Calculus Placement Exam   Brandeis University

Instructions. This is a multiple choice exam. There are five possible answers to each question and only one answer is correct. If you do not know the answer to a question, you should **not** guess.

Take the exam without referring to any books or notes. **Do not use a calculator.** There is no time limit for the exam. If you didn't study calculus in high school, take Part A of the exam only. If you have studied calculus, take Part A and then do as much of Parts B and C as you can.

PART A

1. Simplify $(-2x^3)^4$.

(a) $-8x^{12}$ (b) $-16x^7$ (c) $-16x^{12}$
(d) $16x^7$ (e) $16x^{12}$

2. Simplify $\dfrac{x^{-5}y^2z}{x^{-3}yz}$. Note: It's OK to have a negative exponent in your answer.

(a) $x^{-2}y$ (b) $x^{-2}yz$ (c) $x^{-8}y^3z^2$
(d) $x^{15}y^3z$ (e) $x^{-2}z^2$

3. Simplify $\dfrac{1}{x+1} + \dfrac{4}{x-3}$.

(a) $\dfrac{1}{x^2-2x-3}$ (b) $5x+1$ (c) $\dfrac{5x}{x^2-2x-3}$
(d) $\dfrac{5x+1}{x^2-2x-3}$ (e) none of the above

4. Factor $2x^2+7x-4$.

(a) $(2x-1)(x-4)$ (b) $(2x+1)(x-4)$ (c) $(2x+2)(x-2)$
(d) $(2x+4)(x-1)$ (e) none of the above

5. Simplify $\dfrac{\frac{1}{x} - \frac{1}{x+h}}{h}$.

(a) 1 (b) -1 (c) $\dfrac{1}{x(x+h)}$
(d) $\dfrac{h^2}{x(x+h)}$ (e) $x(x+h)$

6. Evaluate $16^{-\frac{3}{4}}$.

(a) -12 (b) 8 (c) -8
(d) $\frac{1}{8}$ (e) $-\frac{1}{8}$

7. Find the solution(s) of the equation $x^2 + 5x + 1 = 0$.

(a) -2 and -8 (b) -1 and -6 (c) $\dfrac{-5 \pm \sqrt{21}}{2}$

(d) $-5 \pm \dfrac{\sqrt{21}}{2}$ (e) no real solutions

8. Find the solution(s) of the equation $\dfrac{x+3}{x^2 - 16} = 0$.

(a) -3, 0 (b) -4, 4 (c) -3, 4, -4

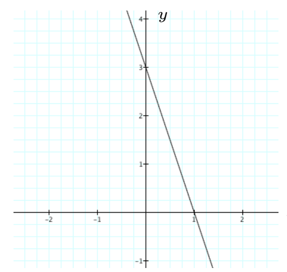
(d) -3 (e) no real solutions

9. Find the equation of the line through the points $(-4, 3)$ and $(2, 0)$.

(a) $y = -2x$ (b) $y = -2x + 2$ (c) $y = -\frac{1}{2}x + 1$

(d) $y = -\frac{1}{2}x + 2$ (e) $y = -\frac{1}{2}x$

10. What is the slope of the line shown here?



(a) 3 (b) -3 (c) $\frac{1}{3}$

(d) $-\frac{1}{3}$ (e) -1

11. Which of the graphs shown below best represents the graph of $y = -(x-1)^2 + 2$?

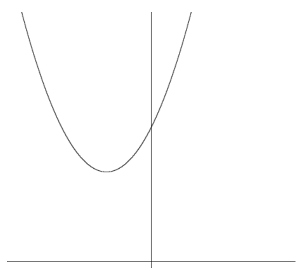
(a) Graph 1

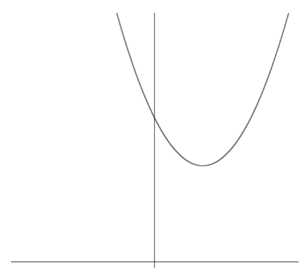
(b) Graph 2

(c) Graph 3

(d) Graph 4

(e) Graph 5


Graph 1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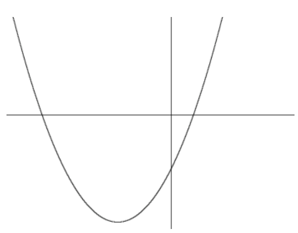
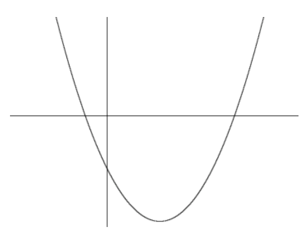
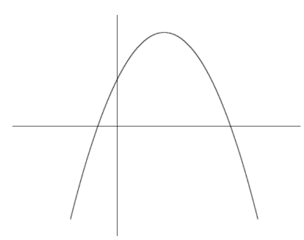

Graph 3


Graph 4


Graph 5

12. Let $f(x) = \begin{cases} x^2 - 5, & \text{if } x \le -1 \\ x + 4, & \text{if } -1 < x < 2 \\ 3, & \text{if } x \ge 2 \end{cases}$. Find $f(2)$.

