DEFORMATIONS OF REPRESENTATIONS AND COHOMOLOGY

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The Atiyah-Patodi-Singer eta invariant associates to an oriented odd-dimensional Riemannian manifold M, and a unitary representation α of its fundamental group $\pi = \pi_1(M)$, a real number $\eta_\alpha(M)$. Moreover, if one "reduces" the eta invariant by subtracting its value at the trivial representation, the resulting invariant $\rho_\alpha(M)$ is independent of the Riemannian structure and is, therefore, a "topological" invariant of M. The question of when $\rho_\alpha(M)$ is a homotopy invariant of M has been the object of some interest (see [6] and [7]). One approach to understanding the ρ invariant has been to examine its behavior as α varies in the variety $\mathcal{R}_k(M)$ of (k-dimensional) unitary representations of π. From this point of view, one sees that ρ decomposes into a continuous and a discrete part. The former is easily understood, up to an indeterminacy in the form of a locally constant function from $\mathcal{R}_k(M)$ to $\mathbb{R}/\mathbb{Z}$ (see [2]), but the discrete part, which corresponds to the *spectral flow* of an associated self- adjoint elliptic differential operator on M presents a deeper problem. Recently there have been two solutions to the problem of describing this discrete part of ρ. In [2] the present authors associate to any germ α_t of an analytic path in $\mathcal{R}_k(M)$ a purely homotopy-theoretic form on the cohomology of M, twisted by α_t, and give a formula for the spectral flow in terms of signature invariants of this form. In a series of papers-[3], [4], [5]- Kirk and Klassen also give a formula for the spectral flow in terms of signatures of Hermitian pairings defined on a sequence of subquotients $\mathcal{G}_i$ of the deRham cohomology $\mathcal{G}_0$ of M. Each $\mathcal{G}_i$ is equipped with a coboundary operator δ_i defined, via higher Massey products, from a path of *signature* operators corresponding to a path of flat connexions. $\mathcal{G}_{i+1}$ is the cohomology of $(\mathcal{G}_i,\ \delta_i)$ and the Hermitian pairing is defined from δ_i and the Riemannian structure on M.

It is the aim of the present work to reformulate the signature invariants of [2] using only the cohomology of M at α_0 and a cochain of π defined by the deformation α_t. This will also provide a topological version of the Kirk-Klassen scheme, demonstrating its equivalence to the Farber-Levine scheme.

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1. Cohomology of a space twisted by a path of representations

Let X be a connected Poincare complex of formal dimension n and fundamental group $\pi = \pi_1(X)$. In other words, for any left $\mathbb{Z}\pi$-module A, the cap product

$$[X]\cap : C^p(X;A) \to \ C_{n-p}(X;\overline{A})$$

is a chain homotopy equivalence. Recall $C^p(X;A) = \mathrm{Hom}_{\mathbb{Z}\pi}(C_p(\tilde{X}), A)$ and $C_q(X;\overline{A}) = \overline{A} \otimes_{\mathbb{Z}\pi} C_q(\tilde{X})$, where $\tilde{X}$ is the universal covering space of X, $C_q(\tilde{X})$ is the singular chain complex of $\tilde{X}$ and $\overline{A}$ denotes the right $\mathbb{Z}\pi$- module defined by A with $\alpha \cdot g = g^{-1} \cdot \alpha$. We also recall the general cap product pairing

$$C_p(X;\overline{A}) \otimes C^q(X;B) \to C_{p-q}(X;\overline{C})$$

where A, B and C are left $\mathbb{Z}\pi$-modules equipped with a $\mathbb{Z}$ homomorphism $\phi : A \otimes_{\mathbb{Z}} B \to C$ and $A \otimes_{\mathbb{Z}} B$ has the diagonal π-action: $g \cdot (\alpha \otimes \beta) = g \cdot \alpha \otimes g \cdot \beta$. The cap product is defined by the formula:

$$(\alpha \otimes \sigma) \cap c = \phi(\alpha \otimes c(\sigma')) \otimes \sigma''$$

for any $\alpha \in A$ and σ a p-simplex in $\tilde{X}$, with $\sigma' = \sigma \mid$"front q-face "and $\sigma'' = \sigma \mid$"back (p-q)-face ".

There is a Liebnitz formula. If $c \in C_p(X;\overline{A})$ and $u \in C^q(X;B)$ then

$$\partial c \cap u = c \cap \delta u + (-1)^q \partial(c \cap u)$$

Now suppose α_t is a *formal analytic germ* of a path in $\mathcal{R}_k(\pi) =$ the real algebraic variety of representations of π into the unitary group $\mathcal{U}_k$. By this we mean only that α_t is a homomorphism $\pi \to \mathcal{U}_k(P)$, where $P = \mathbb{C}[[t]]$, the ring of power series over $\mathbb{C}$, and $\mathcal{U}_k(P)$ is the group of $(k \times k)$ matrices M over P which satisfy the formula $M\overline{M}^t = I$ ($\overline{M}^t$ is the conjugate transpose of M- conjugation in P means conjugate every coefficient). We denote by α_0 the ordinary unitary representation obtained by setting $t = 0$. Note that we impose no convergence requirement on α_t. We can use α_t to define a local coefficient system over X. Let V_t denote the free P-module of rank k specifically identified as $(k \times 1)$-column vectors with entries in P. Then V_t is a left module over $\mathcal{M}_k(P) =$ the ring of all $(k \times k)$-matrices over P, and so, via α_t, a left module over $\mathbb{C}\pi$. We can now define $H^*(X;\alpha_t)$ (or $H^*(X;V_t)$) to be the cohomology of the cochain complex $\mathrm{Hom}_{\mathbb{Z}\pi}(C_k(\tilde{X}), V_t)$ so that $H^*(X;\alpha_t)$ is a P-module. We can also use α_0 to define V_t as a local coefficient system over X by the natural inclusion $\mathcal{U}_k(\mathbb{C}) \subset \mathcal{U}_k(P)$.

There is a cup product pairing

$$C^*(X;\alpha_t) \times C^*(X;\alpha_t) \to C^*(X;P)$$

since α_t is a unitary representation. Generally, given left $\mathbb{Z}\pi$-modules A, B, C and a π-homomorphism $\phi : A \otimes_{\mathbb{Z}} B \to C$ (with the diagonal π-action) we define a cup product pairing by

$$(u \cup v)(\sigma) = \phi(u(\sigma') \otimes v(\sigma''))$$

It satisfies the usual Liebnitz formula $\delta(u \cup v) = \delta u \cup v + (-1)^p u \cup \delta v$ and, therefore, induces a pairing in cohomology. The necessary $\phi : V_t \otimes V_t \to P$ is defined by the usual scalar product on column vectors and ϕ is a homomorphism precisely because α_t is unitary. Since X is an n-dimensional Poincare complex, it follows, just as in classical Poincare duality, since P is a principal ideal domain, that the induced cup product

$$H^p(X; \alpha_t) \times H^{n-p}(X; \alpha_t) \to H^n(X; P) \cong P$$

is non-singular on the P-torsionfree quotients. There is also an induced non-singular pairing on the P-torsion submodules

$$tH^p(X; \alpha_t) \times tH^{n-1-p}(X; \alpha_t) \to \hat{P}/P \tag{1}$$

where $\hat{P}$ is the field of Laurent series over $\mathbb{C}$ (i.e. the quotient field of P). This pairing is defined in the usual way. Given $\alpha \in tH^p(X; \alpha_t), \beta \in tH^{n-1-p}(X; \alpha_t)$, and suppose $t^m \alpha = 0$. Then we can choose $\tilde{\alpha} \in H^{p-1}(X; V_t/t^m V_t)$ such that $\delta^*(\tilde{\alpha}) = \alpha$, where δ^* is the Bockstein defined by the exact coefficient sequence

$$0 \to V_t \xrightarrow{t^m} V_t \to V_t/t^m V_t \to 0$$

and define

$$< \alpha, \beta > \equiv t^{-m}(\alpha' \cup \beta) \pmod{P} \tag{2}$$

where $\alpha' \cup \beta \in H^n(X; P/t^m P) \cong P/t^m P$ and we use the obvious pairing $(V_t/t^m V_t) \otimes V_t \to P/t^m P$ induced by the scalar product. The pairing $<,>$ is used in [2] in the formula for the spectral jump at α_0 along the path α_t (when it has a positive radius of convergence). Our aim is to give a computation of $H^*(X; \alpha_t)$ and $<,>$ from $H^*(X; \alpha_0)$ and certain endomorphisms defined by homological extraction from α_t.

