Rigidity of Eigenvalues for β Ensemble in Multi-cut Regime

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Definition and some background of the β ensemble

A β ensemble is a probability measure on $\mathbb{R}^N$:

$$\mu_N = \frac{1}{Z} \cdot e^{-\frac{N\beta}{2}[V(x_1)+\cdots+V(x_N)]} \prod_{1\le i<j\le N} |x_i - x_j|^\beta dx_1 \cdots dx_N$$

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- Z is the normalization constant.

Definition and some background of the β ensemble

Interest: the behavior of $x_1,\ldots,x_N$ as $N \to +\infty$.

Definition and some background of the β ensemble

Recall the Gaussian Orthogonal Ensemble (GOE):

$$M_N = \begin{pmatrix} \sqrt{2}\, a_{11} & a_{12} & \cdots & a_{1N} \\ a_{12} & \sqrt{2}\, a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & \cdots & \sqrt{2}\, a_{NN} \end{pmatrix}$$

where $a_{11}, a_{12}, a_{22}, a_{13}, \dots$ are i.i.d. standard normal random variables.

Since M_N is symmetric, all its eigenvalues are real.

Definition and some background of the β ensemble

Theorem

If $\beta = 1$ and $V(x) = \frac{x^2}{2}$, then μ_N is the law of eigenvalues of $\frac{1}{\sqrt{N}} M_N$.

More precisely, if $\lambda_1 \leq \cdots \leq \lambda_N$ are eigenvalues of $\frac{1}{\sqrt{N}} M_N$, then

$$\mathbb{P}\big((\lambda_1, \ldots, \lambda_N) \in A\big) = N! \cdot \mu_N(A) \qquad \forall A \subset \mathbb{R}^N \cap \{x_1 \leq \cdots \leq x_N\}.$$

Definition and some background of the β ensemble

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  (GUE: Gaussian Unitary Ensemble)

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  $$\mu_N \xrightarrow{\beta=4} \text{ random Hermitian quaternionic matrix } \xrightarrow{V(x)=\frac{x^2}{2}} \text{GSE}$$

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- $$\mu_N \xrightarrow{V(x)=ax^2+bx+c} \text{random tridiagonal matrix}$$

  (Dumitriu,Edelman (2002))

Equilibrium measure of μ_N

For $(x_1, \ldots, x_N) \in \mathbb{R}^N$, define the empirical measure

$$L_N = \frac{1}{N} \sum_{i=1}^{N} \delta(t - x_i).$$

For example, $\int_{\mathbb{R}} \sin t \, dL_N(t) = \frac{1}{N} \sum_{i=1}^{N} \sin x_i$.

Theorem

Under some technical conditions, L_N converges to a limit $\rho(t)dt$ in the following sense:

$$\mathbb{E}^{\mu_N}\left[\int_{\mathbb{R}} f(t) \, dL_N(t) \right] \xrightarrow{N \to +\infty} \int_{\mathbb{R}} f(t)\rho(t) \, dt$$

for all bounded continuous function $f(t)$.
We call $\rho(t)dt$ the equilibrium measure.

Equilibrium measure of μ_N

Fact:

- $\rho(t)$ is continuous.

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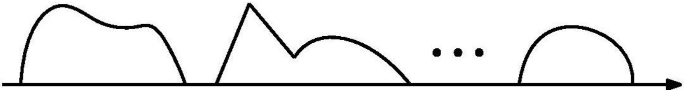
Fact:

- $\rho(t)$ is continuous.
- $\rho(t)$ has a compact support which is the union of finite many intervals:

$$\text{supp}(\rho) = [A_1, B_1] \cup \cdots \cup [A_q, B_q].$$



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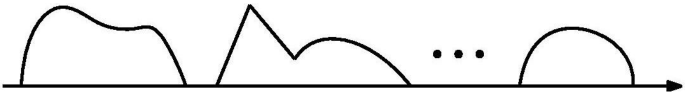
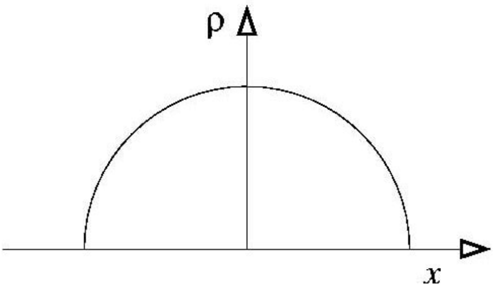


If $q = 1$, then we say that the β ensemble is in one-cut regime.
If $q > 1$, then we say that the β ensemble is in multi-cut regime.
Each $[A_i, B_i]$ is called a cut.

