Joint probability for the Pearcey process

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Contents

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The results in this paper form a step in the direction of understanding the behavior of non-intersecting Brownian motions on $\mathbb{R}$ (Dyson's Brownian motions), when the number of particles tends to ∞. Consider n Brownian particles leaving from points $a_1 < \ldots < a_p$ and forced to end up at $b_1 < \ldots < b_q$ at time $t = 1$. It is clear that, when $n \to \infty$, the equilibrium measure for $t \sim 0$ has its support on p intervals and for $t \sim 1$ on q intervals. It is also clear that, when t evolves, intervals must merge, must disappear and be created, leading to various *phase transitions*, depending on the respective fraction of particles leaving from the points a_i and arriving at the points b_j. Therefore the region $\mathcal{R}$ in the space-time strip (x,t) formed by the support ($\subset \mathbb{R}$) of the equilibrium measure as a function of time $0 \leq t \leq 1$ will typically present singularities of different types.

Near the moments, where a phase transition takes place, one expects to find in the limit $n \nearrow \infty$ an infinite-dimensional diffusion, *a Markov cloud*, having some universality properties. Universality here means that the infinite-dimensional diffusion is to depend on the type of singularity only. These Markov clouds are infinite-dimensional diffusions, which 'in principle' could be described by an infinite-dimensional Laplacian with a drift term. We conjecture that each of the Markov clouds obtained in this fashion is related to some *integrable system*, which enables one to derive a non-linear (finite-dimensional) PDE, satisfied by the joint probabilities. The purpose of this paper is to show, for a simple model leading to a cusp, that the joint probabilities at different times, do satisfy such a non-linear PDE. The interrelation between all such equations and the "initial" and "final" ($t \to \pm\infty$) conditions are interesting and challenging open problems. Moreover, special cases have shown an intimate connection between the integrable system and the Riemann-Hilbert problem associated with the singularity.

The **first question** is the study of the finite Brownian motion model, which, as will be explained in section 2, hinges on the study of the coupled Gaussian Hermitian random matrix ensemble $\mathcal{H}_n$ with external source A, given by coupling terms $c_1, \ldots, c_m$ and the diagonal matrix (set $n = k_1 + k_2$)

$$A := \begin{pmatrix} \alpha & & & & & \\ & \ddots & & & \mathbf{O} & \\ & & \alpha & & & \\ & & & -\alpha & & \\ & \mathbf{O} & & & \ddots & \\ & & & & & -\alpha \end{pmatrix} \begin{matrix} \updownarrow k_1 \\ \\ \\ \\ \updownarrow k_2 \end{matrix} . \tag{0.1}$$

The probability of such an ensemble is defined by

$$\mathbb{P}_n(\alpha, c_1, \ldots, c_m; E_1, \ldots, E_m) := \frac{1}{Z_n} \int_{\prod_{\ell=1}^m \mathcal{H}_n(E_\ell)} e^{-\frac{1}{2}\operatorname{Tr}(M_1^2 + \ldots + M_m^2 - 2c_1 M_1 M_2 - \ldots - 2c_{m-1} M_{m-1} M_m - 2AM_m)} dM_1 \ldots dM_m.$$

Given a disjoint union of intervals and the associated algebra of differential operators

$$E_\ell := \bigcup_{i=1}^{r_\ell} [b_{2i-1}^{(\ell)}, b_{2i}^{(\ell)}] \subset \mathbb{R} \quad \text{and} \quad \mathcal{D}_k(E_\ell) = \sum_{i=1}^{2r} (b_i^{(\ell)})^{k+1} \frac{\partial}{\partial b_i^{(\ell)}}, \qquad 1 \le \ell \le m$$

and given the tridiagonal matrix

$$J^{-1} := J^{-1}(c_1, \ldots, c_{m-1}) := \begin{pmatrix} -1 & c_1 & & & \\ & & \ddots & & \mathbf{O} \\ c_1 & -1 & & \ddots & \\ & \ddots & \ddots & & \ddots \\ & & \ddots & -1 & c_{m-1} \\ & \mathbf{O} & \ddots & & \\ & & & c_{m-1} & -1 \end{pmatrix}, \tag{0.2}$$

define the following differential operators:

$$\begin{aligned}
\bar{\mathcal{A}}_1^{\pm} &:= -\frac{1}{2}\left(\sum_{j=1}^{m} J_{mj}\mathcal{D}_{-1}(E_j) \pm \frac{\partial}{\partial\alpha}\right)\\
\bar{\mathcal{C}}_1 &:= \sum_{j=1}^{m} J_{1j}\mathcal{D}_{-1}(E_j)\\
\bar{\mathcal{A}}_2 &:= \frac{1}{2}\left(\mathcal{D}_0(E_m) - \alpha\frac{\partial}{\partial\alpha} - c_{m-1}\frac{\partial}{\partial c_{m-1}}\right)\\
\bar{\mathcal{C}}_2 &:= -\mathcal{D}_0(E_1) + c_1\frac{\partial}{\partial c_1}.
\end{aligned}$$

Theorem 0.1 *The log of the probability $\mathbb{P}_n(\alpha; c_1, \ldots, c_{m-1}; E_1, \ldots, E_m)$ satisfies a fourth-order PDE in α, $c_1, \ldots, c_{m-1}$ and in the endpoints $b_1^{(\ell)}, ..., b_{2r}^{(\ell)}$ of the sets E_ℓ, with quartic non-linearity*[1]*:*

$$\begin{aligned}
0 &= \left(F^+\bar{\mathcal{C}}_1G^- + F^-\bar{\mathcal{C}}_1G^+\right)\{F^-, F^+\}_{\bar{\mathcal{C}}_1} - \left(F^+G^- + F^-G^+\right)\bar{\mathcal{C}}_1\{F^-, F^+\}_{\bar{\mathcal{C}}_1}\\
&= \det\begin{pmatrix} -G^+ & \bar{\mathcal{C}}_1F^+ & -F^+ & 0\\ G^- & \bar{\mathcal{C}}_1F^- & -F^- & 0\\ -\bar{\mathcal{C}}_1G^+ & \bar{\mathcal{C}}_1^2F^+ & 0 & -F^+\\ \bar{\mathcal{C}}_1G^- & \bar{\mathcal{C}}_1^2F^- & 0 & -F^- \end{pmatrix}
\end{aligned} \tag{0.3}$$

where

$$\begin{aligned}
F^{\pm} &:= \bar{\mathcal{A}}_1^{\pm}\bar{\mathcal{C}}_1\log\mathbb{P}_n + k_{\left\{ {1 \atop 2} \right\}}J_{1m},\\
G^{\pm} &:= \left\{\left(\bar{\mathcal{A}}_2\bar{\mathcal{C}}_1 \pm J_{1m}\frac{\partial}{\partial\alpha}\right)\log\mathbb{P}_n \mp K_{\left\{ {1 \atop 2} \right\}}\ ,\ F^{\pm}\right\}_{\bar{\mathcal{C}}_1}\\
&\qquad + \left\{\left(\bar{\mathcal{C}}_2 \pm 2\alpha J_{1m}\bar{\mathcal{C}}_1\right)\mathcal{A}_1^{\pm}\log\mathbb{P}_n\ ,\ F^{\pm}\right\}_{\bar{\mathcal{A}}_1^{\pm}}
\end{aligned}$$

with $K_{\left\{ {1 \atop 2} \right\}}$ a constant, depending on α, c_i, k_1 and k_2,

$$K_{\left\{ {1 \atop 2} \right\}} := J_{1m}\left(2k_{\left\{ {1 \atop 2} \right\}}\alpha J_{mm} - \frac{k_1k_2}{\alpha}\right).$$

Note the *robustness* of these equations: the equations always have the same form (0.3), regardless of the length of the chain of matrices; only the quantities $F^{\pm}$, $G^{\pm}$ and $H_i^{\pm}$ change, via some minors of the matrix J.

[1] in terms of the Wronskians $\{f, g\}_X = gXf - fXg$.

The **second question** concerns a simple model of non-intersecting Brownian motions on $\mathbb{R}$ and their behavior, when the number of particles tends to ∞.

Consider $n = 2k$ non-intersecting Brownian motions on $\mathbb{R}$, all starting at the origin, such that the k left paths end up at $-a$ and the k right paths end up at $+a$ at time $t = 1$; see [7, 21, 18, 5]. Inspired by [15, 16, 7], the Karlin-McGregor formula [17] enables one to express the transition probability $\mathbb{P}_0^{\pm a}$ in terms of the Gaussian Hermitian random matrices in a chain $\mathbb{P}_n(a; E)$ with external source, explained above; this will be done in section 2.

Let now the number $n = 2k$ of particles go to infinity, and let the points a and $-a$ go to $\pm\infty$. This forces the left k particles to $-\infty$ at $t = 1$ and the right k particles to $+\infty$ at $t = 1$. Since the particles all leave from the origin at $t = 0$, it is natural to believe that for small times the equilibrium measure (mean density of particles) is supported by *one interval*, and for times close to 1, the equilibrium measure is supported by *two intervals*. With a precise scaling, $t = 1/2$ is critical in the sense that for $t < 1/2$, the equilibrium measure for the particles is indeed supported by one, and for $t > 1/2$, by two intervals. The heart-shaped region $\mathcal{R}$ formed by the support of the equilibrium measure as a function of time $0 \le t \le 1$ has thus a cusp at $t = 1/2$. The *Pearcey process* $\mathcal{P}(s)$ is now defined as the motion of an infinite number of non-intersecting Brownian paths, just around time $t = 1/2$, with a precise scaling; see [7, 21, 18, 5]. The joint probability that the Pearcey process avoids the windows $E_1, \dots, E_m$ at times $s_1, \dots, s_m$ is defined by

$$\mathbb{P}\begin{pmatrix} \mathcal{P}(s_1) \cap E_1 = \emptyset \\ \vdots \\ \mathcal{P}(s_m) \cap E_m = \emptyset \end{pmatrix}$$

$$:= \lim_{z\to 0} \mathbb{P}_0^{\pm 1/z^2} \left. \begin{pmatrix} \text{all } x_j\left(\frac{1+s_1 z^2}{2}\right) \notin zE_1 & \\ \vdots & ;\ 1 \le j \le n \\ \text{all } x_j\left(\frac{1+s_m z^2}{2}\right) \notin zE_m & \end{pmatrix} \right|_{n=\frac{2}{z^4}},$$

where $\mathbb{P}_0^{\pm a}$ was defined above. The main result of this paper is to show that the infinite-dimensional diffusion equation for the Pearcey process can be replaced by a finite-dimensional non-linear PDE, which is intimately related to the 3-component KP hierarchy and which we now describe.

Given $E_\ell := \bigcup_{i=1}^{r_\ell}[x_{2i-1}^{(\ell)}, x_{2i}^{(\ell)}] \subset \mathbb{R}$, define the space and time gradients

$$\mathcal{X}_{-1} := \sum_{\ell=1}^{m}\sum_{i=1}^{2r_\ell}\frac{\partial}{\partial x_i^{(\ell)}}, \quad \mathcal{T}_{-1} = \sum_{\ell=1}^{m}\frac{\partial}{\partial s_\ell},$$

space and time *Euler operators* $\tilde{\mathcal{X}}_0$ and $\tilde{\mathcal{T}}_0$ and a mixed space-time operator $\tilde{\mathcal{X}}_{-1}$,

$$\mathcal{X}_0 := \sum_{\ell=1}^{m}\sum_{i=1}^{2r_\ell} x_i^{(\ell)}\frac{\partial}{\partial x_i^{(\ell)}}, \quad \mathcal{T}_0 = \sum_{\ell=1}^{m} s_\ell\frac{\partial}{\partial s_\ell}, \quad \tilde{\mathcal{X}}_{-1} = \sum_{\ell=1}^{m} s_\ell\sum_{i=1}^{2r_\ell}\frac{\partial}{\partial x_i^{(\ell)}}.$$

Theorem 0.2 *Then*

$$\mathbb{Q}(s_1,\dots,s_m;E_1,\dots,E_m) := \log\mathbb{P}\begin{pmatrix}\mathcal{P}(s_1)\cap E_1=\emptyset\\ \vdots\\ \mathcal{P}(s_m)\cap E_m=\emptyset\end{pmatrix}$$

satisfies a 4th order and 3rd degree PDE, which can be written as a single Wronskian in the gradient $\mathcal{X}_{-1}$*:*

$$\Big\{\mathcal{X}_{-1}^2\mathcal{T}_{-1}\mathbb{Q}\ ,\ \frac{1}{8}\left\{\mathcal{X}_{-1}\mathcal{T}_{-1}\mathbb{Q}\ ,\ \mathcal{X}_{-1}^2\mathbb{Q}\right\}_{\mathcal{X}_{-1}} + (\mathcal{X}_0+2\mathcal{T}_0-2)\,\mathcal{X}_{-1}^2\mathbb{Q} - 4(\tilde{\mathcal{X}}_{-1}\mathcal{X}_{-1}-\mathcal{T}_{-1}^2)\mathcal{T}_{-1}\mathbb{Q}\Big\}_{\mathcal{X}_{-1}} = 0. \tag{0.4}$$

In particular $\mathbb{Q}(s;E) = \log\mathbb{P}\left(\mathcal{P}(s)\cap E=\emptyset\right)$ *satisfies*

$$\left\{\mathcal{X}_{-1}^2\frac{\partial\mathbb{Q}}{\partial s}\ ,\ \frac{1}{8}\left\{\mathcal{X}_{-1}\frac{\partial\mathbb{Q}}{\partial s}\ ,\ \mathcal{X}_{-1}^2\mathbb{Q}\right\}_{\mathcal{X}_{-1}} + (\mathcal{X}_0-2)\mathcal{X}_{-1}^2\mathbb{Q} + 4\frac{\partial^3\mathbb{Q}}{\partial s^3}\right\}_{\mathcal{X}_{-1}} = 0.$$

Notice here as well the robustness of the equations. The shape of the equation (0.4) is the same, regardless of the number of times one considers. Moreover the equations are "*commutative*": the times and windows can be permuted simultaneously. Notice that the term containing $\tilde{\mathcal{X}}_{-1}$ is the only one which ties up the time s_i with the precise set E_i. We expect that this equation can be used to derive large-time asymptotics, when $t\to\pm\infty$. Also one expects that the PDE's for the sine and Airy processes [4] can be obtained from this equation by an appropriate scaling limit. These questions remain challenging open problems.

1 Gaussian Hermitian random matrices coupled in a chain with external source

The present paper studies m Gaussian Hermitian random matrices $M_i \in \mathcal{H}_n$, coupled in a chain with external source A, given by the diagonal matrix (set $n = k_1 + k_2$)

$$A := \left(\begin{array}{cccccc} \alpha & & & & & \\ & \ddots & & & \mathbf{O} & \\ & & \alpha & & & \\ & & & -\alpha & & \\ & \mathbf{O} & & & \ddots & \\ & & & & & -\alpha \end{array} \right) \begin{array}{c} \updownarrow k_1 \\ \\ \\ \updownarrow k_2 \end{array}, \tag{1.1}$$

and given by the coupling terms $c_1, \ldots, c_{m-1}$; its density is given by

$$\frac{1}{Z_n} e^{-\frac{1}{2}\operatorname{Tr}(M_1^2 + \ldots + M_m^2 - 2c_1 M_1 M_2 - \ldots - 2c_{m-1} M_{m-1} M_m - 2AM_m)} dM_1 \ldots dM_m \tag{1.2}$$

For each index $1 \leq \ell \leq m$, consider a disjoint union of intervals $E_\ell := \bigcup_{i=1}^r [b_{2i-1}^{(\ell)}, b_{2i}^{(\ell)}] \subset \mathbb{R}$, and define the associated algebra of differential operators

$$\mathcal{D}_k(E_\ell) = \sum_{i=1}^{2r} (b_i^{(\ell)})^{k+1} \frac{\partial}{\partial b_i^{(\ell)}}, \qquad 1 \leq \ell \leq m. \tag{1.3}$$

Consider the following probability:

$$\begin{aligned} & \mathbb{P}_n(\alpha; c_1, \ldots, c_{m-1}; E_1, \ldots, E_m) \\ := \ & \mathbb{P}\left((M_1, \ldots, M_m) \in \prod_{\ell=1}^m \mathcal{H}_n \text{ , with } \left\{ \begin{array}{c} M_1\text{-spectrum in } E_1 \\ \vdots \\ M_m\text{-spectrum in } E_m \end{array} \right\} \right) \\ = \ & \frac{1}{Z_n'} \int_{\prod_{\ell=1}^m \mathcal{H}_n(E_\ell)} e^{-\frac{1}{2}\operatorname{Tr}(M_1^2 + \ldots + M_m^2 - 2c_1 M_1 M_2 - \ldots - 2c_{m-1} M_{m-1} M_m - 2AM_m)} dM_1 \ldots dM_m \end{aligned} \tag{1.4}$$

To be clear, this integral is to be taken over the space of m-uples of Hermitian matrices, with M_1-spectrum in $E_1, \ldots,$ M_m-spectrum in E_m , and Z_n' is the above integral with all the E_i replaced by $\mathbb{R}$.

Proposition 1.1 . *The following holds*[2]*:*

$$
\begin{aligned}
&\mathbb{P}_n\left(\alpha; c_1,\dots,c_{m-1}; E_1,\dots,E_m\right)\\
&= \frac{1}{Z_n}\int_{\prod_1^m E_i^{k_1+k_2}} \Delta_{k_1+k_2}(y^{(1)})\\
&\qquad \Delta_{k_1}(y^{(m)\prime})\prod_{i=1}^{k_1} e^{-\frac12\sum_{\ell=1}^m y_i^{(\ell)2}+\sum_{\ell=1}^{m-1}c_\ell y_i^{(\ell)}y_i^{(\ell+1)}+\alpha y_i^{(m)}}\prod_{\ell=1}^m dy_i^{(\ell)}\\
&\qquad \Delta_{k_2}(y^{(m)\prime\prime})\prod_{i=k_1+1}^{k_1+k_2} e^{-\frac12\sum_{\ell=1}^m y_i^{(\ell)2}+\sum_{\ell=1}^{m-1}c_\ell y_i^{(\ell)}y_i^{(\ell+1)}-\alpha y_i^{(m)}}\prod_{\ell=1}^m dy_i^{(\ell)},
\end{aligned}
$$

where

$$
y^{(m)\prime} := (y_1^{(m)},\dots,y_{k_1}^{(m)}) \quad \textit{and} \quad y^{(m)\prime\prime} := (y_{k_1+1}^{(m)},\dots,y_{k_1+k_2}^{(m)}).
$$

The *proof* of this statement is a standard application of the Harish-Chandra-Bessis-Itzykson-Zuber formula, combined with the techniques explained in the next section.