Let us, for the time being, place ourselves in a more general situation in which we have a representation $\alpha_t : \pi \to \mathcal{G}l_k(P)$, where $\mathcal{G}l_k(P)$ is the group of invertible $(k \times k)$ matrices over P. As above, we have the cochain complexes $C^*(X; \alpha_t) = \mathrm{Hom}_\pi(C(\tilde{X}), V_t)$ using α_t and $C^*(X; \alpha_0)_t = \mathrm{Hom}_\pi(C(\tilde{X}), V_t)$ using α_0. It will be convenient to replace $C(\tilde{X})$ by a slightly smaller, but chain-homotopy equivalent π-subcomplex, assuming X is connected. Choose a base-point $x_0 \in X$ and consider the π- subcomplex of $C(\tilde{X})$ generated by the set $\mathcal{S}(\tilde{X})$ of singular simplices σ such that $\sigma(v)$ lies over x_0, for every vertex v. The usual proof shows that this subcomplex is a π-equivariant chain deformation retract of $C(\tilde{X})$. We will, from now on, use $C(\tilde{X})$ to denote this subcomplex.

Our next step is to set up a P-isomorphism between the cochain complexes $C^*(X; \alpha_t)$ and $C^*(X; \alpha_0)_t$. To do this we choose a base-point $\tilde{x}_0 \in \tilde{X}$ lying over x_0 and define $\mathcal{S}_0(\tilde{X}) \subset \mathcal{S}(\tilde{X})$ to be the set of singular simplices $\sigma \in \mathcal{S}(\tilde{X})$ satisfying $\sigma(v_0) = \tilde{x}_0$, where v_0 is the initial vertex in the canonical domain simplex of σ. Clearly $\mathcal{S}_0(\tilde{X})$ is a basis for the free $\mathbb{Z}\pi$-module $C(\tilde{X})$ and so an element of $C^*(X; \alpha_t)$ or $C^*(X; \alpha_0)_t$ is determined by its value on the elements of $\mathcal{S}_0(\tilde{X})$. We will agree to identify $C^*(X; \alpha_t)$ and $C^*(X; \alpha_0)_t$ by identifying cochains which take identical values on $\mathcal{S}_0(\tilde{X})$. Clearly this defines an isomorphism of P-modules.

2. Cochains defined by a path of representations

We will need a description of the coboundary operator δ_t in $C^*(X;\alpha_t)$ under this identification. In order to achieve this we need to digress to examine some group cochains associated to any P-representation of a discrete group.

Given a representation $\alpha_t : \pi \to \mathcal{G}l_k(P)$, we define $\phi_t : \pi \to \mathcal{G}l_k(P)$ and $\tilde{\phi}_t : \pi \to \mathcal{M}_k(P)$ by the formulae

$$\phi_t(g) = \alpha_t(g)\alpha_0(g)^{-1} \text{ and } \tilde{\phi}_t(g) = \phi_t(g) - I \tag{3}$$

where $\mathcal{M}_k(P)$ is the ring of all $(k \times k)$ matrices over P. We wish to regard ϕ_t and $\tilde{\phi}_t$ as 1-cochains in π with coefficients in $\mathcal{M}_k(P)$. We consider $\mathcal{M}_k(P)$ to have the left π-module structure defined by

$$gM = \alpha_0(g)M\alpha_0(g)^{-1}$$

Recall the bar resolution for a group π (see [1]). $C_n(\pi)$ is the free left $\mathbb{Z}\pi$-module with a basis consisting of n-tuples $[g_1|\dots|g_n]$ and boundary operator defined by

$$\partial\ [g_1|\dots|g_n] = g_1[g_2|\dots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\dots|g_ig_{i+1}|\dots|g_n] + (-1)^n[g_1|\dots|g_{n-1}]$$

Then, if A is a left $\mathbb{Z}\pi$-module $C^*(\pi;A) = \mathrm{Hom}_{\mathbb{Z}\pi}(C(\pi),A)$. We will also need the standard cup product structure in $C^*(\pi;A)$. Suppose we have left π-modules A, B, C and a π-homomorphism $\mu : A \otimes_{\mathbb{Z}} B \to C$, where $A \otimes_{\mathbb{Z}} B$ has the usual diagonal π-action. Let $u \in C^p(\pi;A), v \in C^q(\pi;B)$; then $u \cup v \in C^{p+q}(\pi;C)$ is defined by

$$(u \cup v)[g_1|\dots|g_{p+q}] = \mu(u([g_1|\dots|g_p]), g_1\dots g_p v([g_{p+1}|\dots|g_{p+q}])$$

(Note that we have chosen a different sign convention than [1], but it still satisfies the usual Liebnitz formula $\delta(u \cup v) = \delta u \cup v + (-1)^p u \cup \delta v$).

Theorem 2.1. *Consider the 1-cochain $\tilde{\phi}_t \in C^1(\pi;\mathcal{M}_k(P))$ defined, as above, from a representation $\alpha_t : \pi \to \mathcal{G}l_k(P)$ and the π-structure on $\mathcal{M}_k(P)$ defined as above by α_0. Then the usual matrix multiplication on $\mathcal{M}_k(P)$ defines a cup product on $C^*(\pi;\mathcal{M}_k(P))$ and we have $\delta\tilde{\phi}_t = -\tilde{\phi}_t \cup \tilde{\phi}_t$.*

Proof. The multiplicative property $\alpha_t(gh) = \alpha_t(g)\alpha_t(h)$ translates to $\phi_t(gh) = \phi_t(g) \cdot g\phi_t(h)$ (g acting as conjugation by $\alpha_0(g)$) and so

$$\tilde{\phi}_t(gh) = \tilde{\phi}_t(g) \cdot g\tilde{\phi}_t(h) + \tilde{\phi}_t(g) + g\tilde{\phi}_t(h)$$

This gives the desired formula. □

Since $\mathcal{M}_k(P) = \prod_{i\geq 0} t^i \mathcal{M}_k(\mathbb{C})$, we may write ϕ_t in the form $\phi_t = \sum_i t^i\phi_i$, thereby defining $\phi_i \in C^1(\pi;\mathcal{M}_k(\mathbb{C}))$. Then we may reformulate Theorem (2.1) as:

Theorem 2.2. *If we write* $\tilde{\phi}_t = \sum_{i=1}^{\infty} \phi_i t^i$, $\phi_i \in C^1(\pi; \mathcal{M}_k(\mathbb{C}))$, *then*

$$\delta\phi_r = -\sum_{i=1}^{r-1} \phi_i \cup \phi_{r-i}$$

.

