL HYPERSURFACES AND PERCOLATION

PETER SARNAK

JOINT WORK WITH IGOR WIGMAN.
One reason for studying such functions is that one expects that nodal sets of highly excited states of the quantization of a chaotic Hamiltonian to behave like a random "monochromatic wave".

\[
\begin{align*}
\text{Single variable (Kac, Rice, \ldots)} \\
&f(x) = \sum_{j=0}^{t} a_j x^j, \quad a_j \in \mathbb{R} \\
&V(f) = \{ x : f(x) = 0 \}.
\end{align*}
\]

Topology of \( V(f) \) is the number of zeros \( |V(f)| \).

Let \( W_{1,t} \) be the real vector space of such \( f \)'s.
3

WHAT IS RANDOM?

WE STICK TO CENTERED GAUSSIAN ENSEMBLES ON (FINITE) DIMENSIONAL VECTOR SPACES

\[ \leftrightarrow \text{ GIVING AN INNER PRODUCT.} \]

"NAIVE" ENSEMBLE:

\[ W_{1,t} \quad , \quad (f,g) = \sum_{j=0}^{t} a_j b_j \]

THIS IS THE SAME AS CHOOSING THE COEFFICIENTS \( a_j \) AS STANDARD GAUSSIANS AND INDEPENDENTLY.

* NOT SO NATURAL SINCE IT SINGLES OUT THE POINTS \( \pm 1 \) AS TO WHERE MOST ZEROS ARE LOCATED.
REAL FUBINI STUDY ENSEMBLE (RFS):

TURN $\mathbb{P}^1(\mathbb{R})$ INTO A HOMOGENEOUS SPACE SO THAT ALL POINTS ARE FAVORED EQUALLY.

**Homogenize:**

$$f(x_0, x_1) = \sum_{j=1}^{t} a_j x_0^j x_1^{t-j}$$

$$(g, f) = \int_{\mathbb{R}^2} f(x) g(x) e^{-\frac{1}{2}x^2} \, dx$$

$$= \ast \int_{\mathbb{P}^1(\mathbb{R})} f(x) g(x) \, d\sigma(x)$$

$$= \{x : |x| = 1, \exists \, \pm 1 \}$$

$d\sigma$ are length on the round circle.

In this ensemble $x_0^j x_1^{t-j}$ are not orthogonal; an o.n.b. consists of $\sin n\theta, \cos n\theta$, $0 \leq n \leq t$. 

The Kac-Rice formula gives the expected number of zeros:

$$Z = \vert V(f) \vert$$

<table>
<thead>
<tr>
<th>Ensemble</th>
<th>Degree</th>
<th>$\text{Exp.(Z)}$</th>
<th>$P(Z=0)$</th>
<th>$P(Z=1)$</th>
<th>$P(Z=2)$</th>
<th>$P(Z=3)$</th>
<th>$\ldots$</th>
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<tbody>
<tr>
<td>Naive</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>RFS</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>Naive</td>
<td>2</td>
<td>1.297</td>
<td>0.301</td>
<td>0</td>
<td>0.699</td>
<td>0</td>
<td>0</td>
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<tr>
<td>RFS</td>
<td>2</td>
<td>1.632</td>
<td>0.184</td>
<td>0</td>
<td>0.816</td>
<td>0</td>
<td>0</td>
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<tr>
<td>Naive</td>
<td>3</td>
<td>1.492</td>
<td>0</td>
<td>0.754</td>
<td>0</td>
<td>0.246</td>
<td>0</td>
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<tr>
<td>RFS</td>
<td>3</td>
<td>2.236</td>
<td>0</td>
<td>0.382</td>
<td>0</td>
<td>0.618</td>
<td>0</td>
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</table>

$$t \sim \log t$$

$$t \sim \left(\frac{t(t+2)}{3}\right)^{1/2}$$
WORD ABOUT KAC-RICE:

\( f(x) \in W_{1,t} \) for each \( x \), \( f(x) \) is a centered Gaussian and so determined by its variance.