2 Non-intersecting Brownian motions

Consider $n = k_1 + k_2$ non-intersecting Brownian motions on $\mathbb{R}$ (Dyson's Brownian motions), all starting at the origin, such that the k_2 left paths end up at $-a$ and the k_1 right paths end up at $+a$ at time $t = 1$:

$$
\begin{aligned}
&\mathbb{P}_0^{\pm a}\left(\text{all } x_j(t)\in E\right)\\
&:= \mathbb{P}\left(\text{all } x_j(t)\in E \;\middle|\; \begin{array}{c}\text{all } x_j(0)=0\\ k_2 \text{ left paths end up at } -a \text{ at time } t=1,\\ k_1 \text{ right paths end up at } +a \text{ at time } t=1\end{array}\right)
\end{aligned}
\tag{2.1}
$$

In the Proposition below we shall be using the Karlin-McGregor formula for non-intersecting Brownian motions $x_j(t)$ for $0 < t < 1$:

[2] Throughout this paper, $\Delta_n(x) = \prod_{1\le i<j\le n}(x_i - x_j)$ is the Vandermonde.

$$\mathbb{P}\left(\text{all } x_i(t) \in E,\ 1 \le i \le n \,\middle|\, \begin{array}{l} \text{given } x_i(0) = \gamma_i \\ \text{given } x_i(1) = \delta_i \end{array}\right)$$
$$= \int_{E^n} \frac{1}{Z_n} \det(p(t;\gamma_i, x_j))_{1\le i,j\le n} \det(p(1-t; x_{i'}, \delta_{j'}))_{1\le i',j'\le n} \prod_1^n dx_i$$

for

$$p(t,x,y) := \frac{1}{\sqrt{\pi t}}\, e^{-\frac{(y-x)^2}{t}}. \tag{2.2}$$

Consider now the Brownian motions at different times

$$0 = t_0 < t_1 < t_2 < \ldots < t_{m-1} < t_m < t_{m+1} = 1$$

and set

$$\tau_i = t_{i+1} - t_i \ \text{ and } \ \frac{1}{\sigma_j} = \frac{1}{t_j - t_{j-1}} + \frac{1}{t_{j+1} - t_j}, \quad \text{for } 0 \le j \le m.$$

Considering m disjoint unions of intervals $E_\ell := \bigcup_{i=1}^r [\tilde{b}_{2i-1}^{(\ell)}, \tilde{b}_{2i}^{(\ell)}] \subset \mathbb{R}$ for $1 \le \ell \le m$, we show that the two probabilities $\mathbb{P}_0^{\pm a}$ and $\mathbb{P}_n$, as in (1.4) and (2.1), are related by a mere change of variables:

Proposition 2.1 *For* $0 = t_0 < t_1 < t_2 < \ldots < t_{m-1} < t_m < t_{m+1} = 1$,

$$\mathbb{P}_0^{\pm a}(\text{all } x_i(t_1) \in \tilde{E}_1, \ldots, \text{all } x_i(t_m) \in \tilde{E}_m) \;=\; \mathbb{P}_n(\alpha; c_1, \ldots, c_{m-1}; E_1, \ldots, E_m)$$

upon setting

$$E_\ell = \tilde{E}_\ell \sqrt{\frac{2(t_{\ell+1} - t_{\ell-1})}{(t_{\ell+1} - t_\ell)(t_\ell - t_{\ell-1})}}\ ,\ c_j = \sqrt{\frac{(t_{j+2} - t_{j+1})(t_j - t_{j-1})}{(t_{j+2} - t_j)(t_{j+1} - t_{j-1})}} \tag{2.3}$$

and

$$\alpha = a\sqrt{\frac{2(t_m - t_{m-1})}{(1 - t_m)(1 - t_{m-1})}}. \tag{2.4}$$

Proof: In the following computation, we shall be using the notation

$$x^{(m)\prime} = (x_1^{(m)}, \ldots, x_{k_1}^{(m)}), \quad x^{(m)\prime\prime} = (x_{k_1+1}^{(m)}, \ldots, x_{k_1+k_2}^{(m)}).$$

Remembering $p(t,x,y)$ is the Brownian transition probability (2.2), one computes:

$\mathbb{P}_0^{\pm a}(\text{all } x_i(t_1) \in \tilde{E}_1, \ldots, \text{all } x_i(t_m) \in \tilde{E}_m)$

$$= \lim_{\substack{x_i^{(0)} \to 0 \\ a_1,\ldots,a_{k_1} \to a \\ a_{k_1+1},\ldots,a_n \to -a}} \frac{1}{Z_n} \int_{\tilde{E}_1^n \times \ldots \times \tilde{E}_m^n} \prod_{\substack{1\le i\le n \\ 1\le j\le m}} dx_i^{(j)}$$

$$\det\left(p(t_1, x_i^{(0)}, x_j^{(1)})\right)_{1\le i,j\le n} \det\left(p(t_2-t_1, x_i^{(1)}, x_j^{(2)})\right)_{1\le i,j\le n}$$

$$\ldots \det\left(p(t_m - t_{m-1}, x_i^{(m-1)}, x_j^{(m)})\right)_{1\le i,j\le n} \det\left(p(1-t_m, x_i^{(m)}, a_j)\right)_{1\le i,j\le n}$$

$$= \lim_{\substack{a_1,\ldots,a_{k_1} \to a \\ a_{k_1+1},\ldots,a_n \to -a}} \frac{1}{Z_n'} \int_{\tilde{E}_1^n \times \ldots \times \tilde{E}_m^n} \prod_{\substack{1\le i\le n \\ 1\le j\le m}} dx_i^{(j)} \Delta_n(x^{(1)}) \prod_{j=1}^m e^{-\sum_{i=1}^n x_i^{(j)2}\left(\frac{1}{t_j - t_{j-1}} + \frac{1}{t_{j+1}-t_j}\right)}$$

$$\det\left(e^{\frac{2x_i^{(1)}x_j^{(2)}}{t_2-t_1}}\right)_{1\le i,j\le n} \ldots \det\left(e^{\frac{2x_i^{(m-1)}x_j^{(m)}}{t_m - t_{m-1}}}\right)_{1\le i,j\le n} \det\left(e^{\frac{2x_i^{(m)}a_j}{1-t_m}}\right)_{1\le i,j\le n}$$

$$\stackrel{*}{=} \frac{(n!)^m}{Z_n''} \int_{\tilde{E}_1^n \times \ldots \times \tilde{E}_m^n} \prod_{\substack{1\le i\le n \\ 1\le j\le m}} dx_i^{(j)} \Delta_n(x^{(1)})$$

$$\prod_{i=1}^{k_1} (x_i^{(m)})^{i-1} e^{-\sum_{j=1}^m \frac{x_i^{(j)2}}{\sigma_j} + \sum_{j=1}^{m-1} \frac{2x_i^{(j)}x_i^{(j+1)}}{t_{j+1}-t_j} + \frac{2ax_i^{(m)}}{1-t_m}}$$

$$\prod_{i=k_1+1}^{k_1+k_2} (x_i^{(m)})^{i-k_1-1} e^{-\sum_{j=1}^m \frac{x_i^{(j)2}}{\sigma_j} + \sum_{j=1}^{m-1} \frac{2x_i^{(j)}x_i^{(j+1)}}{t_{j+1}-t_j} - \frac{2ax_i^{(m)}}{1-t_m}}$$

$$\stackrel{**}{=} \frac{(n!)^m}{(k_1!)(k_2)!Z_n''} \int_{\tilde{E}_1^n \times \ldots \times \tilde{E}_m^n} \Delta_n(x^{(1)})$$

$$\Delta_{k_1}(x^{(m)\prime}) \prod_{i=1}^{k_1} e^{-\sum_{j=1}^m \frac{x_i^{(j)2}}{\sigma_j} + \sum_{j=1}^{m-1} \frac{2x_i^{(j)}x_i^{(j+1)}}{t_{j+1}-t_j} + \frac{2ax_i^{(m)}}{1-t_m}} \prod_{j=1}^m dx_i^{(j)}$$

$$\Delta_{k_2}(x^{(m)\prime\prime}) \prod_{i=k_1+1}^{k_1+k_2} e^{-\sum_{j=1}^m \frac{x_i^{(j)2}}{\sigma_j} + \sum_{j=1}^{m-1} \frac{2x_i^{(j)}x_i^{(j+1)}}{t_{j+1}-t_j} - \frac{2ax_i^{(m)}}{1-t_m}} \prod_{j=1}^m dx_i^{(j)}$$

$$= \frac{(n!)^m}{Z_n'' k_1! k_2!} \int_{\prod_1^m E_i^n} \prod_{\substack{1\le i\le n\\ 1\le j\le m}} dy_i^{(j)} \Delta_n(y^{(1)})$$

$$\Delta_{k_1}(y^{(m)\prime}) \prod_{i=1}^{k_1} e^{-\frac{1}{2}\sum_{j=1}^m y_i^{(j)2} + \sum_{j=1}^{m-1} c_j y_i^{(j)} y_i^{(j+1)} + \alpha y_i^{(m)}}$$

$$\Delta_{k_2}(y^{(m)\prime\prime}) \prod_{i=k_1+1}^{k_1+k_2} e^{-\frac{1}{2}\sum_{j=1}^m y_i^{(j)2} + \sum_{j=1}^{m-1} c_j y_i^{(j)} y_i^{(j+1)} - \alpha y_i^{(m)}}$$

by setting, for $1 \le i \le n,\ 1 \le j \le m-1$,

$$\frac{x_i^{(\ell)}}{\sqrt{\sigma_\ell}} = \frac{y_i^{(\ell)}}{\sqrt{2}}, \qquad c_j = \frac{\sqrt{\sigma_j \sigma_{j+1}}}{t_{j+1} - t_j} \text{ and } \alpha = \frac{\sqrt{2\sigma_m}}{1 - t_m} a;$$

the change of variables $x^{(\ell)} \mapsto y^{(\ell)}$ induces a change of variables for the boundary terms of the integrals

$$E_\ell = \tilde{E}_\ell \sqrt{\frac{2}{\sigma_\ell}},$$

thus confirming (2.3) and (2.4).

Identity $\overset{*}{=}$ in the previous set of identities is established by means of the following argument, which we explain for the indices $j = 1, 2$:

$$\prod_{\substack{1\le i\le n\\ 1\le j\le 2}} dx_i^{(j)} \Delta_n(x^{(1)}) \det\left(e^{\frac{2x_i^{(1)} x_j^{(2)}}{\tau_1}}\right)_{1\le i,j\le n} \det\left(e^{\frac{2x_i^{(2)} x_j^{(3)}}{\tau_2}}\right)_{1\le i,j\le n}$$

$$= \prod_{\substack{1\le i\le n\\ 1\le j\le 2}} dx_i^{(j)} \Delta_n(x^{(1)}) \sum_\pi (-1)^\pi \prod_1^n e^{\frac{2x_{\pi(i)}^{(1)} x_i^{(2)}}{\tau_1}} \sum_{\pi'} (-1)^{\pi'} \prod_1^n e^{\frac{2x_{\pi'(j)}^{(2)} x_j^{(3)}}{\tau_2}}$$

Upon setting

$$x_{\pi'(j)}^{(2)} \longmapsto x_j^{(2)},$$

$$x_{\pi(\pi'(i))}^{(1)} \longmapsto x_i^{(1)},$$

this expression turns into

$$
\begin{aligned}
&\longmapsto \prod_{\substack{1\le i\le n\\ 1\le j\le 2}} dx_i^{(j)} \sum_{\pi,\pi'} (-1)^{\pi+\pi'} \Delta_n(x^{(1)}_{(\pi\circ\pi')^{-1}(1)},\dots,x^{(1)}_{(\pi\circ\pi')^{-1}(n)}) \prod_1^n e^{\frac{2x_i^{(1)}x_i^{(2)}}{\tau_1}} \prod_1^n e^{\frac{2x_j^{(2)}x_j^{(3)}}{\tau_2}} \\
&= \quad (n!)^2 \Delta_n(x^{(1)}) \prod_{\substack{1\le i\le n\\ 1\le j\le 2}} dx_i^{(j)} \prod_1^n e^{\frac{2x_i^{(1)}x_i^{(2)}}{\tau_1}+\frac{2x_i^{(2)}x_i^{(3)}}{\tau_2}}.
\end{aligned}
$$

Then one uses the symmetry of the integration ranges vis-à-vis i. In general, one makes not 2 but m synchronized changes of variables.

Identity $\stackrel{**}{=}$ follows by simultaneously setting

$$
\begin{aligned}
x_i^{(\ell)} &\mapsto x^{(\ell)}_{\pi'(i)}, \;\; 1\le i\le k_1,\; 1\le \ell\le m,\; \pi'\in S_{k_1} \\
x^{(\ell)}_{k_1+j} &\mapsto x^{(\ell)}_{k_1+\pi''(j)}, \;\; 1\le j\le k_2,\; 1\le \ell\le m,\; \pi''\in S_{k_2},
\end{aligned}
$$

subsequently summing over $\pi'\in S_{k_1},\ \pi''\in S_{k_2}$, giving rise to $\Delta_{n_1}(x^{(m)'})$ and $\Delta_{n_2}(x^{(m)''})$, because of the presence of $\Delta_n(x^{(1)})$, and then dividing by $k_1!k_2!$, thus ending the proof of Proposition 2.1. ■

When taking the limit for $n\to\infty$, we shall need the following scaling (see [7]), assuming $k=k_1=k_2$:

$$
n = 2k = \frac{2}{z^4}, \quad \pm a = \pm\frac{1}{z^2}, \quad x_i \mapsto x_i z, \quad t_j = \frac{1}{2}(1+s_j z^2), \quad \text{for } z\to 0. \tag{2.5}
$$

Proposition 2.2 . *Given $\tilde E_\ell = \bigcup_{i=1}^r [\tilde b^{(\ell)}_{2i-1}, \tilde b^{(\ell)}_{2i}] \subset \mathbb{R}$, the following holds:*

$$
\begin{aligned}
&\mathbb{P}_0^{\pm a}(\text{all } x_i(t_1)\in \tilde E_1,\dots,\text{all } x_i(t_m)\in \tilde E_m)\Bigg|_{\substack{t_j=\frac12(1+s_jz^2)\\ \tilde b_j^{(\ell)}=u_j^{(\ell)}z\\ a=\frac{1}{z^2}\\ n=\frac{2}{z^4}}} \\
&= \mathbb{P}_n\left(\alpha; c_1,\dots,c_{m-1}; b^{(1)},\dots,b^{(m)}\right)\Big|_{n=\frac{2}{z^4}},
\end{aligned} \tag{2.6}
$$

with

$$c_j = \frac{\sqrt{\sigma_j \sigma_{j+1}}}{t_{j+1} - t_j} = \sqrt{\frac{(t_j - t_{j-1})(t_{j+2} - t_{j+1})}{(t_{j+1} - t_{j-1})(t_{j+2} - t_j)}}$$

$$= \begin{cases} \sqrt{\frac{(1+s_1 z^2)(s_3 - s_2)}{(1+s_2 z^2)(s_3 - s_1)}} & \text{for } j = 1 \\ \sqrt{\frac{(s_j - s_{j-1})(s_{j+2} - s_{j+1})}{(s_{j+1} - s_{j-1})(s_{j+2} - s_j)}} & \text{for } 2 \le j \le m-2 \\ \sqrt{\frac{(s_{m-1} - s_{m-2})(1 - s_m z^2)}{(s_m - s_{m-2})(1 - s_{m-1} z^2)}} & \text{for } j = m-1 \end{cases}$$

$$\alpha = \frac{a\sqrt{2\sigma_m}}{1 - t_m} = \frac{\sqrt{2}}{z^2} \frac{\sqrt{t_m - t_{m-1}}}{\sqrt{(1 - t_m)(1 - t_{m-1})}} = \frac{2}{z} \frac{\sqrt{s_m - s_{m-1}}}{\sqrt{(1 - s_m z^2)(1 - s_{m-1} z^2)}}$$

$$b_i^{(\ell)} = \tilde{b}_i^{(\ell)} \sqrt{\frac{2}{\sigma_\ell}} = \tilde{b}_i^{(\ell)} \sqrt{\frac{2(t_{\ell+1} - t_{\ell-1})}{(t_\ell - t_{\ell-1})(t_{\ell+1} - t_\ell)}}$$

$$= \begin{cases} 2u_i^{(1)} \sqrt{\frac{1+s_2 z^2}{(1+s_1 z^2)(s_2 - s_1)}}, & \text{for } \ell = 1 \\ 2u_i^{(\ell)} \sqrt{\frac{s_{\ell+1} - s_{\ell-1}}{(s_\ell - s_{\ell-1})(s_{\ell+1} - s_\ell)}}, & \text{for } 2 \le \ell \le m-1 \\ 2u_i^{(m)} \sqrt{\frac{1 - s_{m-1} z^2}{(s_m - s_{m-1})(1 - s_m z^2)}}, & \text{for } \ell = m \end{cases}$$

Proof: Straightforward from Proposition 2.1, combined with the scaling, appearing in (2.6). ∎

3 The inverse of a tridiagonal matrix and its derivatives

Consider the $(k+1)\times(k+1)$ tridiagonal matrix, with non-diagonal entries $c_1,\ldots,c_k$:

$$J^{-1}(c_1,\ldots,c_k) = \begin{pmatrix} -1 & & c_1 & & & \\ & & & \ddots & & \mathbf{O} \\ c_1 & & -1 & & \ddots & \\ & \ddots & & \ddots & & \ddots \\ & & \ddots & & -1 & & c_k \\ & \mathbf{O} & & \ddots & & \\ & & & & c_k & & -1 \end{pmatrix}$$

with

$$D(c_1,\ldots,c_k) := \begin{cases} \det J^{-1}(c_1,\ldots,c_k), & \text{for } k \geq 1 \\ -1 & \text{for } k = 0 \\ 1 & \text{for } k = -1. \end{cases} \tag{3.1}$$

Then one checks, that for $1 \leq j \leq m$,

$$J_{1j}(c_1,\ldots,c_{m-1}) = (-1)^{j-1} c_1 \ldots c_{j-1} \frac{D(c_{j+1},\ldots,c_{m-1})}{D(c_1,\ldots,c_{m-1})}$$

and

$$J_{mj}(c_1,\ldots,c_{m-1}) = (-1)^{m-j} c_j \ldots c_{m-1} \frac{D(c_1,\ldots,c_{j-2})}{D(c_1,\ldots,c_{m-1})}. \tag{3.2}$$

Define

$$\begin{aligned} r_j &:= \tfrac{J_{mj}}{J_{1j}} \quad \text{for } 1 \leq j \leq m \\ &= 0 \qquad \text{for } j = 0. \end{aligned} \tag{3.3}$$

For later use, we shall need the following identities:

Lemma 3.1

$$c_1\frac{\partial}{\partial c_1}\frac{J_{m1}J_{mi}}{J_{1i}}=-2J_{m1}^2 \quad , \quad c_1\frac{\partial}{\partial c_1}\log\frac{J_{1m}}{J_{1i}}=\delta_{1i}\ ,\ c_1\frac{\partial}{\partial c_1}J_{mm}=-2J_{m1}^2$$

$$\begin{aligned}
c_{m-1}\frac{\partial}{\partial c_{m-1}}\log\frac{J_{m1}J_{mi}}{J_{1i}} &= 2\left(-1-2J_{mm}+\frac{J_{m1}J_{mi}}{J_{1i}}\right)\\
c_{m-1}\frac{\partial}{\partial c_{m-1}}\log\frac{J_{m1}}{J_{1i}} &= -1-\frac{2D(c_{i+1},\dots,c_{m-2})}{D(c_{i+1},\dots,c_{m-1})}=-1-2\frac{J_{mm}J_{1i}-J_{m1}J_{mi}}{J_{1i}}\\
c_{m-1}\frac{\partial}{\partial c_{m-1}}\log J_{mi} &= 1-2c_{m-1}J_{m,m-1}-\delta_{im}=-2J_{mm}-1-\delta_{im}
\end{aligned}$$

Proof: At first notice that

$$\begin{aligned}
c_1\frac{\partial}{\partial c_1}D(c_1,\dots,c_j) &= -2c_1^2D(c_3,\dots,c_j)\\
&= 2\left(D(c_1,\dots,c_j)+D(c_2,\dots,c_j)\right)
\end{aligned}$$

$$\begin{aligned}
c_{m-1}\frac{\partial}{\partial c_{m-1}}D(c_j,\dots,c_{m-1}) &= -2c_{m-1}^2D(c_j,\dots,c_{m-3})\\
&= 2\Big(D(c_j,\dots,c_{m-1})+D(c_j,\dots,c_{m-2})\Big)
\end{aligned}$$
$$, \tag{3.4}$$

and

$$D(c_2,\dots,c_{m-1})D(c_1,\dots,c_{i-2})-D(c_1,\dots,c_{m-1})D(c_2,\dots,c_{i-2})=D(c_{i+1},\dots,c_{m-1})\prod_1^{i-1}c_k^2.$$

$$D(c_1,\dots,c_{m-2})D(c_{i+1},\dots,c_{m-1})-D(c_1,\dots,c_{m-1})D(c_{i+1},\dots,c_{m-2})=D(c_1,\dots,c_{i-2})\prod_i^{m-1}c_k^2. \tag{3.5}$$

Then one checks, by (3.2) and (3.5), that

$$\begin{aligned}
\frac{J_{m1}J_{mi}}{J_{1i}} &= \frac{c_i^2\dots c_{m-1}^2D(c_1,\dots,c_{i-2})}{D(c_{i+1},\dots,c_{m-1})D(c_1,\dots,c_{m-1})}\\
&= \frac{D(c_1,\dots,c_{m-2})}{D(c_1,\dots,c_{m-1})}-\frac{D(c_{i+1},\dots,c_{m-2})}{D(c_{i+1},\dots,c_{m-1})}\\
&= J_{mm}-\frac{D(c_{i+1},\dots,c_{m-2})}{D(c_{i+1},\dots,c_{m-1})}.
\end{aligned} \tag{3.6}$$