There are two inconveniences associated to the cochains $\tilde{\phi}_t$ or ϕ_i.

a) The ϕ_i are not cocycles (except for ϕ_1) and so a homological descrioption of the situation is not apparent.

b) The conditions on $\tilde{\phi}_t$ corresponding to α_t being unitary are unnatural. The formula $\phi_t(g)\overline{\phi_t}(g)^t = I$ becomes

$$\phi_r(g) + \overline{\phi}_r(g)^t + \sum_{i=1}^{r-1} \phi_i(g)\overline{\phi}_{r-i}(g)^t = 0 \tag{4}$$

which does not break up into independent conditions on each ϕ_i.

One way to deal with these difficulties is to concentrate on the lowest order terms of the deformation.

Proposition 2.3. *Suppose* $\alpha_t : \pi \to \mathcal{G}l_k(P)$ *and* ϕ_t *is as defined in (3). Suppose* $\phi_i = 0$ *for* $i < r$. *Then:*

a) ϕ_i *is a cocycle for* $i < 2r$ *and*

b) *If* α_t *is unitary, then* ϕ_i *is skew-Hermitian for* $i < 2r$, *i.e.* $\phi_i \in \mathcal{SH}_k(\mathbb{C})$, *the subspace of skew- Hermitian matrices, for every* $g \in \pi$.

Proof. These assertions follow easily from Theorem (2.2) and equation (4). □

As an interesting corollary of this Proposition we have the following. Suppose $\alpha_0 : \pi \to \mathcal{G}l_k(\mathbb{C})$ and $\tau \in H^1(\pi; \mathcal{M}_k(\mathbb{C}))$ where π acts on $\mathcal{M}_k(\mathbb{C})$ via the adjoint representation of α_0. $H^1(\pi; \mathcal{M}_k(\mathbb{C}))$ is the formal tangent space to the representation variety at α_0. We can ask whether there is a formal deformation, $\alpha_t : \pi \to \mathcal{G}l_k(P)$, of α_0 which is 'tangent to τ', i.e. so that ϕ_1 is a cocycle representative of τ . We have:

Corollary 2.4. *If* $\tau \cup \tau = 0$ *(a* necessary *condition) and every element of* $H^2(\pi; \mathcal{M}_k(\mathbb{C}))$ *can be written in the form* $\tau \cup \xi + \xi \cup \tau$ *for some* $\xi \in H^1(\pi; \mathcal{M}_k(\mathbb{C}))$, *then* α_t *exists.*

Proof. Suppose inductively that α_t exists mod t^r, for some $r \geq 2$,i.e. we have $\alpha_t : \pi \to \mathcal{G}l_k(P)$ such that $\alpha_t(gh) \equiv \alpha_t(g)\alpha_t(h) \pmod{t^r}$,for all $g, h \in \pi$, and ϕ_1 represents τ. For the inductive step we need to find $\gamma : \pi \to \mathcal{M}_k(\mathbb{C})$ so that $\alpha_t' = \alpha_t + t^r\gamma\alpha_0^{-1}$ is a homomorphism mod t^{r+1}. Thus we change ϕ_t to ${\phi_t}' = \phi_t + t^r\gamma$. The t^r coefficient of the required equation: ${\phi_t}'(gh) \equiv {\phi_t}'(g) \cdot g{\phi_t}'(h) \pmod{t^{r+1}}$ is the equation:

$$\gamma(gh) + \phi_r(gh) = \gamma(g) + g\gamma(h) + \phi_r(g) + g\phi_r(h) + \sum_{i=1}^{r-1} \phi_i(g) \cdot g\phi_{r-i}(h)$$

If we consider $\gamma, \phi_i \in C^1(\pi; \mathcal{M}_k(\mathbb{C}))$, this expression can be rewritten

$$-\delta\gamma - \delta\phi_r = \phi_1 \cup \phi_{r-1} + \phi_{r-1} \cup \phi_1 + \sum_{i=2}^{r-2} \phi_i \cup \phi_{r-i} \tag{5}$$

If $r = 2$, we can choose γ to satisfy (5) since $\tau \cup \tau = 0$. Suppose $r > 2$. We first observe that we may change ϕ_{r-1} to $\phi_{r-1} + u$ for any *cocycle* $u \in C^1(\pi; \mathcal{M}_k(\mathbb{C}))$ without disturbing the fact that α_t is a homomorphism mod t^r. This is clear since the equations which express this property are, for every $l < r$:

$$\delta\phi_l = \sum_{i=1}^{l-1} \phi_i \cup \phi_{l-i}$$

So now we can rewrite (5) as

$$-\delta\gamma = \phi_1 \cup u + u \cup \phi_1 + (\delta\phi_r + \sum_{i=1}^{r-1} \phi_i \cup \phi_{r-i}) \tag{6}$$

But since the term in parentheses is a cocycle our hypothesis says that we can choose u, γ to satisfy (6). $\square$

Remark. To obtain an analogous result for unitary representations we need to consider cohomology with coefficients in a Lie algebra. We discuss this below.

Another way to ameliorate these difficulties is to consider the *logarithm*:

$$\lambda_t(g) = \log\, \phi_t(g) = \sum_{i=1}^{\infty} (-1)^i \tilde{\phi}_t(g)^i / i \tag{7}$$

This is also a well-defined element of $C^1(\pi; \mathcal{M}_k(P))$ and, of course, $\tilde{\phi}_t$ can be recovered from it by exponentiating.

Proposition 2.5.
a) $\delta\lambda_t = -1/2[\lambda_t, \lambda_t{}'] - 1/12\,([[\lambda_t, \lambda_t{}'], \lambda_t{}'] - [[\lambda_t, \lambda_t{}'], \lambda_t]) + \cdots$
where $\lambda_t{}' = \lambda_t$
b) α_t *is unitary iff.* α_0 *is unitary and* λ_t *is skew-Hermitian.*

Explanations.
a) We view $\mathcal{M}_k(P)$ as a Lie algebra with the usual bracket $[M, N] = MN - NM$. The terms on the right are various "cup products" defined by using iterated brackets. If L is a Lie algebra with a left π-module structure satisfying $g[a, b] = [ga, gb]$ then any formal bracket in two variables defines a "cup product" $C^p(X; L) \times C^q(X; L) \to C^{p+q}(X; L)$. For example the formal bracket $[[x, y], x]$ defines $(u, v) \to w$ where $w(\sigma) = [[u(\sigma'), v(\sigma''], u(\sigma')]$ for any $\sigma \in \mathcal{S}(\tilde{X})$. Note that these cup products are not bilinear except for the special case of $[x, y]$. This particular cup product also satisfies the Liebnitz formula and so induces a cup product on cohomology. In the right-hand formula of (2.8a) we introduce an alias

$\lambda_t{}'$ of λ_t to indicate which formal brackets are used for the various cup products. The entire formula is just the usual Campbell-Baker-Haussdorf formula.

b) Let $\mathcal{S}H_k(P) \subseteq \mathcal{M}_k(P)$ be the Lie subalgebra and, assuming α_0 is unitary, π-submodule consisting of skew-Hermitian matrices, i.e. $M \in \mathcal{S}H_k(P)$ iff. $\overline{M}^t = -M$. So $C^*(X; \mathcal{S}H_k(P)) \subseteq C^*(X; \mathcal{M}_k(P))$, a subcomplex. We are asserting that $\lambda_t \in C^*(\pi; \mathcal{S}H_k(P))$ iff. α_t is unitary or, equivalently, if $\lambda_t = \sum \lambda_i t^i$ ($\lambda_i \in C^*(\pi; \mathcal{M}_k(\mathbb{C}))$), then α_t is unitary iff. α_0 is unitary and every $\lambda_i \in C^*(\pi; \mathcal{S}H_k(\mathbb{C}))$.

