Introduction to the Theory of Spin Glasses

Chandan Dasgupta
Department of Physics
Indian Institute of Science
cdgupta@physics.iisc.ernet.in

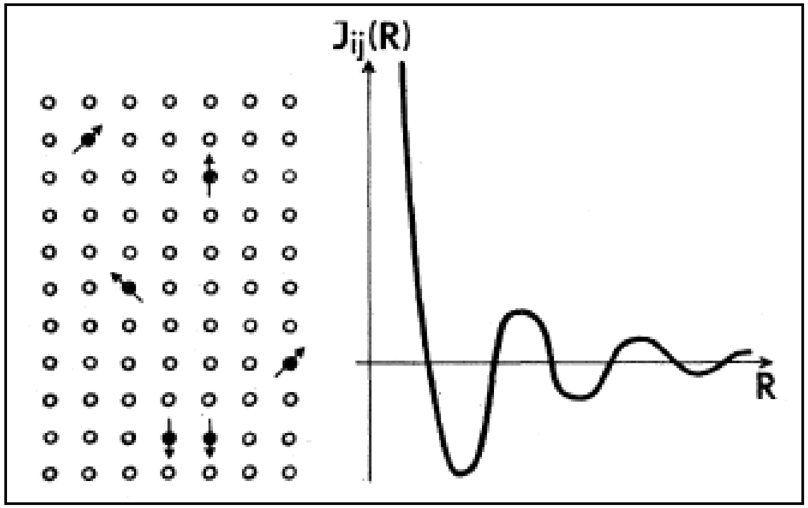
What are Spin Glasses?

- Magnetic systems with **quenched disorder**.
- Competition between ferromagnetic and antiferromagnetic interactions.