Define

\[ \text{cov}(x, y) = \text{Exp}_{f}(f(x)f(y)) = K_{t}(x, y) \]

To compute \( |V(f)| \)

\[
\begin{align*}
\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} & \left| \left\{ x : |f(x)| > \varepsilon \right\} \right| \\
& = \sum_{a \in V(f)} \frac{1}{|f'(a)|}
\end{align*}
\]

So

\[ \text{Exp}_{f}(|V(f)|) = \text{Exp}_{f} \left( \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{|f(y)| < \varepsilon} |f'(y)| \, dy \right) \]

Switch orders gives \( \text{Exp}_{f}(|V(f)|) \) in terms of \( K_{t}(x, y) \).

\( \Rightarrow \) Study \( K_{t}(x, y) \) as \( t \to \infty \).
Several variables:

\[ f(x_0, x_1, \ldots, x_n) \text{ homogeneous of degree } t \]

Denote \( W_{n,t} \) the linear space.

For typical \( f \), \( V(f) \) is a compact smooth \( n-1 \) manifold \( \subset \mathbb{P}^n(\mathbb{R}) \)

\[ V(f) = \{ x : f(x) = 0 \} \]

For a Gaussian ensemble, the Kac-Rice formula allows for the explicit computation of the expected values of local quantities:

**E.G.:** \( |V(f)| \) the induced \( (n-1) \) volume of \( V(f) \),

- Euler characteristic,
- \# of critical points of \( f \)

The question of the (global) topology of \( V(f) \) is much more difficult.
Natural Ensembles on $W_{n,t}$

(i) Naive

$$(f,g) = \sum_{\|J\| = t} a_J b_J$$

Where

$$f(x) = \sum_{\|J\| = t} a_J x^J, \quad J = (j_0, \ldots, j_n) \quad \|J\| = \sum j_k$$

(ii) Complex Fubini Study

$$(f,g) = \int f(z) \overline{g(z)} \, d\sigma(z) = \sum_{\|J\| = t} \left( \frac{t^{|J|}}{|J|!} \right) a_J b_J$$

$\mathbb{P}^n(\mathbb{C})$

(iii) Real Fubini Study (RFS)

$$(f,g) = \int f(x) \, g(x) \, d\sigma_1(x)$$

$\mathbb{P}^n(\mathbb{R})$

By the way for $n = 1$

For $\text{FSF(i)}$ above

$$\text{Exp}_f(1_{\{|\Psi|\leq 1\}}) = \sqrt{t} \quad (\text{Kac-Rice})$$
RANDOM BAND LIMITED FUNCTIONS

\[ M \text{ a compact Riemannian (smooth) } \pi\text{-dimensional manifold.} \]

\[ \phi_j, \ j = 0, 1, 2, \ldots \]

An orthonormal basis of eigenfunctions of the Laplacian \( \Delta \)

\[ \Delta \phi_j + \lambda_j \phi_j = 0. \]

Fix \( 0 \leq \alpha \leq 1 \), the \( \alpha \)-band limited ensemble \( E_{M,\alpha}(T) \) is the Gaussian

\[ f(x) = \sum_{\alpha T \leq j \leq T} c_j \phi_j(x) \]

where \( c_j \)'s are iid standard Gaussians.

- If \( \alpha = 1 \) we mean \( t - \eta(T) \leq t_j \leq T \), with \( \eta(T) = o(T) \), \( \eta(T) \to \infty \).
NB: (a) If $(M, ds)$ is $(\mathbb{P}^n(\mathbb{R}), \sigma)$ then the $\phi_j$'s are homogeneous polynomials (spherical harmonics) so $x = 0$ is the real Fubini Study ensemble in this case.

(b) $\alpha = 1$ is monochromatic random wave.

- $V(f) \subset M$ the zero set (nodal set).
- Let $C(f)$ denote the connected components of $V(f)$:
  
  $V(f) = \bigcup_{c \in C(f)} c$

- $M \setminus V(f) = \bigcup_{w \in \nu(f)} u$, $w$ the connected components of "nodal domains".