Remark. If $\alpha_t : \pi \to \mathcal{S}U_k(P)$, the special unitary group, then $\lambda_t : \pi \to \mathcal{S}H_k^0(P)$, skew-Hermitian matrices with zero trace.

Proof. a) Recall the formula $\phi_t(gh) = \phi_t(g) \cdot g\phi_t(h)$ from the proof of Theorem 2.1. Taking logarithms and applying the Campbell-Baker-Haussdorf formula to the right side gives the following:

$$\lambda_t(gh) = \lambda_t(g) + \log(g\phi_t(h)) + 1/2\,[\lambda_t(g), \log(g\phi_t(h)] + \cdots$$

To obtain the formula in (a) we only need to point out that $\log(g\phi_t(h)) = g{\cdot}\log\,\phi_t(h)$ since $(g\tilde{\phi_t}(h))^i = g\tilde{\phi_t}(h)^i$ because the action of π is defined by the adjoint action under α_0.

b) The Campbell-Baker-Haussdorf formula gives

$$\log(\phi_t(g)\overline{\phi_t(g)}^t) = \lambda_t(g) + \overline{\lambda_t(g)}^t - 1/2\,[\lambda_t(g), \overline{\lambda_t(g)}^t] + \ldots$$

since $\log \overline{\phi_t(g)}^t = \overline{\log\,\phi_t(g)}^t$. Now suppose $\lambda_t(g) + \overline{\lambda_t(g)}^t = t^r\sigma_t(g)$ where $\sigma_0(g) \neq 0$. If we substitute $\overline{\lambda_t(g)}^t = t^r\sigma_t(g) - \lambda_t(g)$ into all the bracket terms and use the fact that $[\lambda_t(g), \lambda_t(g)] = 0$, and $\lambda_0(g) = 0$, we find that $\log(\phi_t(g)\overline{\phi_t(g)}^t) \equiv t^r\sigma_t(g) \pmod{t^{r+1}}$. Thus we conclude that $\log(\phi_t(g)\overline{\phi_t(g)}^t) = 0$ iff. $\lambda_t(g) + \overline{\lambda_t(g)}^t = 0$. □

Corollary 2.6. *If $k = 1$, then every λ_i is a cocycle.*

Now suppose we have a unitary representation $\alpha_0 : \pi \to \mathcal{U}_k(P)$ and a "tangent vector at α_0", $\tau \in H^1(\pi; \mathcal{S}H_k(P))$. We ask whether there is a formal unitary deformation α_t of α_0 so that if $\lambda_t = \log\,\phi_t$ then λ_1 represents τ.

Analogous to Corollary (2.4) we have:

Corollary 2.7. *Suppose $[\tau, \tau] = 0$ and every element of $H^2(\pi; \mathcal{S}H_k(P))$ can be written in the form $[\tau, \xi] + [\xi, \tau]$ for some $\xi \in H^1(\pi; \mathcal{S}H_k(P))$. Then α_t exists.*

Proof. By (7) such a deformation α_t corresponds to a function $\lambda_t : \pi \to \mathcal{S}H_k(P)$ satisfying $\lambda_0 = 0$ and

$$\lambda_t(gh) = \lambda_t(g) + g\lambda_t(h) + 1/2\,[\lambda_t(g), g\lambda_t(h)] + \cdots \tag{8}$$

where the right side is the Campbell-Baker-Haussdorf formula. Suppose we have λ_t which satisfies this equation mod $t^r, r \geq 2$. We will, as above, try to find

$\lambda_t{}' = \lambda_t + t^r\xi$, where $\xi : \pi \to \mathcal{S}H_k(\mathbb{C})$. The equation for the coefficient of t^r becomes:

$$\lambda_r(gh) + \xi(gh) = \\ \lambda_r(g) + g\lambda_r(h) + \xi(h) + 1/2\left([\lambda_1(g), g\lambda_{r-1}(h)] + [\lambda_{r-1}(g), g\lambda_1(h)]\right) + \cdots$$

where the omitted terms involve neither λ_1 nor λ_{r-1} . Interpreting ξ and λ_i as cochains, this becomes

$$\delta\lambda_r + \delta\xi = 1/2\left([\lambda_1, \lambda_{r-1}] + [\lambda_{r-1}, \lambda_1]\right) + \cdots \tag{9}$$

If $r = 2$, there are no omitted terms and $[\tau, \tau] = 0$ assures that we can choose ξ appropriately. If $r > 2$ we allow ourselves to replace λ_{r-1} by $\lambda_{r-1} + u$, where u is a cocycle. As before this does not affect the fact that λ_t satisfies (8) mod t^r. Now equation (9) becomes

$$\delta\lambda_r + \delta\xi = 1/2\left([\lambda_1, u] + [u, \lambda_1]\right) + \cdots \tag{10}$$

where the omitted terms involve neither ξ nor u. The existence of ξ and u satisfying (10) now follows from the hypothesis. □

Remark. The theorem also holds for special unitary representations using $\mathcal{S}H^0_k(P)$.

Example: Consider representations into $\mathcal{S}U_2$. Then $\mathcal{S}H^0_2(\mathbb{C}) \cong \mathbb{R} \oplus \mathbb{C}$ via

$$\begin{pmatrix} it & z \\ -\overline{z} & -it \end{pmatrix} \leftrightarrow (t, z) \quad \text{with the bracket operation defined by the formula:}$$

$$[(t, z), (s, w)] = (i(z\overline{w} - \overline{z}w), 2i(tw - sz)).$$

Suppose α_0 is *reducible*, i.e. $\alpha_0(g) = \begin{pmatrix} \omega(g) & 0 \\ 0 & \overline{\omega(g)} \end{pmatrix}$ for some homomorphism $\omega : \pi \to S^1$. Then

$$H^*(\pi; \mathcal{S}H^0_2(\mathbb{C})) \cong H^*(\pi; \mathbb{R}) \oplus H^*(\pi; \mathbb{C}_{\omega^2})$$

where the first term has untwisted coefficients and, in the second, π acts on $\mathbb{C}$ by $g \to \omega(g)^2$. We can see easily that the cup product

$$H^1(\pi; \mathcal{S}H^0_2(\mathbb{C})) \times H^1(\pi; \mathcal{S}H^0_2(\mathbb{C})) \to H^2(\pi; \mathcal{S}H^0_2(\mathbb{C}))$$

is described as follows. If $\xi, \xi' \in H^1(\pi; \mathbb{R})$ and $\eta, \eta' \in H^1(\pi; \mathbb{C}_{\omega^2})$, then

$$[\xi, \xi'] = 0, \quad [\eta, \eta'] = [\eta', \eta] = \eta \cup \eta' \in H^2(\pi; \mathbb{R})$$

where the last term is the cup product defined by the pairing $\mathbb{C}_{\omega^2} \otimes \mathbb{C}_{\omega^2} \to \mathbb{R}, \quad (z, w) \to i(z\overline{w} - w\overline{z})$. and

$$[\xi, \eta] = [\eta, \xi] = 2i(\xi \cup \eta) \in H^2(\pi; \mathbb{C}_{\omega^2})$$

where the last term refers to the cup product pairing defined by $\mathbb{R}\otimes\mathbb{C}_{\omega^2}\to\mathbb{C}_{\omega^2}$, $(t,z)\to tz$. The commutativity of these cup products follows from the usual super-commutativity of cup products and the fact that the pairing $\mathbb{R}\otimes\mathbb{C}_{\omega^2}\to\mathbb{C}_{\omega^2}$ is the negative of the relevant pairing $\mathbb{C}_{\omega^2}\otimes\mathbb{R}\to\mathbb{C}_{\omega^2}$ while $\mathbb{C}_{\omega^2}\otimes\mathbb{C}_{\omega^2}\to\mathbb{R}$ is anti-commutative.