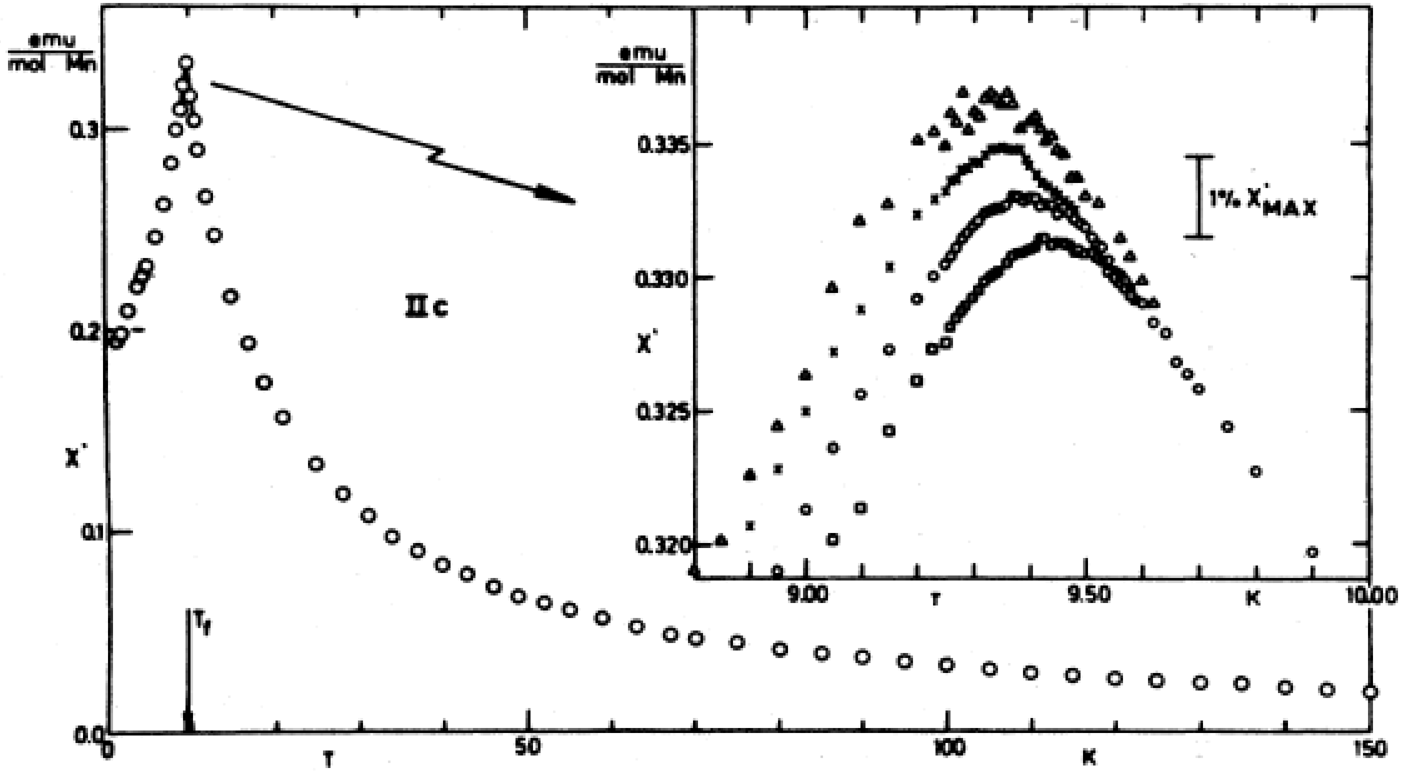
Example: CuMn, AuFe, …

$$J(r) = J_0 \frac{\cos(2k_F r + \phi_0)}{(k_F r)^3}$$

RKKY Interaction between localized spins



Experimental results: (1) Cusp in the magnetic susceptibility



Susceptibility of CuMn as a function of temperature

Figures from K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).

Experimental results: (2) Slow dynamics at low temperatures

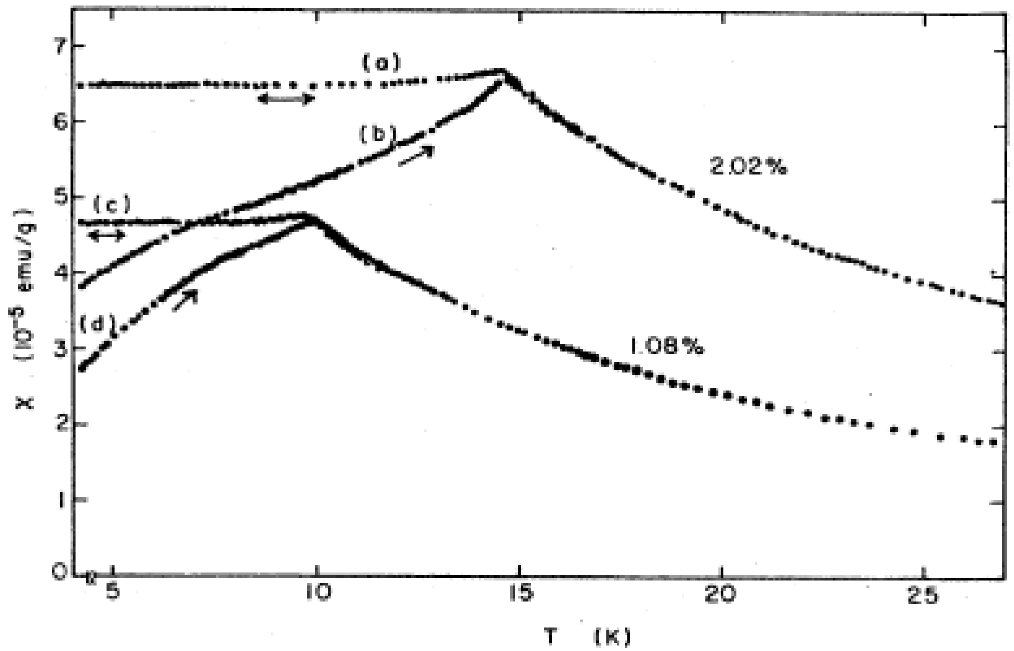


FIG. 7. Static susceptibilities of *Cu*Mn vs temperature for 1.08 and 2.02 at. % Mn. After zero-field cooling ($H < 0.05$ Oe), initial susceptibilities (b) and (d) were taken for increasing temperature in a field of $H = 5.9$ Oe. The susceptibilities (a) and (c) were obtained in the field $H = 5.9$ Oe, which was applied above T_f before cooling the samples. From Nagata *et al.* (1979).