(a) 3 (b) 6 (c) -1

(d) 6 and 3 (e) none of the above

13. Let $f(x) = \dfrac{1}{x^3 - 1}$ and $g(x) = 7x - 11$. Find $f(g(2))$.

(a) $\frac{3}{7}$ (b) $\frac{1}{26}$ (c) $\frac{1}{8}$
(d) -10 (e) none of the above

14. Let $f(x) = 3x^2 - x$. Find $f(x+h)$ and simplify.

(a) $3x^2 + 3h^2 - x - h$ (b) $3x^2 + 6xh + 3h^2 - x - h$ (c) $3x^2 + 3h^2 - x + h$
(d) $3x^2 + h^2 - x$ (e) $3x^2 + 3h^2 - x$

15. Suppose that the angle θ has measure $135°$. What is the radian measure of θ?

(a) $\frac{3\pi}{4}$ radians (b) $\frac{\pi}{4}$ radians (c) $\frac{2\pi}{3}$ radians
(d) π radians (e) none of the above

16. Let θ be an angle. Which of the following statements are true?

I. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ II. $\csc\theta = \dfrac{1}{\sin\theta}$ III. $\sec\theta = \dfrac{1}{\cos\theta}$ IV. $\cot\theta = \dfrac{1}{\tan\theta}$

(a) only I is true (b) only I and IV are true (c) only II and III are true
(d) none of them is true (e) all four statements are true

17. Which of the following statements are true?

I. $\sin\frac{\pi}{4} = \frac{1}{2}$ II. $\cos\frac{2\pi}{3} = -\frac{1}{2}$ III. $\tan\pi = 0$

(a) only III is true (b) only I and III are true (c) only II and III are true
(d) none of them is true (e) all three statements are true

18. How many solutions does the equation $2\sin\theta = -1$ have if $0 \le \theta \le 2\pi$?

(a) two solutions (b) one solution (c) four solutions
(d) eight solutions (e) no solutions

19. Evaluate $\log_2 \frac{1}{8}$.

(a) 3 (b) -3 (c) $\frac{1}{4}$
(d) -2 (e) none of the above

20. Write the expression $3\ln x - \ln(x+1)$ as a single logarithm.

(a) $\ln\big(x^3(x+1)\big)$ (b) $\ln\big(3x(x+1)\big)$ (c) $\ln\Big[\Big(\dfrac{x}{x+1}\Big)^3\Big]$

(d) $\ln\Big(\dfrac{x^3}{x+1}\Big)$ (e) $\ln\Big(\dfrac{3x}{x+1}\Big)$

21. Find the solution(s) to the equation $\log x = 0$.

(a) 0 (b) 1 (c) 10
(d) $\frac{1}{10}$ (e) no real solutions

22. Evaluate $\ln(\sqrt[3]{e})$.

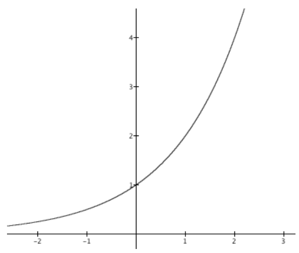
(a) can't be evaluated (b) $\sqrt[3]{e}$ (c) e
(d) 3 (e) $\frac{1}{3}$

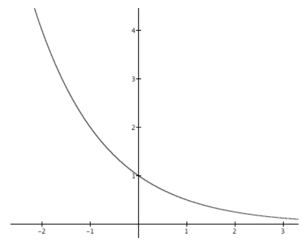
23. Find the solution(s) to the equation $\ln(x+2) = 3$.