Suppose π is a knot group. Then $H^2(\pi;\mathbb{R})=0$ and the only non-trivial cup products are $[\xi,\eta]=[\eta,\xi]$. It is well-known that $H^1(\pi;\mathbb{C}_{\omega^2})=H^2(\pi;\mathbb{C}_{\omega^2})=0$ unless ω^2 is a root of the Alexander polynomial. Since

$$H^1(\pi;\mathbb{R})\otimes_{\mathbb{R}}H^1(\pi;\mathbb{C}_{\omega^2})\to H^2(\pi;\mathbb{C}_{\omega^2})$$

is an isomorphism, by duality, we conclude that, for $\tau=\xi+\eta$ to be the tangent vector of a deformation we need $\xi\cup\eta=0$ which requires that either $\xi=0$ or $\eta=0$. If $\eta=0$ and $\xi\neq 0$, then $\xi\cup H^1(\pi;\mathbb{C}_{\omega^2})=H^2(\pi;\mathbb{C}_{\omega^2})$ so the formal deformation exists. On the other hand this is easy to see directly, since the circle of reducible representations gives such a deformation. If $\xi=0$, it is false that $H^1(\pi;\mathbb{R})\cup\eta=H^2(\pi;\mathbb{C}_{\omega^2})$.

We now examine the effect of changing the deformation $\alpha_t:\pi\to\mathcal{G}l_k(P)$ by a conjugation. Let $\theta_t\in\mathcal{G}l_k(P)$ with θ_0 =identity and consider $\alpha_t':\pi\to\mathcal{G}l_k(P)$ defined by $\alpha_t'=\theta_t\alpha_t(g)\theta_t^{-1}$ (If $\alpha_t:\pi\to\mathcal{U}_k(P)$ we would demand that $\theta_k\in\mathcal{U}_k(P)$) Let ${\phi_t}'(g)=\alpha_t'(g)\alpha_0(g)^{-1}$.

Proposition 2.8. *Given $\alpha_t,\alpha_t',\theta_t$ as above:*

a) *Suppose $\phi_i=0$ for $i<r$. Then $\phi_i'=0$ for $i<r$ and, if $\theta_i=0$ for $0<i<l$, then ϕ_i and ϕ_i' are cohomologous (more specifically $\phi_i'-\phi_i=\delta\theta_i$) for $i<r+l$. In particular ϕ_r and ϕ_r' are always cohomologous. If θ_t is unitary, then $\theta_i\in\mathcal{S}H_k(\mathbb{C})$ for $i\le 2l$. In other words, the "gauge equivalence" class of α_t determines the cohomology class of ϕ_i in $H^1(\pi;\mathcal{M}_k(\mathbb{C}))$ for $i\le r+l$ and, in the unitary case, in $H^1(\pi;\mathcal{S}H_k(\mathbb{C}))$ for $i\le l+min\{r,l\}$*

b) *Suppose α_t and θ_t are unitary. Set $\lambda_t=\log\phi_t,{\lambda_t}'=\log{\phi_t}'$ and $\eta_t=\log\theta_t$. Then, for any $g\in\pi$*

$$\lambda_t(g)-{\lambda_t}'(g)=(\delta\eta_t)(g)+\text{ brackets involving }\eta_t\text{ or }g\eta_t$$

Corollary 2.9. *If $k=1$, then λ_t and ${\lambda_t}'$ are cohomologous.*

Proof. (a) We have

$$\phi_t{}'(g)=\theta_t\phi_t(g)g\theta_t^{-1} \tag{11}$$

If $\theta_t=1+\Omega_t$, then $\theta_t^{-1}\equiv 1-\Omega_t \pmod{t^2}$ and so:

$$\begin{aligned}{\phi_t}'&\equiv\phi_t(g)+\Omega_t\phi_t(g)-\phi_t(g)\cdot g\Omega_t &&\pmod{t^{2l+r}}\\ &\equiv\phi_t(g)+\Omega_t-g\Omega_t &&\pmod{t^{r+l}}\end{aligned}$$

If we consider $\Omega_t\in C^0(\pi;\mathcal{M}_k(P))$, then this equation becomes

$$\phi_t(g)-{\phi_t}'(g)\equiv(\delta\Omega_t)(g)\pmod{t^{r+l}}$$

(b) If we apply the Campbell-Baker-Haussdorf formula to (11) with the observation that $\log(g\theta_t^{-1})=g\log(\theta_t^{-1})=-g\log\theta_t$ we get

$${\lambda_t}'(g)=\lambda_t(g)+(1-g)\eta_t+1/2\left([(1+g)\eta_t,\lambda_t(g)]-[\eta_t,g\eta_t]\right)+\cdots$$

So, if we consider $\eta_t\in C^0(\pi;\mathcal{S}H_k(P))$, then $(g-1)\eta_t=(\delta\eta_t)(g)$. □

3. Relating the cohomologies

We can now return to the question of the relation between the coboundary operators δ and δ_t in $C^*(X;\alpha_0)_t \cong C^*(X;\alpha_t)$, respectively.Suppose $\pi = \pi_1(X, x_0)$, or more generally, we have a homomorphism
$\pi_1(X, x_0) \to \pi$. Then we have an induced map $X \to B\pi$ and so a chain map $\xi : C(\tilde{X}) \to C(\pi)$. Using the bar resolution this can be described as follows. Suppose x_0 and $\tilde{x}_0$ are chosen as above and $\sigma \in \mathcal{S}(\tilde{X})$. Then each vertex v_i of the standard simplex determines $h_i \in \pi$ by $\sigma(v_i) = h_i(\tilde{x}_0)$. Then $\xi(\sigma) = h_0[g_1|\dots|g_n]$ where $g_i = h_{i-1}^{-1}h_i$. It is straightforward to check that this is a chain-map and a $\mathbb{Z}\pi$-homomorphism. Recall ([1]) that $C(\pi)$ can be described as the ordered simplicial chain complex of the simplex Δ whose vertices are the elements of π and the action of π is the obvious simplicial one (it is not free on Δ if π has elements of finite order). Then ξ is induced by $\sigma \to$the ordered simplex $< h_0, \dots h_n >$. Define the 1-cochain $\psi_t = \xi^\sharp(\tilde{\phi}_t) \in C^*(X; \mathcal{M}_k(P))$, π-action defined by α_0. Then, from Theorem (2.1) we have:

$$\delta\psi_t = -\psi_t \cup \psi_t \tag{12}$$

Now there is also a cup product

$$C^*(X; \mathcal{M}_k(P)) \times C^*(X;\alpha_0)_t \to C^*(X;\alpha_0)_t$$

defined by the pairing $\mathcal{M}_k(P) \times V_t \to V_t$ given by matrix multiplication (The π-actions defined by α_0). The next result says that the cohomological effect of the deformation α_t on X is determined entirely by the cocycle ψ_t.

