Airy processes with wanderers and new universality classes

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August 8, 2009

Abstract

Consider $n+m$ non-intersecting Brownian bridges, with n of them leaving from 0 at time $t=-1$ and returning to 0 at time $t=1$, while the m remaining ones (wanderers) go from m points a_i to m points b_i. First we keep m fixed and we scale a_i, b_i appropriately with n. In the large-n limit we obtain a new Airy process with wanderers, in the neighborhood of $\sqrt{2n}$, the approximate location of the rightmost particle in the absence of wanderers. This new process is governed by an Airy-type kernel, with a rational perturbation.

Letting the number m of wanderers tend to infinity as well, leads to two Pearcey processes about two cusps, a closing and an opening cusp, the location of the tips being related by an elliptic curve. Upon tuning the starting and target points, one can let the two tips of the cusps grow very close; this leads to a new process, which we conjecture to be governed by a kernel, represented as a double integral involving the exponential of a quintic polynomial in the integration variables.

MSC 2000. Primary: 60G60, 60G65, 35Q53; secondary: 60G10, 35Q58.
Key words and Phrases: Dyson's Brownian motion, Airy process, Pearcey process, extended kernels, random Hermitian ensembles, quintic kernel, coupled random matrices.

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1 Introduction

Consider $n+m$ non-intersecting Brownian motions (Brownian bridges) on $\mathbb{R}$ depending on time $t \in [-1,1]$, with n of them leaving from and returning to 0, while the m remaining ones leave from $a_m \leq \ldots \leq a_1$ and are forced to end up at $b_m \leq \ldots \leq b_1$. We denote by $x_i(t)$ the position at time t of the ith largest Brownian particle among the $n+m$ non-intersecting Brownian bridges. Denote by $\mathcal{D}$ the conditioning event defined by the following conditions:
(i) non-intersecting paths: $x_1(t) > x_2(t) > \ldots > x_{m+n}(t)$, $t \in (-1,1)$,
(ii) n bridges from 0 to 0: $x_i(-1) = x_i(1) = 0$ for $i = m+1, \ldots, m+n$,
(iii) m *wanderers* from a_i to b_i: $x_i(-1) = a_i$, $x_i(1) = b_i$ for $i = 1, \ldots, m$.
Then, denote the conditional probability under $\mathcal{D}$ by $\mathbb{P}_{\rm ab}$, i.e.,

$$\mathbb{P}_{\rm ab}(\cdot) = \mathbb{P}(\cdot|\mathcal{D}). \tag{1.1}$$

The interest in non-intersecting Brownian motions stems from a paper by Dyson [19], who made the important observation that putting dynamics into the GUE-random matrix model (Ornstein-Uhlenbeck Processes on the real and imaginary parts of the entries) leads to finitely many non-intersecting Brownian motions on $\mathbb{R}$ for the eigenvalues (stationary process). A space-time transformation enables one to map the above Dyson process into non-intersecting Brownian motions starting from 0 and returning to 0;

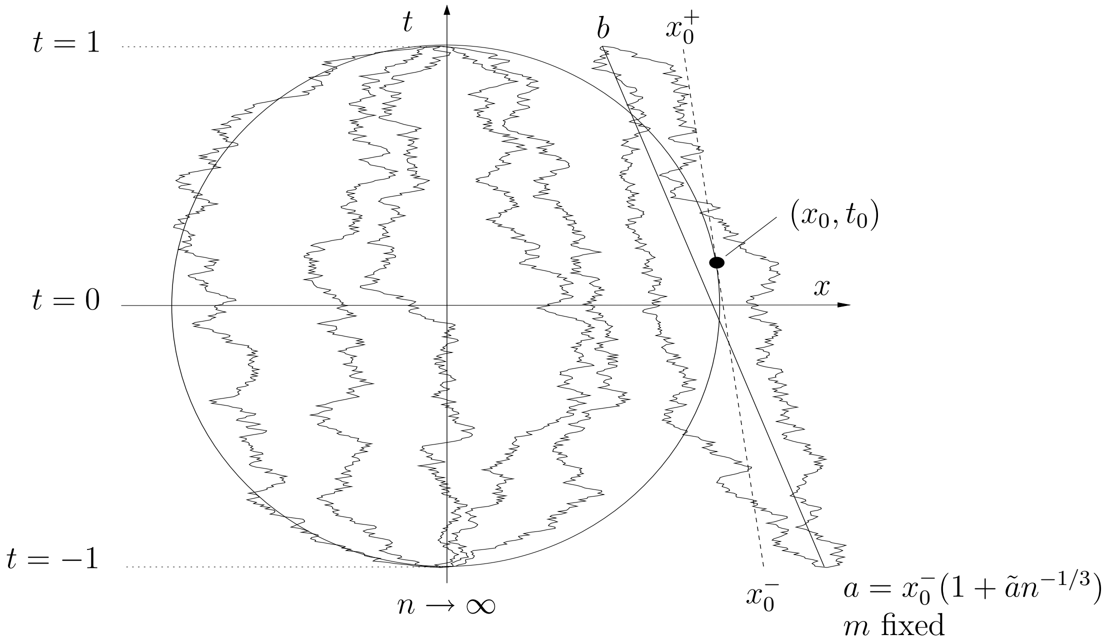

Figure 1: Non-intersecting Brownian bridges with m wanderers, leaving from $a = x_0^+(1 + \tilde{a}n^{-1/3})$ and forced to $b = x_0^-(1 - \tilde{b}n^{-1/3})$, with $\tilde{a} < \tilde{b}$, where $x_0^- = \sqrt{2n}\sqrt{\frac{1+t_0}{1-t_0}}$, $x_0^+ = \sqrt{2n}\sqrt{\frac{1-t_0}{1+t_0}}$. The dotted line linking $(x_0^-, -1)$ to $(x_0^+, 1)$ is tangent to the curve $x = \sqrt{2n(1-t^2)}$ at the point (x_0, t_0).

see formula (1.7) in [1]. In their work on coincidence probabilities, Karlin-McGregor [30] found a determinantal formula for the transition probability of non-intersecting Brownian motions. The relationship between non-intersecting Brownian motions, matrix models and random matrix theory has been developped starting with Johansson [26] and has led to the Airy and other processes [2–4, 17, 38, 39], when the number of particles tend to infinity, see also [18].

At first, consider the motion of the non-intersecting Brownian particles above, but with $m = 0$, and let n become very large. The *Airy process* $\mathcal{A}(\tau)$ describes this cloud of particles ("infinite-dimensional diffusion"), but viewed from any point on the "edge" $\mathcal{C}:\ x = \sqrt{2n(1-t^2)}$ of the set of particles, with time and space properly rescaled; the Airy process will be independent of the point chosen and will be governed by the Airy kernel. This process was found by Prähofer and Spohn [36] in the context of stochastic growth models and further investigated in [2, 22, 27, 28, 38].

Assume now a fixed and finite $m \geq 1$ and all $a_i = 0$, with the target points all equal to b scaled as $b = \rho_0\sqrt{2n} > 0$. Does it affect the Brownian fluctuations along the curve $\mathcal{C}$ for large n? No new process appears as long as one considers points $(y,t) \in \mathcal{C}$, below the point of tangency of the tangent to the curve passing through $(\rho_0\sqrt{2n}, 1)$. At this tangency point the fluctuations obey a new statistics, which we call the *Airy process with m outliers* $\mathcal{A}_m^b(\tau)$, governed by a rational perturbation of the Airy kernel, see [1]. This kernel was already considered by Baik-Ben Arous-Péché [5] and Péché [35] in the context of multivariate statistics.

The first result in this paper concerns the limiting process, described in (1.1), in the large-n limit, while keeping m fixed; this process is denoted by $\mathcal{A}_m^{(\tilde{a},\tilde{b})}(\tau)$. This paper deals with the statistical fluctuations of the edge of the cloud of particles near any point on the curve $\mathcal{C}:\ x = \sqrt{2n(1-t^2)}$, in the presence of wanderers. To do so, consider the tangent line to the curve $\mathcal{C}$, with point of tangency (x_0, t_0), as in Figure 1; this tangent intersects the lines $t = -1$ and $t = 1$ at the points $x_0^- = \frac{x_0}{1-t_0} = \sqrt{2n}\sqrt{\frac{1+t_0}{1-t_0}}$ and $x_0^+ = \frac{x_0}{1+t_0} = \sqrt{2n}\sqrt{\frac{1-t_0}{1+t_0}}$ respectively. Consider now m wanderers leaving from neighboring points (when n gets large) of the point x_0^- at time $t = -1$ and forced to neighboring points of x_0^+ at time $t = 1$. A first part of this paper is to show that the fluctuations near the edge of the cloud and near the point (x_0, t_0) obeys a new statistic, independent of the point (x_0, t_0) chosen on the curve above, showing *universality* within that class.

At a first stage (Theorems 1.1 and 1.2), the result will be shown for a vertical line tangent to $\mathcal{C}$ at the point $(\sqrt{2n}, 0)$, whereas Theorem 1.3 deals with the universality result. The non-intersecting nature of the first n bridges implies that the largest one will again reach a height of about $\sqrt{2n}$. So, it is natural to consider the following scaling of the starting and the target points

$$a_i = \sqrt{2n}\left(1 + \frac{\tilde{a}_i}{n^{1/3}}\right) \quad \text{and} \quad b_i = \sqrt{2n}\left(1 - \frac{\tilde{b}_i}{n^{1/3}}\right). \tag{1.2}$$

With this scaling, the m wanderers will interact with the bulk (of n particles, with n very large) in a non-trivial way, upon considering regions close to $x = \sqrt{2n}$ and $t = 0$, namely at space-time positions (x,t) which scale like

$$t = \tau n^{-1/3}, \quad x = \sqrt{2n} + \frac{\xi - \tau^2}{\sqrt{2}n^{1/6}}. \tag{1.3}$$

This will only be so under some geometric condition: the lines connecting the starting and target points in (x,t)-space must pass to the left of $\sqrt{2n}$ at $t = 0$; see Figure 1. Then, the first result concerns the gap probability at a given time τ for very large n and keeping m finite and fixed, i.e., the probability that a set is not visited by any of the $n+m$ Brownian bridges at time τ. Thus, in Theorems 1.1 and 1.2, a different (non-trivial) process $\mathcal{A}_m^{(\tilde{a},\tilde{b})}(\tau)$ will appear due to the interaction of the m wanderers with the Airy field in the neighborhood of $(x,t) = (\sqrt{2n}, 0)$. Note that in the absence of wanderers the particles must look, near the edge, like the Airy process. This also explains why the kernel (1.5) obtained below is another rational extension of the Airy kernel.

Theorem 1.1. *Consider points a_i and b_i, as in (1.2), with $\tilde{a}_m \le \ldots \le \tilde{a}_1 < \tilde{b}_1 \le \ldots \le \tilde{b}_m$ on the real line*[1]*. Given any compact set $E \subset \mathbb{R}$, the gap probability at rescaled time-space (1.3) is given, in the large-n limit, by*

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}_{\mathrm{ab}}\left(\left\{\textit{all } x_i\left(\frac{\tau}{n^{1/3}}\right) \in \sqrt{2n} + \frac{E^c - \tau^2}{\sqrt{2}n^{1/6}}\right\}\right) &= \det(\mathbb{1} - \chi_E K_m^{\tilde{a},\tilde{b}} \chi_E)_{L^2(\mathbb{R})} \\ &= \mathbb{P}(\mathcal{A}_m^{(\tilde{a},\tilde{b})}(\tau) \cap E = \emptyset), \quad (1.4)\end{aligned}$$

where $\chi_E(\xi) = \mathbb{1}(\xi \in E)$, where det *denotes the Fredholm determinant on $L^2(\mathbb{R})$ and where the kernel $K_m^{\tilde{a},\tilde{b}}$ is given by*

$$K_m^{\tilde{a},\tilde{b}}(\tau;\xi_1,\xi_2) = \frac{1}{(2\pi i)^2}\int_{\Gamma_{\tilde{a}-\tau>}} d\omega \int_{\Gamma_{<\tilde{b}-\tau}} d\widetilde{\omega}\, \frac{e^{-\frac{\omega^3}{3}+\xi_2\omega}}{e^{-\frac{\widetilde{\omega}^3}{3}+\xi_1\widetilde{\omega}}} \frac{\prod_{k=1}^m \left(\frac{\widetilde{\omega}-\tilde{a}_k+\tau}{\omega-\tilde{a}_k+\tau}\right)\left(\frac{\omega-\tilde{b}_k+\tau}{\widetilde{\omega}-\tilde{b}_k+\tau}\right)}{\omega-\widetilde{\omega}}. \tag{1.5}$$

The integration contours are as follows: $\Gamma_{\tilde{a}>}$ goes from $e^{-2\pi i/3}\infty$ to $e^{2\pi i/3}\infty$, and passes on the right of all the $\tilde{a}_i - \tau$, while $\Gamma_{<\tilde{b}}$ goes from $e^{\pi i/3}\infty$ to $e^{-\pi i/3}\infty$, and passes to the left of all $\tilde{b}_i - \tau$. Moreover, the two contours do not intersect; see Figure 2 for an illustration.

This kernel has also appeared in recent work of Borodin and Péché [15], as a limit of a directed percolation in a quadrant with defective rows and columns, itself a generalization of a kernel of Baik-Ben Arous-Péché [5] and Péché [35] and considered in [1] in the context of non-intersecting Brownian motions. The same limit process occurs in the asymmetric exclusion process, see [14, 23]. The proof of Theorem 1.1 will be given in Section 2, when the points a_i and the points b_i are all different. When the a_i's all coincide, and similarly the b_i's, the proof of Theorem 1.1 breaks down and must be replaced by another one; two approaches are being discussed here (see Section 4): (1) using a certain moment matrix, (2) using biorthogonal polynomials.

[1]The inequalities that all the $\tilde{a}_i$ be smaller than all the $\tilde{b}_i$'s means geometrically that the lines connecting corresponding points intersect the x-axis to the left of $x = \sqrt{2n}$; see Figure 1.

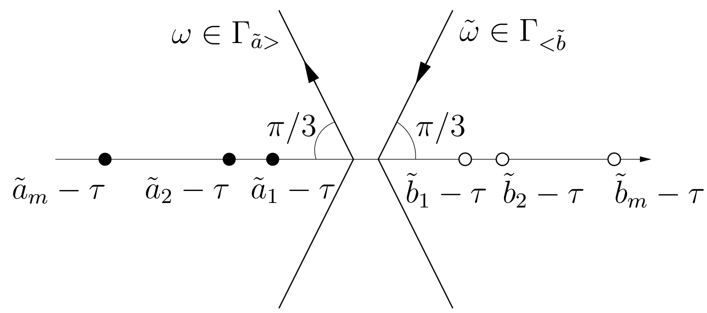

Figure 2: Integration paths of the kernel $K_m^{\tilde a,\tilde b}$ of (1.5).

In Theorem 1.2 (see Section 3) the first result will be extended to the joint gap probabilities at different (rescaled) times $\tau_1, \ldots, \tau_\ell$. Obviously Theorem 1.1 is the specialization of Theorem 1.2 to the one-time case.