Difference between zero-field-cooled and field-cooled magnetization for T<T(cusp)

Frustration

All pair interactions can not be satisfied simultaneously

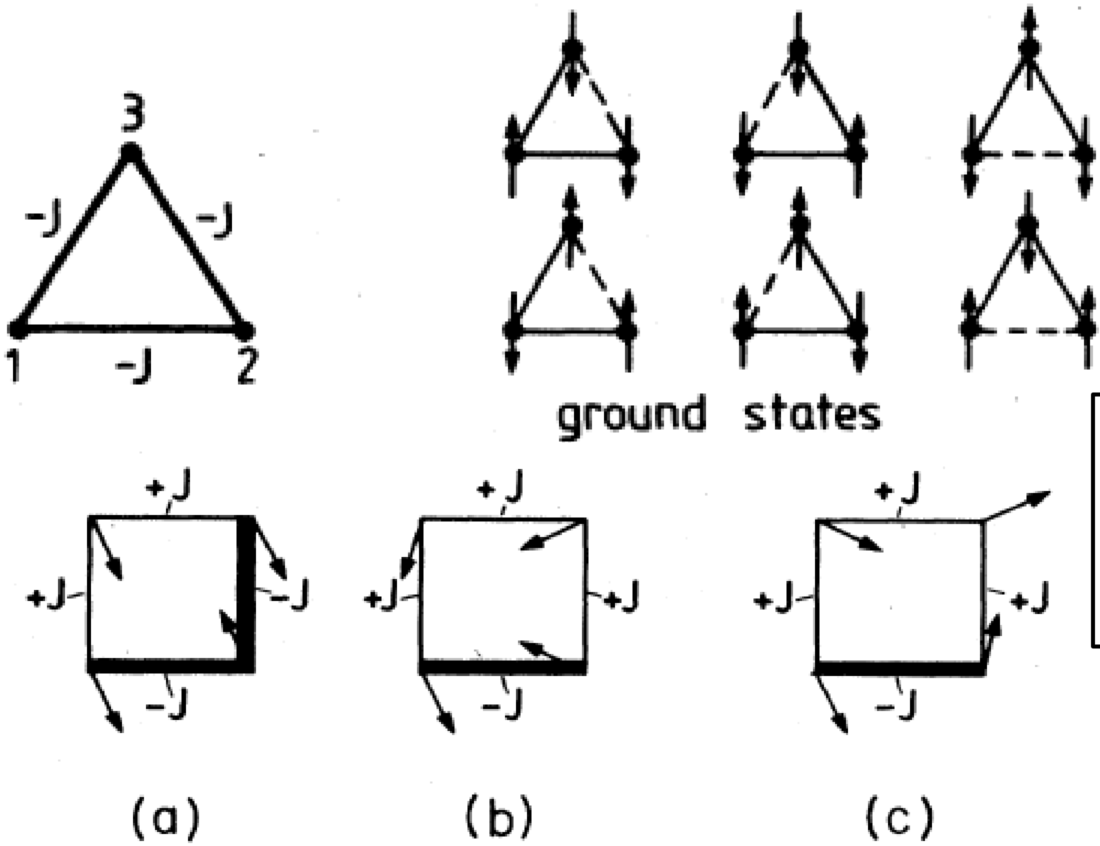


FIG. 41. Classical ground state of a set of four spins in the *XY* model with interactions $\pm J$ (thick bonds are antiferromagnetic, thin bonds are ferromagnetic). (a) Nonfrustrated plaquette; (b) frustrated plaquette, chirality $\tau=+1$; (c) frustrated plaquette, chirality $\tau=-1$.