Hence

$$\frac{D(c_{i+1},\dots,c_{m-2})}{D(c_{i+1},\dots,c_{m-1})}=\frac{J_{mm}J_{1i}-J_{m1}J_{mi}}{J_{1i}},$$

and by (3.2),

$$\frac{J_{m1}}{J_{1i}}=\frac{(-1)^{m-i}c_i\dots c_{m-1}}{D(c_{i+1},\dots,c_{m-1})}. \tag{3.7}$$

Moreover, explicit differentiation of (3.6) and using the identities (3.5), (3.2), one is led to

$$\begin{aligned}
&c_1\frac{\partial}{\partial c_1}\frac{J_{m1}J_{mi}}{J_{1i}}\\
&= \frac{-2c_i^2\dots c_{m-1}^2\left(D(c_2,\dots,c_{m-1})D(c_1,\dots,c_{i-2})-D(c_2,\dots,c_{i-2})D(c_1,\dots,c_{m-1})\right)}{D(c_{i+1},\dots,c_{m-1})D^2(c_1,\dots,c_{m-1})}\\
&= \frac{-2\prod_1^{m-1}c_i^2}{D^2(c_1,\dots,c_{m-1})}\\
&= -2J_{m1}^2,
\end{aligned}$$

and, setting $i=m$, yields at once the last identity on the first line of the statement of Lemma 3.1. Also by (3.6) and (3.5)

$$\begin{aligned}
&\frac{1}{2}c_{m-1}\frac{\partial}{\partial c_{m-1}}\log\frac{J_{m1}J_{mi}}{J_{1i}}\\
&= -1-2J_{mm}+\frac{D(c_1,\dots,c_{m-2})D(c_{i+1},\dots,c_{m-1})-D(c_1,\dots,c_{m-1})D(c_{i+1},\dots,c_{m-2})}{D(c_1,\dots,c_{m-1})D(c_{i+1},\dots,c_{m-1})}\\
&= -1-2J_{mm}+\frac{J_{m1}J_{mi}}{J_{1i}}
\end{aligned}$$

From (3.7) compute at once

$$c_1\frac{\partial}{\partial c_1}\log\frac{J_{1m}}{J_{1i}}=\delta_{1i}$$

and from (3.7), (3.4) and (3.6),

$$\begin{aligned}
c_{m-1}\frac{\partial}{\partial c_{m-1}}\log\frac{J_{1m}}{J_{1i}} &= 1-2\frac{D(c_{i+1},\dots,c_{m-1})+D(c_{i+1},\dots,c_{m-2})}{D(c_{i+1},\dots,c_{m-1})}\\
&= -1-2\frac{D(c_{i+1},\dots,c_{m-2})}{D(c_{i+1},\dots,c_{m-1})}\\
&= -1-2\frac{J_{mm}J_{1i}-J_{m1}J_{mi}}{J_{1i}}.
\end{aligned}$$

Also, using

$$D(c_1,\ldots,c_{m-1}) = -D(c_1,\ldots,c_{m-2}) - c_{m-1}^2 D(c_1,\ldots,c_{m-3})$$

and the explicit formula (3.2) for J_{mi}, one computes, using (3.4),

$$\begin{aligned}
c_{m-1}\frac{\partial}{\partial c_{m-1}}\log J_{mi} &= c_{m-1}\frac{\partial}{\partial c_{m-1}}\log\left((-1)^{m-i}c_i\ldots c_{m-1}\frac{D(c_1,\ldots,c_{i-2})}{D(c_1,\ldots,c_{m-1})}\right)\\
&= 1+2c_{m-1}^2\frac{D(c_1,\ldots,c_{m-3})}{D(c_1,\ldots,c_{m-1})}-\delta_{im}\\
&= 1-2c_{m-1}J_{m,m-1}-\delta_{im}\\
&= 2\left(1-c_{m-1}J_{m,m-1}\right)-1-\delta_{im}\\
&= 2\left(\frac{D(c_1,\ldots,c_{m-1})+c_{m-1}^2D(c_1,\ldots,c_{m-3})}{D(c_1,\ldots,c_{m-1})}\right)-1-\delta_{im}\\
&= -2\frac{D(c_1,\ldots,c_{m-2})}{D(c_1,\ldots,c_{m-1})}-1-\delta_{im}\\
&= -2J_{mm}-1-\delta_{im},
\end{aligned}$$

confirming the last formula in the statement of Lemma 3.1. ∎

Proposition 3.2 *For arbitrary $z\in\mathbb{C}$, consider the map of Proposition 2.2, namely*

$$((s_1,\ldots,s_m),(u^{(1)},\ldots,u^{(m)}))\longmapsto((c_1,\ldots,c_{m-1},\alpha),(b^{(1)},\ldots,b^{(m)}))$$

where

$$\begin{aligned}
c_j &= \begin{cases}\sqrt{\frac{(1+s_1z^2)(s_3-s_2)}{(1+s_2z^2)(s_3-s_1)}} & \text{for } j=1\\[2ex] \sqrt{\frac{(s_j-s_{j-1})(s_{j+2}-s_{j+1})}{(s_{j+1}-s_{j-1})(s_{j+2}-s_j)}} & \text{for } 2\le j\le m-2\\[2ex] \sqrt{\frac{(s_{m-1}-s_{m-2})(1-s_mz^2)}{(s_m-s_{m-2})(1-s_{m-1}z^2)}} & \text{for } j=m-1\end{cases}\\
\alpha &= \frac{2}{z}\sqrt{\frac{s_m-s_{m-1}}{(1-s_{m-1}z^2)(1-s_mz^2)}}\\
b^{(i)} &= \frac{u^{(i)}}{U_i(s,z)}\qquad 1\le i\le m,
\end{aligned}\tag{3.8}$$

with

$$U_i(s;z) := \begin{cases} \frac{1}{2}\sqrt{\frac{(1+s_1z^2)(s_2-s_1)}{1+s_2z^2}}, & \textit{for } i=1, \\ \frac{1}{2}\sqrt{\frac{(s_i-s_{i-1})(s_{i+1}-s_i)}{s_{i+1}-s_{i-1}}}, & \textit{for } 2\le i\le m-1, \\ \frac{1}{2}\sqrt{\frac{(1-s_mz^2)(s_m-s_{m-1})}{1-s_{m-1}z^2}}, & \textit{for } i=m, \end{cases}$$

The inverse map involves the tridiagonal matrix J^{-1} and can be expressed as a fractional linear map in $\frac{J_{m1}J_{mi}}{J_{1i}}$ for $1 \le i \le m$

$$\begin{aligned} s_i &= s_i(\alpha, c; z) = \frac{1}{z^2}\frac{\alpha^2 z^4 \frac{J_{m1}J_{mi}}{J_{1i}} + 2}{\alpha^2 z^4 \frac{J_{m1}J_{mi}}{J_{1i}} - 2} \\ u^{(i)} &= b^{(i)} U_i(s(\alpha, c; z); z) \end{aligned} \tag{3.9}$$

with

$$U_i(s(\alpha,c;z);z) = \frac{-\alpha z \ J_{m1}}{\alpha^2 z^4 J_{m1}J_{mi} - 2J_{1i}} = \frac{-1}{J_{mi}}\left(\frac{\alpha z \ \frac{J_{m1}J_{mi}}{J_{1i}}}{\alpha^2 z^4 \frac{J_{m1}J_{mi}}{J_{1i}} - 2}\right)$$

Note that also the entries of J can be expressed as functions of s_i:

$$\begin{aligned} J_{1i} &= \frac{-(1-z^2 s_i)}{4z^2 U_i(s,z)}\sqrt{\frac{(1+s_1z^2)(1+s_2z^2)}{s_2-s_1}} \\ J_{mi} &= \frac{-(1+z^2 s_i)}{4z^2 U_i(s,z)}\sqrt{\frac{(1-s_{m-1}z^2)(1-s_mz^2)}{s_m-s_{m-1}}} \end{aligned} \tag{3.10}$$

with $U_i(s,z)$ as in (3.8).

Proof: Step 1: Upon expanding $D(c_1, \ldots, c_{m-1})$ along the j-th row, one checks by (3.2) and (3.3), that for $1 \leq j \leq m-1$,

$$
\begin{aligned}
r_{j+1} - r_j &= \frac{\prod_1^{m-1}(-c_i)}{c_1^2 \ldots c_j^2} \\
&\quad \frac{D(c_1, \ldots, c_{j-1}) D(c_{j+1}, \ldots, c_{m-1}) - c_j^2 D(c_{j+2}, \ldots, c_{m-1}) D(c_1, \ldots, c_{j-2})}{D(c_{j+2}, \ldots, c_{m-1}) D(c_{j+1}, \ldots, c_{m-1})} \\
&= \frac{\prod_1^{m-1}(-c_i)}{\prod_1^{j} c_i^2} \cdot \frac{D(c_1, \ldots, c_{m-1})}{D(c_{j+1}, \ldots, c_{m-1}) D(c_{j+2}, \ldots, c_{m-1})},
\end{aligned}
\tag{3.11}
$$

and similarly, for $1 \leq j \leq m-2$,

$$
\begin{aligned}
r_{j+2} - r_j &= \frac{\prod_1^{m-1}(-c_j)}{\prod_1^{j+1} c_i^2} \\
&\quad \frac{D(c_1, \ldots, c_j) D(c_{j+1}, \ldots, c_{m-1}) - c_j^2 c_{j+1}^2 D(c_{j+3}, \ldots, c_{m-1}) D(c_1, \ldots, c_{j-2})}{D(c_{j+3}, \ldots, c_{m-1}) D(c_{j+1}, \ldots, c_{m-1})} \\
&= \frac{\prod_1^{m-1}(-c_i)}{\prod_1^{j+1} c_i^2} \frac{D(c_1, \ldots, c_{m-1})}{D(c_{j+1}, \ldots, c_{m-1}) D(c_{j+3}, \ldots, c_{m-1})}.
\end{aligned}
$$

These identities then lead to:

$$
\frac{(r_i - r_{i-1})(r_{i+2} - r_{i+1})}{(r_{i+1} - r_{i-1})(r_{i+2} - r_i)} = c_i^2, \text{ for } 1 \leq i \leq m-2
$$

$$
\frac{(r_i - r_{i-1})(r_{i+1} - r_i)}{(r_{i+1} - r_{i-1})} = -\frac{J_{m1}}{J_{1i}^2}, \text{ for } 1 \leq i \leq m-1
$$

and

$$\frac{r_{m-1}-r_{m-2}}{r_m-r_{m-2}} = c_{m-1}^2. \tag{3.12}$$

$$r_m - r_{m-1} = -\frac{1}{J_{1m}}$$

<u>Step 2</u>: It is easier to show that the inverse map of (3.9) is given by (3.8). So, from inverting the fractional linear map, appearing in (3.9), one computes

$$J_{m1}r_i = \frac{J_{m1}J_{mi}}{J_{1i}} = -\frac{2}{\alpha^2 z^4}\left(\frac{1+z^2 s_i}{1-z^2 s_i}\right) \tag{3.13}$$

and so, inverting this map, one computes, in cascade,

$$\begin{aligned}
1-z^2 s_i &= -\frac{4}{\alpha^2 z^4 r_i J_{m1}-2}, \qquad 1+z^2 s_i = \frac{2\alpha^2 z^4 r_i J_{m1}}{\alpha^2 z^4 r_i J_{m1}-2} \\
s_i - s_{i-1} &= \frac{4\alpha^2 z^4 (r_{i-1}-r_i)J_{m1}}{(\alpha^2 z^4 r_i J_{m1}-2)(\alpha^2 z^4 r_{i-1} J_{m1}-2)} \\
s_{i+1} - s_{i-1} &= \frac{4\alpha^2 z^2 (r_{i-1}-r_{i+1})J_{m1}}{(\alpha^2 z^4 r_{i-1} J_{m1}-2)(\alpha^2 z^4 r_{i+1} J_{m1}+2)}.
\end{aligned}$$

Therefore, using (3.12), one checks

$$\begin{aligned}
\frac{(s_i-s_{i-1})(s_{i+2}-s_{i+1})}{(s_{i+1}-s_{i-1})(s_{i+2}-s_i)} &= \frac{(r_i-r_{i-1})(r_{i+2}-r_{i+1})}{(r_{i+1}-r_{i-1})(r_{i+2}-r_i)} = c_i^2 \text{ for } 2 \le i \le m-2 \\
\frac{(1+s_1 z^2)(s_3-s_2)}{(1+s_2 z^2)(s_3-s_1)} &= \frac{r_1(r_3-r_2)}{r_2(r_3-r_1)} = c_1^2 \\
\frac{(1-s_m z^2)(s_{m-1}-s_{m-2})}{(1-s_{m-1}z^2)(1-s_m z^2)} &= \frac{r_{m-1}-r_{m-2}}{r_m-r_{m-2}} = c_{m-1}^2 \\
\frac{4}{z^2}\frac{s_m-s_{m-1}}{(1-s_{m-1}z^2)(1-s_m z^2)} &= -J_{m1}(r_m-r_{m-1})\alpha^2 = \alpha^2
\end{aligned}$$

and similarly for the expressions U_i in (3.8). The signs are all specified from $\sqrt{s_{i+1}-s_i}$ and $\sqrt{s_{i+2}-s_i}$.

Identity (3.10) is obtained by solving

$$s_i = s_i(\alpha, c; z) = \frac{1}{z^2}\frac{\alpha^2 z^4 \frac{J_{m1}J_{mi}}{J_{1i}}+2}{\alpha^2 z^4 \frac{J_{m1}J_{mi}}{J_{1i}}-2}$$

for $\frac{J_{m1}J_{mi}}{J_{1i}}$, by substituting the result in (3.9), i.e.,

$$U_i(s(\alpha,c;z);z) = -\frac{\alpha z\ \frac{J_{m1}}{J_{1i}}}{\alpha^2 z^4 \frac{J_{m1}J_{mi}}{J_{1i}} - 2},$$

and then solving for $\frac{J_{m1}}{J_{1i}}$. Finally expressing $J_{mi} = \frac{\frac{J_{m1}J_{mi}}{J_{1i}}}{\frac{J_{m1}}{J_{1i}}}$ leads to the second relation (3.10). The first relation (3.10) is obtained from the ratio $J_{1i} = \frac{J_{m1}}{J_{m1}/J_{1i}}$ and using the result previously obtained for J_{mi} at $i = 1$. ■

Lemma 3.3 *The following identities hold*

$$\alpha\frac{\partial}{\partial\alpha}s_i(\alpha,c;z) = -8U_i^2(s,z)\frac{J_{mi}J_{1i}}{J_{m1}} = \frac{1}{z^2} - s_i^2 z^2$$

$$\alpha\frac{\partial}{\partial\alpha}\log U_i(s(\alpha,c;z),z) = -z^2 s_i$$

$$c_1\frac{\partial}{\partial c_1}s_i(\alpha,c;z) = 8U_i^2(s,z)J_{1i}^2 = \frac{(1-s_i z^2)^2(1+s_1 z^2)(1+s_2 z^2)}{2z^4(s_2-s_1)}$$

$$\begin{aligned} c_1\frac{\partial}{\partial c_1}\log U_i(s(\alpha,c;z),z) &= \delta_{i1} - 2\alpha z^3 J_{m1}J_{1i}U_i(s,z) \\ &= \delta_{i1} - \frac{1}{2z^2(s_2-s_1)}(1+s_1z^2)(1+s_2z^2)(1-s_iz^2) \end{aligned}$$

$$\begin{aligned} & c_{m-1}\frac{\partial}{\partial c_{m-1}}s_i(\alpha,c;z) \\ &\quad = 8U_i^2(s,z)\frac{J_{mi}J_{1i}}{J_{m1}}\left(1+2J_{mm}-\frac{J_{mi}J_{m1}}{J_{1i}}\right) \\ &\quad = -\frac{1+s_iz^2}{2z^4(s_m-s_{m-1})} \\ &\qquad \left(\left(1-s_mz^2\right)\left(1-s_{m-1}z^2\right)\left(1+s_iz^2\right) - 2\left(1-s_ms_{m-1}z^4\right)\left(1-s_iz^2\right)\right) \end{aligned}$$

$$\begin{aligned} & c_{m-1}\frac{\partial}{\partial c_{m-1}}\log U_i(s_i(\alpha,c;z),z) \\ &\quad = -1 - 2\frac{D(c_{i+1},\ldots,c_{m-2})}{D(c_{i+1},\ldots,c_{m-1})} - 2\alpha z^3 U_i(s,z)J_{mi}\left(1+2J_{mm}-\frac{J_{mi}J_{m1}}{J_{1i}}\right) \end{aligned}$$

where

$$
\begin{aligned}
1+2J_{mm}-\frac{J_{mi}J_{m1}}{J_{1i}}&\\
=\ &\frac{1}{2z^2(s_m-s_{m-1})(1-s_iz^2)}\\
&\times\Big((1-s_mz^2)(1-s_{m-1}z^2)(1+s_iz^2)-2(1-s_ms_{m-1}z^4)(1-s_iz^2)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\frac{D(c_{i+1},\dots,c_{m-2})}{D(c_{i+1},\dots,c_{m-1})} &= \frac{J_{mm}J_{1i}-J_{m1}J_{mi}}{J_{1i}}\\
&= -\frac{(s_m-s_i)(1-z^2s_{m-1})}{(s_m-s_{m-1})(1-z^2s_i)}\\
J_{mm} &= -\frac{(1-s_{m-1}z^2)(1+s_mz^2)}{2z^2(s_m-s_{m-1})}\\
J_{m1} &= -\frac{1}{2z^2}\sqrt{\frac{(1+s_1z^2)(1+s_2z^2)(1-s_{m-1}z^2)(1-s_mz^2)}{(s_2-s_1)(s_m-s_{m-1})}}\\
\frac{J_{mi}J_{m1}}{J_{1i}} &= -\frac{(1-s_mz^2)\,(1-s_{m-1}\,z^2)\,\,(1+s_iz^2)}{2z^2\,(s_m-s_{m-1})\,(1-s_iz^2)}\\
J_{mi}U_i\alpha &= \frac{-(1+s_iz^2)}{2z^3}\\
J_{1i}U_i &= -\frac{1-s_iz^2}{4z^2}\sqrt{\frac{(1+s_1z^2)(1+s_2z^2)}{s_2-s_1}}.
\end{aligned}
$$

Proof: Differentiating the first identity (3.9) with regard to a and using the second expression (3.9) yield

$$
a\frac{\partial}{\partial a}s_i(a,c;z)=-\frac{8a^2z^2\,\frac{J_{m1}J_{mi}}{J_{1i}}}{\Big(a^2z^4\,\frac{J_{m1}J_{mi}}{J_{1i}}-2\Big)^2}=-8U_i^2(s,z)\frac{J_{mi}J_{1i}}{J_{m1}}.
$$