Theorem 1.2. *Consider ℓ distinct times $\tau_1, \tau_2, \ldots, \tau_\ell$ and compact sets $E_1, \ldots, E_\ell \subset \mathbb{R}$. Then,*

$$\lim_{n\to\infty} \mathbb{P}_{\rm ab}\left(\bigcap_{k=1}^{\ell}\left\{\text{all } x_i\left(\frac{\tau_k}{n^{1/3}}\right) \in \sqrt{2n} + \frac{E_k^c - \tau_k^2}{\sqrt{2}n^{1/6}}\right\}\right)$$
$$= \det(\mathbb{1} - \chi_E K_m^{\tilde a,\tilde b}\chi_E) = \mathbb{P}\left(\bigcap_{k=1}^{\ell}\left\{\mathcal{A}_m^{(\tilde a,\tilde b)}(\tau_k) \cap E_k = \emptyset\right\}\right), \qquad (1.6)$$

where $\chi_E(\tau_k, \xi) := \mathbb{1}(\xi \in E_k)$. Here det *denotes the (matrix) Fredholm determinant on the space $L^2(\{\tau_1, \ldots \tau_\ell\} \times \mathbb{R})$ and the extended kernel $K_m^{\tilde a,\tilde b}$ is given by*

$$K_m^{\tilde a,\tilde b}(\tau_1, \xi_1; \tau_2, \xi_2) = -\frac{\mathbb{1}(\tau_2 > \tau_1)}{\sqrt{4\pi(\tau_2 - \tau_1)}} e^{-\frac{(\xi_2-\xi_1)^2}{4(\tau_2-\tau_1)} - \frac{1}{2}(\tau_2-\tau_1)(\xi_2+\xi_1) + \frac{1}{12}(\tau_2-\tau_1)^3}$$
$$+ \frac{1}{(2\pi i)^2}\int_{\Gamma_{\tilde a - \tau_2>}} d\omega \int_{\Gamma_{<\tilde b-\tau_1}} d\widetilde\omega \, \frac{e^{-\omega^3/3+\xi_2\omega}}{e^{-\widetilde\omega^3/3+\xi_1\widetilde\omega}} \frac{\prod_{k=1}^m \left(\frac{\widetilde\omega - \tilde a_k + \tau_1}{\omega - \tilde a_k + \tau_2}\right)\left(\frac{\omega - \tilde b_k + \tau_2}{\widetilde\omega - \tilde b_k + \tau_1}\right)}{(\omega + \tau_2) - (\widetilde\omega + \tau_1)}. \qquad (1.7)$$

The integration contours are as in Figure 2, but with $\tilde a_k - \tau$ replaced by $\tilde a_k - \tau_2$ and $\tilde b_k - \tau$ replaced by $\tilde b_k - \tau_1$.

A similar statement can then be made along any point (x_0, t_0) of the curve $x = \sqrt{2n(1-t^2)}$, with tangent intersecting the lines $t = -1$ and $t = 1$ at the points

$$x_0^- = \frac{x_0}{1-t_0} = \sqrt{2n}\sqrt{\frac{1+t_0}{1-t_0}} \quad \text{and} \quad x_0^+ = \frac{x_0}{1+t_0} = \sqrt{2n}\sqrt{\frac{1-t_0}{1+t_0}}, \qquad (1.8)$$

respectively.

Theorem 1.3. (Universality statement) *As before, consider ℓ distinct times $\tau_1, \tau_2, \ldots, \tau_\ell$ and compact sets $E_1, \ldots, E_\ell \subset \mathbb{R}$. Also, consider m Brownian wanderers, now leaving from the points $a_\ell = x_0^-(1+\frac{\tilde{a}_\ell}{n^{1/3}})$ and forced to $b_\ell = x_0^+(1-\frac{\tilde{b}_\ell}{n^{1/3}})$, with the condition*[2] $\tilde{a}_m \leq \ldots \leq \tilde{a}_1 < \tilde{b}_1 \leq \ldots \leq \tilde{b}_m$. *For n large, pick ℓ points in a $n^{-1/3}$-neighborhood of (x_0, t_0), lying on the curve $x = \sqrt{2n(1-t^2)}$,*

$$x_k := \sqrt{2n(1-t_k^2)}, \quad \textit{with } t_k := t_0 + \frac{(1-t_0^2)\tau_k}{n^{1/3}}, \quad 1 \leq k \leq \ell. \tag{1.9}$$

Then, the following limit holds[3]*:*

$$\lim_{n\to\infty} \mathbb{P}_{\mathrm{ab}} \left(\bigcap_{k=1}^{\ell} \left\{ \textit{all } x_i\,(t_k) \in \left(1+\frac{E_k^c}{2n^{2/3}}\right) x_k \right\} \right)$$
$$= \mathbb{P}\left(\bigcap_{k=1}^{\ell} \left\{ \mathcal{A}_m^{(\tilde{a},\tilde{b})}(\tau_k) \cap E_k = \emptyset \right\} \right). \tag{1.10}$$

Remark 1.4. For $(x_0, t_0) = (\sqrt{2n}, 0)$, this statement reduces to Theorem 1.2, as can seen from Footnote 3.

In view of the new process $\mathcal{A}_m^{(\tilde{a},\tilde{b})}(\tau)$, it seems natural to let the number of wanderers m to go infinity. For simplicity, consider the case where the m wanderers all start from the same point $\tilde{a}$, and end up at the same point $\tilde{b}$, with $\tilde{a} < \tilde{b}$, with the scaling

$$\tilde{a} = \alpha m^{1/3}, \quad \tilde{b} = \beta m^{1/3}, \quad \text{with } \alpha < \beta. \tag{1.11}$$

Under this scaling, the set of m wanderers itself produces an Airy field which then interact with the one already present after the $n \to \infty$ limit. Thus, we might expect that there will be two regions where the Pearcey process arises. Indeed, the first Pearcey process occurs when the "Pearcey cusp" closes, while the second does when the cusp opens, as illustrated in Figure 3.

[2]Here also, the inequalities that all the $\tilde{a}_i$ be smaller than all the $\tilde{b}_i$'s means geometrically that the lines connecting corresponding points intersect the horizontal line through (x_0, t_0) to the left of (x_0, t_0); see Figure 1.

[3]Expanded out, $\left(1+\frac{E_k^c}{2n^{2/3}}\right) x_k$ reads

$$\sqrt{1-t_0^2}\sqrt{2n}\left(1 - \frac{\tau_k t_0}{n^{1/3}} + \frac{E_k^c - \tau_k^2}{2n^{2/3}} - t_0\tau\frac{E_k^c + \tau_k^2}{2n}\right) + O\left(\frac{1}{n^{5/6}}\right).$$

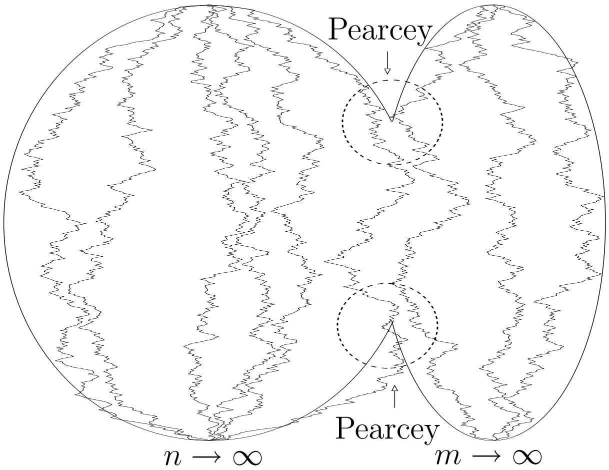

Figure 3: Illustration of the two Pearcey processes, arising around the two cusps.

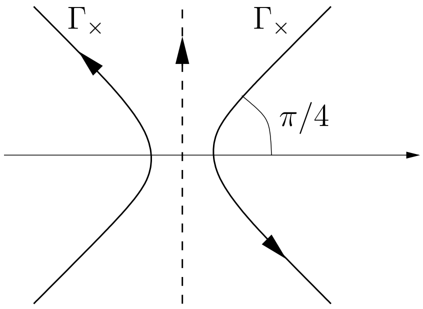

Figure 4: Integration paths of the Pearcey kernel $K_u^{\mathcal{P}}$ defined in (1.12). The two solid lines form together $\Gamma_\times$, the dashed line is the z-integration path.

The reader is reminded of the extended Pearcey kernel $K^{\mathcal{P}}(\theta_1, v_1; \theta_2, v_2)$ with space-time parameters (θ_i, v_i), which is given by

$$\begin{aligned} K^{\mathcal{P}}(\theta_1, v_1; \theta_2, v_2) = &-\frac{\mathbb{1}(\theta_2 > \theta_1)}{\sqrt{2\pi(\theta_2 - \theta_1)}} e^{-(v_2-v_1)^2/2(\theta_2-\theta_1)} \\ &+ \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} dz \int_{\Gamma_\times} d\tilde{z} \frac{1}{z - \tilde{z}} \frac{e^{-z^4/4+\theta_2 z^2/2-v_2 z}}{e^{-\tilde{z}^4/4+\theta_1 \tilde{z}^2/2-v_1 \tilde{z}}}, \end{aligned} \tag{1.12}$$

where the path $\Gamma_\times$ is illustrated in Figure 4, see Tracy-Widom [39]. This leads to Theorem 1.5, established in Section 5.

Theorem 1.5. *Let the starting point $\tilde{a}$ and the target point $\tilde{b}$ of the m wanderers for the Airy process with wanderers (1.10) grow with m, as $\tilde{a} = \alpha m^{1/3}$ and $\tilde{b} = \beta m^{1/3}$ with arbitrary $\alpha < \beta$. Given $\alpha < \beta$, the following equations,*

$$\beta - \alpha = \frac{4\sigma^4 x^3}{2-x}, \quad \textit{with } (x,\sigma) \in \mathcal{E} : \; 4\sigma^6 x^4 - 2x + 3 = 0, \textit{(elliptic curve)} \tag{1.13}$$

have a

$$\begin{array}{lcl} \textit{unique solution } (x,\sigma) & := & (x,\sigma_+) \in \left((\tfrac{3}{2},2) \times (-\tfrac{1}{2},0)\right) \;\; \textbf{\textit{(opening cusp)}} \\ \textit{unique solution } (x,\sigma) & := & (x,\sigma_-) \in \left((\tfrac{3}{2},2) \times (0,\tfrac{1}{2})\right) \;\; \textbf{\textit{(closing cusp)}}. \end{array}$$

Then, the Airy process with m wanderers $\mathcal{A}_m^{(\tilde{a},\tilde{b})}(\tau)$ properly rescaled as $m \to \infty$, converges to two (identical) Pearcey processes $\mathcal{P}(\theta)$ about two cusps, one ***opening cusp*** *(T_+) and one* ***closing cusp*** *(T_-) about* [4]

$$\tau \sim T_\pm m^{1/3}, \quad \xi \sim X m^{2/3}, \quad \textit{with } T_\pm := \frac{\alpha+\beta}{2} - \frac{2\sigma_\pm}{2-x}, \quad X := \sigma_\pm^2(1-2x), \tag{1.14}$$

with $T_- < \frac{\alpha+\beta}{2} < T_+$. To be precise, upon using the two different scalings (1.16) below, depending on the opening or closing cusp, one has, for any $\ell = 1,2,\ldots$, that the limit of the gap probability of the sets $\widetilde{E}_1,\ldots,\widetilde{E}_\ell$ at times $\tau_1,\ldots,\tau_\ell$ is given by the same (matrix) Fredholm determinant,

$$\begin{aligned} &\lim_{m\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{\ell} \left\{ \mathcal{A}_m^{(\alpha m^{1/3},\beta m^{1/3})}(\tau_k) \cap \widetilde{E}_k = \emptyset \right\} \right) \\ &= \det\left(\mathbb{1} - \chi_E K^{\mathcal{P}} \chi_E\right)_{L^2((\theta_1,\ldots,\theta_\ell)\times\mathbb{R})} =: \mathbb{P}\left(\bigcap_{k=1}^{\ell} \{\mathcal{P}(\theta_k) \cap E_k = \emptyset\} \right), \end{aligned} \tag{1.15}$$

where the rescaling from the space-time variables $(\tilde{E}_i,\tau_i)$ to the new space-time variables (E_i,θ_i) is imposed by the initial scaling (1.14), to yield

$$\begin{aligned} \tau_i &= T_\pm m^{1/3} + \frac{1}{2}\kappa^2 \theta_i m^{-1/6}, \quad \kappa := \left(\frac{2(x-1)}{|\sigma_\pm| x^2}\right)^{1/4}, \\ \widetilde{E}_i &= X m^{2/3} - \kappa^2 \sigma_\pm \theta_i m^{1/6} - \kappa E_i m^{-1/12}. \end{aligned} \tag{1.16}$$

Remark 1.6. Note that the involution: $v_1 \leftrightarrow v_2$, $\theta_1 \leftrightarrow -\theta_2$, $T_+ \leftrightarrow T_-$, $\sigma_+ \leftrightarrow \sigma_- = -\sigma_+$, where $v_k \in E_k$, maps the opening cusp into the closing cusp and, in particular, acts on the kernel (1.12) to produce the kernel going with the closing cusp.

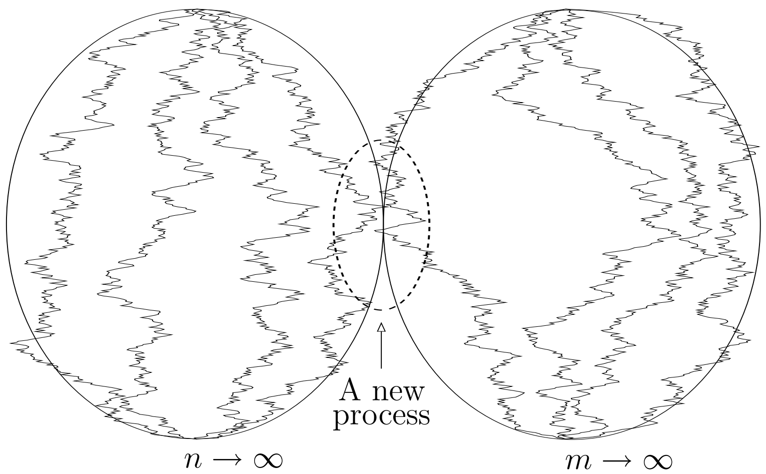

Figure 5: When two Pearcey cusps touch, there will be a new process.

The tips of the two cusps in Theorem 1.5 come together, when $\alpha,\ \beta \to 0$, and hence $x \to 3/2$, $\sigma_\pm \to 0$ and $T_\pm \to 0$; this is not the only way for this to happen, as will be mentioned below. At the very point where the two cusps meet, a new process will emerge (as in Figure 5), which we *conjecture* to be governed by a *"quintic kernel"*.

Conjecture 1.7. *The gap probability for the new process appearing in Figure 5 is given by the Fredholm determinant of the following* quintic kernel*:*

$$K^{\mathcal{Q}}(\theta,\eta;x,y) = \frac{1}{(2\pi i)^2}\int_{\mathcal{C}} dz \int_{\tilde{\mathcal{C}}} d\tilde{z} \frac{1}{z-\tilde{z}} \frac{e^{2z^5/5-\theta z^3/3-\eta z^2+zx}}{e^{2\tilde{z}^5/5-\theta\tilde{z}^3/3-\eta\tilde{z}^2+\tilde{z}y}}, \tag{1.17}$$

where the z and $\tilde{z}$-integration paths are given by the z and $\tilde{z}$-paths in Figure 6, with the orientation indicated.