Frustration leads to a multiplicity of ground states of the spin system

Edwards-Anderson Model

S. F. Edwards and P. W. Anderson, J. Phys. F **5**, 965 (1975).

$$\mathcal{H} = -\sum_{<ij>} J_{ij}\sigma_i\sigma_j \quad \sigma_i = \pm 1$$

Ising spins on a regular lattice
Nearest-neighbor interactions
Quenched disorder

$$\tilde{P}(\{J_{ij}\}) = \Pi_{<ij>} P(J_{ij})$$

$$P(J_{ij}) = \frac{1}{\sqrt{2\pi J^2}} \exp[-J_{ij}^2/2J^2]$$

or

$$P(J_{ij}) = \frac{1}{2}[\delta(J_{ij} + J) + \delta(J_{ij} - J)]$$

$$[J_{ij}]_{av} = 0, \quad [J_{ij}^2]_{av} = J^2$$

No ferromagnetic or antiferromagnetic phase is possible

Spin Glass Phase

High-temperature paramagnetic phase $\langle \sigma_i \rangle = 0$ $\quad M \equiv \frac{1}{N}\sum_{i=1}^{N} \langle \sigma_i \rangle = 0$

Low-temperature spin glass phase $\langle \sigma_i \rangle \neq 0$ $\quad M \equiv \frac{1}{N}\sum_{i=1}^{N} \langle \sigma_i \rangle = 0$

$$q \equiv \frac{1}{N}\sum_{i=1}^{N} (\langle \sigma_i \rangle)^2 \neq 0$$

Temporal autocorrelation function

$$C(t) \equiv \frac{1}{N}\sum_{i=1}^{N} \langle \sigma_i(t)\sigma_i(0) \rangle$$

$$C(t)|_{t\to\infty} = \frac{1}{N}\sum_{i=1}^{N} (\langle \sigma_i \rangle)^2 = q$$

Spin glass transition : "Freezing" of the spins in random orientations

The Replica Method Disorder-averaged Free Energy

$$F = Nf = -T[\ln Z(\{J_{ij}\})]_{av}$$
$$= -T \int \Pi_{<ij>} dJ_{ij} \tilde{P}(\{J_{ij}\}) \ln Z(\{J_{ij}\})$$

Mathematical identity: $\ln(x) = \lim_{n \to 0} \frac{x^n - 1}{n}$

$$[\ln Z(\{J_{ij}\})]_{av} = \lim_{n \to 0} \frac{[Z^n(\{J_{ij}\})]_{av} - 1}{n}$$

$$[Z^n(\{J_{ij}\})]_{av} = [\mathrm{Tr}_{\{\sigma_i^\alpha\}} \exp[-\sum_{\alpha=1}^{n} \mathcal{H}(\{\sigma_i^\alpha\}, \{J_{ij}\})/T]]_{av}$$
$$= \mathrm{Tr}_{\{\sigma_i^\alpha\}} \exp[-\mathcal{H}_{eff}(\{\sigma_i^\alpha\})/T]$$

$$\mathcal{H}_{eff}(\{\sigma_i^\alpha\}) = -T \ln[\int \Pi_{<ij>} dJ_{ij} \tilde{P}(\{J_{ij}\})$$
$$\times \exp[-\sum_{\alpha=1}^{n} \mathcal{H}(\{\sigma_i^\alpha\}, \{J_{ij}\})/T]]$$

$\mathcal{H}_{eff}(\{\sigma_i^\alpha\})$ **does not** have any quenched disorder

Use standard methods to treat the replicated (n-component) spin model described by $\mathcal{H}_{eff}(\{\sigma_i^\alpha\})$

Take n → 0 limit at the end of the calculation

Edwards-Anderson (Spin Glass) Order Parameter

$$q = [\langle\sigma_i\rangle^2]_{av} = \langle\sigma_i^\alpha\sigma_i^\beta\rangle_{\mathcal{H}_{eff}},\ \alpha \neq \beta$$

The spin glass transition is from the paramagnetic state with q=0 to a spin glass state with nonzero q as the temperature is decreased.