Differentiating U_i as in (3.9), and using the expression (3.9) for s_i, yield:

$$
\alpha\frac{\partial}{\partial\alpha}\log U_i(\alpha,c;z)=-\frac{\alpha^2z^4\,\frac{J_{m1}J_{mi}}{J_{1i}}+2}{\alpha^2z^4\,\frac{J_{m1}J_{mi}}{J_{1i}}-2}=-z^2s_i,
$$

while (3.9) and Lemma 3.1 yield

$$
\begin{aligned}
c_1 \frac{\partial}{\partial c_1} s_i(\alpha, c; z) &= \frac{-4\alpha^2 z^2}{\left(\alpha^2 \frac{J_{m1} J_{mi}}{J_{1i}} z^4 - 2\right)^2} c_1 \frac{\partial}{\partial c_1} \frac{J_{m1} J_{mi}}{J_{1i}} \\
&= \frac{8\alpha^2 z^2 J_{m1}^2}{\left(\alpha^2 \frac{J_{m1} J_{mi}}{J_{1i}} z^4 - 2\right)^2} \\
&= 8U_i(s,z)^2 J_{1i}^2
\end{aligned}
$$

$$
\begin{aligned}
c_{m-1} \frac{\partial}{\partial c_{m-1}} s_i(\alpha, c; z) &= \frac{-4\alpha^2 z^2}{\left(\alpha^2 \frac{J_{m1} J_{mi}}{J_{1i}} z^4 - 2\right)^2} c_{m-1} \frac{\partial}{\partial c_{m-1}} \frac{J_{m1} J_{mi}}{J_{1i}} \\
&= 8U_i(s,z)^2 \frac{J_{mi} J_{1i}}{J_{m1}} \left(1 + 2J_{mm} - \frac{J_{m1} J_{mi}}{J_{1i}}\right)
\end{aligned}
$$

$$
\begin{aligned}
c_1 \frac{\partial}{\partial c_1} \log U_i(s; z) &= c_1 \frac{\partial}{\partial c_1} \log \frac{J_{m1}}{J_{1i}} - \frac{\alpha^2 z^4}{\alpha^2 \frac{J_{m1} J_{mi}}{J_{1i}} z^4 - 2} c_1 \frac{\partial}{\partial c_1} \frac{J_{m1} J_{mi}}{J_{1i}} \\
&= \delta_{1i} - 2\alpha z^3 J_{m1} J_{i1} U_i(s,z)
\end{aligned}
$$

$$
\begin{aligned}
&c_{m-1} \frac{\partial}{\partial c_{m-1}} \log U_i(s; z) \\
&\quad = c_{m-1} \frac{\partial}{\partial c_{m-1}} \log \frac{J_{m1}}{J_{1i}} - \frac{\alpha^2 z^4}{\alpha^2 \frac{J_{m1} J_{mi}}{J_{1i}} z^4 - 2} c_{m-1} \frac{\partial}{\partial c_{m-1}} \frac{J_{m1} J_{mi}}{J_{1i}} \\
&\quad = -1 - 2\frac{D(c_{i+1}, \ldots, c_{m-2})}{D(c_{i+1}, \ldots, c_{m-1})} \\
&\qquad\qquad -2\alpha z^3 U_i(s,z) J_{mi} \left(1 + 2J_{mm} - \frac{J_{mi} J_{m1}}{J_{1i}}\right)
\end{aligned}
$$

yielding most of the differential identities of Lemma 3.3. The remaining relations are a consequence of (3.8), (3.9), (3.10) and (3.13). ∎

4 Integrable deformations and the Virasoro constraints

In order to compute the differential equation for

$$\begin{aligned}
&\mathbb{P}_n\left(\alpha; c_1, \ldots, c_{m-1}; E_1 \times \ldots \times E_m\right) \\
&= \frac{1}{Z_n} \int_{\prod_1^m E_i^{k_1+k_2}} \Delta_{k_1+k_2}(x^{(1)}) \prod_{i=1}^{k_1+k_2} \prod_{\ell=1}^{m} dx_i^{(\ell)} \\
&\quad \Delta_{k_1}(x^{(m)\prime}) \prod_{i=1}^{k_1} e^{-\frac{1}{2}\sum_{\ell=1}^m x_i^{(\ell)2} + \sum_{\ell=1}^{m-1} c_\ell x_i^{(\ell)} x_i^{(\ell+1)} + \alpha x_i^{(m)}} \\
&\quad \Delta_{k_2}(x^{(m)\prime\prime}) \prod_{i=k_1+1}^{k_1+k_2} e^{-\frac{1}{2}\sum_{\ell=1}^m x_i^{(\ell)2} + \sum_{\ell=1}^{m-1} c_\ell x_i^{(\ell)} x_i^{(\ell+1)} - \alpha x_i^{(m)}},
\end{aligned} \tag{4.1}$$

we need to add to the numerator of $\mathbb{P}_n$ many auxiliary variables

$$\begin{cases} \bar{t} := (\bar{t}_1, \bar{t}_2, \ldots), \bar{s} := (\bar{s}_1, \bar{s}_2, \ldots), \bar{u} := (\bar{u}_1, \bar{u}_2, \ldots) \text{ and } \beta, \\ \gamma^{(\ell)} := (\gamma_1^{(\ell)}, \gamma_2^{(\ell)}, \ldots) \text{ for } 2 \le \ell \le m-1 \\ c^{(\ell)} := (c_{p,q}^{(\ell)})_{p,q\ge 1} \text{ for } 1 \le \ell \le m-1 \end{cases}.$$

Note that the time variables $\bar t,\ \bar s,\ \bar u$ are totally different from the t-variables appearing in the Brownian motion. yielding the following integral[3]

$$
\begin{aligned}
&\tau_{k_1k_2}(\bar t,\bar s,\bar u;\beta,\gamma^{(2)},\dots,\gamma^{(m-1)},c^{(1)},\dots,c^{(m-1)},\alpha,E_1\times\dots\times E_m)\\
&= \frac{1}{k_1!k_2!}\int_{\prod_1^m E_i^{k_1+k_2}}\Delta_{k_1+k_2}(x^{(1)})\prod_{i=1}^{k_1+k_2}\left(e^{\sum_{j=1}^\infty \bar t_j x_i^{(1)j}}\prod_{\ell=1}^m dx_i^{(\ell)}\right)\\
&\quad\Delta_{k_1}(x_1^{(m)},\dots,x_{k_1}^{(m)})\\
&\quad\prod_{i=1}^{k_1} e^{-\frac12\sum_{\ell=1}^m x_i^{(\ell)2}+\alpha x_i^{(m)}+\beta x_i^{(m)2}-\sum_{j=1}^\infty \bar s_j (x_i^{(m)})^j+\sum_{p,q\ge1}\sum_{\ell=1}^{m-1}c_{p,q}^{(\ell)}(x_i^{(\ell)})^p(x_i^{(\ell+1)})^q+\sum_{\ell=2}^{m-1}\sum_{r=1}^\infty\gamma_r^{(\ell)}(x_i^{(\ell)})^r}\\
&\quad\Delta_{k_2}(x_{k_1+1}^{(m)},\dots,x_{k_1+k_2}^{(m)})\\
&\quad\prod_{i=k_1+1}^{k_1+k_2} e^{-\frac12\sum_{\ell=1}^m x_i^{(\ell)2}-\alpha x_i^{(m)}-\beta x_i^{(m)2}-\sum_{j=1}^\infty \bar u_j (x_i^{(m)})^j+\sum_{p,q\ge1}\sum_{\ell=1}^{m-1}c_{p,q}^{(\ell)}(x_i^{(\ell)})^p(x_i^{(\ell+1)})^q+\sum_{\ell=2}^{m-1}\sum_{r=1}^\infty\gamma_r^{(\ell)}(x_i^{(\ell)})^r}\\
&= \det\begin{pmatrix}(\mu_{ij}^+)_{1\le i\le k_1,\ 1\le j\le k_1+k_2}\\ \\ (\mu_{ij}^-)_{1\le i\le k_2,\ 1\le j\le k_1+k_2}\end{pmatrix}
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
\mu_{ij}^{\pm} &= \int_{\prod_1^m E_i}\left(\prod_{\ell=1}^m dx^{(\ell)}\right)e^{\sum_1^\infty\left(\bar t_k x^{(1)k}-\binom{\bar s_k}{\bar u_k}x^{(m)k}\right)}(x^{(1)})^{j-1}(x^{(m)})^{i-1}\\
&\qquad e^{-\frac12\sum_{\ell=1}^m x^{(\ell)2}\pm\alpha x^{(m)}\pm\beta x^{(m)2}+\sum_{p,q\ge1}\sum_{\ell=1}^{m-1}c_{pq}^{(\ell)}(x^{(\ell)})^p(x^{(\ell+1)})^q+\sum_{\ell=2}^{m-1}\sum_{r=1}^\infty\gamma_r^{(\ell)}(x^{(\ell)})^r}
\end{aligned}
\tag{4.3}
$$

This is to say, the integral (4.2), along the locus

$$
\mathcal{L} := \left\{\begin{array}{l}\bar t_i=0,\ \bar s_i=0,\bar u_i=0,\beta=0,\ \gamma_r^{(\ell)}=0,\\ c_{11}^{(\ell)}=c_\ell\ \text{ and }\ c_{ij}^{(\ell)}=0\ \text{ for }\ i,j\ge1\text{ with }(i,j)\neq(1,1)\end{array}\right\}
\tag{4.4}
$$

[3]If $m=1$ or 2, the formulae below must be reinterpreted; e.g., for $m=1$, the c_i's are absent and for $m=2$, γ is not present.

yields the integral (4.1); also for the sake of brevity, set $\gamma_\ell := \gamma_1^{(\ell)}$. The following locus

$$\mathcal{L}_\beta := \left\{ \begin{array}{l} \bar t_i = 0,\ \bar s_i = 0, \bar u_i = 0,\ \gamma_r^{(\ell)} = 0, \\ c_{11}^{(\ell)} = c_\ell \text{ and } c_{ij}^{(\ell)} = 0 \text{ for } i,j \geq 1 \text{ with } (i,j) \neq (1,1) \end{array} \right\} \tag{4.5}$$

will also be used. Then the following statement holds:

Proposition 4.1 *Given a disjoint union of intervals and the associated algebra of differential operators*

$$E_\ell := \bigcup_{i=1}^{r} [b_{2i-1}^{(\ell)}, b_{2i}^{(\ell)}] \subset \mathbb{R} \quad \textit{and} \quad \mathcal{D}_k(E_\ell) = \sum_{i=1}^{2r} (b_i^{(\ell)})^{k+1} \frac{\partial}{\partial b_i^{(\ell)}}, \qquad 1 \leq \ell \leq m$$

the integrals (4.2) satisfy, besides the trivial relations,

$$-\frac{\partial}{\partial \bar s_1} + \frac{\partial}{\partial \bar u_1} = \frac{\partial}{\partial \alpha}, \qquad -\frac{\partial}{\partial \bar s_2} + \frac{\partial}{\partial \bar u_2} = \frac{\partial}{\partial \beta}, \tag{4.6}$$

and, upon setting

$$\gamma_1 := \bar t_1, \textit{ and } \quad \frac{\partial}{\partial \gamma_m} := -\frac{\partial}{\partial \bar s_1} - \frac{\partial}{\partial \bar u_1},$$

the following Virasoro relations ($2 \leq \ell \leq m-1$)*:*

$$\mathcal{D}_{-1}(E_1)\tau_{k_1k_2} \;=\; \left\{ \begin{array}{l} \overbrace{-\frac{\partial}{\partial \bar t_1}}^{-\frac{\partial}{\partial \gamma_1}} + c_1 \frac{\partial}{\partial \gamma_2} + (k_1+k_2)\bar t_1 \\ \\ + \sum_{i\geq 2} i \bar t_i \frac{\partial}{\partial \bar t_{i-1}} + \sum_{\substack{i\geq 2\\ j\geq 1}} i c_{ij}^{(1)} \frac{\partial}{\partial c_{i-1,j}^{(1)}} + \sum_{j\geq 2} c_{1j}^{(1)} \frac{\partial}{\partial \gamma_j^{(2)}} \end{array} \right\} \tau_{k_1k_2}$$

$$\vdots$$

$$
\mathcal{D}_{-1}(E_\ell)\tau_{k_1k_2} = \left\{
\begin{array}{l}
c_{\ell-1}\dfrac{\partial}{\partial\gamma_{\ell-1}} - \dfrac{\partial}{\partial\gamma_\ell} + c_\ell\dfrac{\partial}{\partial\gamma_{\ell+1}} + (k_1+k_2)\gamma_\ell \\[3mm]
+\displaystyle\sum_{\substack{i\geq 1\\ j\geq 2}} jc_{ij}^{(\ell-1)}\dfrac{\partial}{\partial c_{i,j-1}^{(\ell-1)}} + \displaystyle\sum_{\substack{i\geq 2\\ j\geq 1}} ic_{ij}^{(\ell)}\dfrac{\partial}{\partial c_{i-1,j}^{(\ell)}} \\[3mm]
+\sum_{i\geq 2} c_{i1}^{(\ell-1)}\frac{\partial}{\partial\gamma_i^{(\ell-1)}} + \sum_{j\geq 2} c_{1j}^{(\ell)}\frac{\partial}{\partial\gamma_j^{(\ell+1)}} \\[3mm]
+\sum_{r\geq 2} r\gamma_r^{(\ell)}\frac{\partial}{\partial\gamma_{r-1}^{(\ell)}}
\end{array}
\right\}\tau_{k_1k_2}
$$

$$\vdots$$

$$
\mathcal{D}_{-1}(E_m)\tau_{k_1k_2} = \left\{
\begin{array}{l}
c_{m-1}\dfrac{\partial}{\partial\gamma_{m-1}} + \overbrace{\left(\dfrac{\partial}{\partial\bar{s}_1}+\dfrac{\partial}{\partial\bar{u}_1}\right)}^{-\frac{\partial}{\partial\gamma_m}} + 2\beta\,\overbrace{\left(-\dfrac{\partial}{\partial\bar{s}_1}+\dfrac{\partial}{\partial\bar{u}_1}\right)}^{\frac{\partial}{\partial\alpha}} \\[3mm]
-k_1(\bar{s}_1-\alpha) - k_2(\bar{u}_1+\alpha) \\[3mm]
+\displaystyle\sum_{i\geq 2} i\left(\bar{s}_i\dfrac{\partial}{\partial\bar{s}_{i-1}} + \bar{u}_i\dfrac{\partial}{\partial\bar{u}_{i-1}}\right) \\
+\displaystyle\sum_{i\geq 2} c_{i1}^{(m-1)}\dfrac{\partial}{\partial\gamma_i^{(m-1)}} + \displaystyle\sum_{\substack{i\geq 1\\ j\geq 2}} jc_{ij}^{(m-1)}\dfrac{\partial}{\partial c_{i,j-1}^{(m-1)}}
\end{array}
\right\}\tau_{k_1k_2}
\tag{4.7}
$$

and (only needed for $\ell = 1, m$*)*

$$
\mathcal{D}_0(E_1)\tau_{k_1k_2} = \left(
\begin{array}{l}
-\dfrac{\partial}{\partial\bar{t}_2} + c_1\dfrac{\partial}{\partial c_1} + \dfrac{(k_1+k_2)(k_1+k_2+1)}{2} \\[3mm]
\displaystyle\sum_{i\geq 1} i\bar{t}_i\dfrac{\partial}{\partial\bar{t}_i} + \displaystyle\sum_{\substack{i,j\geq 1\\ (i,j)\neq(1,1)}} ic_{ij}^{(1)}\dfrac{\partial}{\partial c_{i,j}^{(1)}}
\end{array}
\right)\tau_{k_1k_2}
$$

$$\mathcal{D}_0(E_m)\tau_{k_1k_2} = \left(\begin{array}{l} \alpha\overbrace{\left(-\frac{\partial}{\partial \bar s_1}+\frac{\partial}{\partial \bar u_1}\right)}^{\frac{\partial}{\partial\alpha}} + c_{m-1}\frac{\partial}{\partial c_{m-1}} + \left(\frac{\partial}{\partial \bar s_2}+\frac{\partial}{\partial \bar u_2}\right) \\ +2\beta\overbrace{\left(-\frac{\partial}{\partial \bar s_2}+\frac{\partial}{\partial \bar u_2}\right)}^{\frac{\partial}{\partial\beta}} + \frac{k_1(k_1+1)}{2} + \frac{k_2(k_2+1)}{2} \\ \sum_{i\geq 1}\left(i\bar s_i\frac{\partial}{\partial \bar s_i} + i\bar u_i \frac{\partial}{\partial \bar u_i}\right) + \sum_{\substack{i,j\geq 1\\(i,j)\neq(1,1)}} jc_{ij}^{(m-1)}\frac{\partial}{\partial c_{i,j}^{(m-1)}} \end{array}\right)\tau_{k_1k_2} \tag{4.8}$$

Before giving the proof of Proposition 4.1, we need the following lemma, concerning the expressions:

$$dI_n := \Delta_n(x)\prod_{k=1}^n dx_k e^{\sum_{i=1}^\infty \bar t_i x_k^i}$$

$$\mathcal{E}(x,y) := \prod_{k=1}^n e^{\sum_1^\infty \bar t_i x_k^i + \sum_{i,j\geq 1} c_{ij}x_k^i y_k^j + \sum_1^\infty \bar s_i y_k^i}$$

Lemma 4.2

$$\frac{\partial}{\partial\varepsilon}\prod_{i=1}^n d(x_i+\varepsilon x_i^{k+1})\Big|_{\varepsilon=0} = \begin{cases} 0, & k=-1 \\ n\prod_{i=1}^n dx_i, & k=0\end{cases}$$

$$\frac{\partial}{\partial\varepsilon} dI_n(x+\varepsilon x^{k+1})\Big|_{\varepsilon=0} = \begin{cases} \left(\sum_{i\geq 2} i\bar t_i \frac{\partial}{\partial \bar t_{i-1}} + n\bar t_1\right) dI_n(x), & k=-1 \\ \\ \left(\sum_{i\geq 1} i\bar t_i\frac{\partial}{\partial \bar t_i} + \frac{n(n+1)}{2}\right) dI_n(x), & k=0\end{cases}$$

$$
\frac{\partial}{\partial\varepsilon}\mathcal{E}(x+\varepsilon x^{k+1},y)\Big|_{\varepsilon=0}
= \begin{cases} \left(\sum_{i\geq 2} i\bar t_i \frac{\partial}{\partial \bar t_{i-1}} + n\bar t_1 + \sum_{i\geq 2, j\geq 1} i c_{ij}\frac{\partial}{\partial c_{i-1,j}} + \sum_{j\geq 1} c_{1j}\frac{\partial}{\partial \bar s_j}\right)\mathcal{E}(x,y), & k=-1 \\ \left(\sum_{i\geq 1} i t_i \frac{\partial}{\partial \bar t_i} + \sum_{i,j\geq 1} i c_{ij}\frac{\partial}{\partial c_{ij}}\right)\mathcal{E}(x,y), & k=0 \end{cases}
$$

$$
\frac{\partial}{\partial\varepsilon}\mathcal{E}(x,y+\varepsilon y^{k+1})\Big|_{\varepsilon=0}
= \begin{cases} \left(\sum_{i\geq 2} i\bar s_i \frac{\partial}{\partial \bar s_{i-1}} + n\bar s_1 + \sum_{i\geq 1, j\geq 2} j c_{ij}\frac{\partial}{\partial c_{i,j-1}} + \sum_{i\geq 1} c_{i1}\frac{\partial}{\partial \bar t_i}\right)\mathcal{E}(x,y), & k=-1 \\ \left(\sum_{i\geq 1} i \bar s_i \frac{\partial}{\partial \bar s_i} + \sum_{i,j\geq 1} j c_{ij}\frac{\partial}{\partial c_{ij}}\right)\mathcal{E}(x,y), & k=0 \end{cases}
$$

Proof: This is obtained by setting $x_i \mapsto x_i + \varepsilon x_i^{k+1}$ and then $y_i \mapsto y_i + \varepsilon y_i^{k+1}$; then take the derivative with regard to ε and set $\varepsilon = 0$.

Proof of Proposition 4.1:
Case 1: Performing the infinitesimal change of variables, for all $1 \leq i \leq n$,

$$
x_i^{(1)} \longmapsto x_i^{(1)} + \varepsilon {x_i^{(1)}}^{k+1}
$$

in the integral (4.2) involves the following integral only

$$
\prod_{i=1}^{k_1+k_2} dx_i^{(1)} \Delta_{k_1+k_2}(x^{(1)}) \prod_{i=1}^{k_1+k_2} e^{\sum_{j=1}^{\infty} \bar t_j (x_i^{(1)})^j} \prod_{i=1}^{k_1} e^{-\frac{1}{2}x_i^{(1)2} + \sum_{p,q\geq 1} c_{p,q}^{(1)} (x_i^{(1)})^p (x_i^{(2)})^q + \sum_{r=1}^{\infty} \gamma_r^{(2)} (x_i^{(2)})^r}
$$

and leads to the first Virasoro constraints in (4.7), upon differentiation with regard to ε, setting $\varepsilon = 0$, applying Lemma 4.2 and taking into account the variation of the boundary term in the integral, $b_j^{(1)} \mapsto b_j^{(1)} - \varepsilon(b_j^{(1)})^{k+1} + O(\varepsilon^2)$. The shifts in the time parameter produces the terms in the first line of the Virasoro constraint.