To give some evidence to this conjecture, we first notice that the curve $\mathcal{E}$ (introduced in (1.13)) contains another real point (besides the real segments introduced just after (1.13)) namely at $(x,\sigma) = (\infty, 0)$, for which $(\alpha,\beta) = (2^{1/3}, -2^{1/3})$; there the critical point w_c of the associated steepest-descent F-function becomes order 5, with $(X,T) = (-2^{2/3}, 0)$, rather than order 4 as in the Pearcey case; this expresses the fact that the two tips come together. For this choice of $(\alpha,\beta) = (2^{1/3}, -2^{1/3})$, the source and the target points

$$a = \sqrt{2n}\left(1+\frac{\tilde{a}}{n^{1/3}}\right), \quad b = \sqrt{2n}\left(1-\frac{\tilde{b}}{n^{1/3}}\right), \text{ with } \tilde{a} = \alpha m^{1/3} \text{ and } \tilde{b} = \beta m^{1/3}$$

[4] T_+ corresponds to $\sigma_+ < 0$ and T_- corresponds to $\sigma_- > 0$, with obviously $\sigma_+ = -\sigma_-$.

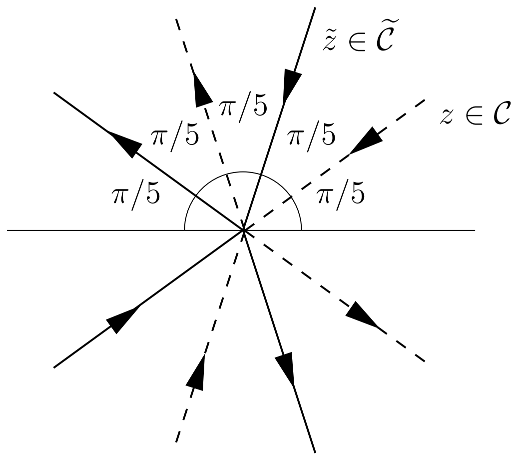

Figure 6: Integration path $\mathcal{C}$ (dashed line) and $\widetilde{\mathcal{C}}$ (solid line) of the quintic kernel.

do not, of course, satisfy the inequality $\tilde{a} < \tilde{b}$, but rather the opposite inequality. We then perform an analytic continuation of the (one-time) kernel[5]

$$K_m^{\tilde{a},\tilde{b}}(\xi_1;\xi_2) = \frac{1}{(2\pi i)^2}\int_{\Gamma_{\tilde{a}>}} d\omega \int_{\Gamma_{<\tilde{b}}} d\widetilde{\omega}\, \frac{e^{-\omega^3/3+\xi_2\omega}}{e^{-\widetilde{\omega}^3/3+\xi_1\widetilde{\omega}}} \frac{\left(\frac{\widetilde{\omega}-\tilde{a}}{\omega-\tilde{a}}\right)^m \left(\frac{\omega-\tilde{b}}{\widetilde{\omega}-\tilde{b}}\right)^m}{\omega-\widetilde{\omega}}. \quad (1.18)$$

by moving $\tilde{a}$ and $\tilde{b}$ in the complex plane from their original position $\tilde{a} < \tilde{b}$ to a new position $\tilde{b} < \tilde{a}$ on the real line. Then by picking $\tilde{a} = \alpha m^{1/3}$ and $\tilde{b} = \beta m^{1/3}$, with $(\alpha, \beta) = (2^{1/3}, -2^{1/3})$ and letting $m \to \infty$, we show the kernel (1.18) tends to the quintic kernel (1.17) with the precise contour of integration in the figure above. Some evidence in favor of this conjecture is given in Section 6, which contains two rigorous statements, with proofs. However this does not suffice to prove the conjecture; e.g., it is still unknown whether the Fredholm determinant of the quintic kernel (1.17) determines a probability. For numerical methods, see, for instance, Bornemann [9].

Remark 1.8. It is interesting to put the three kernels in parallel, Airy, Pearcey and quintic together with their appropriate contours, as in Figure 7:

[5]In the one-time case, one can just absorb the time τ in the $\tilde{a}$ and $\tilde{b}$.

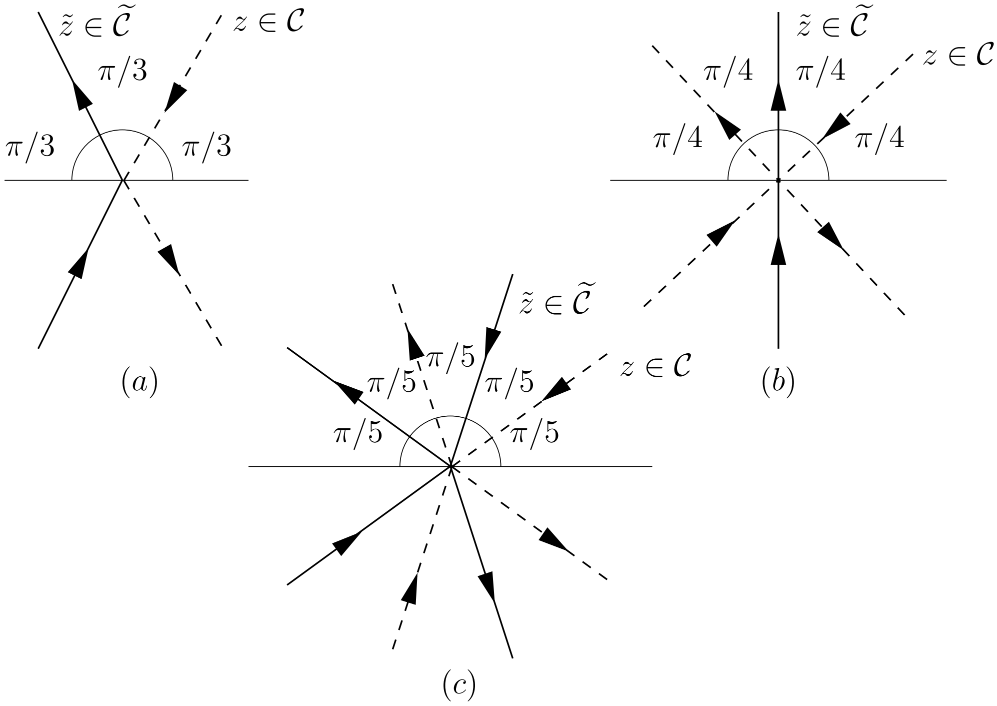

Figure 7: Integration paths $\mathcal{C}$ (dashed line) and $\widetilde{\mathcal{C}}$ (solid line) of the (a) Airy kernel $K^{\mathcal{A}}$, (b) Pearcey kernel $K^{\mathcal{P}}$, and (c) Quintic kernel $K^{\mathcal{Q}}$.

$$
\begin{aligned}
K^{\mathcal{A}}(\tau_1,\xi_1;\tau_2,\xi_2) &= \frac{1}{(2\pi i)^2}\int_{\tilde{\mathcal{C}}} d\tilde z \int_{\mathcal{C}} dz \;\frac{1}{(\tilde z+\tau_2)-(z+\tau_1)}\frac{e^{z^3/3-\xi_1 z}}{e^{\tilde z^3/3-\xi_2\tilde z}} \\
&\quad - \frac{\mathbb{1}(\tau_2>\tau_1)}{\sqrt{4\pi(\tau_2-\tau_1)}} e^{-\frac{(\xi_2-\xi_1)^2}{4(\tau_2-\tau_1)}-\frac{1}{2}(\tau_2-\tau_1)(\xi_2+\xi_1)+\frac{1}{12}(\tau_2-\tau_1)^3} \\
K^{\mathcal{P}}(\tau_1,\xi_1;\tau_2,\xi_2) &= \frac{1}{(2\pi i)^2}\int_{\tilde{\mathcal{C}}} d\tilde z \int_{\mathcal{C}} dz \frac{1}{\tilde z - z}\;\frac{e^{z^4/4-\tau_1 z^2/2+\xi_1 z}}{e^{\tilde z^4/4-\tau_2\tilde z^2/2+\xi_2\tilde z}} \\
&\quad - \frac{\mathbb{1}(\tau_2>\tau_1)}{\sqrt{2\pi(\tau_2-\tau_1)}} e^{-(\xi_2-\xi_1)^2/2(\tau_2-\tau_1)} \\
K^{\mathcal{Q}}(\tau,\eta,\xi_1;\tau,\eta;\xi_2) &= \frac{1}{(2\pi i)^2}\int_{\tilde{\mathcal{C}}} d\tilde z \int_{\mathcal{C}} dz \frac{1}{z-\tilde z}\;\frac{e^{2z^5/5-\tau z^3/3-\eta z^2+\xi_1 z}}{e^{2\tilde z^5/5-\tau\tilde z^3/3-\eta\tilde z^2+\xi_2\tilde z}}.
\end{aligned}
$$

Acknowledgments: The support of a National Science Foundation grant # DMS-07-04271 is gratefully acknowledged (AM, PvM). Also, a European Science Foundation grant (MISGAM), a Marie Curie Grant (ENIGMA), a FNRS grant and a "Interuniversity Attraction Pole" grant are gratefully acknowledged. PLF started the work while being at the Weierstrass Institute for Applied Analysis and Stochastics (WIAS), Berlin.

2 Airy process with wanderers leaving from and going to distinct points

The aim of this section is to prove Theorem 1.1 in the case that all points $\tilde{a}_i$ are distinct and all $\tilde{b}_i$ as well. We first present the case $\tau = 0$. The multi-time joint gap probabilities will be discussed in the next section, implying the case of the one-time process; i.e., general τ beyond $\tau = 0$. However, first presenting the one-time case will prove useful for understanding the basic structure.

Denote by $p(x, y; t)$ the one-particle Brownian motion transition from x to y during a time interval t, namely

$$p(x, y; t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}. \tag{2.1}$$

Let us consider $n + m$ Brownian bridges leaving at $t = -1$ from $a_{m+n} < \ldots < a_{m+1} < a_m < \ldots < a_1$ and ending at $t = 1$ at positions $b_{m+n} < \ldots < b_{m+1} < b_m < \ldots < b_1$. The positions of these particles at time t are denoted by $\mathbf{x}(t) = \{x_1(t), \ldots, x_{m+n}(t)\}$. Then, the probability density that $\mathbf{x}(t) = \mathbf{x}$, conditioned that the Brownian bridges do not intersect in $t \in (-1, 1)$, is given by Karlin-McGregor formula [30], namely

$$P(\mathbf{x}(t) = \mathbf{x}) = \frac{1}{Z} \det(p(a_i, x_j; 1 + t))_{1\le i,j\le m+n} \det(p(x_i, b_j; 1 - t))_{1\le i,j\le m+n}, \tag{2.2}$$

with Z the normalization constant, which is equal to the probability that the $m+n$ paths do not intersect, given the initial and final conditions at $t = \pm 1$.

It is known that a measure on $x = (x_1, \ldots, x_{m+n})$ of the form (2.2) has determinantal correlation functions (see e.g. Proposition 2.2 of [11], or for information on determinantal processes [7, 25, 31, 40, 41]).

As mentioned before we restrict the discussion in this section to the case $t = 0$. Then, the k-point correlation functions $\rho^{(k)}$ are given by

$$\rho^{(k)}(x_1, \ldots, x_k) = \det\left(K(x_i, x_j)\right)_{1\le i,j\le k}, \tag{2.3}$$

with the kernel K explicitly given by

$$K(x, y) = \sum_{i,j=1}^{m+n} p(x, b_i; 1)[B^{-1}]_{i,j} p(a_j, y; 1), \tag{2.4}$$

where

$$B = [B_{i,j}]_{1\le i,j\le m+n}, \quad B_{i,j} = \int_{\mathbb{R}} dx\, p(a_i, x; 1)p(x, b_j; 1). \tag{2.5}$$

In particular, the gap probability of a set E, i.e., the probability that none of the $x_1, \ldots, x_{n+m}$ belongs to the set E, is given in terms of a Fredholm determinant,

$$\mathbb{P}(\text{none of the } x_i \in E) = \det(\mathbb{1} - \chi_E K \chi_E)_{L^2(\mathbb{R})}, \quad \chi_E(x) = \mathbb{1}(x \in E). \tag{2.6}$$

The structure of the measure does not change when taking the limit of one of more of the Brownian bridges starting and/or leaving from the same position. Thus the determinantal structure of correlation still holds, yielding:

Proposition 2.1. *Consider* $a_{m+1} = \ldots = a_{m+n} = 0$ *and* $b_{m+1} = \ldots = b_{m+n} = 0$ *and the other* m *Brownian bridges from* a_i *to* b_i*, with* $0 < a_m < \ldots < a_1$ *and* $0 < b_m < \ldots < b_1$*. Then,*

$$\mathbb{P}(\boldsymbol{x}(0) \notin E) = \det\left(\mathbb{1} - \chi_E K_{n,m} \chi_E\right), \tag{2.7}$$

where the kernel $K_{n,m}$ *is given by*

$$K_{n,m}(x, y) = K_n^{\text{Hermite}}(x, y) + \sum_{i,j=1}^{m} \psi_i^{(n)}(x)(\mu^{-1})_{ij}\varphi_j^{(n)}(y). \tag{2.8}$$

The Hermite kernel K_n^{Hermite} *is defined by the classical Hermite polynomials and their* L^2*-norms* [6]

$$K_n^{\text{Hermite}}(x, y) = e^{-(x^2+y^2)/2} \sum_{i=0}^{n-1} \frac{1}{c_i^2} H_i(x)H_i(y); \tag{2.9}$$

the functions $\psi_k^{(n)}$ *and* $\varphi_k^{(n)}$ *are defined as follows for* $1 \le k \le m$*,*

$$\varphi_k^{(n)}(x) = \frac{e^{-x^2/2}}{2\pi i} \oint_{\Gamma_{0,a/2}} dz \frac{e^{-z^2+2xz}}{z^n(z - \frac{a_k}{2})}, \quad \psi_k^{(n)}(x) = \frac{e^{-x^2/2}}{2\pi i} \oint_{\Gamma_{0,b/2}} dz \frac{e^{-z^2+2xz}}{z^n(z - \frac{b_k}{2})}, \tag{2.10}$$

where $\Gamma_{0,a/2}$ *denotes any contour containing the points* $z = 0, a_1/2, \ldots, a_m/2$*, and similarly for* $\Gamma_{0,b/2}$*. Finally, the entries of the matrix of inner products*

$$\mu = (\mu_{k\ell})_{1\le k,\ell\le m} \text{ with } \mu_{k\ell} = \langle \varphi_k^{(n)}, \psi_\ell^{(n)} \rangle \equiv \int_{\mathbb{R}} dx \varphi_k^{(n)}(x)\psi_\ell^{(n)}(x) \tag{2.11}$$

[6] $\int_{\mathbb{R}} dx H_k(x)H_\ell(x)e^{-x^2} = \delta_{k,\ell}\, c_k^2 = \delta_{k,\ell} 2^k k! \sqrt{\pi}$.

can be written [7]

$$\mu_{k\ell} = \frac{\sqrt{\pi}\ 2^n}{2\pi i} \oint_{\Gamma_{0,a_k b_\ell/2}} dz \frac{e^z}{z^n\,(z - a_k b_\ell/2)}. \tag{2.12}$$