Equilibrium measure of μ_N

For example, if $\beta = 1$ and $V(x) = \frac{x^2}{2}$, then $\rho(x)dx$ is the semicircle distribution:

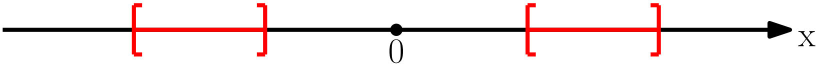
$$\rho(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\mathbb{1}_{|x|\leq 2}$$



Equilibrium measure of μ_N

For example, if $\beta > 0$ and $V(x) = \frac{1}{2}x^4 - \frac{3\sqrt{2}\beta}{2}x^2$, then $\rho(x)dx$ is supported on:

$$\left[-\sqrt{2\sqrt{2}\beta}, -\sqrt{\sqrt{2}\beta}\right] \cup \left[\sqrt{\sqrt{2}\beta}, \sqrt{2\sqrt{2}\beta}\right]$$

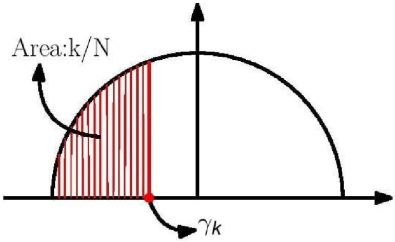


Rigidity of particles for β ensemble in one-cut regime

Under μ_N, $(x_1, \ldots, x_N)$ is a random elemant in $\mathbb{R}^N$. So each coordinate is a random element in $\mathbb{R}$. We call each x_i a particle or an eigenvalue.

"classical location" of the k-th largest particle:

$$\gamma_k = \gamma_k(N) = \inf_{t \in \mathbb{R}} \int_{-\infty}^{t} \rho(s)ds = \frac{k}{N} \qquad k = 1, \ldots, N$$



Rigidity of particles for β ensemble in one-cut regime

- Assume that the β ensemble is in the one-cut regime.
- For each $x = (x_1, \ldots, x_N)$, use $\bar{x}_k$ to denote its k-th largest particle.

Rigidity for particles in the bulk:

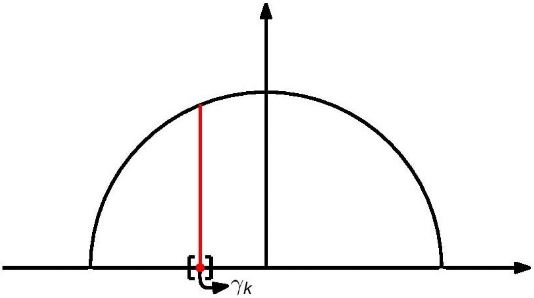
Theorem (Bourgade-Erdös-Yau (2011))

Suppose $a > 0$ and $\epsilon > 0$. Then under some technical conditions, there exists $C > 0$ such that

$$\mathbb{P}^{\mu_N}\left(|\bar{x}_k - \gamma_k| \le N^{-1+\epsilon} \text{ for all } k \in [aN, (1-a)N]\right) \ge 1 - e^{-N^C}$$

when N is large enough.

Rigidity of particles for β ensemble in one-cut regime



$\bar{x}_k$, i.e., the k-th largest particle, is basically located in an interval of size N^{-1} around γ_k.