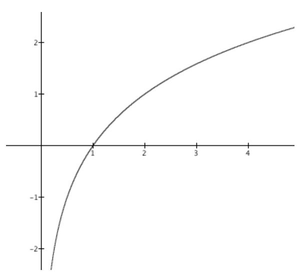
(a) 1 (b) $3 - \ln 2$ (c) $e^{3-\ln 2}$
(d) $e^3 - 2$ (e) -2

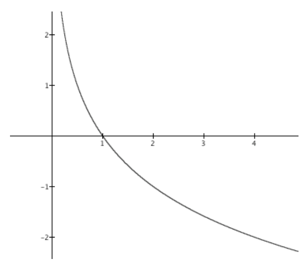
24. Which of the graphs shown below best represents the graph of $y = \log_2 x$?

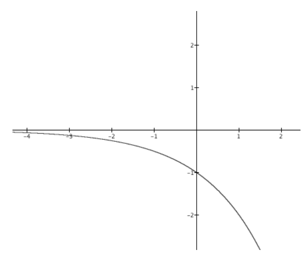
(a) Graph 1
(b) Graph 2
(c) Graph 3
(d) Graph 4
(e) Graph 5


Graph 1


Graph 2


Graph 3


Graph 4


Graph 5

PART B

1. Find $\lim_{x \to 2} \dfrac{x-2}{x^2 - x - 2}$.

(a) 0 (b) 3 (c) $\frac{1}{2}$
(d) $\frac{1}{3}$ (e) doesn't exist

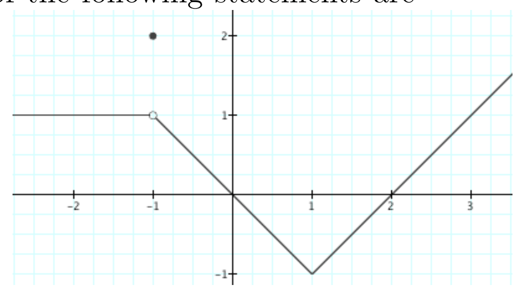
2. Find $\lim_{x \to +\infty} \dfrac{x^3 - 5}{4x^3 + x + 1}$.

(a) $\frac{3}{4}$ (b) $\frac{1}{4}$ (c) 1
(d) 0 (e) $+\infty$

3. Find $\lim\limits_{x\to-\infty} e^x$.

(a) 0 (b) e (c) 1
(d) $+\infty$ (e) $-\infty$

4. The graph of a function $g(x)$ is shown below. Which of the following statements are true of $g(x)$?



I. $\lim\limits_{x\to-1} g(x) = 2$ II. $\lim\limits_{x\to-1} g(x) = 1$
III. $\lim\limits_{x\to-1} g(x)$ does not exist IV. $\lim\limits_{x\to1} g(x) = -1$
V. $\lim\limits_{x\to1} g(x)$ does not exist

(a) only I and IV are true (b) only I and V are true (c) only II and IV are true
(d) only II and V are true (e) only III and V are true

5. $f'(x)$ can be defined as:

(a) $\lim\limits_{\Delta x\to0} \dfrac{f(x+\Delta x)-f(x)}{x}$ (b) $\lim\limits_{x\to0} \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ (c) $\lim\limits_{x\to\Delta x} \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

(d) $\lim\limits_{\Delta x\to0} \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ (e) none of the above

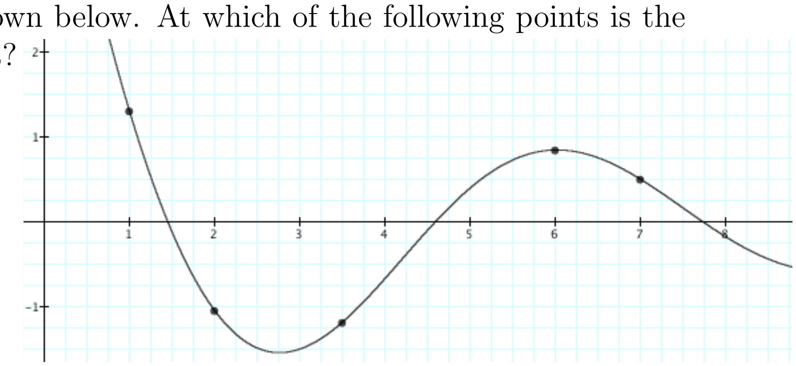
6. Consider again the function $g(x)$ whose graph is shown in problem 4. Which of the following statements are true of $g(x)$?