Proof. We start from the setting (2.2)-(2.6) and take the limit when the $2n$ points $a_{m+n}, \ldots, a_{m+1} \to 0$ and $b_{m+n}, \ldots, b_{m+1} \to 0$, and leaving the $2m$ points $a_m < \ldots < a_1$ and $b_m < \ldots < b_1$ fixed. Then, the probability density on the x_i's becomes

$$P(\mathbf{x}(0)\!=\!\mathbf{x}) = \frac{1}{Z'} \det \left(\begin{array}{c} \left(e^{a_i x_j - x_j^2/2}\right)_{\substack{1\le i\le m\\ 1\le j\le m+n}} \\ \left(x_j^{i-1} e^{-x_j^2/2}\right)_{\substack{1\le i\le n\\ 1\le j\le m+n}} \end{array} \right) \det \left(\begin{array}{c} \left(e^{b_i x_j - x_j^2/2}\right)_{\substack{1\le i\le m\\ 1\le j\le m+n}} \\ \left(x_j^{i-1} e^{-x_j^2/2}\right)_{\substack{1\le i\le n\\ 1\le j\le m+n}} \end{array} \right), \tag{2.13}$$

where Z' is a normalization constant. Consider any set of functions $\{\varphi_k^{(n)}(x), k = 1, \ldots, n+m\}$ spanning the vector space

$$V(a_1, \ldots, a_m) = \mathrm{span}\{e^{a_i x - x^2/2}, 1 \le i \le m,\ \ x^{j-1} e^{-x^2/2}, 1 \le j \le n\}, \tag{2.14}$$

and similarly a set of functions $\{\psi_k^{(n)}(x), k = 1, \ldots, n+m\}$ spanning $V(b_1, \ldots, b_m)$. Then,

$$P(\mathbf{x}(0)\!=\!\mathbf{x}) = \frac{1}{Z''} \det \left(\varphi_i^{(n)}(x_j)\right)_{1\le i,j\le n+m} \det \left(\psi_i^{(n)}(x_j)\right)_{1\le i,j\le n+m}. \tag{2.15}$$

As mentioned above, this measure defines a determinantal point process with defining kernel

$$K(x,y) = \sum_{i,j=1}^{n+m} \psi_i^{(n)}(x) [B^{-1}]_{i,j} \varphi_j^{(n)}(y), \tag{2.16}$$

where $B = [B_{i,j}]_{1\le i,j\le n+m}$ has entries $B_{i,j} = \langle \varphi_i^{(n)}, \psi_j^{(n)} \rangle$. Thus the goal is to find nice functions $\psi_k^{(n)}$ and $\varphi_k^{(n)}$ such that the inverse of the matrix B is manageable; usually one looks for a set of functions such that B becomes the identity matrix (bi-orthogonalization). In this instance, it is more convenient for doing asymptotics to find functions such that the matrix B has the form

$$B = \begin{pmatrix} \mu & 0 \\ 0 & \mathbb{1}_n \end{pmatrix}. \tag{2.17}$$

[7] Similarly $\Gamma_{0,\frac{a_k b_\ell}{2}}$ denotes a contour containing 0 and $a_k b_\ell/2$. Note that $\frac{1}{2\pi i}\oint_{\Gamma_{0,u}} \frac{e^z dz}{z^n (z-u)} = \frac{1}{u^n}(\sum_{k=n}^{\infty} \frac{u^k}{k!})$.

As will be shown below, the choice of functions for which this is the case is as follows:

$$
\begin{aligned}
\varphi_k^{(n)}(x) &= \frac{e^{-x^2/2}}{2\pi i}\oint_{\Gamma_{0,a_k/2}} dz \frac{e^{-z^2+2xz}}{z^n(z-a_k/2)}, && 1 \le k \le m,\\
\varphi_{m+k}^{(n)}(x) &= \frac{(k-1)!}{c_{k-1}}\frac{e^{-x^2/2}}{2\pi i}\oint_{\Gamma_0} dz \frac{e^{-z^2+2xz}}{z^k} = \frac{H_{k-1}(x)}{c_{k-1}}e^{-x^2/2}, && 1 \le k \le n.
\end{aligned}
\tag{2.18}
$$

The $H_k(x)$ are the classical Hermite polynomials, with generating function

$$
e^{-z^2+2xz} = \sum_{j=0}^{\infty} \frac{z^j}{j!}H_j(x), \quad \text{and thus} \quad \frac{1}{2\pi i}\oint_{\Gamma_0} e^{-z^2+2xz}\frac{dz}{z^{j+1}} = \frac{H_j(x)}{j!},
\tag{2.19}
$$

and with orthogonality relations

$$
\int_{\mathbb{R}} dx H_k(x)H_\ell(x)e^{-x^2} = \delta_{k,\ell}\, c_k^2, \text{ with } c_k = \sqrt{2^k k!\sqrt[4]{\pi}}.
\tag{2.20}
$$

By the residue theorem it follows that

$$
\frac{e^{-x^2/2}}{2\pi i}\oint_{\Gamma_{0,a/2}} dz \frac{e^{-z^2+2xz}}{z^n(z-a_k/2)} \in \operatorname{span}(e^{a_k x}, H_0, \dots, H_{n-1})e^{-x^2/2}.
\tag{2.21}
$$

Similarly one defines the functions $\psi_k^{(n)}(x)$ upon replacing a_k by b_k in (2.18). Thus the set of functions $\{\varphi_k^{(n)}(x), k = 1, \dots, n+m\}$ spans the vector space $V(a_1, \dots, a_m)$, and the set $\{\psi_k^{(n)}(x), k = 1, \dots, n+m\}$ the vector space $V(b_1, \dots, b_m)$, as defined in (2.14).

The last step is to show that with our choice we actually obtain (2.17). From the representation (2.18) of the $\varphi_k^{(n)}(x), \psi_\ell^{(n)}(y)$ in terms of Hermite polynomials, it follows immediately that

$$
\mu_{k\ell} = \langle \varphi_k^{(n)}, \psi_\ell^{(n)} \rangle = \delta_{k\ell} \quad \text{for} \quad m+1 \le k, \ell \le m+n.
\tag{2.22}
$$

Next we show that

$$
\begin{aligned}
\langle \varphi_k^{(n)}, \psi_\ell^{(n)} \rangle = 0 \quad &\text{for} \quad 1 \le k \le m, m+1 \le \ell \le m+n\\
&\text{and} \quad m+1 \le k \le m+n, 1 \le \ell \le m.
\end{aligned}
$$

Indeed for $1 \leq k \leq m$ and $m+1 \leq \ell \leq m+n$, we have

$$\begin{aligned}
\langle \varphi_k^{(n)}, \psi_\ell^{(n)} \rangle &= \frac{\mathrm{const}}{(2\pi i)^2} \oint_{\Gamma_{0,a/2}} \frac{dz\, e^{-z^2}}{z^n(z-a_k/2)} \oint_{\Gamma_0} dw \frac{e^{-w^2}}{w^{\ell-m}} \int_{-\infty}^{\infty} dx\; e^{-x^2+2x(w+z)} \\
&= \frac{\mathrm{const}\sqrt{\pi}}{(2\pi i)^2} \oint_{\Gamma_{0,a/2}} \frac{dz}{z^n(z-a_k/2)} \oint_{\Gamma_0} \frac{dw}{w^{\ell-m}} e^{2zw} \\
&= \frac{\mathrm{const}\sqrt{\pi}}{2\pi i} \oint_{\Gamma_{0,a/2}} \frac{dz\; P_{\ell-m-1}(z)}{z^n(z-a_k/2)} = 0,
\end{aligned} \tag{2.23}$$

where $P_i(x)$ is a polynomial of degree i. The result is zero because for $\ell - m - 1 \leq n-1$ the residue at infinity is zero.

Finally, for $1 \leq k, \ell \leq m$, by the same argument one gets

$$\mu_{k,\ell} = \langle \varphi_k^{(n)}, \psi_\ell^{(n)} \rangle = \frac{\sqrt{\pi}}{(2\pi i)^2} \oint_{\Gamma_{0,a_k/2}} \frac{dz}{z^n(z-a_k/2)} \oint_{\Gamma_{0,b_\ell/2}} \frac{dw\; e^{2zw}}{w^n(w-b_\ell/2)}. \tag{2.24}$$

By the residue theorem, the contribution of the pole at $w=0$ is a polynomial of degree $n-1$ in z. Thus the integral over z is zero, because the residue at infinity is zero. Thus, it remains to compute the contribution of the pole at $w = b_\ell/2$, namely

$$\mu_{k,\ell} = \frac{\sqrt{\pi}}{(2\pi i)^2} \oint_{\Gamma_{0,a_k/2}} dz \frac{e^{zb_\ell}}{z^n(z-a_k/2)} \frac{2^n}{b_\ell^n} = \frac{\sqrt{\pi}2^n}{(2\pi i)^2} \oint_{\Gamma_{0,a_k b_\ell/2}} dz \frac{e^z}{z^n(z-a_k b_\ell/2)}. \tag{2.25}$$

This ends the proof of Proposition 2.1. □

The next step in showing Theorem 1.1 for $\tau = 0$ is to determine the $n \to \infty$ limit of the kernel under the space scaling

$$x = \sqrt{2n} + \frac{\xi_1}{\sqrt{2}n^{1/6}}, \quad y = \sqrt{2n} + \frac{\xi_2}{\sqrt{2}n^{1/6}}, \tag{2.26}$$

with a_i, b_i scaled as in (1.2),

$$a_i = \sqrt{2n}\left(1 + \frac{\tilde{a}_i}{n^{1/3}}\right) \quad \text{and} \quad b_i = \sqrt{2n}\left(1 - \frac{\tilde{b}_i}{n^{1/3}}\right) \tag{2.27}$$

and with the assumption

$$\tilde{a}_i < \tilde{b}_j, \quad 1 \leq i,j \leq m. \tag{2.28}$$

Thus, we have to show that for ξ_1, ξ_2 in a bounded set,

$$\lim_{n\to\infty} \frac{1}{\sqrt{2}n^{1/6}} K_{n,m}(x,y) = K_m^{\tilde{a},\tilde{b}}(0;\xi_1,\xi_2). \tag{2.29}$$

It is well known that the Hermite kernel under the above scaling, for ξ_1, ξ_2 in a bounded set, converges to the Airy kernel $K_{\mathcal{A}}$ (see e.g. App. A.7 of [21])

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{\sqrt{2}n^{1/6}} K_n^{\mathrm{Hermite}}(x,y) &= \frac{1}{(2\pi i)^2} \int_{\Gamma_>} d\omega \int_{\Gamma_<} d\widetilde{\omega} \frac{1}{\omega - \widetilde{\omega}} \frac{e^{-\omega^3/3+\xi_2\omega}}{e^{-\widetilde{\omega}^3/3+\xi_1\widetilde{\omega}}} \\ &=: K_{\mathcal{A}}(\xi_1,\xi_2), \end{aligned} \tag{2.30}$$

where the path $\Gamma_>$ goes from $e^{-2\pi i/3}\infty$ to $e^{2\pi i/3}\infty$, the path $\Gamma_<$ from $e^{\pi i/3}\infty$ to $e^{-\pi i/3}\infty$, with $\Gamma_>$ and $\Gamma_<$ not intersecting each other.

What remains is to compute the limit of the last term in (2.8). Since m remains finite, one can take the $n \to \infty$ limit inside the sum. Below we compute the asymptotics for $\psi_i^{(n)}$, $\varphi_j^{(n)}$, and $\mu_{i,j}^{-1}$ separately. Let us start with the matrix μ, as defined in (2.17).

Lemma 2.2. *The following asymptotics holds for the inverse of the $m \times m$ matrix:*

$$\lim_{n\to\infty} \frac{1}{\sqrt{2}n^{1/6}} \left(\frac{2e}{n}\right)^n \mu^{-1} = -A^{-1}, \quad \textit{where } A = \left(\frac{1}{\tilde{a}_k - \tilde{b}_\ell}\right)_{1\le k,\ell\le m}. \tag{2.31}$$

Proof. Using the scaling (1.2), the quantity

$$\frac{a_k b_\ell}{2} = n\left(1 + \frac{\tilde{a}_k - \tilde{b}_\ell}{n^{1/3}} + \mathcal{O}\left(\frac{1}{n^{2/3}}\right)\right) \tag{2.32}$$

is, for n large enough, strictly less than n by assumption (2.28). We use (2.12) and make the change of variable $z = u\,n$

$$\begin{aligned}\mu_{k\ell} &= \frac{\sqrt{\pi}\ 2^n}{2\pi i} \oint_{\Gamma_{0,\frac{a_k b_\ell}{2}}} \frac{dz\ e^z}{z^n\left(z - \frac{a_k b_\ell}{2}\right)} \\ &= \frac{\sqrt{\pi}\ (2/n)^n}{2\pi i} \oint_{|u|=1} du \frac{e^{nF(u)}}{u - 1 - (\tilde{a}_k - \tilde{b}_\ell)n^{-1/3} + \mathcal{O}(n^{-2/3})}\end{aligned} \tag{2.33}$$

where

$$F(u) := u - \ln u = 1 + \tfrac{1}{2}(u-1)^2 + \mathcal{O}\left((u-1)^3\right), \tag{2.34}$$

with

$$\mathrm{Re}(F(u)) = \mathrm{Re}(u) - \ln(|u|). \tag{2.35}$$

Thus, we can deform the path $|u| = 1$ into $\gamma_\delta = \{1 + iy, -\delta \le y \le \delta\}$ plus a circle segment γ' centered at zero joining the extremities of γ_δ. By (2.35), the path $\gamma_\delta \vee \gamma'$ is a steep descent path for F with maximum at $u = 1$, $F(1) = 1$. We choose $\delta = n^{-2/5}$, then, the contribution of the integral in (2.33) from γ' is of order $\mathcal{O}(e^{-cn^{1/5}})$ smaller than the main contribution, coming from γ_δ, for some $c>0$. Thus, continuing (2.33),

$$\mu_{k\ell} = \left(\frac{2e}{n}\right)^n \frac{\sqrt{\pi}}{2\pi i}\int_{1-in^{-2/5}}^{1+in^{-2/5}} du \frac{e^{n(u-1)^2/2+n\mathcal{O}\left((u-1)^3\right)}}{u-1-(\tilde a_k-\tilde b_\ell)n^{-\frac13}+\mathcal{O}(n^{-\frac23})}\left(1+\mathcal{O}(e^{-cn^{1/5}})\right). \tag{2.36}$$

By the change of variable $\omega=(u-1)\sqrt{n}$, the last integral becomes

$$\frac{\sqrt{\pi}}{2\pi i}\int_{-in^{1/10}}^{in^{1/10}} d\omega \frac{e^{\frac12\omega^2(1+\mathcal{O}(n^{-2/5}))}}{\omega-(\tilde a_k-\tilde b_\ell)n^{1/6}+\mathcal{O}(n^{-1/6})}. \tag{2.37}$$

In the $n\to\infty$ limit we finally have

$$\lim_{n\to\infty}\sqrt{2}n^{1/6}(2.37) = \frac{-1}{\tilde a_k-\tilde b_\ell}. \tag{2.38}$$

Thus, we have shown that

$$\lim_{n\to\infty}\sqrt{2}n^{1/6}\left(\frac{n}{2e}\right)^n \mu_{k,l} = \frac{-1}{\tilde a_k-\tilde b_\ell} = -A_{k,l}. \tag{2.39}$$

This suffices to prove Lemma 2.2, since the dimension of the matrix does not depend on n. $\square$

The next item is to determine the asymptotics of $\varphi_k^{(n)}$ and $\psi_k^{(n)}$.