Rigidity of particles for β ensemble in one-cut regime

Rigidity for particles in the edge:

Theorem (Bourgade-Erdös-Yau (2013))

Suppose $\epsilon > 0$. Then under some technical conditions, there exists $C > 0$ such that

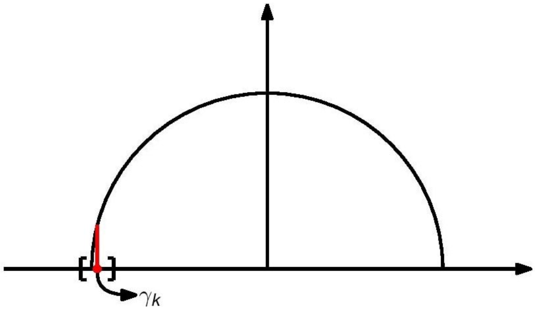
$$\mathbb{P}^{\mu_N}\left(|\bar{x}_k - \gamma_k| \leq N^{-\frac{2}{3}+\epsilon} \cdot \hat{k}^{-\frac{1}{3}} \text{ for all } k \in [1, N]\right) \geq 1 - e^{-N^C}$$

when N is large enough. Here $\hat{k} := \min(k, N+1-k)$.

Remark

- When $k \in [aN, (1-a)N]$ for some $a > 0$, then $N^{-\frac{2}{3}+\epsilon} \cdot \hat{k}^{-\frac{1}{3}} \sim N^{-1+\epsilon}$.
- When k is close to 1 or close to N, then $N^{-\frac{2}{3}+\epsilon} \cdot \hat{k}^{-\frac{1}{3}} \sim N^{-\frac{2}{3}+\epsilon}$.

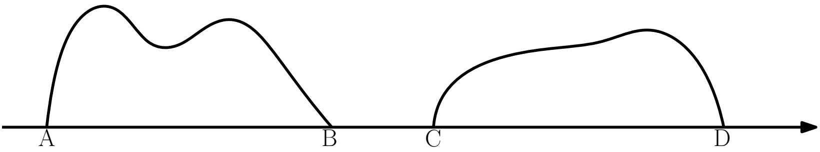
Rigidity of particles for β ensemble in one-cut regime



If k is close to 1 or close to N, i.e., near the edge, then the k-th largest particle $\bar{x}_k$ is basically located in an interval of size $N^{-2/3}$ around γ_k.

Rigidity of particles for β ensemble in multi-cut regime

The equilibrium measure $\rho(x)dx$ is supported on more than one intervals:



For convenience, suppose $\int_A^B \rho(x)dx = \int_C^D \rho(x)dx = \frac{1}{2}$.

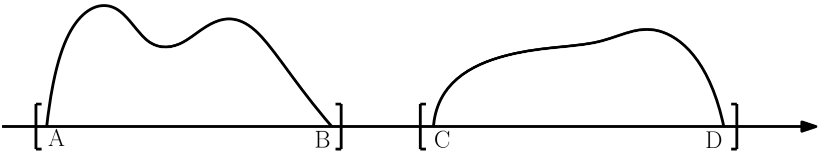
Rigidity of particles for β ensemble in multi-cut regime

Theorem

Suppose $\epsilon > 0$. Under some technical conditions, there exists $\delta > 0$ such that

$$\mathbb{P}^{\mu_N}\Big(\textit{all particles are in } [A-\epsilon, B+\epsilon] \cup [C-\epsilon, D+\epsilon]\Big) \geq 1 - e^{-N^{\delta}}$$

for N large enough.



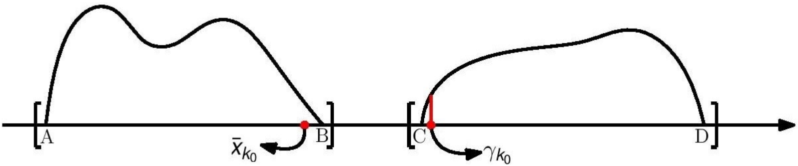
Rigidity of particles for β ensemble in multi-cut regime

Suppose $k_0 = k_0(N) = \lfloor \frac{N}{2} \rfloor + 1$. Then γ_{k_0} is the first classical location in $[C, D]$.

Theorem (Bekerman (2015))

Suppose $\epsilon > 0$. Under some technical conditions we have that

$$\overline{\lim}\, \mathbb{P}^{\mu_N}\left(\bar{x}_{k_0} \in [A-\epsilon, B+\epsilon]\right) := C_V > 0 \quad \textit{as } N \to \infty.$$



Therefore, there is no rigidity near B and C.