I. g is continuous at $x = -1$ II. g is continuous at $x = 1$
III. g is differentiable at $x = -1$ IV. g is differentiable at $x = 0$
V. g is differentiable at $x = 1$

(a) only I and IV are true (b) only II and IV are true (c) only I and V are true
(d) only II and V are true (e) only III and V are true

7. The graph of a function $f(x)$ is shown below. At which of the following points is the value of the derivative $f'(x)$ biggest?

(a) at $x = 1$
(b) at $x = 2$
(c) at $x = 3.5$
(d) at $x = 6$
(e) at $x = 7$



8. Consider again the function $f(x)$ whose graph is shown in problem 7. At which points is the second derivative $f''(x)$ negative?

(a) at $x = 2$ and $x = 3.5$ (b) at $x = 1$, $x = 2$ and $x = 3.5$ (c) at $x = 6$ only
(d) at $x = 7$ only (e) at $x = 6$ and $x = 7$

9. Let $f(x) = \ln x \cdot \cos x$. Find $f'(x)$.

(a) $\dfrac{1}{\cos x} \cdot (-\sin x)$ (b) $\dfrac{1}{x} \cdot (-\sin x)$ (c) $\dfrac{1}{x} \cos x - \ln x \sin x$

(d) $\dfrac{1}{x \cos x}(\cos x - x \sin x)$ (e) none of the above

10. Let $y = 4e^{\tan x}$. Find $\dfrac{dy}{dx}$.

(a) $4e^{\tan x} \cdot \sec^2 x$ (b) $4e^{\tan x} \cdot \dfrac{1}{1+x^2}$ (c) $4e^{\tan x}$

(d) $4e^{\tan x} \cdot \cot x$ (e) $4 \sec^2 x$

11. Let $f(x) = \sin^{-1} x$. Find $f'(0)$.

(a) π (b) 1 (c) $\frac{1}{2}$

(d) 0 (e) none of the above

12. The equation of the line tangent to the graph of $f(x) = x^2 + 5x$ at the point with x-coordinate $x = 2$ is:

(a) $y = 9x - 14$ (b) $y = 9x$ (c) $y = 9x - 4$

(d) $y = -\frac{1}{9}x - \frac{2}{9}$ (e) none of the above

13. Let $f(x) = x^3 - 3x$. Which of the following statements are true?

I. $f(x)$ has local maxima at both $x = -1$ and $x = 1$.

II. $f(x)$ has a local minimum at $x = 1$ and an inflection point at $x = 0$.

III. $f(x)$ has both a local minimum and an inflection point at $x = 0$.

(a) only I is true (b) only II is true (c) only III is true

(d) only I and III are true (e) none of the statements is true

14. A commercial nursery has 1000 yards of fencing which the owners plan to use to enclose as large a rectangular garden as possible. The garden will be bounded on one side by a barn, so no fencing is needed on that side. How large will the garden be (in square yards)?

(a) 125,000 sq yds (b) 250,000 sq yds (c) 111,088.89 sq yds

(d) 62,500 sq yds (e) none of the above

15. The width of a rectangle is increasing at a rate of 2 cm/sec, and its length is increasing at a rate of 3 cm/sec. At what rate is the area of the rectangle increasing when its width is 4 cm and its length is 5 cm?

(a) 31 cm^2/sec (b) 23 cm^2/sec (c) 5 cm^2/sec

(d) 22 cm^2/sec (e) none of the above

16. A rock is dropped from a height of 400 feet and falls toward the earth in a straight line; t seconds after it is dropped, it has fallen a distance of $s(t) = 16t^2$ feet. At what speed is the rock traveling when it hits the ground?

(a) 20 ft/sec (b) 32 ft/sec (c) 640 ft/sec
(d) 320 ft/sec (e) 160 ft/sec

PART C

1. Which of the following gives the area between the curves $y = x^2$ and $y = 2x$ over the interval $[-2, 2]$?

(a) $\int_{-2}^{2} (x^2 - 2x)\,dx$ (b) $\int_{-2}^{2} (2x - x^2)\,dx$ (c) $\int_{-2}^{0} (x^2 - 2x)\,dx + \int_{0}^{2} (2x - x^2)\,dx$
(d) $\left| \int_{-2}^{2} (x^2 - 2x)\,dx \right|$ (e) none of the above

2. Suppose that $f(x)$ is a continuous function with the following properties: $f''(x) = \cos x$, $f'(\pi) = 2$ and $f(0) = 4$. What is $f(\pi)$?