Lemma 2.3. *Consider the scaling (2.26) and (1.2), with ξ_1,ξ_2 in a bounded set. Then,*

$$\varphi_k(\xi_2) := \lim_{n\to\infty}\left(\frac{n}{2e}\right)^{n/2}\varphi_k^{(n)}\left(\sqrt{2n}+\frac{\xi_2}{\sqrt{2}\,n^{1/6}}\right) = \frac{1}{2\pi i}\int_{\Gamma_{\tilde a_k>}} d\omega \frac{e^{-\omega^3/3+\xi_2\omega}}{\omega-\tilde a_k}, \tag{2.40}$$

where $\Gamma_{\tilde a_k>}$ is a simple path from $e^{-2\pi i/3}\infty$ to $e^{2\pi i/3}\infty$ and passing onto the right of $\tilde a_k$. Similarly,

$$\psi_k(\xi_1) := \lim_{n\to\infty}\left(\frac{n}{2e}\right)^{n/2}\psi_k^{(n)}\left(\sqrt{2n}+\frac{\xi_1}{\sqrt{2}\,n^{1/6}}\right) = \frac{1}{2\pi i}\int_{\Gamma_{<\tilde b_k}} d\widetilde\omega \frac{e^{\widetilde\omega^3/3-\xi_1\widetilde\omega}}{\widetilde\omega-\tilde b_k}, \tag{2.41}$$

where $\Gamma_{<\tilde b_k}$ is a simple path from $e^{\pi i/3}\infty$ to $e^{-\pi i/3}\infty$ and passing onto the left of $\tilde b_k$ (similar to Figure 2).

Proof. The plan is to compute the large n behavior of

$$\varphi_k^{(n)}(x) = \frac{e^{-x^2/2}}{2\pi i}\oint_{\Gamma_{0,a_k/2}} dz \frac{e^{-z^2+2xz}}{z^n(z-a_k/2)}, \tag{2.42}$$

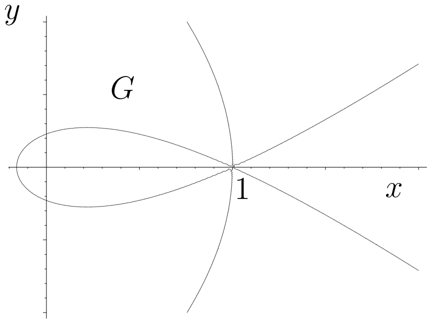

Figure 8: Lines with $\mathrm{Re}(F(x+iy)) = \mathrm{Re}(F(1))$.

with

$$a_k = \sqrt{2n}\left(1 + \frac{\tilde{a}_k}{n^{1/3}}\right), \quad x = \sqrt{2n} + \frac{\xi}{\sqrt{2}\, n^{1/6}}. \tag{2.43}$$

Rescaling the integration variable $z = u\sqrt{n/2}$, one gets

$$\varphi_k^{(n)}(x) = \left(\frac{2}{n}\right)^{n/2} \frac{e^{-n-\xi n^{1/3}+\mathcal{O}(n^{-1/3})}}{2\pi i} \oint_{\Gamma_{0,1+\tilde{a}_k/n^{1/3}}} du \frac{e^{nF(u)+u\xi n^{1/3}}}{u - 1 - \tilde{a}_k/n^{1/3}} \tag{2.44}$$

where $F(u) = -u^2/2 + 2u - \ln(u)$. The leading contribution comes from the neighborhood of the double critical point of $F(u)$ at $u = 1$, where we have

$$F(u) = \frac{3}{2} - \frac{1}{3}(u-1)^3 + \mathcal{O}((u-1)^4). \tag{2.45}$$

As integration path one can choose any path passing through $u = 1 + \alpha n^{-\frac{1}{3}}$, with $\tilde{a}_k < \alpha$, locally following the directions $e^{\pm 2\pi i/3}$, and which remain inside the region G of Figure 8. Then, the integration away from a δ-neighborhood of $u = 1 + \alpha n^{-\frac{1}{3}}$ (where $\delta = n^{-\varepsilon}$, with $0 < \varepsilon < 1/3$) will be of order $\mathcal{O}(e^{-cn})$ smaller than the leading term, with $0 < c \sim \delta^3$ for small δ. Then, in a δ-neighborhood of $u = 1$ we can use series expansion and after the change of variable $\omega = n^{1/3}(u-1)$, one finds

$$\begin{aligned} &\varphi_k^{(n)}\left(\sqrt{2n} + \frac{\xi}{\sqrt{2}\, n^{1/6}}\right) \\ &= \left(\frac{2e}{n}\right)^{n/2} \left[\frac{1}{2\pi i}\left(1 + \mathcal{O}\left(\frac{1}{n^{1/3}}\right)\right) \int d\omega \frac{e^{-\omega^3/3+\xi\omega}}{\omega - \tilde{a}_k}\right] \left(1 + \mathcal{O}(e^{-cn})\right), \end{aligned}$$

where the integral goes from $e^{-2\pi i/3}\delta n^{1/3}$ to $e^{2\pi i/3}\delta n^{1/3}$, and passing on the right of $\tilde{a}_k$. From this, the $n \to \infty$ limit in (2.40) holds.

The asymptotic for $\psi_k^{(n)}(x)$ is essentially the same, except that $\omega \mapsto -\omega$ and $\tilde{a}_k \mapsto -\tilde{b}_k$, ending the proof of Lemma 2.3. $\square$

We shall also need:

Lemma 2.4. *Given the matrix*

$$A := \left(\left(\frac{1}{\tilde{a}_i - \tilde{b}_j} \right)_{1 \le i,j \le m} \right), \tag{2.46}$$

the following identity holds:

$$\sum_{1 \le i,j \le m} \frac{(A^{\top -1})_{ij}}{(z - \tilde{a}_i)(w - \tilde{b}_j)} = \frac{1}{w - z} \left(\prod_{k=1}^{m} \left(\frac{w - \tilde{a}_k}{z - \tilde{a}_k} \right) \left(\frac{z - \tilde{b}_k}{w - \tilde{b}_k} \right) - 1 \right). \tag{2.47}$$

Proof. Since

$$\det A = \frac{\Delta(\tilde{a})\Delta(\tilde{b})}{\prod\limits_{1 \le i,j \le m} (\tilde{a}_i - \tilde{b}_j)}, \tag{2.48}$$

one checks the identity (2.47), by computing the residue on the right hand side at the points $z = \tilde{a}_i$, $w = \tilde{b}_j$ and identifying with $(A^{\top -1})_{ij}$ using Cramer's rule and repeatedly using (2.48). $\square$

Proof of Theorem 1.1. Assembling the asymptotic result (2.30), Proposition 2.1, Lemma 2.2 and Lemma 2.3, one obtains Theorem 1.1 in the special case $\tau = 0$, with distinct $\tilde{a}_i$, $\tilde{b}_i$, under the condition $\tilde{a}_i < \tilde{b}_j$. Upon using the scaling (2.26) and (2.27), the limit kernel is thus given by the limit of the sum of the kernels in (2.8); i.e., the sum of the Airy kernel $K_{\mathcal{A}}$, defined in (2.30), and a new kernel:

$$\begin{aligned} \lim_{n \to \infty} \frac{1}{\sqrt{2} n^{1/6}} K_{n,m}(x, y) &= K_{\mathcal{A}}(\xi_1, \xi_2) - \sum_{i,j=1}^{m} \psi_i(\xi_1) [A^{-1}]_{i,j} \varphi_j(\xi_2) \\ &= \frac{1}{(2\pi i)^2} \int_{\Gamma_>} d\omega \int_{\Gamma_<} d\widetilde{\omega} \, \frac{e^{-\omega^3/3 + \xi_2 \omega}}{e^{-\widetilde{\omega}^3/3 + \xi_1 \widetilde{\omega}}} \frac{1}{\omega - \widetilde{\omega}} \\ &\quad - \frac{1}{(2\pi i)^2} \int_{\Gamma_{\tilde{a}>}} d\omega \int_{\Gamma_{<\tilde{b}}} d\widetilde{\omega} \, \frac{e^{-\omega^3/3 + \xi_2 \omega}}{e^{-\widetilde{\omega}^3/3 + \xi_1 \widetilde{\omega}}} \sum_{i,j=1}^{m} \frac{[A^{-1}]_{i,j}}{(\widetilde{\omega} - \tilde{b}_i)(\omega - \tilde{a}_j)}. \end{aligned} \tag{2.49}$$

The fact that this expression actually equals the kernel $K_m^{\tilde{a},\tilde{b}}(0; \xi_1, \xi_2)$, as defined in Theorem 1.1, follows from Lemma 2.4. $\square$

3 Extended kernel for the Airy process with wanderers

In this section, we will prove Theorem 1.2. For this purpose, we need to know the measure, defined on the positions of the Brownian bridges at different times $-1 < T_1 < T_2 < \ldots < T_\ell < 1$. Set $\mathbf{x}(T_i) := (x_1(T_i), \ldots, x_{m+n}(T_i))$. Then, by Karlin-McGregor applied to these different times, the measure obtained by the non-intersecting condition on the Brownian bridges is given by

$$P(\mathbf{x}(T_1) = \mathbf{x}^1, \ldots, \mathbf{x}(T_\ell) = \mathbf{x}^\ell) = \frac{1}{Z}\det(p(a_i, x_j^1, T_1 + 1))_{1\le i,j\le n+m}$$
$$\times \left(\prod_{k=1}^{\ell-1} \det(p(x_i^k, x_j^{k+1}, T_{k+1} - T_k))_{1\le i,j\le n+m}\right) \det(p(x_i^\ell, b_j, 1-T_\ell))_{1\le i,j\le n+m}. \tag{3.1}$$

It is well known that this measure, a generalization of (2.2) to multi-times, or any measure of this form has determinantal correlations in space-time [16,20, 24,32,37] (even in cases when the size of the determinant is increasing [12,13]).

Proposition 3.1. *Any measure on $\{x_i^{(n)}, 1 \le i \le N, 1 \le n \le \ell\}$ of the form*[8]

$$\frac{1}{Z}\det(\phi(T_0, a_i; T_1, x_j^{(1)}))_{1\le i,j\le N} \left(\prod_{n=1}^{\ell-1} \det(\phi(T_n, x_i^{(n)}; T_{n+1}, x_j^{(n+1)}))_{1\le i,j\le N}\right)$$
$$\times \det(\phi(T_\ell, x_i^{(\ell)}; T_{\ell+1}, b_j))_{1\le i,j\le N}, \tag{3.2}$$

has, assuming $Z \neq 0$, the following k-point correlation functions for $t_1, \ldots, t_k \in \{T_1, \ldots, T_\ell\}$:

$$\rho^{(k)}(t_1, x_1, \ldots, t_k, x_k) = \det\left(K(t_i, x_i; t_j, x_j)\right)_{1\le i,j\le k}, \tag{3.3}$$

where the space-time kernel (often called extended kernel) is given by

$$\begin{aligned} K(t_1, x_1; t_2, x_2) &= -\phi(t_1, x_1; t_2, x_2)\mathbb{1}(t_2 > t_1) \\ &+ \sum_{i,j=1}^{N} \phi(t_1, x_1; T_{\ell+1}, b_i)[B^{-1}]_{i,j}\phi(T_0, a_j; t_2, x_2) \end{aligned} \tag{3.4}$$

with ($*$ means integration with regard to the consecutive dots)

$$\phi(T_r, x; T_s, y) = \begin{cases} \phi(T_r, x; T_{r+1}, \cdot) * \cdots * \phi(T_{s-1}, \cdot; T_s, y), & \textit{if } T_r < T_s, \\ 0, & \textit{if } T_r \ge T_s, \end{cases} \tag{3.5}$$

[8] The functions $\phi(T_n, x; T_{n+1}, y)$ themselves may in fact vary with n above.

and with the $N \times N$ matrix B having entries $B_{i,j} = \phi(T_0, a_i; T_{\ell+1}, b_j)$. Remark that $(N!)^\ell \det(B) = Z$, so that B^{-1} exists as soon as $Z \neq 0$.

We now apply this general fact to the non-intersecting Brownian motion formula (3.1): here $x_i^{(n)}$ denotes the position $x_i(T_n)$ of the ith Brownian motion at time T_n, while one sets $T_0 = -1$, $T_{\ell+1} = 1$, and one sets

$$\phi(t, x; t', x') := p(x, x', t' - t). \tag{3.6}$$

As for the one-time case, the structure is unchanged, even after letting $a_{n+m}, \ldots, a_{m+1} \to 0$ and $b_{n+m}, \ldots, b_{m+1} \to 0$, keeping $a_m < \ldots < a_1$ and $b_m < \ldots < b_1$ fixed. The only difference is that the entries on the first and last determinants in (3.2) will be different (together with a different normalization constant Z). Indeed, the first determinant in (3.2) is just replaced by

$$\det \begin{pmatrix} \left(e^{a_i x_j^{(1)}/(1+T_1)} p(0, x_j^{(1)}, T_1 + 1) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m+n}} \\ \left(\left(\frac{x_j^{(1)}}{1+T_1}\right)^{i-1} p(0, x_j^{(1)}, T_1 + 1) \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m+n}} \end{pmatrix}, \tag{3.7}$$

while the last determinant is replaced by

$$\det \begin{pmatrix} \left(e^{b_i x_j^{(\ell)}/(1-T_\ell)} p(x_j^{(\ell)}, 0, 1 - T_\ell) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m+n}} \\ \left(\left(\frac{x_j^{(\ell)}}{1-T_\ell}\right)^{i-1} p(x_j^{(\ell)}, 0, 1 - T_\ell) \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m+n}} \end{pmatrix}. \tag{3.8}$$

As for the one-time situation, one looks for sets of functions generating the same vector spaces as the functions in (3.7) and (3.8), namely one searches for functions $\varphi_k^{(n)}(T_1, x)$ and $\psi_k^{(n)}(T_\ell, x)$, such that

$$\begin{aligned} (3.7) = &\quad \text{const} \times \det(\varphi_i^{(n)}(T_1, x_j^1))_{1 \leq i,j \leq n+m}, \\ (3.8) = &\quad \text{const} \times \det(\psi_i^{(n)}(T_\ell, x_j^\ell))_{1 \leq i,j \leq n+m}, \end{aligned} \tag{3.9}$$

and such that the matrix B has the same form (2.17) as before. Setting

$$\gamma(t) := \sqrt{\frac{1-t}{1+t}}, \quad \sigma(t) := \sqrt{1+t}, \tag{3.10}$$

one picks, for $1 \leq k \leq m$,

$$\begin{aligned} \varphi_k^{(n)}(t, x) &:= \frac{e^{-x^2/2\sigma(t)^2}}{\sigma(t)} \frac{1}{2\pi i} \oint_{\Gamma_{0,a_k/2}} dz \frac{e^{-z^2\gamma(t)^2 + 2xz/\sigma^2(t)}}{z^n\,(z - a_k/2)}, \\ \psi_k^{(n)}(t, x) &:= \frac{e^{-x^2/2\sigma(-t)^2}}{\sigma(-t)} \frac{1}{2\pi i} \oint_{\Gamma_{0,b_k/2}} dz \frac{e^{-z^2\gamma(-t)^2 + 2xz/\sigma^2(-t)}}{z^n\,(z - b_k/2)}, \end{aligned} \tag{3.11}$$

and for $1 \leq k \leq n$,

$$
\begin{aligned}
\varphi^{(n)}_{m+k}(t,x) &:= \frac{(4\pi)^{1/4}}{\sqrt{(k-1)!2^{k-1}}}\gamma(t)^{k-1}H_{k-1}\left(\frac{x}{\gamma(t)\sigma^2(t)}\right)p(0,x;t+1),\\
\psi^{(n)}_{m+k}(t,x) &:= \frac{(4\pi)^{1/4}}{\sqrt{(k-1)!2^{k-1}}}\gamma(-t)^{k-1}H_{k-1}\left(\frac{x}{\gamma(-t)\sigma^2(-t)}\right)p(x,0;1-t).
\end{aligned}
\tag{3.12}
$$

