Introduction to the Theory of Spin Glasses

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What are Spin Glasses?

- Magnetic systems with **quenched disorder**.
- Competition between ferromagnetic and antiferromagnetic interactions.

Example: CuMn, AuFe, ...

\[ J(r) = J_0 \frac{\cos(2k_F r + \phi_0)}{(k_F r)^3} \]

RKKY Interaction between localized spins
Experimental results: (1) Cusp in the magnetic susceptibility

Susceptibility of CuMn as a function of temperature

Experimental results: (2) Slow dynamics at low temperatures

FIG. 7. Static susceptibilities of CuMn vs temperature for 1.08 and 2.02 at. % Mn. After zero-field cooling ($H < 0.05$ Oe), initial susceptibilities (b) and (d) were taken for increasing temperature in a field of $H = 5.9$ Oe. The susceptibilities (a) and (c) were obtained in the field $H = 5.9$ Oe, which was applied above $T_f$ before cooling the samples. From Nagata et al. (1979).
Frustration
All pair interactions can not be satisfied simultaneously

Frustration leads to a multiplicity of ground states of the spin system

FIG. 41. Classical ground state of a set of four spins in the XY model with interactions $\pm J$ (thick bonds are antiferromagnetic, thin bonds are ferromagnetic). (a) Nonfrustrated plaquette; (b) frustrated plaquette, chirality $\tau=+1$; (c) frustrated plaquette, chirality $\tau=-1$. 
Edwards-Anderson Model


\[ H = - \sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j \quad \sigma_i = \pm 1 \]

Ising spins on a regular lattice
Nearest-neighbor interactions
Quenched disorder

\[ \tilde{P}(\{J_{ij}\}) = \prod_{\langle ij \rangle} P(J_{ij}) \]

\[ P(J_{ij}) = \frac{1}{\sqrt{2\pi} J^2} \exp\left[-\frac{J_{ij}^2}{2J^2}\right] \]

or

\[ P(J_{ij}) = \frac{1}{2} [\delta(J_{ij} + J) + \delta(J_{ij} - J)] \]

\[ [J_{ij}]_{av} = 0, \quad [J_{ij}^2]_{av} = J^2 \]

No ferromagnetic or antiferromagnetic phase is possible
Spin Glass Phase

High-temperature paramagnetic phase
\[ \langle \sigma_i \rangle = 0 \quad M \equiv \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i \rangle = 0 \]

Low-temperature spin glass phase
\[ \langle \sigma_i \rangle \neq 0 \quad M \equiv \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i \rangle = 0 \]
\[ q \equiv \frac{1}{N} \sum_{i=1}^{N} (\langle \sigma_i \rangle)^2 \neq 0 \]

Temporal autocorrelation function
\[ C(t) \equiv \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i(t)\sigma_i(0) \rangle \]
\[ C(t)|_{t\to\infty} = \frac{1}{N} \sum_{i=1}^{N} (\langle \sigma_i \rangle)^2 = q \]

Spin glass transition: “Freezing” of the spins in random orientations
The Replica Method  Disorder-averaged Free Energy

\[ F = N f = -T \left[ \ln Z(\{J_{ij}\}) \right]_{av} \]
\[ = -T \int \prod_{<ij>} dJ_{ij} \tilde{P}(\{J_{ij}\}) \ln Z(\{J_{ij}\}) \]

Mathematical identity: \( \ln(x) = \lim_{n \to 0} \frac{x^n - 1}{n} \)

\[ [\ln Z(\{J_{ij}\})]_{av} = \lim_{n \to 0} \frac{[Z^n(\{J_{ij}\})]_{av} - 1}{n} \]
\[ [Z^n(\{J_{ij}\})]_{av} = [\text{Tr} \{\sigma_i^\alpha\} \exp\left[-\sum_{\alpha=1}^{n} \mathcal{H}(\{\sigma_i^\alpha\}, \{J_{ij}\})/T\right]]_{av} \]
\[ = \text{Tr} \{\sigma_i^\alpha\} \exp\left[-\mathcal{H}_{eff}(\{\sigma_i^\alpha\})/T\right] \]
\[ \mathcal{H}_{eff}(\{\sigma_i^\alpha\}) = -T \ln\left[\int \prod_{<ij>} dJ_{ij} \tilde{P}(\{J_{ij}\}) \right. \]
\[ \times \exp\left[-\sum_{\alpha=1}^{n} \mathcal{H}(\{\sigma_i^\alpha\}, \{J_{ij}\})/T\right] \]
Edwards-Anderson (Spin Glass) Order Parameter

$$q = \left[ \langle \sigma_i \rangle^2 \right]_{av} = \langle \sigma_i^\alpha \sigma_i^\beta \rangle \mathcal{H}_{eff}, \ \alpha \neq \beta$$