Case 2: Performing the infinitesimal change of variables, for all $1 \le i \le n$,

$$x_i^{(\ell)} \longmapsto x_i^{(\ell)} + \varepsilon {x_i^{(\ell)}}^{k+1}$$

in the integral (4.2) involves the following integral only

$$\prod_{i=1}^{k_1+k_2} dx_i^{(\ell)} \prod_{i=1}^{k_1} e^{\sum_{r=1}^{\infty} \gamma_r^{(\ell)} (x_i^{(\ell)})^r - \frac{1}{2} x_i^{(\ell)2} + \sum_{r=1}^{\infty} \gamma_r^{(\ell-1)} (x_i^{(\ell-1)})^r + \sum_{p,q\ge 1} c_{p,q}^{(\ell-1)} (x_i^{(\ell-1)})^p (x_i^{(\ell)})^q}$$
$$e^{\sum_{r=1}^{\infty} \gamma_r^{(\ell+1)} (x_i^{(\ell+1)})^r + \sum_{p,q\ge 1} c_{p,q}^{(\ell)} (x_i^{(\ell)})^p (x_i^{(\ell+1)})^q}$$

and leads to the Virasoro constraints for $2 \le \ell \le m-1$ in (4.7), upon differentiation with regard to ε, setting $\varepsilon = 0$, applying Lemma 4.2 and taking into account the variation of the boundary term in the integral, $b_j^{(\ell)} \mapsto b_j^{(\ell)} - \varepsilon (b_j^{(\ell)})^{k+1} + O(\varepsilon^2)$. Also here one must take into account the shifts.
Case 3: Performing the infinitesimal change of variables, for all $1 \le i \le n$,

$$x_i^{(m)} \longmapsto x_i^{(m)} + \varepsilon {x_i^{(m)}}^{k+1}$$

in the integral (4.2) involves the following integral only

$$\prod_{i=1}^{k_1+k_2} dx_i^{(m)} \Delta_{k_1}(x_1^{(m)}, \dots, x_{k_1}^{(m)}) \Delta_{k_2}(x_{k_1+1}^{(m)}, \dots, x_{k_1+k_2}^{(m)})$$

$$\prod_{i=1}^{k_1} e^{-\sum_{j=1}^{\infty} [\bar{s}_j - \delta_{j1}\alpha + \delta_{j2}(\frac{1}{2}-\beta)](x_i^{(m)})^j + \sum_{r=1}^{\infty} \gamma_r^{(m-1)} (x_i^{(m-1)})^r + \sum_{p,q\ge 1} c_{p,q}^{(m-1)} (x_i^{(m-1)})^p (x_i^{(m)})^q}$$

$$\prod_{i=k_1}^{k_1+k_2} e^{-\sum_{j=1}^{\infty} [\bar{u}_j + \delta_{j1}\alpha + \delta_{j2}(\frac{1}{2}+\beta)](x_i^{(m)})^j + \sum_{r=1}^{\infty} \gamma_r^{(m-1)} (x_i^{(m-1)})^r + \sum_{p,q\ge 1} c_{p,q}^{(m-1)} (x_i^{(m-1)})^p (x_i^{(m)})^q},$$

and leads to the last Virasoro constraint for $\ell = m$ with a similar argument; this ends the proof of Proposition 4.1. The relations (4.6) follow at once by inspection of the integral (4.2). ∎

Setting

$$J^{-1} := J^{-1}(c_1, \ldots, c_{m-1}) := \begin{pmatrix} -1 & & c_1 & & & & \\ & & & \ddots & & \mathbf{O} & \\ c_1 & & -1 & & \ddots & & \\ & \ddots & & \ddots & & \ddots & \\ & & \ddots & & -1 & & c_{m-1} \\ & \mathbf{O} & & \ddots & & & \\ & & & & c_{m-1} & & -1 \end{pmatrix}, \tag{4.9}$$

define the differential operators $\mathcal{A}_i^{\pm}, \mathcal{C}_i$ and $\bar{\mathcal{A}}_i, \bar{\mathcal{C}}_i$:

$$\begin{aligned} \mathcal{A}_1^{\pm} &:= -\frac{1}{2}\sum_{j=1}^{m} J_{mj}\left(\mathcal{D}_{-1}(b^{(j)}) - 2\delta_{jm}\beta\frac{\partial}{\partial\alpha}\right) \mp \frac{1}{2}\frac{\partial}{\partial\alpha} =: \bar{\mathcal{A}}_1^{\pm} + \beta J_{mm}\frac{\partial}{\partial\alpha} \\ \mathcal{C}_1 &:= \sum_{j=1}^{m} J_{1j}\left(\mathcal{D}_{-1}(b^{(j)}) - 2\delta_{jm}\beta\frac{\partial}{\partial\alpha}\right) =: \bar{\mathcal{C}}_1 - 2\beta J_{1m}\frac{\partial}{\partial\alpha} \end{aligned}$$

$$\begin{aligned} \mathcal{A}_2^{\pm} &:= \frac{1}{2}\left(\mathcal{D}_0(b^{(m)}) - \alpha\frac{\partial}{\partial\alpha} - c_{m-1}\frac{\partial}{\partial c_{m-1}}\right) - (\beta \pm \frac{1}{2})\frac{\partial}{\partial\beta} =: \bar{\mathcal{A}}_2 \mp \frac{1}{2}\frac{\partial}{\partial\beta} - \beta\frac{\partial}{\partial\beta} \\ \mathcal{C}_2 &:= -\mathcal{D}_0(b^{(1)}) + c_1\frac{\partial}{\partial c_1} =: \bar{\mathcal{C}}_2 \end{aligned}$$

We now state:

Proposition 4.3 *Along the locus $\mathcal{L}$, the partials and second partials of $f := \log\tau_{k_1k_2}$ with regard to $\bar{t}_i, \bar{s}_i, \bar{u}_i$ can be expressed in terms of the operators $\bar{\mathcal{A}}_i, \bar{\mathcal{C}}_i$ and $\partial/\partial\beta$:*

$$
\begin{aligned}
\bar{\mathcal{A}}_1^+ f &= \frac{\partial f}{\partial \bar{s}_1} + \frac{\alpha\,(k_2-k_1)}{2} J_{mm} \\
\bar{\mathcal{A}}_1^- f &= \frac{\partial f}{\partial \bar{u}_1} + \frac{\alpha\ (k_2-k_1)}{2} J_{mm} \\
\bar{\mathcal{C}}_1 f &= \frac{\partial f}{\partial \bar{t}_1} - \alpha(k_2-k_1) J_{m1} \\
\left(\bar{\mathcal{A}}_2 - \frac{1}{2}\frac{\partial}{\partial\beta}\right) f &= \frac{\partial f}{\partial \bar{s}_2} + \frac{1}{4}({k_2}^2+k_2+{k_1}^2+k_1) \\
\left(\bar{\mathcal{A}}_2 + \frac{1}{2}\frac{\partial}{\partial\beta}\right) f &= \frac{\partial f}{\partial \bar{u}_2} + \frac{1}{4}({k_2}^2+k_2+{k_1}^2+k_1) \\
\bar{\mathcal{C}}_2 f &= \frac{\partial f}{\partial \bar{t}_2} - \frac{1}{2}(k_2+k_1)\,(k_2+k_1+1) \\
\bar{\mathcal{A}}_1^+ \bar{\mathcal{C}}_1 f &= \frac{\partial^2 f}{\partial \bar{t}_1 \partial \bar{s}_1} - k_1 J_{1m} \\
\left(\bar{\mathcal{A}}_2 \bar{\mathcal{C}}_1 + J_{1m}\frac{\partial}{\partial\alpha} - \frac{1}{2}\bar{\mathcal{C}}_1\frac{\partial}{\partial\beta}\right) f &= \frac{\partial^2 f}{\partial \bar{t}_1 \partial \bar{s}_2} - \alpha(k_2-k_1) J_{1m} J_{mm} \\
\bar{\mathcal{C}}_2 \bar{\mathcal{A}}_1^+ f &= \frac{\partial^2 f}{\partial \bar{t}_2 \partial \bar{s}_1} - \alpha(k_2-k_1) J_{1m}^2
\end{aligned}
\qquad (4.10)
$$

Proof: The Virasoro relations (4.7) can be written as (remembering $t_1 = \gamma_1$)

$$
\begin{pmatrix}
\mathcal{D}_{-1}(b^{(1)}) \\
\vdots \\
\mathcal{D}_{-1}(b^{(m-1)}) \\
\mathcal{D}_{-1}(b^{(m)}) - 2\beta\frac{\partial}{\partial\alpha} - (k_1-k_2)\alpha
\end{pmatrix} f = J^{-1}
\begin{pmatrix}
\frac{\partial}{\partial\gamma_1} \\
\vdots \\
\frac{\partial}{\partial\gamma_m}
\end{pmatrix} f + O(\mathcal{L}_\beta),
$$

remembering the locus $\mathcal{L}_\beta$ as in (4.5), and the tridiagonal matrix J^{-1} defined in (4.9). The symbol $O(\mathcal{L}_\beta)$ means a term which vanishes along $\mathcal{L}_\beta$.

Therefore, using the trivial relations (4.6), one finds

$$J\begin{pmatrix} \mathcal{D}_{-1}(b^{(1)}) \\ \mathcal{D}_{-1}(b^{(2)}) \\ \vdots \\ \mathcal{D}_{-1}(b^{(m-1)}) \\ \mathcal{D}_{-1}(b^{(m)}) - 2\beta\frac{\partial}{\partial\alpha} - (k_1-k_2)\alpha \end{pmatrix} f = \begin{pmatrix} \frac{\partial}{\partial t_1} \\ \frac{\partial}{\partial \gamma_2} \\ \vdots \\ \frac{\partial}{\partial \gamma_{m-1}} \\ -\frac{\partial}{\partial \bar{s}_1} - \frac{\partial}{\partial \bar{u}_1} \end{pmatrix} f + O(\mathcal{L}_\beta)$$

which combined with

$$-\frac{\partial}{\partial \bar{s}_1} + \frac{\partial}{\partial \bar{u}_1} = \frac{\partial}{\partial \alpha},$$

leads to the following equations for $f = \log \tau_{k_1 k_2}$ along the locus $\mathcal{L}_\beta$:

$$\mathcal{A}_1^+ f = \frac{\partial f}{\partial \bar{s}_1} + \frac{\alpha}{2}(k_2 - k_1) J_{mm}$$

$$\mathcal{A}_1^- f = \frac{\partial f}{\partial \bar{u}_1} + \frac{\alpha}{2}(k_2 - k_1) J_{mm}$$

$$\mathcal{C}_1 f = \frac{\partial f}{\partial \bar{t}_1} - \alpha(k_2 - k_1) J_{m1}$$

The second set of Virasoro relations, combined with

$$-\frac{\partial}{\partial \bar{s}_2} + \frac{\partial}{\partial \bar{u}_2} = \frac{\partial}{\partial \beta},$$

leads to the following equations, also along the locus $\mathcal{L}_\beta$,

$$\begin{aligned} \mathcal{A}_2^+ f &= \frac{\partial f}{\partial \bar{s}_2} + \frac{1}{4}({k_2}^2 + k_2 + {k_1}^2 + k_1) \\ \mathcal{A}_2^- f &= \frac{\partial f}{\partial \bar{u}_2} + \frac{1}{4}({k_2}^2 + k_2 + {k_1}^2 + k_1) \\ \mathcal{C}_2 f &= \frac{\partial f}{\partial \bar{t}_2} - \frac{1}{2}(k_2 + k_1)(k_2 + k_1 + 1) \end{aligned}$$

Moreover

$$\mathcal{A}_1^+\mathcal{C}_1 f = \frac{\partial^2 f}{\partial \bar{t}_1 \partial \bar{s}_1} - k_1 J_{1m} - \beta J_{mm} J_{m1}(k_2 - k_1)$$

$$\mathcal{A}_2^+\mathcal{C}_1 f = \frac{\partial^2 f}{\partial \bar{t}_1 \partial \bar{s}_2} - \alpha(k_2 - k_1) J_{1m} J_{mm}$$

$$\mathcal{C}_2\mathcal{A}_1^+ f = \frac{\partial^2 f}{\partial \bar{t}_2 \partial \bar{s}_1} - \alpha(k_2 - k_1) J_{1m}^2.$$

Indeed, one first needs to check

$$\mathcal{A}_1(\alpha) = J_{mm}\beta - \frac{1}{2},$$

$$\begin{aligned}
\mathcal{A}_2^+(\alpha J_{m1}) &= -\frac{1}{2}\left(\alpha\frac{\partial}{\partial\alpha} + c_{m-1}\frac{\partial}{\partial c_{m-1}}\right)\alpha J_{m1} \\
&= -\frac{1}{2}\left(\alpha J_{m1} + \alpha J_{m1}(-2J_{mm} - 1)\right) \\
&= \alpha J_{m1} J_{mm}
\end{aligned}$$

and

$$\mathcal{C}_2(J_{mm}) = c_1\frac{\partial}{\partial c_1} J_{mm} = -2J_{m1}^2.$$

Then, using the expressions for the $\mathcal{A}_i^{\pm} f$ and $\mathcal{C}_i f$, one computes along the locus[4] $\mathcal{L}_\beta$,

$$\begin{aligned}
\mathcal{A}_1^+\mathcal{C}_1 f &= \mathcal{A}_1^+\left(\frac{\partial f}{\partial \bar{t}_1} - \alpha(k_2 - k_1)J_{m1}\right) \\
&= \frac{\partial}{\partial \bar{t}_1}(\mathcal{A}_1^+ f) - (k_2 - k_1)J_{m1}(\mathcal{A}_1^+\alpha) \\
&= \frac{\partial}{\partial \bar{t}_1}\left(\frac{\partial f}{\partial \bar{s}_1} + \frac{\alpha}{2}(k_2 - k_1)J_{mm} - \frac{\bar{t}_1}{2}J_{m1}(k_1 + k_2)\right) - (k_2 - k_1)J_{m1}(J_{mm}\beta - \frac{1}{2}) \\
&= \frac{\partial^2 f}{\partial \bar{t}_1 \partial \bar{s}_1} - J_{m1}k_1 - \beta J_{mm} J_{m1}(k_2 - k_1)
\end{aligned}$$

[4]Since one takes the derivative with regard to $\bar{t}_1$ in the expression below, one must keep track of the term containing $\bar{t}_1$ in the identity for $\mathcal{A}_1^+ f$ before setting $\bar{t}_1 = 0$.

and

$$\begin{aligned}
\mathcal{A}_2\mathcal{C}_1 f &= \mathcal{A}_2\left(\frac{\partial f}{\partial \bar{t}_1} - \alpha(k_2-k_1)J_{m1}\right)\\
&= \frac{\partial}{\partial \bar{t}_1}\mathcal{A}_2 f - \mathcal{A}_2(\alpha(k_2-k_1)J_{m1})\\
&= \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_2} f - (k_2-k_1)\alpha J_{m1}J_{mm}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_2\mathcal{A}_1^+ f &= \frac{\partial}{\partial \bar{s}_1}\mathcal{C}_2 f + \mathcal{C}_2(\frac{\alpha}{2}(k_2-k_1)J_{mm})\\
&= \frac{\partial^2 f}{\partial \bar{s}_1 \partial \bar{t}_2} - \alpha(k_2-k_1)J_{m1}^2.
\end{aligned}$$

Finally, set $\beta = 0$ in each of these expressions. That means that we can replace all $\mathcal{A}_i$, $\mathcal{C}_i$ by $\bar{\mathcal{A}}_i$, $\bar{\mathcal{C}}_i$, except for $\bar{\mathcal{A}}_2$ and $\bar{\mathcal{B}}_2$, which gives an extra $\partial/\partial\beta$ and the composition

$$\begin{aligned}
\mathcal{A}_2^+\mathcal{C}_1 f\Big|_{\beta=0} &= \left(\bar{\mathcal{A}}_2 - \frac{1}{2}\frac{\partial}{\partial\beta}\right)\left(\bar{\mathcal{C}}_1 - 2J_{1m}\beta\frac{\partial}{\partial\alpha}\right) f\Big|_{\beta=0}\\
&= \left(\bar{\mathcal{A}}_2\bar{\mathcal{C}}_1 + J_{1m}\frac{\partial}{\partial\alpha}\right) f - \frac{1}{2}\bar{\mathcal{C}}_1\frac{\partial f}{\partial\beta}\Big|_{\beta=0}.
\end{aligned}$$

This establishes Proposition 4.3. ■

Remark: The five operators $\bar{\mathcal{A}}_1^{\pm}$, $\bar{\mathcal{A}}_2$, $\bar{\mathcal{C}}_1$, $\bar{\mathcal{C}}_2$ form a Lie algebra (upon using Lemma 3.1):

$$\left[\bar{\mathcal{A}}_1^{\pm}, \bar{\mathcal{C}}_1\right] = 0, \quad \left[\bar{\mathcal{A}}_1^{+}, \bar{\mathcal{A}}_1^{-}\right] = 0, \quad \left[\bar{\mathcal{A}}_2, \bar{\mathcal{C}}_2\right] = 0$$

$$\left[\bar{\mathcal{A}}_1^{\pm}, \bar{\mathcal{A}}_2\right] = -(\frac{1}{2} + J_{mm})\bar{\mathcal{A}}_1^{\pm}, \qquad \left[\bar{\mathcal{A}}_1^{\pm}, \bar{\mathcal{C}}_2\right] = -J_{1m}\bar{\mathcal{C}}_1$$

$$\left[\bar{\mathcal{A}}_2, \bar{\mathcal{C}}_1\right] = -J_{1m}(\bar{\mathcal{A}}_1^{+} + \bar{\mathcal{A}}_1^{-}), \qquad \left[\bar{\mathcal{C}}_2, \bar{\mathcal{C}}_1\right] = -(1 + 2J_{11})\bar{\mathcal{C}}_1$$

5 Integrable deformations and 3-component KP

The following integral

$$\tau_{k_1k_2}(\bar t,\bar s,\bar u;\alpha,\beta,\gamma^{(2)},\dots,\gamma^{(m-1)},c^{(1)},\dots,c^{(m-1)},E_1\times\dots\times E_m)$$

$$= \det\begin{pmatrix} (\mu^{+}_{ij})_{1\le i\le k_1,\ 1\le j\le k_1+k_2} \\ \\ (\mu^{-}_{ij})_{1\le i\le k_2,\ 1\le j\le k_1+k_2}\end{pmatrix} \tag{5.1}$$

where

$$\begin{aligned}\mu^{\pm}_{ij}(\bar t,\bar s,\bar u,c,\gamma) &= \int_{\prod_1^m E_i}(x^{(1)})^{j-1}(x^{(m)})^{i-1}e^{\sum_1^\infty\left(\bar t_k x^{(1)k}-\left(\begin{smallmatrix}\bar s_k\\ \bar u_k\end{smallmatrix}\right)x^{(m)k}\right)}e^{\pm\alpha x_m\pm\beta x_m^2}\\ &\qquad F_m(x^{(1)},\dots,x^{(m)})\prod_{\ell=1}^m dx^{(\ell)}\\ &= \left\langle x^{i-1}e^{-\sum_1^\infty\left(\begin{smallmatrix}\bar s_k\\ \bar u_k\end{smallmatrix}\right)x^k}e^{\pm\alpha x\pm\beta x^2}\ \middle|\ x^{j-1}e^{\sum_1^\infty \bar t_k x^k}\right\rangle_m \end{aligned}\tag{5.2}$$

with regard to the inner-product ($m\ge 2$)

$$\langle f\mid g\rangle_m=\int_{\prod_1^m E_i} f(x_m)g(x_1)F_m(x_1,\dots,x_m)dx_1\dots dx_m,$$

for

$$F_m(x_1,\dots,x_m):=\left(\prod_1^m e^{-\frac{x_\ell^2}{2}}\right)e^{\sum_{p,q\ge1}\sum_{\ell=1}^{m-1}c_{pq}^{(\ell)}x_\ell^p x_{\ell+1}^q+\sum_{\ell=2}^{m-1}\sum_{r=1}^\infty\gamma_r^{(\ell)}x_\ell^r}.$$