Remark that, using the integral representation of the Hermite polynomials, an equivalent expression for (3.12) is

$$
\begin{aligned}
\varphi^{(n)}_{m+k}(t,x) &= \frac{(4\pi)^{1/4}}{\sqrt{(k-1)!2^{k-1}}}\frac{e^{-x^2/2\sigma(t)^2}}{\sigma(t)}\frac{(k-1)!}{2\pi i}\oint_{\Gamma_0} dz \frac{e^{-z^2\gamma(t)^2+2xz/\sigma^2(t)}}{z^k},\\
\psi^{(n)}_{m+k}(t,x) &= \frac{(4\pi)^{1/4}}{\sqrt{(k-1)!2^{k-1}}}\frac{e^{-x^2/2\sigma(-t)^2}}{\sigma(-t)}\frac{(k-1)!}{2\pi i}\oint_{\Gamma_0} dz \frac{e^{-z^2\gamma(-t)^2+2xz/\sigma^2(-t)}}{z^k}.
\end{aligned}
\tag{3.13}
$$

It is immediate to verify that these functions generate at $t = T_1$, resp. $t = T_\ell$, the same space as the function in (3.7), resp. (3.8). So, one defines the functions appearing in the first and last determinant of (3.2) by

$$
\phi(T_0,a_i;T_1,x^{(1)}) := \varphi^{(n)}_i(T_1,x^{(1)}) \text{ and } \phi(T_\ell,x^{(\ell)};T_{\ell+1},b_i) := \psi^{(n)}_i(T_\ell,x^{(\ell)}),
\tag{3.14}
$$

for which we show the following property:

Lemma 3.2. *For any $t_1 < t_2$ and $1 \leq k \leq n+m$, one has*

$$
\begin{aligned}
\int_{\mathbb{R}} dx\, \varphi^{(n)}_k(t_1,x)\, p(x,y;t_2-t_1) &= \varphi^{(n)}_k(t_2,y),\\
\int_{\mathbb{R}} dy\, p(x,y;t_2-t_1)\, \psi^{(n)}_k(t_2,y) &= \psi^{(n)}_k(t_1,x).
\end{aligned}
\tag{3.15}
$$

Proof. Since $\psi^{(n)}_k$ is obtained from $\varphi^{(n)}_k$ by the map $t \mapsto -t$ and $a \mapsto b$, it suffices to present the proof for $\varphi^{(n)}_k$. At first, for $1 \leq k \leq m$, one has

$$
\begin{aligned}
\int_{\mathbb{R}} dx\, \varphi^{(n)}_k(t_1,x)\, p(x,y;t_2-t_1) &= \frac{1}{2\pi i}\oint_{\Gamma_{0,a_k/2}} dz \frac{e^{-z^2\gamma(t_1)^2}}{z^n\,(z-a_k/2)}\\
&\times \int_{\mathbb{R}} dx\, \frac{e^{-x^2/2(1+t_1)}}{\sqrt{1+t_1}} e^{2xz/(1+t_1)} \frac{e^{-(x-y)^2/2(t_2-t_1)}}{\sqrt{2\pi(t_2-t_1)}}
\end{aligned}
\tag{3.16}
$$

and, after performing the Gaussian integration, one has

$$
(3.16) = \frac{e^{-y^2/2\sigma(t_2)^2}}{\sigma(t_2)}\frac{1}{2\pi i}\oint_{\Gamma_{0,a_k/2}} dz \frac{e^{-z^2\gamma(t_2)^2+2yz/\sigma^2(t_2)}}{z^n\,(z-a_k/2)} = \varphi^{(n)}_k(t_2,y).
\tag{3.17}
$$

Secondly, consider $1 \leq k \leq n$. Comparing the representations (3.11) and (3.13), we see immediately that the computations are exactly the same. Indeed, the only difference is a k-dependent prefactor and the denominator in the integrand over z. However, there are not affected by the computations above; thus

$$\int_{\mathbb{R}} dx\, \varphi^{(n)}_{m+k}(t_1, x)\, p(x, y; t_2 - t_1) = \varphi^{(n)}_{m+k}(t_2, y) \tag{3.18}$$

holds, ending the proof of Lemma 3.2. □

Proposition 3.3. *The extended kernel is given by*

$$\begin{aligned} K_{n,m}(t_1, x_1; t_2, x_2) = -p(x_1, x_2; t_2 - t_1)\mathbb{1}(t_2 > t_1) + \sum_{i=1}^{n} \psi^{(n)}_{m+i}(t_1, x_1)\varphi^{(n)}_{m+i}(t_2, x_2) \\ + \sum_{i,j=1}^{m} \psi^{(n)}_i(t_1, x_1)[\mu^{-1}]_{i,j}\varphi^{(n)}_j(t_2, x_2), \end{aligned} \tag{3.19}$$

with $\varphi^{(n)}_j(t, x)$ and $\psi^{(n)}_{m+i}(t, x)$ given by (3.11) and (3.13) and with μ given by (2.24), the same as in the 1-time case.

Proof. Given the definitions (3.6) and (3.5), the first term in the kernel (3.4) is simply

$$-\phi(t_1, x_1; t_2, x_2)\mathbb{1}(t_2 > t_1) = -p(x_1, x_2, t_2 - t_1)\mathbb{1}(t_2 > t_1). \tag{3.20}$$

It remains to be shown that B has the form

$$B = \begin{pmatrix} \mu & 0 \\ 0 & \mathbb{1}_n \end{pmatrix}, \tag{3.21}$$

as in the 1-time case, with μ given in (2.24).

Indeed, for any choice of $1 \leq k \leq \ell - 2$, and for $t_i = T_i$ with $1 \leq i \leq \ell$, one has, using the convolution property of the Brownian transition probability and the convolution property in Lemma 3.2, the property that ($*$ means integration with regard to the common variable)

$$
\begin{aligned}
B_{i,j} &= \phi(T_0, a_i; T_{\ell+1}, b_j)\\
&= \varphi_i^{(n)}(t_1, x^{(1)}) * p(x^{(1)}, x^{(2)}; t_2 - t_1) * \dots\\
&\quad * p(x^{(k)}, x^{(k+1)}; t_{k+1} - t_k) * p(x^{(k+1)}, x^{(k+2)}; t_{k+2} - t_{k+1}) *\\
&\quad \dots * p(x^{(\ell-1)}, x^{(\ell)}; t_\ell - t_{\ell-1}) * \psi_j^{(n)}(t_\ell, x^{(\ell)})\\
&= \Big(\varphi_i^{(n)}(t_1, x^{(1)}) * p(x^{(1)}, x^{(k+1)}; t_{k+1} - t_1)\Big)\\
&\quad * \Big(p(x^{(k+1)}, x^{(\ell)}; t_\ell - t_{k+1}) * \psi_j^{(n)}(t_\ell, x^{(\ell)})\Big)\\
&= \varphi_i^{(n)}(t_{k+1}, x^{(k+1)}) * \psi_j^{(n)}(t_{k+1}, x^{(k+1)}) = \langle \varphi_i^{(n)}(t_{k+1}, .), \psi_j^{(n)}(t_{k+1}, .)\rangle
\end{aligned}
$$

is independent of t_{k+1}; therefore, by setting $t_{k+1} = 0$, it is, in particular, equal to the value μ_{ij} obtained in (2.25) and (2.24). This establishes Proposition 3.3. □

In order to prove Theorem 1.2 (and thus also Theorem 1.1 for generic τ), one needs to compute the $n \to \infty$ asymptotics of the kernel. For convenience, recall the scaling for the starting and ending points of the top m Brownian bridges (1.2) and of the subsequent scaling (1.3) of the space-time region one focuses on:

$$
\begin{aligned}
a_i &= \sqrt{2n} + \sqrt{2}\,\tilde a_i n^{1/6}, \quad & b_i &= \sqrt{2n} - \sqrt{2}\,\tilde b_i n^{1/6}\\
t_i &= \tau_i n^{-1/3}, & x_i &= \sqrt{2n} + \frac{\xi_i - \tau_i^2}{\sqrt{2}n^{1/6}}.
\end{aligned} \tag{3.22}
$$

with $\tilde a_i < \tilde b_j$, $1 \le i, j \le m$. Below we prove that, given the scaling (3.22) and for ξ_1, ξ_2 in a bounded set,

$$
\lim_{n\to\infty} \frac{1}{\sqrt{2}n^{1/6}} K_{n,m}(t_1, x_1; t_2, x_2) \equiv K_m^{\tilde a, \tilde b}(\tau_1, \xi_1; \tau_2, \xi_2). \tag{3.23}
$$

where $\equiv$ we means an equivalent kernel[9].

Proposition 3.4. *With the above scaling, for ξ_1, ξ_2 in a bounded set (and τ_1, τ_2 fixed), in the case where all the $\tilde a_i$ (and $\tilde b_i$) are distinct, one has*

$$
\lim_{n\to\infty} \frac{1}{\sqrt{2}n^{1/6}} K_{n,m}(t_1, x_1; t_2, x_2) = K_m^{\tilde a, \tilde b}(\tau_1, \xi_1; \tau_2, \xi_2) \frac{f(\tau_1, \xi_1)}{f(\tau_2, \xi_2)}, \tag{3.24}
$$

where $f(\tau, \xi) = \exp(\tau^3/3 - \xi\tau)$.

[9]Two kernels are equivalent if they define the same determinantal point process. Namely, if there exists some function $f(x) \neq 0$ such that $K(x, y) = \widetilde K(x, y) f(x)/f(y)$, then K and $\widetilde K$ are equivalent, since all the correlation functions are given by determinants in which the functions f cancel exactly.

Proof. Consider the first two terms in the kernel (3.19). These terms are independent of the $\tilde{a}_i, \tilde{b}_i$ and of m. Indeed, it corresponds exactly to the kernel of the system without wanderers, which can be denoted by $K_{n,0}$. Indeed,

$$\sum_{i=1}^{n} \psi_{m+i}^{(n)}(t_1, x_1)\varphi_{m+i}^{(n)}(t_2, x_2) = \sum_{k=0}^{n-1} \frac{(4\pi)^{1/2}}{k!2^k}\gamma(t_2)^k\gamma(-t_1)^k \\ \times p(x_1, 0; 1-t_1)p(0, x_2; t_2+1)H_k\left(\frac{x_1}{\gamma(-t_1)\sigma^2(-t_1)}\right) H_k\left(\frac{x_2}{\gamma(t_2)\sigma^2(t_2)}\right) \tag{3.25}$$

For fixed τ_1, τ_2, we show below that

$$\lim_{n\to\infty} \frac{1}{\sqrt{2}n^{1/6}} K_{n,0}(t_1, x_1; t_2, x_2) = K_{\mathcal{A}}(\tau_1, \xi_1; \tau_2, \xi_2)\frac{f(\tau_1, \xi_1)}{f(\tau_2, \xi_2)} \tag{3.26}$$

uniformly for ξ_1, ξ_2 in a bounded set, with $K_{\mathcal{A}}$ the extended Airy kernel given by

$$K_{\mathcal{A}}(\tau_1, \xi_1; \tau_2, \xi_2) = \begin{cases} \int_{\mathbb{R}_+} d\lambda\, e^{\lambda(\tau_2-\tau_1)} \operatorname{Ai}(\xi_1+\lambda) \operatorname{Ai}(\xi_2+\lambda), & \tau_1 \geq \tau_2, \\ -\int_{\mathbb{R}_-} d\lambda\, e^{\lambda(\tau_2-\tau_1)} \operatorname{Ai}(\xi_1+\lambda) \operatorname{Ai}(\xi_2+\lambda), & \tau_1 < \tau_2. \end{cases} \tag{3.27}$$

To obtain this result, for ξ_1, ξ_2 in a bounded set, one can just use the asymptotics of the classical Hermite polynomials (see for example App. 7 of [21]). Another, better, way is to first perform the sum over k using two different integral representations for Hermite polynomials, a first one is (3.13) and a second one is an integral over $L + i\mathbb{R}$ (see e.g. sect. 2.2 of [29]) for $L > 0$; namely:

$$H_n(x) = \frac{n!}{2\pi i}\oint_\gamma e^{-z^2+2xz}\frac{dz}{z^{n+1}} = \frac{2^n e^{x^2}}{i\sqrt{\pi}}\int_{L+i\mathbb{R}} e^{w^2-2xw}w^n dw \tag{3.28}$$

Then

$$K_{n,0}(x_1, t_1; x_2, t_2) = -p(x_1, x_2; t_2-t_1)\mathbb{1}(t_2 > t_1) \\ + \frac{2}{(2\pi i)^2}\frac{e^{\frac{x_1^2}{2(1+t_1)}-\frac{x_2^2}{2(1+t_2)}}}{\sqrt{(1+t_1)(1+t_2)}}\int_{L+i\mathbb{R}} dU \oint_\gamma dV \frac{\left(\frac{U}{V}\right)^n - 1}{U-V}\frac{e^{\frac{1-t_1}{1+t_1}U^2-\frac{2x_1U}{1+t_1}}}{e^{\frac{1-t_2}{1+t_2}V^2-\frac{2x_2V}{1+t_2}}}. \tag{3.29}$$

Note that the -1 in $(\frac{U}{V})^n - 1$ (appearing in the integral above) can actually be omitted, because there is no residue at $V = U$. One then makes the substitution to new integration variables $\tilde{U}$ and $\tilde{V}$,

$$U\sqrt{\frac{1-t_1}{1+t_1}} = \tilde{U}\sqrt{\frac{n}{2}}, \quad V\sqrt{\frac{1-t_2}{1+t_2}} = \tilde{V}\sqrt{\frac{n}{2}} \tag{3.30}$$

and uses steepest descent in the integral to get the extended Airy kernel (3.27), which is just (1.7) in which one replaces $\tilde a_k=\tilde b_k=0$ and $m=0$, namely

$$K_{\mathcal A}(\tau_1,\xi_1;\tau_2,\xi_2)=-\frac{\mathbb{1}(\tau_2>\tau_1)}{\sqrt{4\pi(\tau_2-\tau_1)}}e^{-\frac{(\xi_2-\xi_1)^2}{4(\tau_2-\tau_1)}-\frac12(\tau_2-\tau_1)(\xi_2+\xi_1)+\frac1{12}(\tau_2-\tau_1)^3}$$
$$+\frac{1}{(2\pi i)^2}\int_{\Gamma_>}d\omega\int_{\Gamma_<}d\widetilde\omega\,\frac{e^{-\omega^3/3+\xi_2\omega}}{e^{-\widetilde\omega^3/3+\xi_1\widetilde\omega}}\frac{1}{(\omega+\tau_2)-(\widetilde\omega+\tau_1)}.\tag{3.31}$$

What remains is to compute the limit of the third term in (3.19), namely

$$\lim_{n\to\infty}\frac{1}{\sqrt2 n^{1/6}}\sum_{i,j=1}^m\psi_i^{(n)}(t_1,x_1)[\mu^{-1}]_{i,j}\varphi_j^{(n)}(t_2,x_2).\tag{3.32}$$

Since m remains finite, we can take the $n\to\infty$ limit inside the sum. Also, the limit of μ^{-1}, taking into account the prefactor, has already been computed in Lemma 2.2. It remains to determine the asymptotics of $\psi_k^{(n)}(t_1,x_1)$ and $\varphi_k^{(n)}(t_2,x_2)$ (for $1\le k\le m$) under the above scaling.