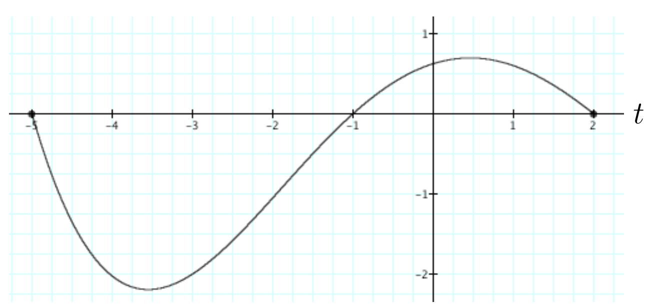
(a) 2 (b) 2π (c) $\pi + 2$
(d) $6 + 2\pi$ (e) 0

3. Suppose that the function $f(x)$ is defined by $f(x) = \int_{1}^{x} \frac{e^t}{t}\,dt$. Find $f'(x)$.

(a) $e^x \ln x$ (b) $e^x \ln x - \frac{e^x}{x}$ (c) $\frac{e^x}{x}$
(d) $\frac{e^x}{x} - e$ (e) the integral can't be computed, so it's impossible to give the answer

4. Let $F(x) = \int_{0}^{x} f(t)\,dt$, where $f(t)$ is the function shown below. Which of the following statements are true?

I. $F(-2) > F(-4)$ II. $F(2) > F(1)$
III. $F(0) > 0$ IV. $F(-1) = 0$



(a) only I is true (b) only II is true (c) only III is true
(d) only I and II are true (e) only II and IV are true

5. Suppose that $f(x) = \dfrac{x}{x^2+1}$. Find $\displaystyle\int_0^2 f'(x)\,dx$.

(a) $\frac{2}{5}$ (b) $-\frac{28}{25}$ (c) $\frac{28}{25}$
(d) 0 (e) none of the above

6. Which of the following statements about indefinite integrals are true?

I. $\displaystyle\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$ II. $\displaystyle\int f(x)g(x)\,dx = \int f(x)\,dx \cdot \int g(x)\,dx$

III. $\displaystyle\int f'(g(x))g'(x)\,dx = f(g(x)) + C$ IV. $\displaystyle\int [f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$

(a) only I and II are true (b) only I and III are true (c) only I and IV are true
(d) only I, II and IV are true (e) only I, III and IV are true

7. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $y = x$ over the interval $[0, 1]$ around the x-axis.

(a) $\displaystyle\int_0^1 \pi(x^2 - x^4)\,dx$ (b) $\displaystyle\int_0^1 \pi(x - x^2)^2\,dx$ (c) $\displaystyle\int_0^1 \pi(x^2 - x^4)^2\,dx$

(d) $\displaystyle\int_0^1 \pi(\sqrt{y} - y)\,dy$ (e) $\displaystyle\int_0^1 \pi(y - y^2)\,dy$

8. The integral $\displaystyle\int \frac{1}{x \ln x}\,dx$ can be found by

(a) making the substitution $u = \ln x$
(b) making the substitution $u = \frac{1}{x}$
(c) using integration by parts, with $u = \ln x$ and $dv = x$
(d) taking the reciprocal of $\displaystyle\int x \ln x\,dx$
(e) none of the above

9. The integral $\displaystyle\int x \sin x\,dx$ can be found by

(a) making the substitution $u = x$
(b) making the substitution $u = \sin x$
(c) using integration by parts, with $u = \sin x$ and $dv = x\,dx$
(d) using integration by parts, with $u = x$ and $dv = \sin x\,dx$
(e) none of the above

10. Find $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx$

(a) $\ln 2$ (b) 1 (c) $\frac{\pi}{12}$
(d) $\frac{\pi}{4}$ (e) 0

11. Find $\lim_{x\to 0} \dfrac{x - \sin x}{x^3}$.

(a) 0 (b) 1 (c) $\frac{1}{6}$
(d) $\frac{1}{3}$ (e) does not exist

12. Find $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$.

(a) 1 (b) 2 (c) 10
(d) 20 (e) the integral diverges

13. Which of the following improper integrals converge to a finite value?

$$\text{(I)} \int_1^\infty e^{-x}\,dx \qquad \text{(II)} \int_{-\infty}^\infty x^3\,dx \qquad \text{(III)} \int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$$