From ([4]), it follows that the function $\tau_{k_1k_2}$ above, which is expressed as the determinant of a moment matrix with regard to two different weights, satisfies the 3-component KP and thus it satisfies in particular, the following PDE's:

$$\begin{aligned}\frac{\partial^2}{\partial\bar s_1\partial\bar t_1}\log\tau_{k_1k_2} &= -\frac{\tau_{k_1+1,k_2}\tau_{k_1-1,k_2}}{\tau^2_{k_1,k_2}}\\ \frac{\partial^2}{\partial\bar u_1\partial\bar t_1}\log\tau_{k_1k_2} &= -\frac{\tau_{k_1,k_2+1}\tau_{k_1,k_2-1}}{\tau^2_{k_1,k_2}}\\ \frac{\partial^2}{\partial\bar s_1\partial\bar u_1}\log\tau_{k_1k_2} &= \frac{\tau_{k_1+1,k_2-1}\tau_{k_1-1,k_2+1}}{\tau^2_{k_1,k_2}}\end{aligned}\tag{5.3}$$

and

$$
\begin{aligned}
\frac{\partial^2}{\partial \bar s_1 \partial \bar t_2} \log \tau_{k_1 k_2} &= -\frac{1}{\tau^2_{k_1 k_2}} \left[\left(\frac{\partial}{\partial \bar t_1} \tau_{k_1+1,k_2} \right) \tau_{k_1-1,k_2} - \tau_{k_1+1,k_2} \left(\frac{\partial}{\partial \bar t_1} \tau_{k_1-1,k_2} \right) \right] \\
\frac{\partial^2}{\partial \bar s_2 \partial \bar t_1} \log \tau_{k_1 k_2} &= -\frac{1}{\tau^2_{k_1 k_2}} \left[\left(\frac{\partial}{\partial \bar s_1} \tau_{k_1-1,k_2} \right) \tau_{k_1+1,k_2} - \tau_{k_1-1,k_2} \left(\frac{\partial}{\partial \bar s_1} \tau_{k_1+1,k_2} \right) \right] \\
\frac{\partial^2}{\partial \bar s_2 \partial \bar u_1} \log \tau_{k_1 k_2} &= \frac{1}{\tau^2_{k_1 k_2}} \left[\left(\frac{\partial}{\partial \bar s_1} \tau_{k_1-1,k_2+1} \right) \tau_{k_1+1,k_2-1} - \tau_{k_1-1,k_2+1} \left(\frac{\partial}{\partial \bar s_1} \tau_{k_1+1,k_2-1} \right) \right].
\end{aligned}
\tag{5.4}
$$

Thus, upon taking the ratio of the first, second and third equations of (5.4) and (5.3), one finds

$$
\begin{aligned}
\frac{\partial}{\partial \bar t_1} \log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} &= \frac{\frac{\partial^2}{\partial \bar t_2 \partial \bar s_1} \log \tau_{k_1,k_2}}{\frac{\partial^2}{\partial \bar t_1 \partial \bar s_1} \log \tau_{k_1,k_2}} \\
\frac{\partial}{\partial \bar s_1} \log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} &= -\frac{\frac{\partial^2}{\partial \bar t_1 \partial \bar s_2} \log \tau_{k_1,k_2}}{\frac{\partial^2}{\partial \bar t_1 \partial \bar s_1} \log \tau_{k_1,k_2}}. \\
\frac{\partial}{\partial \bar s_1} \log \frac{\tau_{k_1+1,k_2-1}}{\tau_{k_1-1,k_2+1}} &= -\frac{\frac{\partial^2}{\partial \bar u_1 \partial \bar s_2} \log \tau_{k_1,k_2}}{\frac{\partial^2}{\partial \bar u_1 \partial \bar s_1} \log \tau_{k_1,k_2}}.
\end{aligned}
\tag{5.5}
$$

6 A PDE for the Gaussian matrices coupled in a chain, with external potential

The purpose of this section is to prove Theorem 0.2. Notice the probability

$$
\begin{aligned}
&\mathbb{P}_n \left(\alpha; c_1, \ldots, c_{m-1}; E_1 \times \ldots \times E_m \right) \\
&= \frac{1}{Z_n} \int_{\prod_1^m E_i^{k_1+k_2}} \Delta_{k_1+k_2}(x^{(1)}) \prod_{i=1}^m dy_i^{(\ell)} \\
&\quad \Delta_{k_1}(x^{(m)\prime}) \prod_{i=1}^{k_1} e^{-\frac{1}{2}\sum_{\ell=1}^m y_i^{(\ell)2} + \sum_{\ell=1}^{m-1} c_\ell y_i^{(\ell)} y_i^{(\ell+1)} + \alpha y_i^{(m)}} \\
&\quad \Delta_{k_2}(x^{(m)\prime\prime}) \prod_{i=k_1+1}^{k_1+k_2} e^{-\frac{1}{2}\sum_{\ell=1}^m y_i^{(\ell)2} + \sum_{\ell=1}^{m-1} c_\ell y_i^{(\ell)} y_i^{(\ell+1)} - \alpha y_i^{(m)}},
\end{aligned}
\tag{6.1}
$$

is invariant under the involution

$$\iota: \quad \alpha \longleftrightarrow -\alpha, \quad \beta \longleftrightarrow -\beta, \quad u_i \longleftrightarrow s_i, \quad k_1 \longleftrightarrow k_2.$$

Proof of Theorem 0.1: The first equality in each of the expressions (6.2) and (6.3) below follows from the expressions for $\bar{\mathcal{A}}_1^+ f$ and $\bar{\mathcal{C}}_1 f$ in (4.10), whereas the second equalities follow from (5.5) and the third equality from the expressions for $\mathcal{A}_1^+\mathcal{C}_1 f$, $\mathcal{A}_2^+\mathcal{C}_1 f$, $\mathcal{C}_2\mathcal{A}_1^+ f$ in (4.10):

$$\begin{aligned}
\bar{\mathcal{C}}_1 \log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} &= \frac{\partial}{\partial \bar{t}_1} \log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} + 2\alpha J_{1m} \\
&= \frac{\frac{\partial^2}{\partial \bar{t}_2 \partial \bar{s}_1} \log \tau_{k_1,k_2}}{\frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_1} \log \tau_{k_1,k_2}} + 2\alpha J_{1m} \\
&= \frac{\left(\frac{\partial^2}{\partial \bar{t}_2 \partial \bar{s}_1} + 2\alpha J_{1m} \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_1} \right) \log \tau_{k_1,k_2}}{\frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_1} \log \tau_{k_1,k_2}} = -\frac{H_2^+}{F^+} \qquad (6.2)
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{A}}_1^+ \log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} &= \frac{\partial}{\partial \bar{s}_1} \log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} - \alpha J_{mm} \\
&= -\frac{\frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_2} \log \tau_{k_1,k_2}}{\frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_1} \log \tau_{k_1,k_2}} - \alpha J_{mm} \\
&= -\frac{H_1^+ - X}{F^+} - \alpha J_{mm} \qquad (6.3)
\end{aligned}$$

where $F^+,\ X,\ H_i^+$ can are functions of $\log \tau_{k_1 k_2}$, which can also be expressed in terms of the actual probability $\mathbb{P}_n$, taking into account Lemma 8.1,

$$\begin{aligned}
F^+ &= \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_1} \log \tau_{k_1,k_2} = \bar{\mathcal{A}}_1^+ \bar{\mathcal{C}}_1 \log \tau_{k_1,k_2} + k_1 J_{1m} = \bar{\mathcal{A}}_1^+ \bar{\mathcal{C}}_1 \log \mathbb{P}_n + k_1 J_{1m} \\
X &= \frac{1}{2} \bar{\mathcal{C}}_1 \frac{\partial}{\partial \beta} \log \tau_{k_1 k_2} = \frac{1}{2} \bar{\mathcal{C}}_1 \frac{\partial}{\partial \beta} \log \mathbb{P}_n \\
H_1^+ &= \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_2} \log \tau_{k_1,k_2} + \frac{1}{2} \frac{\partial}{\partial \beta} \bar{\mathcal{C}}_1 \log \tau_{k_1,k_2} \\
&= \left(\bar{\mathcal{A}}_2 \bar{\mathcal{C}}_1 + J_{1m} \frac{\partial}{\partial \alpha} \right) \log \tau_{k_1,k_2} - \alpha (k_1 - k_2) J_{1m} J_{mm} \\
&= \left(\bar{\mathcal{A}}_2 \bar{\mathcal{C}}_1 + J_{1m} \frac{\partial}{\partial \alpha} \right) \log \mathbb{P}_n - 2\alpha k_1 J_{1m} J_{mm} + \frac{k_1 k_2}{\alpha} J_{1m}
\end{aligned}$$

$$
\begin{aligned}
H_2^+ &= -\left(\frac{\partial^2}{\partial \bar{t}_2 \partial \bar{s}_1} + 2\alpha J_{1m} \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_1}\right) \log \tau_{k_1,k_2} \\
&= -\left(\bar{\mathcal{C}}_2 + 2\alpha J_{1m} \bar{\mathcal{C}}_1\right) \bar{\mathcal{A}}_1^+ \log \tau_{k_1,k_2} - \alpha(k_1+k_2)(J_{1m})^2 \\
&= -\left(\bar{\mathcal{C}}_2 + 2\alpha J_{1m} \bar{\mathcal{C}}_1\right) \bar{\mathcal{A}}_1^+ \log \mathbb{P}_n
\end{aligned}
$$

Thus, $\bar{\mathcal{A}}_1^+$ of the right hand side of (6.2) must equal $\bar{\mathcal{C}}_1$ of the right hand side of (6.3) and furthermore noticing that $\bar{\mathcal{C}}_1 \alpha J_{mm} = 0$, one finds:

$$
\bar{\mathcal{A}}_1^+ \frac{H_2^+}{F^+} = \bar{\mathcal{C}}_1 \left(\frac{H_1^+}{F^+} - \frac{X}{F^+}\right)
$$

or in Wronskian notation

$$
\left\{X, F^+\right\}_{\bar{\mathcal{C}}_1} = \{H_1^+, F^+\}_{\bar{\mathcal{C}}_1} - \{H_2^+, F^+\}_{\bar{\mathcal{A}}_1} = G^+ \tag{6.4}
$$

and

$$
-\left\{X, F^-\right\}_{\bar{\mathcal{C}}_1} = \left\{H_1^-, F^-\right\}_{\bar{\mathcal{C}}_1} - \left\{H_2^-, F^-\right\}_{\bar{\mathcal{B}}_1} = G^- \tag{6.5}
$$

upon applying the involution ι to the first equation. Equations (6.4) and (6.5) yield a linear system of equations in X and $\bar{\mathcal{C}}_1 X$, leading to

$$
\begin{aligned}
X &= \frac{G^- F^+ + G^+ F^-}{-F^-(\bar{\mathcal{C}}_1 F^+) + F^+(\bar{\mathcal{C}}_1 F^-)} \\
\bar{\mathcal{C}}_1 X &= \frac{G^-(\bar{\mathcal{C}}_1 F^+) + G^+(\bar{\mathcal{C}}_1 F^-)}{-F^-(\bar{\mathcal{C}}_1 F^+) + F^+(\bar{\mathcal{C}}_1 F^-)}
\end{aligned}
$$

Subtracting the second equation from $\bar{\mathcal{C}}_1$ of the first equation yields the following:

$$
\begin{aligned}
&\left(F^+ \bar{\mathcal{C}}_1 G^- + F^- \bar{\mathcal{C}}_1 G^+\right)\left(F^+ \bar{\mathcal{C}}_1 F^- - F^- \bar{\mathcal{C}}_1 F^+\right) \\
&-\left(F^+ G^- + F^- G^+\right)\left(F^+ \bar{\mathcal{C}}_1^2 F^- - F^- \bar{\mathcal{C}}_1^2 F^+\right) = 0.
\end{aligned}
$$

The second way of expressing these equations is to write a system of 4 equations, consisting of the system (6.4) and (6.5) and that same system acted upon by $\bar{\mathcal{C}}_1$:

$$
\begin{aligned}
0 &= -\left\{X, F^+\right\}_{\bar{\mathcal{C}}_1} + G^+ \\
0 &= -\left\{X, F^-\right\}_{\bar{\mathcal{C}}_1} - G^- \\
0 &= -\bar{\mathcal{C}}_1 \left\{X, F^+\right\}_{\bar{\mathcal{C}}_1} + \bar{\mathcal{C}}_1 G^+ \\
0 &= -\bar{\mathcal{C}}_1 \left\{X, F^-\right\}_{\bar{\mathcal{C}}_1} - \bar{\mathcal{C}}_1 G^-,
\end{aligned}
$$

or, in matrix notation,

$$\begin{pmatrix} G^+ & \bar{\mathcal{C}}_1 F^+ & -F^+ & 0 \\ -G^- & \bar{\mathcal{C}}_1 F^- & -F^- & 0 \\ \bar{\mathcal{C}}_1 G^+ & \bar{\mathcal{C}}_1^2 F^+ & 0 & -F^+ \\ -\bar{\mathcal{C}}_1 G^- & \bar{\mathcal{C}}_1^2 F^- & 0 & -F^- \end{pmatrix} \begin{pmatrix} 1 \\ X \\ \bar{\mathcal{C}}_1 X \\ \bar{\mathcal{C}}_1^2 X \end{pmatrix} = 0$$

and thus the matrix must be singular, leading to the second formulation and ending the proof of Theorem 0.1. ■

7 The PDE for the transition probability of the Pearcey process

Given $E_\ell := \bigcup_{i=1}^r [x_{2i-1}^{(\ell)}, x_{2i}^{(\ell)}] \subset \mathbb{R}$, define the "*space*" and "*time*" operators $\mathcal{X}_k$ and $\mathcal{T}_k$ related to Brownian motion, together with a mixed "*space-time*" operator $\tilde{\mathcal{X}}_{-1}$,

$$\mathcal{X}_k := \sum_\ell \sum_{i=1}^{2r_\ell} (x_i^{(\ell)})^{k+1} \frac{\partial}{\partial x_i^{(\ell)}}, \quad \mathcal{T}_j = \sum_\ell s_\ell^{j+1} \frac{\partial}{\partial s_\ell}, \quad \tilde{\mathcal{X}}_{-1} = \sum_\ell s_\ell \sum_{i=1}^{2} \frac{\partial}{\partial x_i^{(\ell)}}.$$

We shall also need the intermediate operators

$$\mathcal{X}_k(x^{(\ell)}) := \sum_{i=1}^{2r_\ell} (x_i^{(\ell)})^{k+1} \frac{\partial}{\partial x_i^{(\ell)}}.$$

Define a new function $\mathbb{Q}_z$

$$\log \mathbb{P}_n(\alpha, c_1, \dots, c_{n-1}; b^{(1)}, \dots, b^{(m)}) \Big|_{n=\frac{2}{z^4}} = \mathbb{Q}_z(s_1, \dots, s_m; x^{(1)}, \dots, x^{(m)}) \tag{7.1}$$

by means of the change of variables, defined earlier

$$
\begin{aligned}
\alpha &= \frac{2}{z}\frac{\sqrt{s_m - s_{m-1}}}{\sqrt{(1-s_m z^2)(1-s_{m-1}z^2)}} \\
c_j &= \begin{cases} \sqrt{\frac{(1+s_1 z^2)(s_3-s_2)}{(1+s_2 z^2)(s_3-s_1)}} & \text{for } j=1 \\ \sqrt{\frac{(s_j-s_{j-1})(s_{j+2}-s_{j+1})}{(s_{j+1}-s_{j-1})(s_{j+2}-s_j)}} & \text{for } 2\le j\le m-2 \\ \sqrt{\frac{(s_{m-1}-s_{m-2})(1-s_m z^2)}{(s_m-s_{m-2})(1-s_{m-1}z^2)}} & \text{for } j=m-1 \end{cases} \\
b_i^{(\ell)} &= \begin{cases} 2x_i^{(1)}\sqrt{\frac{1+s_2z^2}{(1+s_1z^2)(s_2-s_1)}}, & \text{for } \ell=1 \\ 2x_i^{(\ell)}\sqrt{\frac{s_{\ell+1}-s_{\ell-1}}{(s_\ell-s_{\ell-1})(s_{\ell+1}-s_\ell)}}, & \text{for } 2\le \ell\le m-1 \\ 2x_i^{(m)}\sqrt{\frac{1-s_{m-1}z^2}{(s_m-s_{m-1})(1-s_m z^2)}}, & \text{for } \ell=m \end{cases}
\end{aligned} \tag{7.2}
$$

with inverse map

$$
\begin{aligned}
s_i &= s_i(\alpha,c;z) = \frac{1}{z^2}\frac{\alpha^2 z^4 \frac{J_{m1}J_{mi}}{J_{1i}}+2}{\alpha^2 z^4 \frac{J_{m1}J_{mi}}{J_{1i}}-2} \\
x_k^{(i)} &= b_k^{(i)} U_i(s(\alpha,c;z);z),
\end{aligned} \tag{7.3}
$$

with

$$
U_i(s(\alpha,c;z);z) = \frac{-1}{J_{mi}}\left(\frac{\alpha z \frac{J_{m1}J_{mi}}{J_{1i}}}{\alpha^2 z^4 \frac{J_{m1}J_{mi}}{J_{1i}}-2}\right).
$$

Then

$$
\begin{aligned}
\frac{\partial x_k^{(i)}}{\partial \alpha} &= \frac{\partial (b_k^{(i)}U_i)}{\partial \alpha} = b_k^{(i)}\frac{\partial U_i}{\partial \alpha} = b_k^{(i)}U_i\frac{\partial \log U_i}{\partial \alpha} = x_k^{(i)}\frac{\partial \log U_i}{\partial \alpha} \\
\frac{\partial x_k^{(i)}}{\partial c_j} &= \frac{\partial (b_k^{(i)}U_i)}{\partial c_j} = b_k^{(i)}\frac{\partial U_i}{\partial c_j} = b_k^{(i)}U_i\frac{\partial \log U_i}{\partial c_j} = x_k^{(i)}\frac{\partial \log U_i}{\partial c_j}.
\end{aligned}
$$

Thus, setting (7.3) in the right hand side of (7.1) and using the chain rule, one computes

$$
\begin{aligned}
\frac{\partial \log \mathbb{P}_n}{\partial \alpha}\Big|_{n=\frac{2}{z^4}} &= \sum_{i=1}^{m} \frac{\partial s_i}{\partial \alpha}\frac{\partial \mathbb{Q}_z}{\partial s_i} + \sum_{i=1}^{m}\sum_{k=1}^{2r_i} \frac{\partial x_k^{(i)}}{\partial \alpha}\frac{\partial \mathbb{Q}_z}{\partial x_k^{(i)}} \\
&= \left(\sum_{i=1}^{m} \frac{\partial s_i}{\partial \alpha}\frac{\partial}{\partial s_i} + \sum_{i=1}^{m} \frac{\partial \log U_i}{\partial \alpha} \sum_{k=1}^{2r_i} x_k^{(i)} \frac{\partial}{\partial x_k^{(i)}}\right)\mathbb{Q}_z \\
&= \sum_{i=1}^{m}\left(\frac{\partial s_i}{\partial \alpha}\frac{\partial}{\partial s_i} + \frac{\partial \log U_i}{\partial \alpha}\mathcal{X}_0(x^{(i)})\right)\mathbb{Q}_z
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \log \mathbb{P}_n}{\partial c_j}\Big|_{n=\frac{2}{z^4}} &= \sum_{i=1}^{m} \frac{\partial s_i}{\partial c_j}\frac{\partial \mathbb{Q}_z}{\partial s_i} + \sum_{i=1}^{m}\sum_{k=1}^{2r_i} \frac{\partial x_k^{(i)}}{\partial c_j}\frac{\partial \mathbb{Q}_z}{\partial x_k^{(i)}} \\
&= \left(\sum_{i=1}^{m} \frac{\partial s_i}{\partial c_j}\frac{\partial}{\partial s_i} + \sum_{i=1}^{m} \frac{\partial \log U_i}{\partial c_j} \sum_{k=1}^{2r} x_k^{(i)} \frac{\partial}{\partial x_k^{(i)}}\right)\mathbb{Q}_z \\
&= \sum_{i=1}^{m}\left(\frac{\partial s_i}{\partial c_j}\frac{\partial}{\partial s_i} + \frac{\partial \log U_i}{\partial c_j}\mathcal{X}_0(x^{(i)})\right)\mathbb{Q}_z
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}_{-1}(E_i)\log \mathbb{P}_n\Big|_{n=\frac{2}{z^4}} &= \sum_{k=1}^{2r_i} \frac{\partial \log \mathbb{P}_n}{\partial b_k^{(i)}}\Big|_{n=\frac{2}{z^4}} \\
&= \sum_{k=1}^{2r_i} \frac{\partial x_k^{(i)}}{\partial b_k^{(i)}}\frac{\partial \mathbb{Q}_z}{\partial x_k^{(i)}} \\
&= \left(U_i \sum_{k=1}^{2r_i} \frac{\partial}{\partial x_k^{(i)}}\right)\mathbb{Q}_z \\
&= U_i \mathcal{X}_{-1}(x^{(i)})\mathbb{Q}_z
\end{aligned}
$$

$$\begin{aligned}
\mathcal{D}_0(E_i)\log\mathbb{P}_n\Big|_{n=\frac{2}{z^4}} &= \sum_{k=1}^{2r_i} b_k^{(i)}\frac{\partial\log\mathbb{P}_n}{\partial b_k^{(i)}}\Big|_{n=\frac{2}{z^4}}\\
&= \sum_{k=1}^{2r_i} b_k^{(i)}\frac{\partial x_k^{(i)}}{\partial b_k^{(i)}}\frac{\partial\mathbb{Q}_z}{\partial x_k^{(i)}}\\
&= \left(\sum_{k=1}^{2r_i} x_k^{(i)}\frac{\partial}{\partial x_k^{(i)}}\right)\mathbb{Q}_z\\
&= \mathcal{X}_0(x^{(i)})\mathbb{Q}_z.
\end{aligned}$$