As will be seen, the computations are very close to the ones for $t=0$ in Lemma 2.3. For convenience, recall the notations $\gamma(t)=\sqrt{(1-t)/(1+t)}$ and $\sigma(t)=\sqrt{1+t}$. From (3.11), after the change of variable $z=w/\gamma(t)$, one gets

$$\varphi_k^{(n)}(t,x)=\frac{e^{-x^2/2\sigma(t)^2}}{\sigma(t)}\gamma(t)^n\frac{1}{2\pi i}\oint_{\Gamma_{0,a_k'/2}}dw\frac{e^{-w^2+2wx'}}{w^n(w-a_k'/2)},\tag{3.33}$$

where x' and a_k' are defined below, together with their asymptotics

$$\begin{aligned}x'&:=\frac{x}{\sigma(t)^2\gamma(t)}=\frac{x}{\sqrt{1-t^2}}=\sqrt{2n}+\frac{\xi}{\sqrt2\,n^{1/6}}+\mathcal O(n^{-5/6})\\ a_k'&:=a_k\gamma(t)=\sqrt{2n}+\sqrt2(\tilde a_k-\tau)n^{1/6}+\mathcal O(n^{-1/6}).\end{aligned}\tag{3.34}$$

Now we benefit from the computation made in the $\tau=0$ case. Indeed, we showed that

$$\frac{1}{2\pi i}\oint_{\Gamma_{0,a/2}}dz\,\frac{e^{-z^2+2zy}}{z^n(z-a/2)}=e^{y^2/2}\left(\frac{2e}{n}\right)^{n/2}\frac{1}{2\pi i}\int_{\Gamma_{\tilde a>}}d\omega\frac{e^{-\omega^3/3+\xi\omega}}{\omega-\tilde a}\left(1+o(1)\right),\tag{3.35}$$

if y and a are scaled as

$$y=\sqrt{2n}+\frac{\xi}{\sqrt2\,n^{1/6}},\quad a=\sqrt{2n}+\sqrt2\,\tilde a n^{1/6}.\tag{3.36}$$

This is exactly our situation with $\tilde{a} = \tilde{a}_k - \tau$. Thus we get

$$\varphi_k^{(n)}(t,x) = \frac{e^{-x^2/2\sigma(t)^2}}{\sigma(t)}\gamma(t)^n e^{x'^2/2}\left(\frac{2e}{n}\right)^{n/2}\frac{1}{2\pi i}\int_{\Gamma_{\tilde{a}_k-\tau>}} d\omega \frac{e^{-\omega^3/3+\xi\omega}}{\omega-\tilde{a}_k+\tau}\left(1+o(1)\right). \tag{3.37}$$

Moreover, the asymptotics of the prefactor reads

$$\frac{e^{-x^2/2\sigma(t)^2}}{\sigma(t)}\gamma(t)^n e^{x'^2/2} = e^{-\tau^3/3+\xi\tau+\mathcal{O}(n^{-1/3})}. \tag{3.38}$$

Thus we have showed that

$$\varphi_k(\tau_2,\xi_2) := \lim_{n\to\infty}\varphi_k^{(n)}(t_2,x_2)\left(\frac{n}{2e}\right)^{n/2} = \frac{1}{2\pi i}\int_{\Gamma_{\tilde{a}_k-\tau_2>}} d\omega \frac{e^{-\omega^3/3+\xi_2\omega}}{\omega-\tilde{a}_k+\tau_2}\frac{1}{f(\tau_2,\xi_2)}. \tag{3.39}$$

and similarly,

$$\psi_k(\tau_1,\xi_1) := \lim_{n\to\infty}\psi_k^{(n)}(t_1,x_1)\left(\frac{n}{2e}\right)^{n/2} = \frac{1}{2\pi i}\int_{\Gamma_{<\tilde{b}_k-\tau_1}} d\widetilde{\omega}\, \frac{e^{\widetilde{\omega}^3/3-\xi_1\widetilde{\omega}}}{\widetilde{\omega}-\tilde{b}_k+\tau_1} f(\tau_1,\xi_1). \tag{3.40}$$

Now we can put together all the pieces, which make up the kernel (3.19), namely (3.26), (3.39), (3.40), and the asymptotics of the inverse of the matrix B in Lemma 2.3. Thus we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{\sqrt{2}n^{1/6}}K_{n,m}(t_1,x_1;t_2,x_2)\frac{f(\tau_2,\xi_2)}{f(\tau_1,\xi_1)} &= K_{\mathcal{A}}(\tau_1,\xi_1;\tau_2,\xi_2)\\ &- \frac{f(\tau_2,\xi_2)}{f(\tau_1,\xi_1)}\sum_{i,j=1}^m \psi_i(\tau_1,\xi_1)[A^{-1}]_{i,j}\varphi_j(\tau_2,\xi_2).\end{aligned} \tag{3.41}$$

The last term in (3.41) (including the minus sign) is equal to

$$-\frac{1}{(2\pi i)^2}\int_{\Gamma_{\tilde{a}-\tau_2>}} d\omega \int_{\Gamma_{<\tilde{b}-\tau_1}} d\widetilde{\omega}\, \frac{e^{-\omega^3/3+\xi_2\omega}}{e^{-\widetilde{\omega}^3/3+\xi_1\widetilde{\omega}}}\sum_{i,j=1}^m \frac{[A^{-1}]_{i,j}}{(\widetilde{\omega}-\tilde{b}_i+\tau_1)(\omega-\tilde{a}_j+\tau_2)}. \tag{3.42}$$

Applying the identity in Lemma 2.4 we get as final result the kernel $K_m^{\tilde{a},\tilde{b}}(\tau_1,\xi_1;\tau_2,\xi_2)$ of Theorem 1.2, and this ends the proof of Proposition 3.4. ☐

Proof of Theorem 1.2: For any bounded set E, the probability (1.10) is given by the Fredholm determinant of the kernel, obtained in Proposition 3.4. Since this kernel is conjugate to the one in Theorem 1.2, their Fredholm determinants are identical. ☐

Proof of Theorem 1.3 (Universality). The proof is a mild variation on the proof of Theorem 1.2 and Proposition 3.4. Referring to the notation used in the statement of Theorem 1.3, one checks that formula (2.32) for $a_k b_\ell/2$ remains the same, since $x_0^- x_0^+ = 2n$ (see (1.8)) and thus also asymptotic formula (2.33) for $\mu_{k\ell}$. Moreover the scaling now reads:

$$\begin{aligned} t &= t_0 + \frac{(1-t_0^2)\tau}{n^{1/3}} \\ x &= \sqrt{2n(1-t^2)}\left(1+\frac{\xi}{2n^{2/3}}\right) \\ &= \sqrt{2n(1-t_0^2)}\left(1-\frac{t_0\tau}{n^{1/3}}+\frac{\xi-\tau^2}{2n^{2/3}}-\frac{t_0\tau}{2n}(\xi+\tau^2)+\mathcal{O}(n^{-4/3})\right). \end{aligned} \tag{3.43}$$

Refering to the notation (3.34), one checks that with x and t as in (3.43) above, one has

$$\begin{aligned} x' &= \frac{x}{\sigma(t)^2\gamma(t)} = \frac{x}{\sqrt{1-t^2}} = \sqrt{2n}\left(1+\frac{\xi}{2n^{2/3}}\right)+\mathcal{O}(n^{-5/6}) \\ a'_k &= a_k\gamma(t) = a_k\sqrt{\frac{1-t}{1+t}} = \sqrt{2n}\left(1-\frac{\tilde a_k-\tau}{n^{1/3}}\right)+\mathcal{O}(n^{-1/6}) \\ b'_k &= b_k\gamma(-t) = b_k\sqrt{\frac{1+t}{1-t}} = \sqrt{2n}\left(1-\frac{\tilde b_k-\tau}{n^{1/3}}\right)+\mathcal{O}(n^{-1/6}). \end{aligned} \tag{3.44}$$

With this information, one checks the following asymptotics, which is the analogue of (3.38), namely

$$\begin{aligned} \frac{e^{-x^2/2\sigma(\pm t)^2}}{\sigma(\pm t)}\gamma(\pm t)^n e^{x'^2/2} &= \frac{e^{\pm\frac{x'^2 t}{2}}}{\sqrt{1\pm t}}\left(\frac{1-t}{1+t}\right)^{\pm n/2} \\ &= \frac{e^{\pm n t_0}}{\sqrt{1\pm t_0}}\left(\frac{1-t_0}{1+t_0}\right)^{\pm n/2} f_n(\tau,\xi)^{\mp 1} e^{\mathcal{O}(n^{-1/3})} \end{aligned} \tag{3.45}$$

with

$$f_n(\tau,\xi) := e^{t_0(n^{2/3}\tau t_0-(\xi-\tau^2)n^{1/3}+t_0\tau(\xi+\tau^2))}e^{\tau^3/3-\xi\tau}. \tag{3.46}$$

Here $f_n(\tau,\xi)$ depends on n, besides τ and ξ. Then we show

$$\begin{aligned} \lim_{n\to\infty}\varphi_k^{(n)}(t_2,x_2)\left(\frac{n}{2e}\right)^{n/2} e^{-nt_0}\left(\frac{1+t_0}{1-t_0}\right)^{n/2}\sqrt{1+t_0}\; f_n(\tau_2,\xi_2) \\ = \frac{1}{2\pi i}\int_{\Gamma_{\tilde a_k-\tau_2>}} d\omega\frac{e^{-\omega^3/3+\xi_2\omega}}{\omega-\tilde a_k+\tau_2}, \end{aligned} \tag{3.47}$$

and similarly,

$$\lim_{n\to\infty} \psi_k^{(n)}(t_1,x_1)\left(\frac{n}{2e}\right)^{n/2} e^{nt_0}\left(\frac{1-t_0}{1+t_0}\right)^{n/2}\sqrt{1-t_0}\;\frac{1}{f_n(\tau_1,\xi_1)}$$
$$= \frac{1}{2\pi i}\int_{<\Gamma_{\tilde b_k-\tau_1}} d\tilde\omega \frac{e^{\tilde\omega^3/3-\xi_1\tilde\omega}}{\tilde\omega-\tilde b_k+\tau_1}. \quad (3.48)$$

Also, as before,

$$\lim_{n\to\infty}\frac{\mu^{-1}}{\sqrt{2}n^{1/6}}\left(\frac{2e}{n}\right)^n = -A^{-1}. \quad (3.49)$$

Then, with x_i, t_i as in (3.43), the limit (3.26) gets replaced by

$$\lim_{n\to\infty}\frac{\sqrt{1-t_0^2}}{\sqrt{2}n^{1/6}}K_{n,0}(t_1,x_1;t_2,x_2)\frac{f_n(\tau_2,\xi_2)}{f_n(\tau_1,\xi_1)} = K_{\mathcal{A}}(\tau_1,\xi_1;\tau_2,\xi_2),$$

with very little change in the steepest descent argument. So, putting all the pieces together, one checks:

$$\lim_{n\to\infty}\frac{\sqrt{1-t_0^2}}{\sqrt{2}n^{1/6}}K_{n,m}(t_1,x_1;t_2,x_2)\frac{f_n(\tau_2,\xi_2)}{f_n(\tau_1,\xi_1)} = K_{\mathcal{A}}(\tau_1,\xi_1;\tau_2,\xi_2)$$
$$-\frac{1}{(2\pi i)^2}\int_{\Gamma_{\tilde a-\tau_2>}} d\omega\int_{\Gamma_{<\tilde b-\tau_1}} d\tilde\omega\,\frac{e^{-\omega^3/3+\xi_2\omega}}{e^{-\tilde\omega^3/3+\xi_1\tilde\omega}}\sum_{i,j=1}^{m}\frac{[A^{-1}]_{i,j}}{(\tilde\omega-\tilde b_i+\tau_1)(\omega-\tilde a_j+\tau_2)}, \quad (3.50)$$

from which one proceeds in the same way as in the proof of Proposition 3.4 and Theorem 1.2. This ends the proof of Theorem 1.3.

4 Airy process with wanderers all leaving from point a and all going to point b.

In this section we prove Theorem 1.2 (and thus also Theorem 1.1) for the case where m wanderers all leave from one point and all are forced to one point; i.e.,

$$\tilde a := \tilde a_m = \ldots = \tilde a_1 < \tilde b_1 = \ldots = \tilde b_m =: \tilde b. \quad (4.1)$$

Thus, the m top Brownian bridges start from a and end at b with

$$a = \sqrt{2n}(1+\tilde a n^{-1/3}), \quad b = \sqrt{2n}(1-\tilde b n^{-1/3}). \quad (4.2)$$

The arguments presented in the previous sections break down. Therefore one should redo the proof, using an argument adapted to this case. It is instructive to shortly present two different approaches. The first follows the approach of the previous section, consisting in computing the inverse of the $m \times m$ matrix μ, and the second approach is to perform the bi-orthogonalization. In principle, with some care because of the $n \to \infty$ limit, one might also be able to do the argument by analytic continuation, since the measure is analytic in the $\tilde{a}_i$, $\tilde{b}_j$ as well as the final kernel (provided the inequality $\tilde{a}_i < \tilde{b}_j$ for all i, j is satisfied).