Magnetic susceptibility

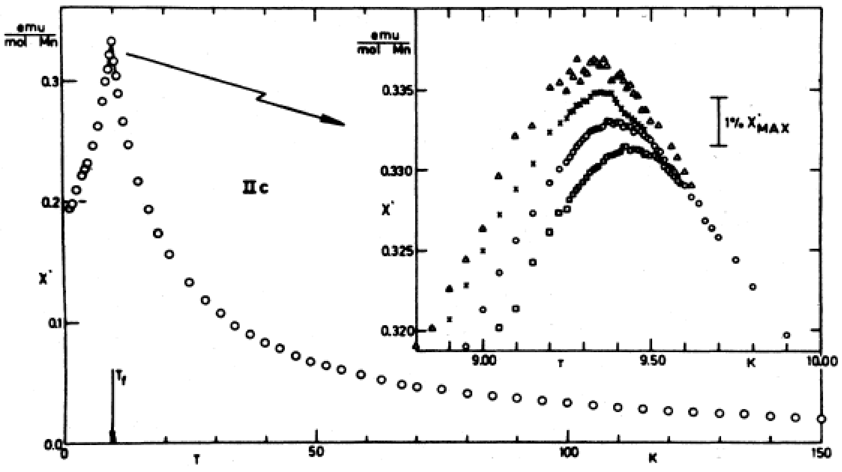
$$\chi(T) = \frac{1}{NT}[\sum_{i,j}(\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle)]_{av}$$

For spin glasses,

$$[\langle\sigma_i\sigma_j\rangle]_{av} = 0 \text{ for } i \neq j, \; = 1 \text{ for } i = j.$$

Also, $[\langle\sigma_i\rangle]_{av} = 0$ and $[\langle\sigma_i\rangle^2]_{av} \neq 0$ in the SG phase

$$\boxed{\chi(T) = \frac{1}{T}(1-q)}$$



The Sherrington-Kirkpatrick Model

D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1972 (1975).

Infinite-range (mean field) model of Ising spin glass

$$\mathcal{H} = -\frac{1}{2}\sum_{i\neq j} J_{ij}\sigma_i\sigma_j = -\sum_{<ij>} J_{ij}\sigma_i\sigma_j.$$

$$P(J_{ij}) = \sqrt{\frac{N}{2\pi J^2}}\exp\left[-\frac{NJ_{ij}^2}{2J^2}\right] \qquad [J_{ij}]_{av} = 0,\ [J_{ij}^2]_{av} = J^2/N.$$

$$[Z^n]_{av} = \mathrm{Tr}_{\{\sigma_i^\alpha\}}\exp\left[\frac{\beta^2 J^2}{2N}\sum_{<ij>}\sum_{\alpha,\beta}\sigma_i^\alpha\sigma_i^\beta\sigma_j^\alpha\sigma_j^\beta\right]$$

Hubbard-Stratanovitch Identity:

$$\exp[\lambda a^2/2] = \sqrt{\frac{\lambda}{2\pi}}\int_{-\infty}^{\infty} dx\,\exp[-\lambda x^2/2+\lambda a x].$$

S-K Model (contd.)

→ $[Z^n]_{av} = \exp\left[\frac{\beta^2 J^2 n N}{4}\right] \int_{-\infty}^{\infty} \Pi_{\alpha<\beta} \sqrt{\frac{N}{2\pi}} \beta J dq_{\alpha\beta}$

$$\times \exp\left[-\frac{N\beta^2 J^2}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 + N \ln \mathrm{Tr}_{\{\sigma^\alpha\}} e^{L(\{q_{\alpha\beta}\},\{\sigma^\alpha\})}\right]$$

where $L(\{q_{\alpha\beta}\}, \{\sigma^\alpha\}) \equiv \beta^2 J^2 \sum_{\alpha<\beta} q_{\alpha\beta}\sigma^\alpha\sigma^\beta$

→ $-\beta f = \lim_{n\to 0}\left[\frac{\beta^2 J^2}{4}(1 - \frac{1}{n}\sum_{\alpha\neq\beta} q_{\alpha\beta}^2) + \frac{1}{n} \ln \mathrm{Tr} e^L\right]$

$q_{\alpha\beta}$ are to be determined from $\frac{\partial f}{\partial q_{\alpha\beta}} = 0$

S-K Model (contd.)