Using the information above, one computes ($\varepsilon = \pm 1$)

$$\begin{aligned}
&\bar{\mathcal{A}}_1^{\varepsilon}\log\mathbb{P}_n(\alpha, c_1,\dots,c_{n-1}; b^{(1)},\dots,b^{(m)})\Big|_{n=\frac{2}{z^4}}\\
&= -\frac{1}{2}\left(\sum_{j=1}^{m} J_{mj}\mathcal{D}_{-1}(E_j)+\varepsilon\frac{\partial}{\partial\alpha}\right)\log\mathbb{P}_n\Big|_{n=\frac{2}{z^4}}\\
&= -\frac{1}{2}\sum_{j=1}^{m}\left(J_{mj}U_j\mathcal{X}_{-1}(x^{(j)})+\varepsilon\frac{\partial s_j}{\partial\alpha}\frac{\partial}{\partial s_j}+\varepsilon\frac{\partial\log U_j}{\partial\alpha}\mathcal{X}_0(x^{(j)})\right)\mathbb{Q}_z\\
&=: A_1^{\varepsilon}\mathbb{Q}_z
\end{aligned} \tag{7.4}$$

$$\begin{aligned}
\bar{\mathcal{C}}_1\log\mathbb{P}_n\Big|_{n=\frac{2}{z^4}} &= \sum_{j=1}^{m} J_{1j}\mathcal{D}_{-1}(Ej)\log\mathbb{P}_n\Big|_{n=\frac{2}{z^4}}\\
&= \left(\sum_{j=1}^{m} J_{1j}U_j\mathcal{X}_{-1}(x^{(j)})\right)\mathbb{Q}_z\\
&=: C_1\mathbb{Q}_z
\end{aligned} \tag{7.5}$$

$$
\begin{aligned}
&\bar{\mathcal{A}}_2 \log \mathbb{P}_n \Big|_{n=\frac{2}{z^4}} \\
&= \frac{1}{2}\left(\mathcal{D}_0(b^{(m)}) - \alpha\frac{\partial}{\partial\alpha} - c_{m-1}\frac{\partial}{\partial c_{m-1}}\right)\log \mathbb{P}_n\Big|_{n=\frac{2}{z^4}} \\
&= -\frac{1}{2}\sum_{j=1}^{m}\left(\begin{array}{l}\left(c_{m-1}\frac{\partial s_j}{\partial c_{m-1}} + \alpha\frac{\partial s_j}{\partial\alpha}\right)\frac{\partial}{\partial s_j} \\ +\left(\left(c_{m-1}\frac{\partial \log U_j}{\partial c_{m-1}} + \alpha\frac{\partial \log U_j}{\partial\alpha}\right) - \delta_{jm}\right)\mathcal{X}_0(x^{(j)})\end{array}\right)\mathbb{Q}_z \\
&=: A_2\mathbb{Q}_z
\end{aligned}
\tag{7.6}
$$

$$
\begin{aligned}
\bar{\mathcal{C}}_2 \log \mathbb{P}_n\Big|_{n=\frac{2}{z^4}} &= \left(-\mathcal{D}_0(b^{(1)}) + c_1\frac{\partial}{\partial c_1}\right)\log \mathbb{P}_n\Big|_{n=\frac{2}{z^4}} \\
&= \sum_{j=1}^{m}\left(c_1\frac{\partial s_j}{\partial c_1}\frac{\partial}{\partial s_j} + \left(c_1\frac{\partial \log U_j}{\partial c_1} - \delta_{j1}\right)\mathcal{X}_0(x^{(j)})\right)\mathbb{Q}_z \\
&=: C_2\mathbb{Q}_z
\end{aligned}
\tag{7.7}
$$

Lemma 7.1 *The following expansions hold near $z \sim 0$:*

$$
\begin{aligned}
\alpha\frac{\partial s_i}{\partial\alpha} &= \frac{1}{z^2} - s_i^2 z^2 \\
\frac{\partial s_i}{\partial\alpha} &= \frac{1}{2z\sqrt{s_m - s_{m-1}}}\left(1 - \frac{z^2}{2}(s_{m-1}+s_m) + O(z^4)\right)
\end{aligned}
$$

$$
\begin{aligned}
\alpha\frac{\partial}{\partial\alpha}\log U_i(s(\alpha,c;z),z) &= -z^2 s_i \\
\frac{\partial}{\partial\alpha}\log U_i(s(\alpha,c;z),z) &= O(z^3)
\end{aligned}
$$

$$
\begin{aligned}
c_1\frac{\partial}{\partial c_1}s_i &= \frac{1}{2(s_2-s_1)}\left(\frac{1}{z^4} + \frac{s_1+s_2-2s_i}{z^2} + O(1)\right) \\
c_{m-1}\frac{\partial}{\partial c_{m-1}}s_i &= \frac{1}{s_m - s_{m-1}}\left(\frac{1}{z^4} + \frac{s_m+s_{m-1}-2s_i}{z^2} + O(1)\right)
\end{aligned}
$$

$$\begin{aligned} c_1\frac{\partial}{\partial c_1}\log U_i &= -\frac{1}{2(s_2-s_1)z^2}+O(1)\\ c_{m-1}\frac{\partial}{\partial c_{m-1}}\log U_i &= -\frac{1}{2(s_m-s_{m-1})z^2}+O(1)\end{aligned}$$

$$\begin{aligned} J_{mi}U_i &= \frac{1}{4\sqrt{s_m-s_{m-1}}}\left(-\frac{1}{z^2}+\frac{1}{2}(s_m+s_{m-1}-2s_i)+O(z^2)\right)\\ J_{1i}U_i &= \frac{1}{4\sqrt{s_2-s_1}}\left(-\frac{1}{z^2}-\frac{1}{2}(s_1+s_2-2s_i)\right.\\ &\qquad\left.+\frac{z^2}{8}((s_2-s_1)^2+4s_1(s_1+s_2))+O(z^4)\right)\end{aligned}$$

$$\alpha=\frac{2}{z}\sqrt{s_m-s_{m-1}}\ (1+\frac{z^2}{2}(s_m+s_{m-1})+O(z^4)).$$

Proof: These series follow from expanding the expressions given in Lemma 3.3 near $z\sim 0$. ∎

Lemma 7.2 *The operators $A_1^\varepsilon,\ A_2,\ ,C_1,\ C_2$, as defined in (7.4), (7.6), (7.5), (7.7) and as acting on the function $\mathbb{Q}_z(s_1,\dots,s_m;x^{(1)},\dots,x^{(m)})$ admit the following expansions in $z\sim 0$, in terms of the operators $\mathcal{X}_k,\ \mathcal{T}_k,\ \bar{\mathcal{X}}_{-1}$,*

$$\begin{aligned} A_1^\varepsilon &= \frac{1}{8\sqrt{s_m-s_{m-1}}}\left\{\begin{array}{l}\frac{1}{z^2}\mathcal{X}_{-1}-\frac{2\varepsilon}{z}\mathcal{T}_{-1}+\left(\tilde{\mathcal{X}}_{-1}-\frac{(s_{m-1}+s_m)}{2}\mathcal{X}_{-1}\right)\\ +\varepsilon z(s_{m-1}+s_m)\mathcal{T}_{-1}+O(z^2)\end{array}\right\}\\ C_1 &= \frac{1}{4\sqrt{s_2-s_1}}\left\{\begin{array}{l}-\frac{1}{z^2}\mathcal{X}_{-1}+\left(\tilde{\mathcal{X}}_{-1}-\frac{(s_1+s_2)}{2}\mathcal{X}_{-1}\right)\\ +z^2\left(\frac{(s_2-s_1)^2}{8}\mathcal{X}_{-1}+\frac{(s_1+s_2)}{2}\tilde{\mathcal{X}}_{-1}\right)+O(z^4)\end{array}\right\}\\ A_2 &= \frac{1}{4(s_m-s_{m-1})}\left\{-\frac{\mathcal{T}_{-1}}{z^4}+\frac{1}{z^2}(\mathcal{X}_0+2\mathcal{T}_0+(s_{m-1}-3s_m)\mathcal{T}_{-1})+O(1)\right\}\\ C_2 &= \frac{1}{2(s_2-s_1)}\left\{\frac{\mathcal{T}_{-1}}{z^4}-\frac{1}{z^2}(\mathcal{X}_0+2\mathcal{T}_0-(s_1+s_2)\mathcal{T}_{-1})+O(1)\right\}.\end{aligned}$$

Proof: These formulae are an immediate consequence of formulae (7.4), (7.6), (7.5), (7.7), combined with the series in Lemma 7.1. ∎

Lemma 7.3 *In the new coordinates $s_1, \ldots, s_m, x^{(1)}, \ldots, x^{(m)}$, the quantities appearing in the basic equation (0.3) for $\mathbb{P}_n$ admit the following expansions in $z \sim 0$, setting $k = n/2 = 1/z^4$:*

$$
\begin{aligned}
F^{\varepsilon} &= A_1^{\varepsilon} C_1 \mathbb{Q}_z + \frac{J_{1m}}{z^4} \\
&= \frac{1}{32\sqrt{(s_2 - s_1)(s_m - s_{m-1})}} \\
&\left\{ \begin{array}{l} -\frac{16}{z^6} - \frac{1}{z^4}(\mathcal{X}_{-1}^2 \mathbb{Q}_z + 8(s_1 + s_2 - s_{m-1} - s_m)) \\ + \frac{2\varepsilon}{z^3} \mathcal{X}_{-1} \mathcal{T}_{-1} \mathbb{Q}_z + \frac{1}{z^2} \left[\frac{1}{2}(s_{m-1} + s_m - s_1 - s_2) \mathcal{X}_{-1}^2 \mathbb{Q}_z \right. \\ \left. + 2((s_2 - s_1)^2 + (s_m - s_{m-1})^2 + 2(s_1 + s_2)(s_{m-1} + s_m)) \right] \\ -\frac{\varepsilon}{z}((s_{m-1} + s_m - s_1 - s_2)\mathcal{X}_{-1} + 2\tilde{\mathcal{X}}_{-1}) \mathcal{T}_{-1} \mathbb{Q}_z + O(1) \end{array} \right\}
\end{aligned} \tag{7.8}
$$

$$
\begin{aligned}
\bar{\mathcal{C}}_1 F^{\varepsilon} &= \frac{1}{128(s_2 - s_1)\sqrt{s_m - s_{m-1}}} \\
&\left\{ \begin{array}{l} \frac{1}{z^6} \mathcal{X}_{-1}^3 \mathbb{Q}_z - \frac{2\varepsilon}{z^5} \mathcal{X}_{-1}^2 \mathcal{T}_{-1} \mathbb{Q}_z \\ + \frac{1}{z^4} \left((s_1 + s_2 - \frac{(s_{m-1} + s_m)}{2}) \mathcal{X}_{-1} - \tilde{\mathcal{X}}_{-1} \right) \mathcal{X}_{-1}^2 \mathbb{Q}_z \\ + \frac{\varepsilon}{z^3}((s_{m-1} + s_m - 2(s_1 + s_2))\mathcal{X}_{-1} + \ 4\tilde{\mathcal{X}}_{-1}) \mathcal{X}_{-1} \mathcal{T}_{-1} \mathbb{Q}_z + O\left(\frac{1}{z^2}\right) \end{array} \right\}
\end{aligned} \tag{7.9}
$$

$$
\begin{aligned}
\bar{\mathcal{C}}_1^2 F^{\varepsilon} &= \frac{1}{512(s_2 - s_1)^{3/2}\sqrt{(s_m - s_{m-1})}} \\
&\qquad \left\{ -\frac{\mathcal{X}_{-1}^4 \mathbb{Q}_z}{z^8} + \frac{2\varepsilon}{z^7} \mathcal{X}_{-1}^3 \mathcal{T}_{-1} \mathbb{Q}_z + O\left(\frac{1}{z^6}\right) \right\}
\end{aligned} \tag{7.10}
$$

$$
\begin{aligned}
\bar{\mathcal{A}}_1 F^{\varepsilon} &= \frac{1}{256(s_m - s_{m-1})\sqrt{(s_2 - s_1)}} \\
&\qquad \left\{ -\frac{\mathcal{X}_{-1}^3 \mathbb{Q}_z}{z^6} + \frac{4\varepsilon}{z^5} \mathcal{X}_{-1}^2 \mathcal{T}_{-1} \mathbb{Q}_z + O\left(\frac{1}{z^4}\right) \right\}
\end{aligned} \tag{7.11}
$$

$$H_1^\varepsilon = \left(A_2C_1 + \varepsilon J_{1m}\frac{\partial}{\partial\alpha}\right)\mathbb{Q}_z - \varepsilon\frac{J_{1m}}{z^4}\left(2\alpha J_{mm} - \frac{1}{z^4\alpha}\right) \tag{7.12}$$

$$= \frac{1}{16(s_m - s_{m-1})\sqrt{s_2 - s_1}} \left\{ \begin{array}{l} -\frac{20\varepsilon}{z^9} - \frac{2\varepsilon}{z^7}(5(s_1+s_2) - 10s_{m-1} + 6s_m) + \frac{1}{z^6}\mathcal{X}_{-1}\mathcal{T}_{-1}\mathbb{Q}_z \\ +\frac{\varepsilon}{2z^5}\left(5(s_2-s_1)^2 + 24s_{m-1}s_m + 4(s_1+s_2)(5s_{m-1} - 3s_m)\right) \\ +\frac{1}{z^4}\left(\begin{array}{l} 2\mathcal{X}_{-1} - \tilde{\mathcal{X}}_{-1}\mathcal{T}_{-1} - 2\mathcal{X}_{-1}\mathcal{T}_0 - \mathcal{X}_{-1}\mathcal{X}_0 \\ +\left(\frac{(s_1+s_2)}{2} - s_{m-1} + 3s_m\right)\mathcal{X}_{-1}\mathcal{T}_{-1} \end{array}\right)\mathbb{Q}_z + O\left(\frac{1}{z^3}\right) \end{array} \right\} \tag{7.13}$$

$$\bar{\mathcal{C}}_1 H_1^\varepsilon = \frac{1}{64(s_2 - s_1)(s_m - s_{m-1})} \left\{ \begin{array}{l} -\frac{1}{z^8}\mathcal{X}_{-1}^2\mathcal{T}_{-1}\mathbb{Q}_z + \frac{1}{z^6}\left(\begin{array}{l} 2\tilde{\mathcal{X}}_{-1}\mathcal{X}_{-1}\mathcal{T}_{-1} + 2\mathcal{X}_{-1}^2\mathcal{T}_0 + \mathcal{X}_0\mathcal{X}_{-1}^2 \\ +\ (s_{m-1} - 3s_m - s_1 - s_2)\mathcal{X}_{-1}^2\mathcal{T}_{-1} \end{array}\right)\mathbb{Q}_z \\ +\ O\left(\frac{1}{z^5}\right) \end{array} \right\} \tag{7.14}$$

$$\begin{aligned} -H_2^\varepsilon &= (C_2 + 2\varepsilon\alpha J_{1m}C_1)A_1^\varepsilon\mathbb{Q}_z \\ &= \frac{1}{16(s_2 - s_1)\sqrt{s_m - s_{m-1}}} \\ &\quad \left\{ \begin{array}{l} \frac{\varepsilon}{z^7}\mathcal{X}_{-1}^2\mathbb{Q}_z - \frac{1}{z^6}\mathcal{X}_{-1}\mathcal{T}_{-1}\mathbb{Q}_z \\ +\frac{\varepsilon}{z^5}(-2\mathcal{T}_{-1}^2 + (s_1 + s_2 - \frac{(s_{m-1}+s_m)}{2})\mathcal{X}_{-1}^2)\mathbb{Q}_z \\ +\frac{1}{z^4}\left(\begin{array}{l} \mathcal{X}_{-1} - \mathcal{X}_0\mathcal{X}_{-1} - 2\mathcal{X}_{-1}\mathcal{T}_0 + 3\tilde{\mathcal{X}}_{-1}\mathcal{T}_{-1} \\ +\left(\frac{1}{2}(s_{m-1}+s_m) - s_1 - s_2\right)\mathcal{X}_{-1}\mathcal{T}_{-1} \end{array}\right)\mathbb{Q}_z \\ +O\left(\frac{1}{z^3}\right) \end{array} \right\} \end{aligned}$$

$$\begin{aligned} -\bar{A}_1^\varepsilon H_2^\varepsilon &= \frac{1}{128(s_2 - s_1)(s_m - s_{m-1})} \\ &\quad \left\{ \begin{array}{l} \frac{\varepsilon}{z^9}\mathcal{X}_{-1}^3\mathbb{Q}_z - \frac{3}{z^8}\mathcal{X}_{-1}^2\mathcal{T}_{-1}\mathbb{Q}_z \\ +\frac{\varepsilon}{z^7}(\tilde{\mathcal{X}}_{-1} + \ (s_1 + s_2 - s_{m-1} - s_m)\mathcal{X}_{-1})\mathcal{X}_{-1}^2\mathbb{Q}_z \\ +\frac{1}{z^6}(-\mathcal{X}_{-1}^2\mathcal{X}_0 - 2\mathcal{X}_{-1}^2\mathcal{T}_0 + 2\tilde{\mathcal{X}}_{-1}\mathcal{X}_{-1}\mathcal{T}_{-1} \\ +\ 4\mathcal{T}_{-1}^3 + 3(s_m + s_{m-1} - s_1 - s_2)\mathcal{X}_{-1}^2\mathcal{T}_{-1})\mathbb{Q}_z \\ +O\left(\frac{1}{z^5}\right) \end{array} \right\} \end{aligned}$$

$$
\begin{aligned}
G^\varepsilon &= \{H_1^\varepsilon, F^\varepsilon\}_{C_1} - \{H_2^\varepsilon, F^\varepsilon\}_{A_1} &(7.15)\\
&= \frac{1}{2\cdot(32)^2((s_2-s_1)(s_m-s_{m-1}))^{3/2}} &(7.16)
\end{aligned}
$$

$$
\left\{\begin{array}{l}
\frac{12\varepsilon}{z^{15}}\mathcal{X}_{-1}^3\mathbb{Q}_z\\
+\frac{\varepsilon}{z^{13}}(-28\tilde{\mathcal{X}}_{-1}+(18(s_1+s_2-s_{m-1})+14s_m)\mathcal{X}_{-1})\mathcal{X}_{-1}^2\mathbb{Q}_z\\
+\frac{1}{z^{12}}\left(\begin{array}{l}
\frac{1}{2}\{\mathcal{X}_{-1}\mathcal{T}_{-1}\mathbb{Q}_z, \mathcal{X}_{-1}^2\mathbb{Q}_z\}_{\mathcal{X}_{-1}}\\
-16(\mathcal{T}_0-1+\frac{1}{2}\mathcal{X}_0)\mathcal{X}_{-1}^2\mathbb{Q}_z\\
+32(\tilde{\mathcal{X}}_{-1}\mathcal{X}_{-1}-\mathcal{T}_{-1}^2)\mathcal{T}_{-1}\mathbb{Q}_z
\end{array}\right)\\
+O\left(\frac{1}{z^{11}}\right)
\end{array}\right\}
$$