(a) I only (b) III only (c) I and II only
(d) I and III only (e) all of them

14. The second order Taylor polynomial at $x = 0$ for $f(x) = e^{-x}$ is

(a) $1 - \dfrac{x^2}{2}$ (b) $1 + \dfrac{x^2}{2}$ (c) $1 + x + \dfrac{x^2}{2}$

(d) $1 - x + x^2$ (e) $1 - x + \dfrac{x^2}{2}$

15. Which of the following series converge?

$$\text{(I)} \sum_{n=1}^\infty \frac{1}{n^2} \qquad \text{(II)} \sum_{n=1}^\infty \frac{1}{n} \qquad \text{(III)} \sum_{n=1}^\infty \frac{n}{2^n}$$

(a) (I) only (b) (III) only (c) (I) and (II) only
(d) (I) and (III) only (e) all of them

16. The radius of convergence of the power series $\displaystyle\sum_{n=0}^\infty x^n$ is

(a) 0 (b) 1 (c) 2
(d) 3 (e) ∞

Math 20a Placement Exam (Part D)

1: Consider the linear system

$$\begin{array}{rrrcr} 3x & +y & & = & 1 \\ & -2y & -6z & = & -8 \\ -2x & & +2z & = & 2 \end{array}$$

The solution set to this system in vector form is given by:

(a) $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} z \right\}$

(b) $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} z \right\}$

(c) $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} z \right\}$

2: Compute the following determinants.

(a) $\begin{pmatrix} 6 & 1 \\ 2 & 3 \end{pmatrix}$

i. 20 ii. 18 iii. 16 iv. 12

(b) $\begin{pmatrix} 3 & 1 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix}$

i. 3 ii. -6 iii. 0 iv. 6

3: Let S be the plane given in parametric form by

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} z \;\middle|\; y, z \in \mathbb{R} \right\}.$$

Does the vector $\begin{pmatrix} -4 \\ 1 \\ 4 \end{pmatrix}$ lie on the plane S?

(a) Yes (b) No

Over

4: Find the dot product and cross product of the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 3\mathbf{k}$.

(a) $\mathbf{a} \cdot \mathbf{b} =$

i. 10 ii. $4\mathbf{i} - 2\mathbf{j} + 1\mathbf{k}$ iii. -2 iv. $(6, 5, 8)$

(b) $\mathbf{a} \times \mathbf{b} =$

i. $-6\mathbf{i} - 5\mathbf{j} - 8\mathbf{k}$ ii. $(6, 5, 8)$ iii. 10 iv. -2

5: Let $\mathbf{v} = \langle 1, \sqrt{3} \rangle$ and $\mathbf{w} = \langle 3, \sqrt{3} \rangle$.

(a) The length of $\mathbf{v}$, $\|\mathbf{v}\| =$

i. 4 ii. $\sqrt{4}$

(b) Compute the dot product $\mathbf{v} \cdot \mathbf{w}$.

i. 10 ii. 6

(c) Compute the interior angle between $\mathbf{v}$ and $\mathbf{w}$.

i. $\pi/6$ ii. $\pi/4$ iii. $\pi/3$ iv. $\pi/2$

6: Write the parametric equation of the line passing through $P = (5, 0, -2)$ and $Q = (1, 1, 7)$.

(a) $\begin{cases} x = 1 + 4t \\ y = 1 - t \\ z = 7 - 9t \end{cases}$ (b) $\begin{cases} x = 5 - 2t \\ y = \frac{t}{2} \\ z = -2 + \frac{9}{2}t \end{cases}$

(c) BOTH OF THE ABOVE (d) NONE OF THE ABOVE

7: Find the equation of the plane containing the three points $P = (-1, 2, 3), Q = (0, 4, 1)$, and $R = (-1, 3, 2)$.

(a) $-x + y - z = 0$ (b) $y + z = 5$

8: For which value(s) of k are the vectors $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ k \\ 2 \end{pmatrix}$ perpendicular?

(a) 7 (b) 0 (c) 7 & 0 (d) -2

9: (a) Find the orthogonal projection of $\langle 4, 1, 0 \rangle$ onto the plane $2x - 2y + 2z = 0$.

i. $\langle 2, 2, 0 \rangle$ ii. $\langle 1, -1, 1 \rangle$ iii. $\langle 3, 2, -1 \rangle$

(b) Find the point on the plane $x - y + z = 0$ which is closest to the point $P = (4, 1, 0)$.

i. $(2, 2, 0)$ ii. $(1, -1, 1)$ iii. $(3, 2, -1)$

(c) Compute the distance between $P = (4, 1, 0)$ and the plane $x - y + z = 0$.

i. 1 ii. $\sqrt{17}$ iii. $\sqrt{3}$