Our interest is in the topologies of the $c$'s in $C(f)$ and $w$'s in $\nu(f)$.
NODAL PORTRAIT
SUM OF SPHERICAL HARMONICS (RANDOM)
OF DEGREE \( \leq 80 \) \( (\alpha = 0 \ \text{RANDOM}
\ \text{FUBINI-STUDY ENSEMBLE}) \)

A. BARNETT.
Nodal Portrait
Random Spherical Harmonic
of Degree 80 (\( \alpha = 1 \) Model)
A. Barnett
Nazarov and Sodin have introduced some powerful (soft) techniques to study this problem. The following can be deduced from their work.

**Theorem (Nazarov-Sodin 2012):**

There are positive constants $\beta_{n,k}$ depending on $n$ and $k$ only such that

$$|E(f)| \sim \beta_{n,k} T^n \quad \text{as} \quad T \to \infty,$$

for most $f$'s in $E_{M,\alpha}(T)$, i.e. with probability tending to 1 as $T \to \infty$.

So there are many connected components and it makes sense to ask about the distribution of the topologies of $E_{\infty}(f)$, in the space of topological types.
Let $\widetilde{H}(m-1)$ denote the discrete space of diffeomorphism types of connected $(m-1)$ compact manifolds.

Let $\widetilde{B}(n)$ be the space of compact (smooth) diffeomorphism types of $n$ dimensional manifolds with smooth boundary.

For $f$ in $\mathcal{E}_{m,n}(T)$ set

$$M_e(f) := \frac{1}{|e(f)|} \sum_{c \in e(f)} \delta_{t(c)}$$

where $t(c)$ is the topological type of $c$ in $\widetilde{H}(m-1)$. So $M_e(f)$ is a probability measure on $\widetilde{H}(m-1)$ giving the distribution of topologies of $e(f)$.

$$M_u(f) := \frac{1}{|u(f)|} \sum_{w \in u(f)} \delta_{t(w)}$$

where $t(w)$ is the topological type of $w$ in $\widetilde{B}(n)$. 
THEOREM (WIGMAN-5 2014)*

There are probability measures

\( \mu_{e,n,\alpha} \) on \( H(n-1) \) the set of topological types of connected compact \((n-1)\) manifolds that embed in \( \mathbb{R}^3 \)

and

\( \mu_{R,n,\alpha} \) on \( B(n) \) the set of topological types which embed in \( \mathbb{R}^n \)

such that:

(i) For \( \varepsilon > 0 \), \( \text{Prob}_{\mathcal{F} \in \mathcal{E}(n)}(T): D(\mu_{e(f)}, \mu_{e,n,\alpha}) \geq \varepsilon) \to 0 \) as \( T \to \infty \), where \( D = \text{discrepancy} \)

\[
D(\mu, \nu) = \sup_{F \in \mathcal{F}(n-1)} |\mu(F) - \nu(F)|.
\]

And similarly for \( \mu_{R}(f) \) and \( \mu_{R,n,\alpha} \).

(ii) \( \text{Support} \mu_{e,n,\alpha} = H(n-1) \), \( \text{Support} \mu_{R,n,\alpha} = B(n) \).
These give universal laws for the distribution of the topologies of the components of the zero sets of band limited functions, in particular for a random algebraic hypersurface ($\alpha = 0$) and a random monochromatic wave ($\alpha = 1$).

Universal laws for the distribution of the vector of Betti numbers of such nodal sets.

**Note:** Upper and lower bounds for the expected Betti numbers $\beta_j(\Omega(f))$ have been given by Gayet and Welschinger (2013) and Larario and Lunberg (2013), Fyodorov, L-L (2014).
N = 2, $H(\alpha)$ is a circle so nothing to say for $\mu_0$.

$B(2)$: Finitely connected planar domains

$w \in B(2)$, $\tau(w) = \text{Connectivity of } w$

$B(2) \subseteq \mathbb{N}$, $\mu_{N,2,\alpha}$ is a (prob) measure on $\mathbb{N}$.