The spin glass transition is from the paramagnetic state with $q=0$ to a spin glass state with nonzero $q$ as the temperature is decreased.
Magnetic susceptibility

\[ \chi(T) = \frac{1}{NT} \left[ \sum_{i,j} (\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle) \right]_{av} \]

For spin glasses,

\[ [\langle \sigma_i \sigma_j \rangle]_{av} = 0 \text{ for } i \neq j, \quad = 1 \text{ for } i = j. \]

Also, \[ [\langle \sigma_i \rangle]_{av} = 0 \text{ and } [\langle \sigma_i \rangle^2]_{av} \neq 0 \text{ in the SG phase} \]
The Sherrington-Kirkpatrick Model


Infinite-range (mean field) model of Ising spin glass

\[ \mathcal{H} = -\frac{1}{2} \sum_{i \neq j} J_{ij} \sigma_i \sigma_j = -\sum_{<ij>} J_{ij} \sigma_i \sigma_j. \]

\[ P(J_{ij}) = \sqrt{\frac{N}{2\pi J^2}} \exp \left[ -\frac{NJ_{ij}^2}{2J^2} \right] \quad [J_{ij}]_{av} = 0, \quad [J_{ij}^2]_{av} = J^2/N. \]

\[ [Z^m]_{av} = \text{Tr} \{ \sigma_i^\alpha \} \exp \left[ \frac{\beta^2 J^2}{2N} \sum_{<ij>} \sum_{\alpha,\beta} \sigma_i^\alpha \sigma_i^\beta \sigma_j^\alpha \sigma_j^\beta \right] \]

Hubbard-Stratanovitch Identity:

\[ \exp[\lambda a^2/2] = \sqrt{\frac{\lambda}{2\pi}} \int_{-\infty}^{\infty} dx \exp[-\lambda x^2/2 + \lambda ax]. \]
\[ [Z^n]_{av} = \exp \left[ \frac{\beta^2 J^2 n N}{4} \right] \int_{-\infty}^{\infty} \prod_{\alpha < \beta} \sqrt{\frac{N}{2\pi}} e^{\beta J d q_{\alpha \beta}} \]

\[ \times \exp \left[ -\frac{N \beta^2 J^2}{2} \sum_{\alpha < \beta} q_{\alpha \beta}^2 + N \ln \text{Tr} \{ \sigma^\alpha \} e^L(\{ q_{\alpha \beta} \}, \{ \sigma^\alpha \}) \right] \]

where \( L(\{ q_{\alpha \beta} \}, \{ \sigma^\alpha \}) \equiv \beta^2 J^2 \sum_{\alpha < \beta} q_{\alpha \beta} \sigma^\alpha \sigma^\beta \)

\[ -\beta f = \lim_{n \to 0} \left[ \frac{\beta^2 J^2}{4} \left( 1 - \frac{1}{n} \sum_{\alpha \neq \beta} q_{\alpha \beta}^2 \right) + \frac{1}{n} \ln \text{Tr} e^L \right] \]

\( q_{\alpha \beta} \) are to be determined from \( \frac{\partial f}{\partial q_{\alpha \beta}} = 0 \)
Replica Symmetry: \( q_{\alpha \beta} = q \) for all \( \alpha \neq \beta \)

Self-consistency equation:

\[
q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \exp(-z^2/2) \tanh^2(\beta J \sqrt{q} z)
\]

\( q \neq 0 \) for \( T < T_c = J \)

Continuous spin glass transition at \( T=J \)
Replica Symmetry Breaking

The replica symmetric solution has unphysical properties for $T < J$.

**Instability of the replica symmetric solution**

$$-\beta f = \lim_{n \to 0} \left[ \frac{\beta^2 J^2}{4} \left(1 - \frac{1}{n} \sum_{\alpha \neq \beta} q_{\alpha \beta}^2 \right) + \frac{1}{n} \ln \text{Tr} e^L \right]$$

**Fluctuations:** $q_{\alpha \beta} = q_0 + \delta q_{\alpha \beta}$

$$\beta f = \beta f(q_0) + \lim_{n \to 0} \frac{1}{2n} \sum_{\alpha < \beta, \gamma < \delta} \mathcal{R}^{\alpha \beta, \gamma \delta} \delta q_{\alpha \beta} \delta q_{\gamma \delta} + \cdots$$

All eigenvalues of $\mathcal{R}$ must be $\geq 0$ for stability and physically meaningful behavior.

