Hyperbolicity in Cube complexes: an Introduction to CAT(0) Geometry

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Classical Hyperbolic Space

CAT(0) Spaces

Cube Complexes

Advantages of CAT(0) geometry
Consider the unit disk in $\mathbb{C}$, with a metric such that the distance between two points is inversely proportional to their distance from the center.
History of Hyperbolic geometry

Mathematicians began studying hyperbolic space in the early nineteenth century when they realized it was possible to construct a logically consistent space that contradicted Euclid’s fifth postulate. In hyperbolic spaces, there are infinitely many lines which are parallel to a given line and which pass though a given point.
Curvature

Another way to think about hyperbolic space is as the Riemannian manifold with constant Gaussian curvature equal to $-1$.

A surface has negative curvature at a saddle point, so hyperbolic space looks like a saddle point at every point.
Importance in Group Theory

- The group of isometries of the Poincaré disk is the Lie group $PSL_2(\mathbb{R})$, so studying hyperbolic geometry can give us information about this group and other related Lie groups.
- Hyperbolic geometry is also used to study surface groups, that is fundamental groups of surfaces of genus at least 2.
Goal

Our goal is to find some spaces with many of the same properties of hyperbolic space that we can use to study a broader class of groups.
An Property of Hyperbolic Space

Triangles in hyperbolic space are thinner! That is the sum of their internal angles is less than $180^\circ$.

You can even have triangles whose internal angle sum is zero. Triangles in hyperbolic space also have bounded area.

\[ \text{Area} = \pi - \text{Angle sum} \]
We begin with a metric space $X$, such that every pair of points $x, y$ is joined by a path $\gamma : [0, \ell] \to X$, where $d(x, y) = \ell$.

This path is called a geodesic, and in general it need not be unique.
Comparison triangles

Suppose we have three points $x, y, z$ in a geodesic metric space. We can construct geodesics between each pair of these points to get a triangle.

We can then consider a triangle in Euclidean space with the same side lengths. This is called the comparison triangle.
**CAT(0) Property**

A geodesic metric space $X$ is CAT(0) if for every set of three points $x, y, z$ in $X$, the triangle formed by $x, y, z$ and every pair of points $a, b$ on this triangle, the distance between $a$ and $b$ is less than or equal to the distance between the corresponding points on the comparison triangle.

This must happen for every triangle in the space!
Examples

- Hyperbolic space
- Euclidean space
- Trees (with metric where each edge has length 1)
- Non Example: Sphere
- Non Example: Anything with nontrivial fundamental group, like a circle or a torus.
Properties of CAT(0) spaces

- Contractible
- Any pair of points $x,y$ has a unique geodesic from $x$ to $y$.
- Any path that is locally geodesic at every point is a global geodesic.
- Projection is well defined.

These properties mean that we can study CAT(0) spaces using many of the same techniques that worked for Euclidean and Hyperbolic spaces.
A cube complex is a collection of cubes in various dimensions glued together so that one dimensional cubes (edges) are glued to one dimensional cubes, two dimensional cubes (squares) are glued to two dimensional cubes and so one.

This is similar to a simplicial complex.
Examples of Cube Complexes in Applications

The space of phylogenetic trees (Billera, Holmes, Vogtmann)
Examples of Cube Complexes in Applications

Configuration space of robot arms (Ardila, Baker, Yatchak, 2014)
A cube complex is a CAT(0) space if it is simply connected and if the link of every vertex is flag.

The link of a vertex is the boundary of a small ball around the vertex. The link is flag if it is “filled in”.

\[
\text{link}(v)
\]
What advantage does this give us over classical hyperbolic space?

- We retain many of the characteristics of classical hyperbolic space and can use many of the same techniques.
- CAT(0) cube complexes encompass a huge variety of spaces with many different properties.
- CAT(0) cube complexes are fairly easy to construct and their geodesics are fairly easy to compute.
What does this tell us about groups

A group which acts geometrically on a CAT(0) space is called a CAT(0) group. CAT(0) groups have many special properties including being biautomatic with solvable word and conjugacy problems.
Examples of CAT(0) groups

- Finite groups
- Free groups
- Groups that act geometrically on classical hyperbolic space
- Coxeter groups (Moussong, 1988)
- Right angled Artin groups (Salvetti, 1987)
- Small cancellation groups (Brady and McCammond, 2011)
- **Conjectured**: Braid groups and other Artin groups
Thank you!

Further Reading