Rigidity of particles for β ensemble in multi-cut regime

Rigidity for particles in the bulk:

Theorem (Li (2016))

Suppose $a > 0$ and $\epsilon > 0$. Then under some technical conditions, there exist $C > 0$ such that

$$\mathbb{P}^{\mu_N}\Big(|\bar{x}_k - \gamma_k| \le N^{-1+\epsilon} \text{ for all } k \in [aN, (\frac{1}{2} - a)N]\Big) \ge 1 - e^{-N^C}$$

when N is large enough. Same conclusion holds for $k \in [(\frac{1}{2} + a)N, (1 - a)N]$

Idea of the proof of bulk rigidity for β ensemble in multi-cut regime

- We cannot directly use the method of Bourgade, Erdös and Yau.
- Their first step is to show that for $\epsilon > 0$, there exist $C > 0$ such that

$$\mathbb{P}^{\mu_N}\Big(|\bar{x}_k - \gamma_k| \leq \epsilon \text{ for all } k \in [1, N]\Big) \geq 1 - e^{-N^C}$$

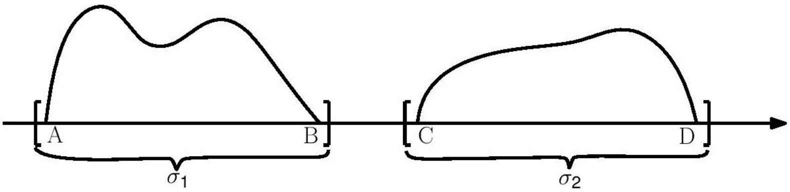
  when N is large enough.
- But in multi-cut case particles may jump.

Idea of the proof of bulk rigidity for β ensemble in multi-cut regime

Idea: decompose the β ensemble in multi-cut regime as a product of β ensembles in one-cut regime.

Consider

- σ_1: a neighbourhood of $[A, B]$;
- σ_2: a neighbourhood of $[C, D]$



Idea of the proof of bulk rigidity for β ensemble in multi-cut regime

Recall that $\mu_N = \frac{1}{Z} \cdot e^{-\frac{N\beta}{2}[V(x_1)+\cdots+V(x_N)]} \prod_{i<j} |x_i - x_j|^\beta dx_1 \cdots dx_N$.

Idea of the proof of bulk rigidity for β ensemble in multi-cut regime

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Define a probability measure on $\left(\sigma_1 \cup \sigma_2\right)^N$:

$$\tilde{\mu}_N = \frac{1}{\tilde{Z}} \cdot e^{-\frac{N\beta}{2} \sum_{i=1}^N V^r(x_i)} \cdot \prod_{\substack{i<j \\ x_i, x_j \in \sigma_1}} |x_i - x_j|^\beta \cdot \prod_{\substack{i<j \\ x_i, x_j \in \sigma_2}} |x_i - x_j|^\beta dx_1 \cdots dx_N$$

where $\tilde{Z}$ is the normalization constant and

$$V^r(x) = \begin{cases} V_1(x) := V(x) - 2\int_{[C,D]} \rho(y) \ln|x-y| dy & \text{if } x \in \sigma_1, \\ V_2(x) := V(x) - 2\int_{[A,B]} \rho(y) \ln|x-y| dy & \text{if } x \in \sigma_2. \end{cases}$$

Idea of the proof of bulk rigidity for β ensemble in multi-cut regime

Fact: $\tilde{\mu}_N$ is very close to μ_N. If an event A_N is exponentially small under $\tilde{\mu}_N$, then it is also exponentially small under μ_N:

$$\mathbb{P}^{\tilde{\mu}_N}(A_N) \geq 1 - e^{-N^C} \Longrightarrow \mathbb{P}^{\mu_N}(A_N) \geq 1 - e^{-N^{C'}}$$

So we only have to show the rigidity with respect to $\tilde{\mu}_N$.