Replica Symmetry: $q_{\alpha\beta} = q \text{ for all } \alpha \neq \beta$

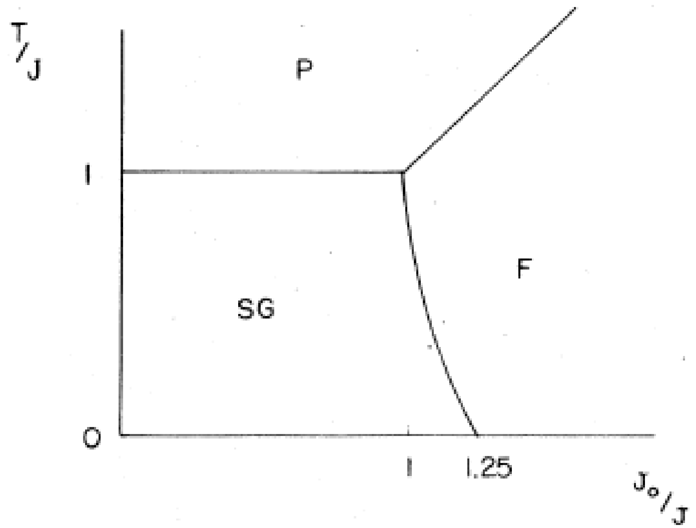
Self-consistency equation:

$$q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \exp(-z^2/2) \tanh^2(\beta J \sqrt{q} z)$$

$$q \neq 0 \text{ for } T < T_c = J$$

Continuous spin glass transition at T=J

Phase diagram



Replica Symmetry Breaking

The replica symmetric solution has unphysical properties for T < J.

Instability of the replica symmetric solution

$$-\beta f = \lim_{n\to 0}\left[\frac{\beta^2 J^2}{4}\left(1-\frac{1}{n}\sum_{\alpha\neq\beta} q_{\alpha\beta}^2\right)+\frac{1}{n}\ln \mathrm{Tr} e^{L}\right]$$

Fluctuations: $q_{\alpha\beta} = q_0 + \delta q_{\alpha\beta}$

$$\beta f = \beta f(q_0) + \lim_{n\to 0}\frac{1}{2n}\sum_{\alpha<\beta,\gamma<\delta}\mathcal{R}^{\alpha\beta,\gamma\delta}\delta q_{\alpha\beta}\delta q_{\gamma\delta} + \cdots$$

All eigenvalues of $\mathcal{R}$ must be ≥ 0 for stability and physically meaningful behavior.

This condition is not satisfied for T < J.

Replica Symmetry Breaking (contd.)