$\alpha = 0$, random RFS plane oval

$\mu_{N,2,0}$ gives the distribution of nodal domains.

<table>
<thead>
<tr>
<th>$\mu_{N,2,0}$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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<tbody>
<tr>
<td>1.0</td>
<td>0.937</td>
<td>0.027</td>
<td>0.009</td>
<td>0.003</td>
<td>0.002</td>
<td>0.002</td>
<td>0.001</td>
<td>0.001</td>
<td>0.0005</td>
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</table>

The Nazarov-Sodin constant $B_{2,0}$ is such that the random oval is about 4% Harnack (i.e., has 4% of its maximal number of ovals).

M. Năstasescu
$\alpha = 1$, the Random Monochromatic Nodal Domain

<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>9</th>
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<tbody>
<tr>
<td>$\mu_{2m}$</td>
<td>.906</td>
<td>.055</td>
<td>.010</td>
<td>.006</td>
<td>.003</td>
<td>.002</td>
<td>.001</td>
<td>.0008</td>
<td>.0004</td>
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</table>

An analysis of these numbers (up to $m = 100$) shows that for $m$ large, $\mu(\Sigma m^2)$ decays like $m^{-\beta}$ with

$\beta \approx 2.149$ for $\alpha = 1$

$\beta \approx 2.057$ for $\alpha = 0$

These are close to the Fisher constant $\frac{187}{91} = 2.0546$

which governs related quantities in critical percolation (Kleban-Ziff).

So the more complex topologies may follow some universal percolation features!
\( n=3 \): H(2) consists of all orientable surfaces \( S \) which are determined by their genus \( g(S) = 9 \).

\[ H(2) \cong G = \{ 0, 1, 2, \ldots \} \]

\( \mu_{e,3,\alpha} \) is a (prob) measure on \( G \).

By a Kac-Rice type computation (Podkorytov 2001).

\[
\text{Exp}_{\deg t} \left( \chi(V(f)) \right) \sim \begin{cases} 
\frac{t^3}{5^{3/2}}, & \alpha = 1 \\
\frac{t^3}{3^{3/2}}, & \alpha = 0 
\end{cases}
\]

\[ \Rightarrow \quad \text{mean} \left( \mu_{e,3,0} \right) \leq 2 + \frac{1}{3^{3/2} \beta_{2,0}} \\
\quad \text{mean} \left( \mu_{e,3,1} \right) \leq 2 + \frac{1}{5^{3/2} \beta_{3,1}} 
\]
SOME COMMENTS ABOUT THE PROOFS:

STEP 1: SEMI LOCALITY COVARIANCE

\[ \exp_{f \in E_{M, \alpha}(T)} [ f(x) f(y) ] \]

\[ = \sum_{\alpha T \leq t_j \leq T} \phi_j(x) \phi_j(y) = K_\alpha(T; x, y) \]

SPECTRAL PROJECTOR.

AS \( T \to \infty \) IS STUDIED BY PARAMETRIX TO WAVE EQN, FOURIER INTEGRAL OPERATORS (LAX, ...)

SAY \( \text{Vol}(M) = 1 \), THEN:

\[ \frac{K_\alpha(T; x, y)}{\dim E_{M, \alpha}(T)} = \begin{cases} 
B_n, \alpha(T \text{d}(x, y)) + O(T^{-1}) & \text{if } T \text{d}(x, y) \ll 1 \\
O(T^{-1}) & \text{if not}
\end{cases} \]
WHERE

\[ B_{m, \alpha}(w) = B_{m, \alpha}(1w1) = \frac{1}{\sqrt{2\pi m, \alpha}} \int e^{i(w, \xi)} d\xi \]

\[ \mathcal{N}_\alpha = \{ w : \alpha \leq |w| \leq 1 \} , \quad d(x, y) = \text{distance } x \text{ to } y . \]

Following the methods of Nazarov and Sodin we show that our quantities are semi local in \( \text{nbh}'s \) of size \( \sim 1/1^T \) in \( M \) (and otherwise independent).