This condition is not satisfied for $T < J$. 
FIG. 48. Plot of the Almeida-Thouless (AT) line for the SK model with $J_0 = 0$. To the right of the line the SK solution with a single order parameter is correct, while to the left of the line the Parisi solution is believed exact. The Parisi solution represents the many-valley structure of phase space and nonergodic behavior. The AT line, therefore, signals the onset of irreversibility.

The Parisi Solution  

Repeat this procedure K times:  
K-step replica symmetry breaking

\[ m_1, m_2, \ldots, m_K; \quad m_0 \geq m_i \geq 1. \]

\[ q(m_0), q(m_1), \ldots, q(m_K) \]
The Parisi Solution (contd.)

\[ K \to \infty : \; m_i \to x, \; 0 \leq x \leq 1, \; q(m_i) \to q(x) \]

\( q(x) \): Order parameter function

Spin glass order parameter:

\[ q = [\langle \sigma_i \rangle^2]_{av} = \int_0^1 q(x) \, dx \]

q(x) at a temperature slightly below the critical temperature
Thouless-Anderson-Palmer Equations


Free energy of the S-K model for a given set of interaction parameters

\[
F = -\frac{1}{2} \sum_{i \neq j} J_{ij} m_i m_j \\
+ \frac{T}{2} \sum_{i} [(1+m_i) \ln\{(1+m_i)/2\} + (1-m_i) \ln\{(1-m_i)/2\}] \\
- \frac{1}{4T} \sum_{i \neq j} J_{ij}^2 (1 - m_i^2)(1 - m_j^2) \quad \text{Onsager Reaction term}
\]

\[
\frac{\partial F}{\partial m_i} = 0 \rightarrow m_i = \tanh[\beta \sum_j J_{ij} m_j - \beta^2 \sum_j J_{ij}^2 (1-m_j^2) m_i]
\]

Local field at site i:

\[
\sum_{j} J_{ij} (m_j - \chi_{jj} J_{ij} m_i) = \sum_{j} J_{ij} m_j - \sum_{j} J_{ij}^2 \beta (1-m_j^2) m_i
\]
TAP Equations (contd.)

Only one solution of the TAP equations, \( m_i = 0 \) for all \( i \), for \( T > J \).

Many solutions with nonzero \( \{m_i\} \) for \( T < J \).

Number of minima with the lowest free energy per spin is not exponential in \( N \).
Free energy barriers between different minima diverge in the thermodynamic limit.

Complex Free Energy Landscape
Physical interpretation of RSB

Large number of “valleys” [“pure states”, “ergodic components”] at temperatures lower than the critical temperature.

\[ P^{(\alpha)}: \text{Probability of the system being in valley } \alpha \]

\[ \langle \sigma_i \rangle = \sum_\alpha P^{(\alpha)} m_i^{(\alpha)} \quad \text{[Average over all valleys]} \]

\[ \frac{1}{N} \sum_i \langle \sigma_i \rangle^2 = \frac{1}{N} \sum_{i=1}^N \sum_{\alpha\beta} P^{(\alpha)} P^{(\beta)} m_i^{(\alpha)} m_i^{(\beta)} \]

Define overlap between valleys \( \alpha \) and \( \beta \),

\[ q_{\alpha\beta} = \frac{1}{N} \sum_{i=1}^N m_i^{(\alpha)} m_i^{(\beta)} \]

Distribution of the overlap:

\[ P(q) = \sum_{\alpha\beta} P^{(\alpha)} P^{(\beta)} \delta(q - q_{\alpha\beta}) \]

Then

\[ \frac{1}{N} \sum_i \langle \sigma_i \rangle^2 = \int_0^1 q P(q) dq \]
Physical interpretation of RSB (contd.)

\[ q = [\langle \sigma_i \rangle^2]_{av} = \int_0^1 q(x) dx = \int q \frac{dx}{dq} dq \]

\[ P(q) = \frac{dx}{dq} \]

Parisi function \( q(x) \) describes the distribution of overlaps between different free-energy minima.

\[ q_{EA} = \frac{1}{N} \sum_i \sum_{\alpha} P^{(\alpha)}[m_i^{(\alpha)}]^2 = q(x = 1) \]

These predictions have been confirmed from simulations.

Correctness of the RSB solution has been established from more rigorous analysis.