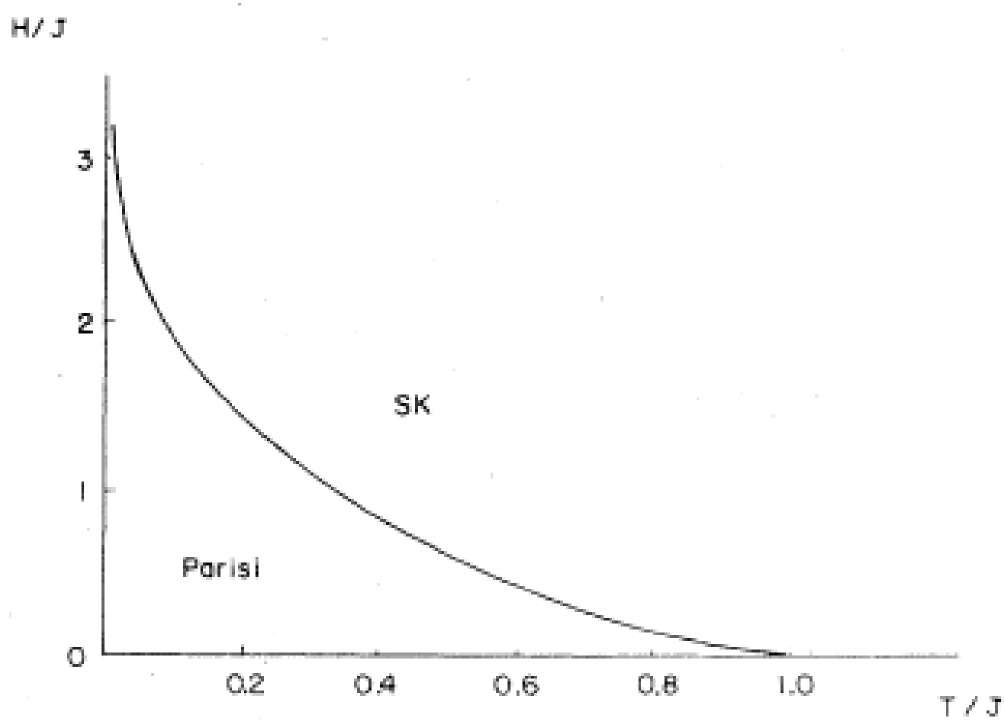
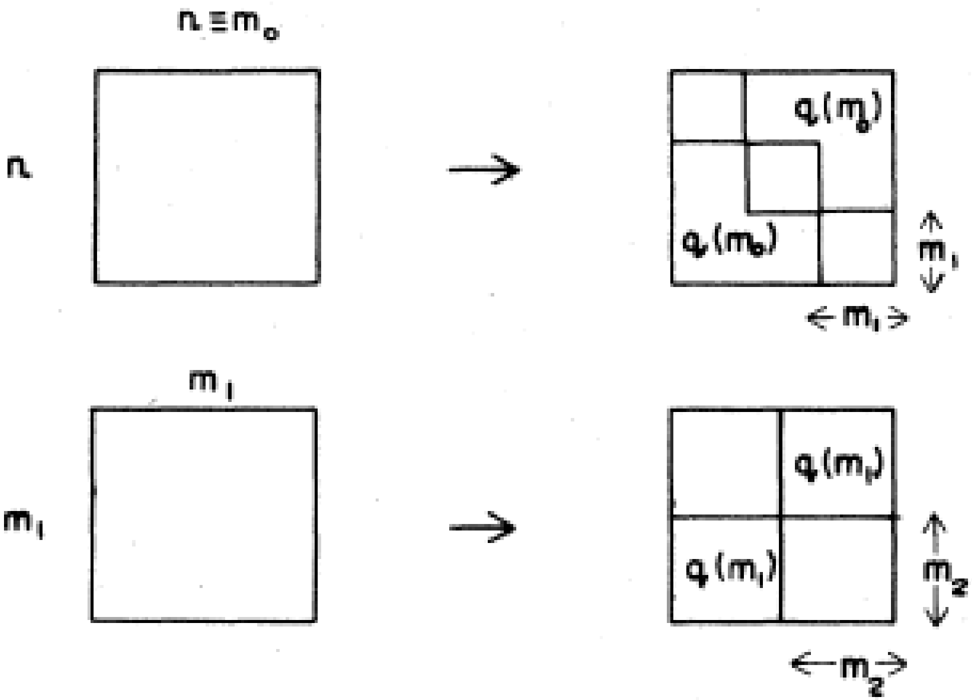


FIG. 48. Plot of the Almeida-Thouless (AT) line for the SK model with $J_0 = 0$. To the right of the line the SK solution with a single order parameter is correct, while to the left of the line the Parisi solution is believed exact. The Parisi solution represents the many-valley structure of phase space and nonergodic behavior. The AT line, therefore, signals the onset of irreversibility.

J.R.L de Almeida and D.J Thouless, J. Phys A **11**, 983 (1978)

The Parisi Solution

G. Parisi, Phys. Rev. Lett. **43**, 1754 (1979)



Iterative scheme

Repeat this procedure K times:

K-step replica symmetry breaking

$$m_1, m_2, \ldots, m_K; \quad m_0 \geq m_i \geq 1.$$

$$q(m_0), q(m_1), \ldots, q(m_K)$$

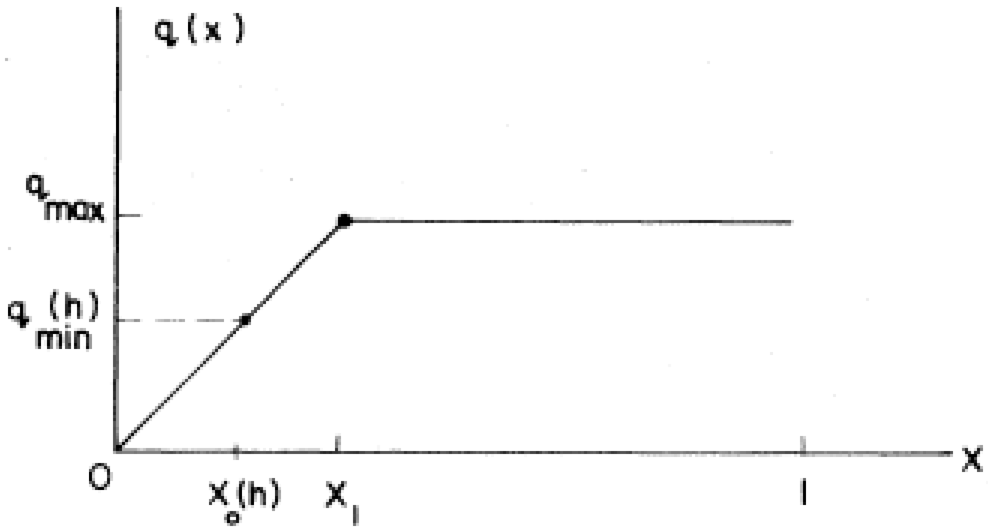
The Parisi Solution (contd.)