Theorem 3.1. *If $u \in C^*(X;\alpha_0)_t$, then $\delta_t u = \delta u + \psi_t \cup u$.*

Proof. It suffices to check the values of both sides when evaluated on an element σ of $\mathcal{S}_0(\tilde{X})$. Write $\partial\sigma = \sum_{i=0}^n (-1)^i \sigma_i$; then $\sigma_i \in \mathcal{S}_0(\tilde{X})$ for $1 \le i \le n$, but we must write $\sigma_0 = g \cdot \overline{\sigma}_0$ where $g \cdot \tilde{x}_0 = \sigma(v_1)$ and $\overline{\sigma}_0 \in \mathcal{S}_0(\tilde{X})$. Thus

$$\delta_t u(\sigma) = \alpha_t(g) \cdot u(\overline{\sigma}_0) + \sum_{i=1}^n (-1)^i u(\sigma_i)$$

and

$$\delta u(\sigma) = \alpha_0(g) \cdot u(\overline{\sigma}_0) + \sum_{i=1}^n (-1)^i u(\sigma_i)$$

Subtracting these, we get

$$(\delta_t u - \delta u)(\sigma) = (\alpha_t(g) - \alpha_0(g)) \cdot u(\overline{\sigma}_0)$$

and so

$$(\delta_t u - \delta u)(\sigma) = \tilde{\phi}_t(g)\alpha_0(g) \cdot u(\overline{\sigma}_0)$$

On the other hand, $(\psi_t \cup u)(\sigma) = \psi_t(\sigma')u(\sigma'')$. One sees readily that $\xi(\sigma') = [g]$, and so $\psi_t(\sigma') = \tilde{\phi}_t(g)$, and $\sigma'' = \sigma_0$, so that $u(\sigma'') = u(g \cdot \overline{\sigma}_0) = \alpha_0(g)u(\overline{\sigma}_0)$ This completes the proof. □

The advantage of reducing the description of the cochain complex $C^*(X;\alpha_t)$ to $C^*(X;\alpha_0)_t$ is that the latter can be identified with the direct product of a countable number of copies of $C^*(X;\alpha_0) = \mathrm{Hom}_\pi(C(\tilde{X}), V_0)$, where V_0 is just $\mathbb{C}^k$ with π-action defined by α_0. If $u \in C^*(X;\alpha_0)_t$, then we may write $u = \sum_{i=0}^\infty u_i t^i$, where $u_i \in C^*(X;\alpha_0)$ and $\delta u = \sum_i (\delta u_i)t^i$. Similarly $C^*(X;\mathcal{M}_k(P))$ is a direct product of a countable number of copies of $C^*(X;\mathcal{M}_k(\mathbb{C}))$. There are cup product pairings

$$C^*(X;\mathcal{M}_k(\mathbb{C})) \times C^*(X;\alpha_0) \to C^*(X;\alpha_0)$$

and

$$C^*(X;\mathcal{M}_k(\mathbb{C})) \times C^*(X;\mathcal{M}_k(\mathbb{C})) \to C^*(X;\mathcal{M}_k(\mathbb{C}))$$

induced by matrix multiplications and, clearly, if $u = \sum_i u_i t^i \in C^*(X;\mathcal{M}_k(P))$ and $v = \sum_i u_i t^i \in C^*(X;\alpha_0)$,then

$$u \cup v = \sum_{r=0}^\infty \sum_{i=0}^r (u_i \cup v_{r-i})t^r$$

Thus we can rephrase Theorem (3.1) as follows.

Theorem 3.2. *If $u \in C^*(X;\alpha_0)_t, u = \sum_{i=0}^\infty u_i t^i$ and $\psi_i = \xi\ (\phi_i)$, then*

$$\delta_t u = \sum_{r=0}^\infty (\delta u_r + \sum_{i=1}^r \psi_i \cup u_{r-i})t^r$$

.

4. Forms on the cohomology at a representation defined by a deformation

We now exploit Theorem (3.1) to describe the torsion part of $H^*(X;\alpha_t)$ by means of a filtration of $H^*(X;\alpha_0)$. This could, alternatively, be done by a simple spectral sequence argument, but we especially want to have a more explicit description of the isomorphisms than one obtains from the spectral sequence approach. These considerations are quite similar to the work of [5] which takes place in an *analytic* context.

First we define a filtration $\{J^i\}$ of $H^*(X;\alpha_0)$. We will say $\alpha \in J^i$ iff. there exists a cochain $u_t = \sum u^i t^i \in C^*(X;\alpha_0)_t$ such that:

(i) u_0 is a cocycle representing α
(ii) $\delta u_t + \psi_t \cup u_t \equiv 0 \pmod{t^{i+1}}$

Obviously

$$H^*(X;\alpha_0) = J^0 \supseteq \cdots \supseteq J^i \supseteq J^{i+1} \supseteq \cdots$$

Now we define homomorphisms

$$\tau_i : J^i \to H^*(X;\alpha_0)/\operatorname{Im}\tau_{i-1} \quad (i \geq 0)$$

recursively, as follows. Set $\tau_{-1} = 0$. For $i \geq 0$ let $\tau_i(\alpha)$ be the cohomology class represented by v, the t^{i+1}-coefficient of $\delta u_t + \psi_t \cup u_t$, where u_t satisfies (ii). In particular $\tau_0(\alpha) = [\psi_1] \cup \alpha$. For $i \geq 1$ we must show:

Claim 4.1.
a) *v is a cocycle.*
b) *a different choice u'_t produces v' so that $v'-v$ represents an element of* $\operatorname{Im}\tau_{i-1}$.
This will show that τ_i is well-defined for all $i \geq 0$. Note that we, somewhat imprecisely, use the notation $\operatorname{Im}\tau_{i-1}$ to refer to the pull-back of the *actual* $\operatorname{Im}\tau_{i-1}$ in $H^*(X;\alpha_0)$. We will also show:
c) $\operatorname{Ker}\tau_i = J^{i+1}$*, for $i \geq 0$.*

Proof. (a)

$$\begin{aligned}\delta(\psi_t \cup u_t) &= \delta\psi_t \cup u_t - \psi_t \cup \delta u_t \\ &\equiv -(\psi_t \cup \psi_t) \cup u_t - \psi_t \cup (-\psi_t \cup u_t) \pmod{t^{i+2}} \\ &\equiv 0 \pmod{t^{i+2}}\end{aligned}$$

(b). It suffices to show that if u_t satisfies (ii) and u_0 is a coboundary, then v represents an element of $\operatorname{Im}\tau_{i-1}$. Let us write $u_t = \delta w + t u'_t$, $w \in C^*(X;\alpha_0)$. Then set $u''_t = u'_t - \psi'_t \cup w$, where $\psi_t = t\psi'_t$. Now we compute:

$$\begin{aligned}\delta u''_t + \psi_t \cup u''_t &= \delta u_t\prime - \delta(\psi'_t \cup w) + \psi_t \cup u'_t - \psi_t \cup (\psi'_t \cup w) \\ &= \delta u'_t - \delta\psi'_t \cup w + \psi'_t \cup \delta w + \psi_t \cup u'_t + \delta\psi'_t \cup w \\ &= \delta u'_t + \psi'_t \cup \delta w + \psi_t \cup u'_t\end{aligned}$$

using (12). Multiplying by t we obtain:

$$\begin{aligned}t(\delta u''_t + \psi_t \cup u''_t) &= \delta(t u'_t + \delta w) + \psi_t \cup (\delta w + t u'_t) \\ &= \delta u_t + \psi_t \cup u_t\end{aligned}$$

To see that u''_0 is a cocycle we show that $\delta u''_t$ is divisible by t.

$$\begin{aligned}t\delta u''_t &= t\delta u'_t - t\delta(\psi'_t \cup w) \\ &= \delta(t u'_t) - \delta\psi_t \cup w + \psi_t \cup \delta w \\ &= \delta u_t + \psi_t \cup (u_t - t u'_t) - \delta\psi_t \cup w \\ &\equiv \delta u_t + \psi_t \cup u_t \pmod{t^2} \\ &\equiv 0 \pmod{t^2}\end{aligned}$$

since $i \geq 1$.