After scaling \( \Rightarrow \)

Gaussian translation invariant, isotropic, infinite dimensional field \( H_{m, \alpha} \) on \( \mathbb{R}^n \) (replacing \( \varepsilon \) locally).
Let \( \psi_j \) be an orthonormal basis of \( L^2 (\mathcal{M}, \mu) \), set

\[ H_n, \mu : f(x) = \sum_{j=1}^{\infty} c_j \hat{\psi}_j (x) \quad \text{on } \mathbb{R}^n \]

\( c_j \) i.i.d. standard Gaussians.

\textbf{NB:} The typical \( f \) is analytic in \( x \) (thanks to decay in \( \hat{\psi}_j (x) \) for \( x \) in a compact).

The existence of a limiting measures as well as convergence in measure follows from soft ergodic theory (of the action of the translation group \( \mathbb{R}^n \)) in a similar fashion to the Nazarov-Sodin asymptotics for connected components.
The properties of $\mu$'s, namely that they are probability measures and "no escape of topology" and that they charge every atom, require analytic, geometric and topological input.

**E.G.:** Support of $\mu_{x,2,1}$ is all $\mathcal{T}$, reduces to showing that there is an $f \in H_{2,1}$ with nesting any (finite) rooted tree $\langle \cdots \rangle$ producing

$$f(x) = \sum_{j=1}^{L} a_j \cdot e(\langle x, \xi_j \rangle)$$  \hspace{1cm} (1)

Trig poly with $|\xi_j| = 1$

and with $e(x(f))$ given.
START WITH

\[ p(x_1, x_2) = \sin \pi x_1 \sin \pi x_2 \]

\[ V(f): \]

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Figure 6. Scaling plot for $s^n n_s(p)$ (left curves), $s^n n_s(1 - p)$ (right curves), and their sum $s^\tau n_s(p)$ (central curves), vs. $z = (p - p_c)s^\sigma$ with $p_c = 1/2$, $\tau = 187/91$ and $\sigma = 36/91$, for $s = 44, 14,$ and $10$ (top to bottom).

Figure 7. $\ln(P_k)$ (the probability of $k$ connections) vs. $\ln k$. Data from the simulations for site percolation on the triangular lattice (triangles), random lattice from Sarnak and Wigman $\alpha = 0$ (squares), and $\alpha = 1$ (circles).

4.1. The Depth. The average depth grows with $L$ as [7]

$$\text{AverageDepth} \sim \frac{1}{2\sqrt{3\pi}} \ln(L/\epsilon) \approx 0.091888 \ln(L/\epsilon)$$

where $\epsilon$ is the lower cutoff ($\sim$ the lattice spacing). This follows simply from the fact that the number of wrapping clusters on a cylinder of circumference 1 is of the order 1 per unit length, and then a conformal transformation converts this to a series of concentric rings with an exponentially growing spacing with the radius. Because of the small coefficient, the average depth grows very slowly with $L$. The maximum depth with be much greater but should also grow logarithmically with $L$. 

Some comments about the proofs:

Step 1: Semi Locality

Covariance

\[ \text{Exp}_{\phi \in E_{M, \alpha}(T)} \left[ f(x)f(y) \right] = \sum_{\alpha T \leq t_j \leq T} \phi_j(x) \phi_j(y) := K_\alpha(T; x, y) \]

Spectral projector.

As \( T \to \infty \) is studied by parametrix to wave eqn, Fourier integral operators (Lax, ...)

Say \( \text{Vol}(M) = 1 \), then:

\[ \frac{K_\alpha(T; x, y)}{\dim E_{M, \alpha}(T)} = \begin{cases} B_{n, \alpha}(Td(x, y)) + O(T^{-1}) & \text{if } Td(x, y) \ll 1 \\ O(T^{-1}) & \text{if not} \end{cases} \]