$$K \to \infty : \; m_i \to x, \; 0 \le x \le 1, \; q(m_i) \to q(x)$$

q(x): Order parameter function

Spin glass order parameter:

$$q = [\langle \sigma_i \rangle^2]_{av} = \int_0^1 q(x) dx$$



q(x) at a temperature slightly below the critical temperature

Thouless-Anderson-Palmer Equations

D.J. Thouless, P.W. Anderson, R.G. Palmer, Phil. Mag. **35**, 593 (1977)

Free energy of the S-K model for a given set of interaction parameters

$$F = -\frac{1}{2}\sum_{i\neq j} J_{ij} m_i m_j + \frac{T}{2}\sum_i [(1+m_i)\ln\{(1+m_i)/2\} + (1-m_i)\ln\{(1-m_i)/2\}] - \frac{1}{4T}\sum_{i\neq j} J_{ij}^2 (1-m_i^2)(1-m_j^2).$$

Onsager Reaction term

$$\frac{\partial F}{\partial m_i} = 0 \rightarrow m_i = \tanh[\beta\sum_j J_{ij} m_j - \beta^2 \sum_j J_{ij}^2 (1-m_j^2) m_i]$$

Local field at site i:

$$\sum_j J_{ij}(m_j - \chi_{jj} J_{ij} m_i) = \sum_j J_{ij} m_j - \sum_j J_{ij}^2 \beta (1-m_j^2) m_i$$

TAP Equations (contd.)

Only one solution of the TAP equations, $m_i = 0$ for all i, for $T > J$.

Many solutions with nonzero $\{ m_i \}$ for $T < J$.

Number of minima with the lowest free energy per spin is not exponential in N.

Free energy barriers between different minima diverge in the thermodynamic limit.

Complex Free Energy Landscape

Physical interpretation of RSB

Large number of "valleys" ["pure states", "ergodic components"] at temperatures lower than the critical temperature.

$P^{(\alpha)}$: Probability of the system being in valley α

$$\langle \sigma_i \rangle = \sum_{\alpha} P^{(\alpha)} m_i^{(\alpha)}$$ [Average over all valleys]

$$\frac{1}{N}\sum_i \langle \sigma_i \rangle^2 = \frac{1}{N}\sum_{i=1}^{N}\sum_{\alpha\beta} P^{(\alpha)} P^{(\beta)} m_i^{(\alpha)} m_i^{(\beta)}$$

Define overlap between valleys α and β,

$$q_{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N} m_i^{(\alpha)} m_i^{(\beta)}$$

Distribution of the overlap: $P(q) = \sum_{\alpha\beta} P^{(\alpha)} P^{(\beta)} \delta(q - q_{\alpha\beta})$

Then $\frac{1}{N}\sum_i \langle \sigma_i \rangle^2 = \int_0^1 q P(q) dq$

Physical interpretation of RSB (contd.)

$$q = [\langle\sigma_i\rangle^2]_{av} = \int_0^1 q(x)dx = \int q\frac{dx}{dq}dq$$

$$\boxed{P(q) = \frac{dx}{dq}}$$

Parisi function q(x) describes the distribution of overlaps between different free-energy minima.

$$q_{EA} = \frac{1}{N}\sum_i\sum_\alpha P^{(\alpha)}[m_i^{(\alpha)}]^2 = q(x=1)$$

These predictions have been confirmed from simulations

Correctness of the RSB solution has been established from more rigorous analysis.