(c). It is clear that $J^{i+1} \subseteq \operatorname{Ker}\tau_i$. For the converse, suppose that $\alpha \in \operatorname{Ker}\tau_i$. This means there exist u_t, u'_t, v so that u_0 represents α and

$$\begin{aligned}\delta u_t + \psi_t \cup u_t &\equiv t^{i+1} v \pmod{t^{i+2}} \\ \delta u'_t + \psi_t \cup u'_t &\equiv t^i (v + \delta w) \pmod{t^{i+1}}\end{aligned}$$

for some w. Therefore, if $u''_t = u_t - t u'_t$, then $\delta u''_t + \psi_t \cup u''_t \equiv -t^{i+1}\delta w \pmod{t^{i+2}}$. Thus $u''_0 = u_0$ represents an element of J^{i+1}. □

We now recast these results in a more spectral-sequence-like format. For this we establish:

Proposition 4.2. $\operatorname{Im}\tau_r \subseteq J^s$ *for any* r, s.

Proof. We proceed inductively. Denote by A(r,s) the assertion of the Proposition for specific given values of r, s. We first observe that A(r,s) only makes sense if A(r-1,s) is assumed to be true. We proceed inductively on r for a given value of s. Suppose $\alpha \in J^r$ is represented by u_0 where $\delta u_t + \psi_t \cup u_t = t^{r+1} v_t$ and so v_0 represents $\tau_r(\alpha)$. It suffices to show that $\delta v_t + \psi_t \cup v_t$ is divisible by t^{s+1}. But, in fact,

$$\begin{aligned} t^{r+1}\delta v_t &= \delta(\delta u_t + \psi_t \cup u_t) \\ &= \delta\psi_t \cup u_t - \psi_t \cup \delta u_t \\ &= -\psi_t \cup \psi_t \cup u_t - \psi_t \cup (t^{r+1} v_t - \psi_t \cup u_t) \\ &= -t^{r+1}(\psi_t \cup v_t) \end{aligned}$$

□

As a consequence of Proposition 4.2 we see that:

$$\operatorname{Im}\tau_i \subseteq J^i / \operatorname{Im}\tau_{i-1} \quad \text{and} \quad \tau_i(\operatorname{Im}\tau_{i-1}) = 0$$

Thus we can define

$$L^i = J^i / \operatorname{Im}\tau_{i-1} = \operatorname{Ker}\tau_{i-1} / \operatorname{Im}\tau_{i-1}$$

Now we see that τ_i induces $\partial_i : L^i \to L^i$ and $\partial_i^2 = 0$. So we can interpret L^{i+1} as the homology of the chain complex (L^i, ∂_i).

We now define some $\pm$-Hermitian pairings on the $\{J^i\}$ and $\{L^i\}$. Suppose α_t is unitary and X is a connected oriented Poincare complex of formal dimension n. We have a cup product

$$H^p(X;\alpha_0) \times H^{n-p}(X;\alpha_0) \to H^n(X;\mathbb{C}) \cong \mathbb{C}$$

defined by the π-homomorphism (using α_0) $V_0 \otimes V_0 \to \mathbb{C}$ coming from the inner product on V_0. Duality tells us that this pairing, which we denote by (,), is non-singular. We also have a cup product

$$C^*(X;\alpha_t) \times C^*(X;\alpha_t) \to C^*(X;P)$$

defined by the π-homomorphism (using α_t) $V_t \otimes_{\mathbb{C}} V_t \to P$ coming from the inner product on V_t. It is, of course, essential here that α_t be unitary.

Proposition 4.3.
a) $(J^i, \operatorname{Im}\tau_{i-1}) = 0$.
b) $(\alpha, \tau_i(\beta)) = (-1)^{p+1}(\tau_i(\alpha), \beta)$ *for any* $\alpha, \beta \in J^i$, *where* $p = \dim\alpha$.

Proof. We prove these together inductively. Let $(a)_i, (b)_i$ denote the assertions for the particular i as stated. First of all, we observe that the terms in $(b)_i$ are only well-defined once we know that $(a)_i$ is true. Second of all, it is immediate that $(b)_i$ implies $(a)_{i+1}$. Thus we only have to prove that $(a)_i$ implies $(b)_i$.

Choose u_t and u'_t so that

$$\delta u_t + \psi_t \cup u_t = t^{i+1} v_t$$
$$\delta u'_t + \psi_t \cup u'_t = t^{i+1} v'_t$$

where u_0 and u'_0 represent α and β, respectively, and so v_0 and v'_0 represent $\tau_i(\alpha)$ and $\tau_i(\beta)$, respectively. These equations take place in $C^*(X;\alpha_0)_t$. If we rewrite them as equations in $C^*(X;\alpha_t)$, then, according to Theorem (3.1), they become $\delta_t u_t = t^{i+1} v_t$ and $\delta_t u'_t = t^{i+1} v'_t$. Using the cup product into $C^*(X;P)$, we then have

$$t^{i+1}(u_t \cup v'_t) = u_t \cup \delta_t u'_t$$

and

$$t^{i+1}(v_t \cup u'_t) = \delta_t u_t \cup u'_t$$

Applying the Liebnitz rule we get

$$\begin{aligned} t^{i+1}(v_t \cup u'_t + (-1)^p u_t \cup v'_t) &= \delta_t u_t \cup u'_t + (-1)^p u_t \cup \delta_t u'_t \\ &= \delta_t (u_t \cup u'_t) \quad \text{in } C^n(X;P) \end{aligned}$$

Therefore $v_t \cup u'_t + (-1)^p u_t \cup v'_t$ represents a cohomology class of order t^{i+1} in $H^n(X;P) \cong P$, which implies that it is, in fact, null-cohomologous. Now if we set $t = 0$ we get a null-cohomologous cocycle which represents $(\tau_i(\alpha), \beta) + (-1)^p(\alpha, \tau_i(\beta))$. □

From Proposition 4.3 it follows that $(,)$ induces a pairing on L^i and that ∂_i is $\pm$-self-adjoint.

We now define a new pairing $\{ , \}_i$ on J^i(and L^i), with values in $\mathbb{C}$, by $\{\alpha, \beta\}_i = (\alpha, \tau_i(\beta))(\text{or}(\alpha, \partial_i(\beta)))$. By Proposition 4.3(b) $\{ , \}_i$ is $\pm$-Hermitian. More precisely

$$\{\alpha, \beta\}_i = (-1)^{(p+1)(n-p)} \overline{\{\beta, \alpha\}_i} \quad \text{where } p = \dim \alpha$$

Note that this pairing is non-trivial only when $\dim \alpha + \dim \beta = n - 1$.