Proof: This is done by straightforward computation, using Lemma 7.2 and the series for the constants appearing in F^ε, H_1^ε, H_2^ε, G^ε, namely

$$
\begin{aligned}
J_{m1}(c) = -\frac{1}{2z^2\sqrt{(s_2-s_1)(s_m-s_{m-1})}}&\left(1+\frac{z^2}{2}(s_1+s_2-s_{m-1}-s_m)\right.\\
&\left.-\frac{z^4}{8}((s_2-s_1)^2+(s_m-s_{m-1})^2+2(s_1+s_2)(s_{m-1}+s_m))+\mathbf{O}(z^4)\right)\\
(\alpha J_{m1}) = -\frac{1}{\sqrt{s_2-s_1}}&\left(\frac{1}{z^3}+\frac{1}{2z}(s_1+s_2)-\frac{z}{8}(s_2-s_1)^2+O(z^3)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{k^2J_{m1}}{\alpha}-2\alpha kJ_{1m}J_{mm}\right)\\
=\ &\frac{1}{4z^9(s_m-s_{m-1})\sqrt{s_2-s_1}}\\
&\left(\begin{array}{l}
-5-\frac{z^2}{2}(5(s_1+s_2)-10s_{m-1}+6s_m)\\
+\frac{z^4}{8}(5(s_2-s_1)^2+24s_{m-1}s_m+4(s_1+s_2)(5s_{m-1}-3s_m))+O(z^6)
\end{array}\right)
\end{aligned}
$$

Proof of Theorem 0.2: In terms of T_i's, defined by

$$
\begin{aligned}
T_1 &:= F^+\bar{\mathcal{C}}_1G^- + F^-\bar{\mathcal{C}}_1G^+ = \bar{\mathcal{C}}_1T_3 - T_5\\
T_2 &:= \{F^-, F^+\}_{\bar{\mathcal{C}}_1}\\
T_3 &:= F^+G_1^- + F^-G_1^+\\
T_4 &:= \bar{\mathcal{C}}_1T_2 = F^+\bar{\mathcal{C}}_1^2F^- - F^-\bar{\mathcal{C}}_1^2F^+\\
T_5 &:= G^-\bar{\mathcal{C}}_1F^+ + G^+\bar{\mathcal{C}}_1F^-,
\end{aligned}
$$

the fundamental equation (0.3) can be written

$$0 = -T_1 \cdot T_2 + T_3 \cdot T_4 = \{T_2, T_3\}_{\bar{\mathcal{C}}_1} + T_2 T_5$$

meaning that T_2, T_3 and T_5 only are needed, which one checks to have the following series in z, using Lemma 7.3,

$$\begin{aligned}
T_2 &= -\frac{\mathcal{X}_{-1}^2 \mathcal{T}_{-1} \mathbb{Q}_z}{64(s_2-s_1)^{3/2}(s_m-s_{m-1})z^{11}} + O\left(\frac{1}{z^9}\right) \\
T_3 &= \frac{1}{128(s_2-s_1)^2(s_m-s_{m-1})^2 z^{18}} \\
&\quad \left\{\begin{array}{l} \frac{1}{16}\{\mathcal{X}_{-1}\mathcal{T}_{-1}\mathbb{Q}_z, \mathcal{X}_{-1}^2\mathbb{Q}_z\}_{\mathcal{X}_{-1}} \\ +(\mathcal{T}_0 - 1 + \frac{1}{2}\mathcal{X}_0)\mathcal{X}_{-1}^2\mathbb{Q}_z \\ -2(\tilde{\mathcal{X}}_{-1}\mathcal{X}_{-1} - \mathcal{T}_{-1}^2)\mathcal{T}_{-1}\mathbb{Q}_z \\ -\frac{3}{32}(\mathcal{X}_{-1}^2\mathbb{Q}_z)(\mathcal{X}_{-1}^2\mathcal{T}_{-1}\mathbb{Q}_z) \end{array}\right\} + O\left(\frac{1}{z^{17}}\right) \\
T_5 &= \frac{3(\mathcal{X}_{-1}^2\mathcal{T}_{-1}\mathbb{Q}_z)(\mathcal{X}_{-1}^3\mathbb{Q}_z)}{128^2(s_2-s_1)^{5/2}(s_m-s_{m-1})^2 z^{20}} + O\left(\frac{1}{z^{19}}\right).
\end{aligned} \tag{7.17}$$

Then remembering from Lemma 7.2,

$$C_1 = \frac{-1}{4\sqrt{s_2-s_1}}\left(\frac{1}{z^2}\mathcal{X}_{-1} + O(1)\right),$$

one easily computes (letting $T_i^{(0)}$ be the leading coefficient of T_i in (7.17))

$$\begin{aligned}
&\{T_2, T_3\}_{C_1} \\
&= \frac{1}{z^{31}}\left\{T_2^0, T_3^0\right\}_{C_1} + O\left(\frac{1}{z^{30}}\right) \\
&= \frac{1}{32^3(s_2-s_1)^4(s_m-s_{m-1})^3 z^{31}} \\
&\quad \Bigg\{\mathcal{X}_{-1}^2\mathcal{T}_{-1}\mathbb{Q}_z, \frac{1}{16}\{\mathcal{X}_{-1}\mathcal{T}_{-1}\mathbb{Q}_z, \mathcal{X}_{-1}^2\mathbb{Q}_z\}_{\mathcal{X}_{-1}} + \left(\mathcal{T}_0 - 1 + \frac{\mathcal{X}_0}{2}\right)\mathcal{X}_{-1}^2\mathbb{Q}_z \\
&\quad -2(\tilde{\mathcal{X}}_{-1}\mathcal{X}_{-1} - \mathcal{T}_{-1}^2)\mathcal{T}_{-1}\mathbb{Q}_z - \frac{3}{32}(\mathcal{X}_{-1}^2\mathcal{T}_{-1}\mathbb{Q}_z)(\mathcal{X}_{-1}^2\mathbb{Q}_z)\Bigg\}_{\mathcal{X}_{-1}} + O\left(\frac{1}{z^{30}}\right)
\end{aligned}$$

and then using below the trivial Wronskian relation $-a^2b' = \{a, ab\}$, one computes

$$
\begin{aligned}
&T_2T_5\\
&= \frac{1}{z^{31}}T_2^0T_5^0 + O\left(\frac{1}{z^{30}}\right)\\
&= \frac{1}{32^3(s_2-s_1)^4(s_m-s_{m-1})^3z^{31}}\left(-\frac{3}{32}(\mathcal{X}_{-1}^2\mathcal{T}_{-1}\mathbb{Q}_z)^2(\mathcal{X}_{-1}^3\mathbb{Q}_z)\right) + O(\frac{1}{z^{30}})\\
&= \frac{1}{32^3(s_2-s_1)^4(s_m-s_{m-1})^3z^{31}}\left\{\mathcal{X}_{-1}^2\mathcal{T}_{-1}\mathbb{Q}_z, \frac{3}{32}(\mathcal{X}_{-1}^2\mathcal{T}_{-1}\mathbb{Q}_z)(\mathcal{X}_{-1}^2\mathbb{Q}_z)\right\}_{\mathcal{X}_{-1}}\\
&\quad +O(\frac{1}{z^{30}}).
\end{aligned}
$$

Adding these two contributions, one finds

$$
\begin{aligned}
0 &= -T_1 \cdot T_2 + T_3 \cdot T_4\\
&= \{T_2, T_3\}_{C_1} + T_2T_5\\
&= \frac{1}{32^3(s_2-s_1)^4(s_m-s_{m-1})^3z^{31}}\\
&\quad \Bigg\{\mathcal{X}_{-1}^2\mathcal{T}_{-1}\mathbb{Q}_z,\ \frac{1}{16}\{\mathcal{X}_{-1}\mathcal{T}_{-1}\mathbb{Q}_z, \mathcal{X}_{-1}^2\mathbb{Q}_z\}_{\mathcal{X}_{-1}}\\
&\quad\quad + \left(\mathcal{T}_0 - 1 + \frac{\mathcal{X}_0}{2}\right)\mathcal{X}_{-1}^2\mathbb{Q}_z - 2(\tilde{\mathcal{X}}_{-1}\mathcal{X}_{-1} - \mathcal{T}_{-1}^2)\mathcal{T}_{-1}\mathbb{Q}_z\Bigg\}_{\mathcal{X}_{-1}} + O(\frac{1}{z^{30}})
\end{aligned}
\tag{7.18}
$$

In [21], Tracy and Widom show that the extended kernel for the non-intersecting Brownian motions tend to the extended Pearcey kernel uniformly in each bounded interval. Since $\mathbb{Q}_z$ is the log of the Fredholm determinant for that kernel, it follows that

$$\lim_{z\to 0}\mathbb{Q}_z = \mathbb{Q}_0.$$

Then taking the limit when $z \to 0$ in (7.18) leads to the PDE for $\mathbb{Q}_0$, ending the proof of Theorem 0.2. ■

8 Appendix: evaluation of the integral over the full range

Lemma 8.1 *The following integral can be evaluated explicitly:*

$$\begin{aligned}
\tau_{k_1k_2}(\mathbb{R}) &= \frac{1}{k_1!k_2!}\int_{(\mathbb{R}^m)^{k_1+k_2}} \Delta_{k_1+k_2}(y^{(1)})\prod_{i=1}^{m} dy_i^{(\ell)} \\
&\quad \Delta_{k_1}(y^{(m)'})\prod_{i=1}^{k_1} e^{-\frac{1}{2}\sum_{\ell=1}^{m} y_i^{(\ell)2}+\sum_{\ell=1}^{m-1} c_\ell y_i^{(\ell)} y_i^{(\ell+1)}+\alpha y_i^{(m)}} \\
&\quad \Delta_{k_2}(y^{(m)''})\prod_{i=k_1+1}^{k_1+k_2} e^{-\frac{1}{2}\sum_{\ell=1}^{m} y_i^{(\ell)2}+\sum_{\ell=1}^{m-1} c_\ell y_i^{(\ell)} y_i^{(\ell+1)}-\alpha y_i^{(m)}} \\
&= \det\begin{pmatrix} (\mu_{i,j}(\alpha))_{\substack{0\le i\le k_1-1\\ 0\le j\le n-1}} \\ \\ (\mu_{i,j}(-\alpha))_{\substack{0\le i\le k_2-1\\ 0\le j\le n-1}} \end{pmatrix} \\
&= c_{k_1k_2}\alpha^{k_1k_2}\left(\prod_1^{m-1} c_i\right)^{-\frac{k_1+k_2}{2}} e^{-\frac{\alpha^2}{2}(k_1+k_2)J_{mm}}\left(J_{1m}\right)^{\frac{1}{2}(k_1+k_2)^2}
\end{aligned} \tag{8.1}$$

where

$$\begin{aligned}
\mu_{ij}(\alpha) &= \int_{\mathbb{R}_m} x_1^j x_m^i e^{-\frac{1}{2}\left(\sum_1^m x_j^2 - 2\sum_1^{m-1} c_j x_j x_{j+1}\right)+\alpha x_m} dx_1\dots dx_m \\
c_{k_1k_2} &= (-2)^{k_1k_2}(2\pi)^{\frac{m}{2}(k_1+k_2)}\prod_0^{k_1-1} j!\prod_0^{k_2-1} j!\ .
\end{aligned}$$

Therefore

$$\begin{aligned}
\left(\bar{\mathcal{A}}_2\bar{\mathcal{C}}_1 + J_{1m}\frac{\partial}{\partial\alpha}\right)\log\tau_{k_1k_2}(\mathbb{R}) &= J_{1m}\frac{\partial}{\partial\alpha}\left(k_1k_2\log\alpha - \frac{\alpha^2}{2}(k_1+k_2)J_{mm}\right) \\
&= J_{1m}\left(\frac{k_1k_2}{\alpha} - \alpha(k_1+k_2)J_{mm}\right),
\end{aligned}$$

$$\begin{aligned}
-\left(\bar{\mathcal{C}}_2 + 2\alpha J_{1m}\bar{\mathcal{C}}_1\right)\mathcal{A}_1\log\tau_{k_1k_2}(\mathbb{R}) &= \frac{1}{2}c_1\frac{\partial}{\partial c_1}\frac{\partial}{\partial\alpha}\left(k_1k_2\log\alpha - \frac{\alpha^2}{2}(k_1+k_2)J_{mm}\right) \\
&= \alpha(k_1+k_2)J_{1m}^2.
\end{aligned}$$

Proof: From the explicit evaluation of the zero moment[5]

$$\begin{aligned}\mu_{00}(\alpha,\gamma) &:= \int_{\mathbb{R}^m} e^{-\frac{1}{2}\left(\sum_1^m x_j^2 - 2\sum_1^{m-1} c_j x_j x_{j+1}\right)+\gamma x_1+\alpha x_m} dx_1\dots dx_m \\ &= \frac{(2\pi)^{m/2}}{\sqrt{\det J^{-1}(c_1,\dots,c_{m-1})}} e^{-\frac{1}{2}(J_{mm}\alpha^2+2\alpha\gamma J_{1m}+J_{11}\gamma^2)},\end{aligned}$$

one deduces the other moments by derivation,

$$\begin{aligned}\mu_{ij}(\pm\alpha) &= \left(\pm\frac{\partial}{\partial\alpha}\right)^i \left(\frac{\partial}{\partial\gamma}\right)^j \mu_{00}(\pm\alpha,\gamma)\Big|_{\gamma=0} \\ &= \mu_{00}(\pm\alpha,\gamma)(\pm 1)^i A^i B_{\pm}^j(1)\Big|_{\gamma=0} \\ &= \mu_{00}(\pm\alpha,\gamma)(\pm 1)^i A^i p_j(\pm\alpha),\end{aligned}$$

where

$$\begin{aligned}A &:= \mu_{00}(\pm\alpha,\gamma)^{-1}\frac{\partial}{\partial\alpha}\mu_{00}(\pm\alpha,\gamma)\Big|_{\gamma=0} \\ &= \frac{\partial}{\partial\alpha} - J_{mm}\alpha \\ B_{\pm} &:= \mu_{00}(\pm\alpha,\gamma)^{-1}\frac{\partial}{\partial\gamma}\mu_{00}(\pm\alpha,\gamma) \\ &= \frac{\partial}{\partial\gamma} - J_{11}\gamma \mp \alpha J_{1m} \\ p_j(\alpha) &:= B_+^j(1)\Big|_{\gamma=0}.\end{aligned}$$

The following holds:

$$p_{2i}(\alpha) = \text{even polynomial}, \qquad p_{2i+1}(\alpha) = \text{odd polynomial of } \alpha,$$

which is used in equality $\overset{**}{=}$ below, and

$$A^n p_k(\alpha) = p_k^{(n)} + \beta_1(\alpha)p_k^{(n-1)} + \beta_2(\alpha)p_k^{(n-2)} + \dots + \beta_n p_k,$$

[5] Using

$$\int_{\mathbb{R}^m} e^{-\frac{1}{2}\langle Qx,x\rangle+\langle \ell,x\rangle} dx_1\dots dx_m = \frac{(2\pi)^{m/2}}{\sqrt{\det Q}} e^{\frac{1}{2}\langle Q^{-1}\ell,\ell\rangle},$$

for $Q := -J^{-1}$ and $\ell := (\gamma, 0, \dots, 0, \alpha)$.

where $p_k^{(n)} := \left(\frac{d}{d\alpha}\right)^n p_k$ and where $\beta_i(\alpha)$ are polynomials in α, independent of k; this feature is used in equality $\overset{*}{=}$ below. The equality $\overset{***}{=}$ hinges on the identity

$$\det \begin{pmatrix} \left(-\alpha^{j-i}\right)_{\substack{1 \le i \le k_1 \\ 1 \le j \le n}} \\ \\ \left(\alpha^{j-i}\right)_{\substack{1 \le i \le k_2 \\ 1 \le j \le n}} \end{pmatrix} = c''_{k_1 k_2} \alpha^{k_1 k_2}$$

for a constant $c''_{k_1 k_2}$ depending on k_1 and k_2 only. Then, setting

$$\mu := \mu_{00}(\pm\alpha, 0) = \frac{(2\pi)^{m/2}}{\sqrt{\det(-J^{-1}(c_1, \ldots, c_{m-1}))}} e^{-\frac{1}{2} J_{mm} \alpha^2},$$

we compute:

$$\begin{aligned}
&\tau_{k_1 k_2}(\mathbb{R})\Big|_{t=s=u=\beta=0} \\
&= \mu^n \det \begin{pmatrix} \left(A^i p_j(\alpha)\right)_{\substack{0 \le i \le k_1 - 1 \\ 0 \le j \le n-1}} \\ \\ \left((-A)^i p_j(-\alpha)\right)_{\substack{0 \le i \le k_2 - 1 \\ 0 \le j \le n-1}} \end{pmatrix} \\
&= \mu^n (-1)^{\frac{k_2(k_2-1)}{2}} \det \begin{pmatrix} \left(A^i p_j(\alpha)\right)_{\substack{0 \le i \le k_1 - 1 \\ 0 \le j \le n-1}} \\ \\ \left(A^i p_j(-\alpha)\right)_{\substack{0 \le i \le k_2 - 1 \\ 0 \le j \le n-1}} \end{pmatrix} \\
&\overset{*}{=} \mu^n (-1)^{\frac{k_2(k_2-1)}{2}} \det \begin{pmatrix} \left(p_j^{(i)}(\alpha)\right)_{\substack{0 \le i \le k_1 - 1 \\ 0 \le j \le n-1}} \\ \\ \left(p_j^{(i)}(-\alpha)\right)_{\substack{0 \le i \le k_2 - 1 \\ 0 \le j \le n-1}} \end{pmatrix} \\
&\overset{**}{=} c_{k_1 k_2} \mu^n \det \begin{pmatrix} \left(((-J_{1m}\alpha)^{j-1})^{(i)}\right)_{\substack{0 \le i \le k_1 - 1 \\ 1 \le j \le n}} \\ \\ \left(((J_{1m}\alpha)^{j-1})^{(i)}\right)_{\substack{0 \le i \le k_2 - 1 \\ 1 \le j \le n}} \end{pmatrix} \\
&= c_{k_1 k_2} \mu^n (J_{1m})^{\frac{n(n-1)}{2}} \det \begin{pmatrix} \left(-\alpha^{j-i}\right)_{\substack{1 \le i \le k_1 \\ 1 \le j \le n}} \\ \\ \left(\alpha^{j-i}\right)_{\substack{1 \le i \le k_2 \\ 1 \le j \le n}} \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&\overset{***}{=} \quad c'_{k_1k_2}\mu^n (J_{1m})^{\frac{n(n-1)}{2}}\alpha^{k_1k_2} \\
&= \quad c'_{k_1k_2}\frac{(2\pi)^{nm/2}}{\Delta(c_1,\dots,c_{m-1})^{n/2}} e^{-\frac{n}{2}J_{mm}\alpha^2}(J_{1m})^{\frac{n(n-1)}{2}}\alpha^{k_1k_2} \\
&= \quad c'_{k_1k_2}(2\pi)^{nm/2}\alpha^{k_1k_2}\left(\prod_1^{m-1} c_i\right)^{-\frac{n}{2}} e^{-\frac{\alpha^2}{2}nJ_{mm}}(J_{1m})^{\frac{1}{2}n^2}.
\end{aligned}
$$

In order to evaluate the integer $c'_{k_1k_2}$, it suffices to notice that this constant is independent of m, so that we may choose $m = 1$, which was done in ([5]). This ends the proof of Lemma 8.1. ■

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