Idea of the proof of bulk rigidity for β ensemble in multi-cut regime

Under $\tilde{\mu}_N$, particles in different cuts don't have interaction.
Basically, $\tilde{\mu}_N = \eta_k \times \xi_s$ where

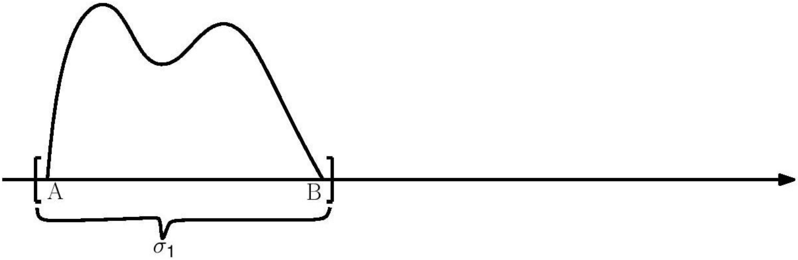
$$\eta_k = \frac{1}{Z_1} e^{-\frac{N\beta}{2}\sum_{i=1}^{k} V_1(x_i)} \prod_{1\le i<j\le k} |x_i - x_j|^\beta \prod_{i=1}^{k} \mathbb{1}_{\sigma_1}(x_i) dx_1 \cdots dx_k$$

$$\xi_s = \frac{1}{Z_2} e^{-\frac{N\beta}{2}\sum_{i=1}^{s} V_2(x_i)} \prod_{1\le i<j\le s} |x_i - x_j|^\beta \prod_{i=1}^{s} \mathbb{1}_{\sigma_2}(x_i) dx_1 \cdots dx_s$$

and $k + s = N$

Idea of the proof of bulk rigidity for β ensemble in multi-cut regime

As $k \to \infty$, the equilibrium measure of η_k is $2\rho(x)\mathbb{1}_{[A,B]}(x)dx$:



Idea of the proof of bulk rigidity for β ensemble in multi-cut regime

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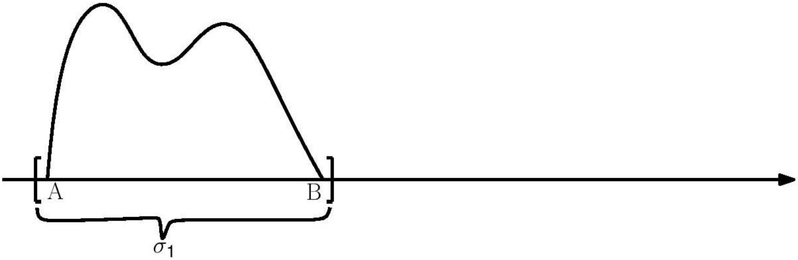
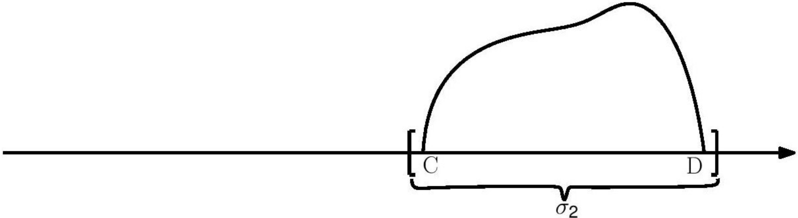


As $s \to \infty$, the equilibrium measure of ξ_s is $2\rho(x)\mathbb{1}_{[C,D]}(x)dx$:



Idea of the proof of bulk rigidity for β ensemble in multi-cut regime

η_k and ξ_s are in one-cut regime

$$\Downarrow$$

rigidity for particles in the bulk with respect to η_k and ξ_s

$$\Downarrow$$

rigidity for particles in the bulk with respect to $\tilde{\mu}_N = \eta_k \times \xi_s$

$$\Downarrow$$

rigidity for particles in the bulk with respect to μ_N

Conjecture: edge rigidity for β ensemble in multi-cut regime

Conjecture:

η_k and ξ_s are in one-cut regime

$\Downarrow$

rigidity for particles near the edges with respect to η_k and ξ_s

$\Downarrow$

rigidity for particles near A and D with respect to $\tilde{\mu}_N = \eta_k \times \xi_s$

$\Downarrow$

rigidity for particles near A and D respect to μ_N

Thank you!