Proposition 4.4. *J^{i+1} is the null-space of $\{ , \}_i$ and so $\{ , \}_i$ induces a non-singular pairing on J^i/J^{i+1}.*

Proof. Since $(,)$ is non-singular,the Proposition will follow from the fact that J^{i+1} and $\operatorname{Im} \tau_i$ are orthogonal complements under $(,)$. In light of Proposition 4.3(a) it is only necessary to prove that $\dim J^{i+1} + \dim \operatorname{Im} \tau_i = \dim H^*(X;\alpha_0)$. But the definitions and Claim 4.1(c) imply that $J^i/J^{i+1} \cong \operatorname{Im} \tau_i / \operatorname{Im} \tau_{i-1}$ and so the quantity $\dim J^{i+1} + \dim \operatorname{Im} \tau_i$ is independent of i. But for $i = -1$, we have $J^0 = H^*(X;\alpha_0)$ and $\tau_{-1} = 0$. □

Since they are $\pm$-Hermitian, the pairings $\{ , \}_i$ have well- defined signatures (which are non-zero only when n is odd). These are the topological version of the signatures used in [5] .

We now relate these constructions in $H^*(X;\alpha_0)$ to the structure of $H^*(X;\alpha_t)$ and the pairing $<, >$ defined in §1, equation (2). Let

$$K_k = \operatorname{Ker} t^k \subseteq H^*(X;\alpha_t) \quad \text{for } k \geq 0$$

Proposition 4.5. *There are isomorphisms $K_k/K_{k-1} \cong J^{k-1}/J^\infty$, for $k \geq 1$ and $J^\infty \subseteq \bigcap_k J^k$ defined below, under which the injection $K_{k+1}/K_k \xrightarrow{t} K_k/K_{k-1}$ corresponds to the inclusion $J^k \subseteq J^{k-1}$. Thus $J^{k-1}/J^k \cong K_k/(K_{k-1} + tK_{k+1})$.*

Proof. The short exact sequences:

$$0 \to V_t \xrightarrow{t^k} V_t \to V_t/t^k V_t \to 0$$

$$0 \to V_t/t^{k-1}V_t \xrightarrow{t} V_t/t^k V_t \to V_0 \to 0$$

yield long exact cohomology sequences which give, respectively, the horizontal and vertical lines of the following commutative diagram:

$$\begin{array}{ccccccc} & & H^*(X; V_t/t^{k+1}V_t) & & & & \\ & & \downarrow t & & & & \\ H^*(X;\alpha_t) & \longrightarrow & H^*(X; V_t/t^k V_t) & \xrightarrow{\delta_k} & K_k & \longrightarrow & 0 \\ \| & & \downarrow e_k & & & & \\ H^*(X;\alpha_t) & \xrightarrow{e_\infty} & H^*(X;\alpha_0) & & & & \end{array} \tag{13}$$

We will show that $\operatorname{Im} e_k = J^{k-1}$ and that $e_k \circ \delta_k^{-1}$ induces the desired isomorphism. $\operatorname{Im} e_k$ is the kernel of the coboundary homomorphism $H^*(X;\alpha_0) \to H^*(X; V_t/t^{k-1}V_t)$ and so $\alpha \in \operatorname{Im} e_k$ iff. there is a cochain $u_t \in C^*(X;\alpha_t)$ such that u_0 represents α and, for some cochain v_t, $\delta_t(u_t - tv_t) \equiv 0 \pmod{t^k}$. Replacing u_t by $u_t - tv_t$, we may just assume that $\delta_t u_t$ is divisible by t^k. By Theorem (3.1), this is exactly the criterion that $\alpha \in J^{k-1}$. Thus, it follows from diagram (13) that $e_k \circ \delta_k^{-1}$ induces an epimorphism $f_k : K_k \to J^{k-1}/J^\infty$, where $J^\infty = \operatorname{Im} e_\infty$. Now the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_t & \xrightarrow{t^{k-1}} & V_t & \longrightarrow & V_t/t^{k-1}V_t & \longrightarrow & 0 \\ & & \downarrow & & \downarrow t & & \downarrow t & & \\ 0 & \longrightarrow & V_t & \xrightarrow{t^k} & V_t & \longrightarrow & V_t/t^k V_t & \longrightarrow & 0 \end{array}$$

induces a commutative diagram:

$$\begin{array}{ccccc} H^*(X; V_t/t^{k-1}V_t) & \xrightarrow{\delta_{k-1}} & K_{k-1} & \longrightarrow & 0 \\ \downarrow t & & \downarrow \text{incl} & & \\ H^*(X; V_t/t^k V_t) & \xrightarrow{\delta_k} & K_k & \longrightarrow & 0 \end{array}$$

from which it follows that $\operatorname{Ker} f_k = K_{k-1}$.

The diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_t & \xrightarrow{t^{k+1}} & V_t & \longrightarrow & V_t/t^{k+1}V_t & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle t} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_t & \xrightarrow{t^{k}} & V_t & \longrightarrow & V_t/t^{k}V_t & \longrightarrow & 0
\end{array}$$

induces a commutative diagram:

$$\begin{array}{ccccc}
H^*(X; V_t/t^{k+1}V_t) & \xrightarrow{\delta_{k+1}} & K_{k+1} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle t} & & \\
H^*(X; V_t/t^{k}V_t) & \xrightarrow{\delta_{k}} & K_{k} & \longrightarrow & 0
\end{array}$$

from which we obtain:

$$\begin{array}{ccc}
K_{k+1} & \xrightarrow{f_{k+1}} & J^k/J^\infty \\
\downarrow{\scriptstyle t} & & \downarrow{\text{incl}} \\
K_k & \longrightarrow & J^{k-1}/J^\infty
\end{array}$$

and the final statement of the Proposition follows. $\square$

5. Relating the forms

We recall the non-singular pairing:

$$<,>: tH^*(X;\alpha_t) \times tH^*(X;\alpha_t) \to \hat{P}/P$$

define above in §1, . In [2] this is used to define the pairing: $<,>_k: K_k \times K_k \to \mathbb{C}$ by the formula $<\alpha,\beta>_k = t^k <\alpha,\beta> |_{t=0}$. Note that $t^k <\alpha,\beta>$ is a well-defined element of P/t^kP. Obviously $<,>_k$ is $\pm$-Hermitian and it is easy to check that $<\alpha,\beta>_k = 0$ if $\alpha \in K_{k-1} + tK_{k+1}$.

Proposition 5.1. *$<,>_k$ corresponds to $\{,\}_{k-1}$ under the isomorphism of Proposition (4.4).*

Proof. Suppose that z_t, w_t are cocycle representatives in $C^*(X;\alpha_t)$ of α, β respectively. Choose $u_t, v_t \in C^*(X;\alpha_t)$ so that $\delta_t u_t = t^k z_t$ and $\delta_t v_t = t^k w_t$. Then $<\alpha,\beta> = t^{-k}(u_t, w_t)_t$ where $(,)_t$ denotes the pairing

$$C^*(X;\alpha_t) \times C^*(X;\alpha_t) \xrightarrow{\text{cup}} C^*(X;P) \xrightarrow{[X]\cap} P$$

and so $<\alpha,\beta>_k = ([u_0],[w_0])$, where $[,]$ denotes the cohomology class in $H^*(X;\alpha_0)$.

Under the isomorphism $C^*(X;\alpha_t) \cong C^*(X;\alpha_0)_t$ and applying Theorem (3.1), the above equations become

$$\delta u_t + \psi_t \cup u_t = t^k z_t$$
$$\delta v_t + \psi_t \cup v_t = t^k w_t$$

Now it is a direct consequence of the definitions that $\alpha \mapsto [u_0], \beta \mapsto [v_0]$ under the map $K_k \to J^{k-1}/J^\infty$ of Proposition (4.4) and $\tau_k[v_0] = [w_0]$.

Thus $<\alpha,\beta> = ([u_0], \tau_k[v_0]) = \{[u_0],[v_0]\}_{k-1}$. $\square$

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