4.1 Via the inversion of the moment matrix

The start is almost the same as in the previous section. The only difference is that the first and last determinant in the measure, instead of (3.7) and (3.8), are now

$$\det \begin{pmatrix} \left((x_j^1)^{i-1} e^{a x_j^1/(1+T_1)} p(0, x_j^1, T_1+1)\right)_{\substack{1\leq i\leq m \\ 1\leq j\leq m+n}} \\ \left((x_j^1)^{i-1} p(0, x_j^1, T_1+1)\right)_{\substack{1\leq i\leq n \\ 1\leq j\leq m+n}} \end{pmatrix}, \tag{4.3}$$

and

$$\det \begin{pmatrix} \left((x_j^\ell)^{i-1} e^{b x_j^\ell/(1-T_\ell)} p(x_j^\ell, 0, 1-T_\ell)\right)_{\substack{1\leq i\leq m \\ 1\leq j\leq m+n}} \\ \left((x_j^\ell)^{i-1} p(x_j^\ell, 0, 1-T_\ell)\right)_{\substack{1\leq i\leq n \\ 1\leq j\leq m+n}} \end{pmatrix}. \tag{4.4}$$

respectively. The functions $\varphi_k^{(n)}$ and $\psi_k^{(n)}$, for $1 \leq k \leq m$, defined by

$$\begin{aligned} \varphi_k^{(n)}(t,x) &= \frac{e^{-x^2/2\sigma(t)^2}}{\sigma(t)} \frac{1}{2\pi i} \oint_{\Gamma_{0,a/2}} dz \frac{e^{-z^2\gamma(t)^2 + 2xz/\sigma^2(t)}}{z^n \left(z-a/2\right)^k}, \\ \psi_k^{(n)}(t,x) &= \frac{e^{-x^2/2\sigma(-t)^2}}{\sigma(-t)} \frac{1}{2\pi i} \oint_{\Gamma_{0,b/2}} dz \frac{e^{-z^2\gamma(-t)^2 + 2xz/\sigma^2(-t)}}{z^n \left(z-b/2\right)^k}, \end{aligned} \tag{4.5}$$

replace those of (3.11), where we recall that $\gamma(t) = \sqrt{\frac{1-t}{1+t}}$, $\sigma(t) = \sqrt{1+t}$. Of course, since the last n rows of the determinants (4.3) and (4.4) are exactly the same as in (3.7) and (3.8), we keep the same choice for the functions $\varphi_{m+k}^{(n)}$ and $\psi_{m+k}^{(n)}$, $1 \leq k \leq n$ as in (3.12)-(3.13). Define the matrix $m \times m$ matrix μ by $\mu_{i,j} = \langle \varphi_i^{(n)}, \psi_j^{(n)} \rangle$, $1 \leq i, j \leq m$. Once again, this choice of $\varphi_k^{(n)}$ and $\psi_k^{(n)}$ generates the same vector space as the function in the above determinants (4.3) and (4.4).

Note that in this section we use the same notations as in the previous section. However, the matrix μ and some of the functions are not the same.

What remains the same is the form of the kernel. Indeed, since Proposition 3.3 holds exactly as before, one has once again:

$$K_{n,m}(t_1,x_1;t_2,x_2) = K_{n,0}(t_1,x_1;t_2,x_2) + \sum_{i,j=1}^{m} \psi_i^{(n)}(t_1,x_1)[\mu^{-1}]_{i,j}\varphi_j^{(n)}(t_2,x_2), \tag{4.6}$$

where $K_{n,0}$ is the kernel without wanderers and in the scaling limit will converge to the extended Airy kernel. Thus, we only have to deal with the double sum below.

Lemma 4.1. *Under the scaling (4.2), we have*

$$\lim_{n\to\infty} \mu_{k,l}\left(\frac{n}{2e}\right)^n \left(n^{1/6}/\sqrt{2}\right)^{k+l-1} = \frac{1}{2}\frac{1}{(\tilde{b}-\tilde{a})^{l+k-1}}\binom{l+k-2}{k-1} \tag{4.7}$$

for $1 \le k,l \le m$.

Proof. For convenience, in the proof we compute $\mu_{k+1,l+1}$ to avoid -1's in the formulas. Since, as before, $\mu_{k\ell}$ is time-independent, we may set $t_1 = t_2 = 0$ in the computation; so, as in (2.24) and after integrating over the x variable, one finds

$$\mu_{k+1,l+1} = \frac{\sqrt{\pi}}{(2\pi i)^2}\oint_{\Gamma_{0,a/2}} dz\, \frac{1}{z^n(z-a/2)^{k+1}}\oint_{\Gamma_{0,b/2}} dw\, \frac{e^{2wz}}{w^n(w-b/2)^{l+1}}. \tag{4.8}$$

Then we apply twice the identity

$$\frac{1}{(z-a/2)^{k+1}} = \frac{2^k}{k!}\left(\frac{\partial}{\partial a}\right)^k \frac{1}{z-a/2} \tag{4.9}$$

and obtain

$$\mu_{k+1,l+1} = \frac{2^{k+l}}{k!l!}\left(\frac{\partial}{\partial a}\right)^k \left(\frac{\partial}{\partial a}\right)^l \mu_{1,1}. \tag{4.10}$$

But $\mu_{1,1}$ was already expressed as a single contour integral; see (2.12). Thus

$$\mu_{k+1,l+1} = \frac{2^{k+l}}{k!l!}\left(\frac{\partial}{\partial a}\right)^k \left(\frac{\partial}{\partial b}\right)^l \frac{\sqrt{\pi}\,2^n}{2\pi i}\oint_{\Gamma_{0,ab/2}} dz\, \frac{e^z}{z^n(z-ab/2)}. \tag{4.11}$$

We now compute the derivatives of $(z-ab/2)^{-1}$ and obtain

$$\frac{2^{k+l}}{k!l!}\left(\frac{\partial}{\partial a}\right)^k \left(\frac{\partial}{\partial a}\right)^l \frac{1}{z-\frac{ab}{2}} = a^l b^k \sum_{j=0}^{\min(k,l)} \frac{(ab/2)^{-j}}{(z-\frac{ab}{2})^{l+k-j+1}}\frac{(l+k-j)!}{(k-j)!(l-j)!j!}, \tag{4.12}$$

and so

$$\mu_{k+1,l+1} = a^l b^k \sum_{j=0}^{\min(k,l)} \frac{1}{(ab/2)^j} \frac{(l+k-j)!}{(k-j)!(l-j)!j!} \times \frac{\sqrt{\pi}\, 2^n}{2\pi i} \oint_{\Gamma_{0,ab/2}} dz\, \frac{e^z}{z^n (z-ab/2)^{l+k-j+1}}. \quad (4.13)$$

Finally, we need to do the asymptotic analysis of the integral, which essentially has already been made in Lemma 2.2. Consider the following small change in (2.33), with $a_k = a$, $b_\ell = b$: replace $(z-ab/2)^{-1}$ by $(z-ab/2)^{-p-1}$ for any *finite* $p = 1, 2, \ldots$. Then the steepest descent analysis is unchanged except for that finite power, which would be present in (2.37) too. This extra power gives a factor $n^{-p/6}$ and we also have an extra factor coming from the change of variables equal to $n^{-p/2}$. In the end, the result is

$$\frac{\sqrt{\pi}\, 2^n}{2\pi i} \oint_{\Gamma_{0,ab/2}} dz\, \frac{e^z}{z^n (z-ab/2)^{p+1}} = \left(\frac{2e}{n}\right)^n \frac{1}{(\tilde{b}-\tilde{a})^{p+1}} \frac{n^{-2p/3}}{\sqrt{2} n^{1/6}} (1+o(1)). \quad (4.14)$$

We put (4.14) into (4.13) and compare the dependence in j of the different terms in the sum. Since $ab/2 = n(1+\mathcal{O}(n^{-1/3}))$, the jth term in the sum (4.13) contains the following power of n, namely:

$$\frac{1}{n^j} n^{2j/3} = n^{-j/3}. \quad (4.15)$$

Therefore, since the sum is finite, in the $n \to \infty$ limit, the leading term is the one with $j = 0$, the other ones being of smaller order. Thus,

$$\begin{aligned} \mu_{k+1,l+1} &= \binom{k+l}{k} \left(\frac{2e}{n}\right)^n \frac{1}{(\tilde{b}-\tilde{a})^{k+l+1}} \frac{a^l b^k n^{-2(k+l)/3}}{\sqrt{2} n^{1/6}} (1+o(1)) \\ &= \frac{1}{2} \binom{k+l}{k} \left(\frac{2e}{n}\right)^n \left(\frac{\sqrt{2}}{n^{1/6}(\tilde{b}-\tilde{a})}\right)^{k+l+1} (1+o(1)), \end{aligned} \quad (4.16)$$

ending the proof of Lemma 4.1. □

Corollary 4.2. *With the same scaling as in Lemma 4.1, we have*

$$\lim_{n\to\infty} [\mu^{-1}]_{k,l} \left(\frac{2e}{n}\right)^n \left(\frac{\sqrt{2}}{n^{1/6}}\right)^{k+l-1} = 2(\tilde{b}-\tilde{a})^{l+k-1} [(L^{-1})^T L^{-1}]_{k,l} \quad (4.17)$$

for $1 \le k, l \le m$, *where the* $m \times m$ *lower-triangular matrix* L^{-1} *has binomial entries*

$$(L^{-1})_{k,l} = (-1)^{k-l} \binom{k-1}{l-1}. \quad (4.18)$$

Proof. From (4.7) it follows that computing the inverse of μ reduces to computing the inverse of the $m \times m$ matrix ν,

$$\nu = (\nu_{k,l})_{1 \le k,l \le m}, \text{ with } \nu_{k,l} = \binom{k+l-2}{k-1}. \tag{4.19}$$

A convenient way of taking the inverse is to compute the $\nu = LL^T$ decomposition, where L is lower-triangular, yielding

$$\nu = LL^T, \quad L_{k,l} = \binom{k-1}{l-1}, \tag{4.20}$$

from which

$$\nu^{-1} = (L^T)^{-1} L^{-1}, \quad (L^{-1})_{k,l} = (-1)^{k-l} \binom{k-1}{l-1}, \tag{4.21}$$

This establishes the asymptotics (4.17). $\square$

We now turn to the asymptotics of the functions $\varphi_k^{(n)}$ and $\psi_k^{(n)}$, defined in (4.5).

Lemma 4.3. *Under the scaling (4.2) and*

$$t_i = \tau_i n^{-1/3}, \quad x_i = \sqrt{2n} + \frac{\xi_i - \tau_i^2}{\sqrt{2}\, n^{1/6}}, \tag{4.22}$$

one has

$$\begin{aligned} \varphi_k(\tau_2, \xi_2) \; &:= \; \lim_{n\to\infty} \varphi_k^{(n)}(t_2, x_2) \left(\frac{n}{2e}\right)^{n/2} (n^{1/6}/\sqrt{2})^{k-1} \\ &= \; \frac{1}{2\pi i} \int_{\Gamma_{\tilde{a}-\tau_2 >}} d\omega\, e^{-\omega^3/3 + \xi_2 \omega} \frac{f(\tau_2, \xi_2)^{-1}}{(\omega - \tilde{a} + \tau_2)^k}, \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} \psi_\ell(\tau_1, \xi_1) \; &:= \; \lim_{n\to\infty} \psi_\ell^{(n)}(t_1, x_1) \left(\frac{n}{2e}\right)^{n/2} (n^{1/6}/\sqrt{2})^{\ell-1} \\ &= \; \frac{(-1)^{\ell-1}}{2\pi i} \int_{\Gamma_{<\tilde{b}-\tau_1}} d\widetilde{\omega}\, e^{\widetilde{\omega}^3/3 - \xi_1 \widetilde{\omega}} \frac{f(\tau_1, \xi_1)}{(\widetilde{\omega} - \tilde{b} + \tau_1)^\ell} \end{aligned} \tag{4.24}$$

Proof. We must compute the asymptotics of

$$\varphi_k^{(n)}(t, x) = \frac{e^{-x^2/2\sigma(t)^2}}{\sigma(t)} \frac{1}{2\pi i} \oint_{\Gamma_{0,a/2}} dz \frac{e^{-z^2\gamma(t)^2 + 2xz/\sigma^2(t)}}{z^n\,(z - a/2)^k} \tag{4.25}$$

and compare this expression with (3.11). One sees that the only differences are that now a replaces a_k and the denominator in $z - a/2$ has a power k instead of power 1. For any *finite* k, the asymptotic analysis for this case has only minor differences with respect to the asymptotic for (3.11). Namely, one picks up some extra factors by the changes of variables: setting $z = u\sqrt{n/2}$, one gets a factor $\sqrt{2/n}^{k-1}$ and then $\omega = n^{1/3}(u-1)$ results in a factor $n^{(k-1)/3}$. In total, an extra factor $(\sqrt{2}/n^{1/6})^{k-1}$ appears. A similar argument holds for $\psi_\ell^{(n)}$. $\square$

Proof of Theorems 1.1 and 1.2 for Brownian bridges starting from a and ending up at b. Putting together Corollary 4.2 and Lemma 4.3, one obtains

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\sqrt{2}\,n^{1/6}} \sum_{i,j=1}^{m} \psi_i^{(n)}(t_1,x_1)[\mu^{-1}]_{i,j}\varphi_j^{(n)}(t_2,x_2) \\
&= \sum_{i,j=1}^{m} \psi_i(\tau_1,\xi_1)(\tilde b-\tilde a)^{i+j-1}[(L^T)^{-1}L^{-1}]_{i,j}\varphi_j(\tau_2,\xi_2)\frac{f(\tau_1,\xi_1)}{f(\tau_2,\xi_2)} \\
&= \frac{1}{(2\pi i)^2}\int_{\Gamma_{\tilde a-\tau_2>}} d\omega \int_{\Gamma_{<\tilde b-\tau_1}} d\widetilde\omega\, \frac{e^{-\omega^3/3+\xi_2\omega}}{e^{-\widetilde\omega^3/3+\xi_1\widetilde\omega}}\frac{f(\tau_1,\xi_1)}{f(\tau_2,\xi_2)} \\
&\quad\times(\tilde a-\tilde b)\sum_{k=1}^{m}\sum_{i,j=1}^{m}\binom{k-1}{i-1}\binom{k-1}{j-1}\frac{(\tilde b-\tilde a)^{i+j-2}(-1)^j}{(\widetilde\omega+\tau_1-\tilde b)^i(\omega+\tau_2-\tilde a)^j}.
\end{aligned}\tag{4.26}$$

Finally, using the fraction decomposition identity

$$\begin{aligned}
&\frac{1}{V-U}\left(\left(\frac{U-a}{U-b}\right)^m\left(\frac{V-b}{V-a}\right)^m-1\right) \\
&=(a-b)\sum_{k=1}^{m}\sum_{i,j=1}^{m}\binom{k-1}{i-1}\binom{k-1}{j-1}\frac{(b-a)^{i+j-2}(-1)^j}{(U-b)^i(V-a)^j}
\end{aligned}\tag{4.27}$$

with $U = \widetilde\omega + \tau_1$ and $V = \omega + \tau_2$, one gets the final result

$$\begin{aligned}
(4.26) = &\frac{1}{(2\pi i)^2}\int_{\Gamma_{\tilde a-\tau_2>}} d\omega \int_{\Gamma_{<\tilde b-\tau_1}} d\widetilde\omega\, \frac{e^{-\omega^3/3+\xi_2\omega}}{e^{-\widetilde\omega^3/3+\xi_1\widetilde\omega}}\frac{1}{(\omega+\tau_2)-(\widetilde\omega+\tau_1)} \\
&\times\left[\left(\frac{(\widetilde\omega-\tilde a+\tau_1)(\omega-\tilde b+\tau_2)}{(\widetilde\omega-\tilde b+\tau_1)(\omega-\tilde a+\tau_2)}\right)^m-1\right]\frac{f(\tau_1,\xi_1)}{f(\tau_2,\xi_2)}.
\end{aligned}\tag{4.28}$$

4.2 Via bi-orthogonal functions

Here we present a slightly different approach, which consists in using biorthogonal functions, instead of the functions defined in (4.5). We use a representation with determinants known from classical orthogonal polynomial theory. Let us first define the polynomials $\tilde{\varphi}_i^{(n)}$, $\tilde{\psi}_j^{(n)}$, and then show they are actually biorthogonal.

Set

$$\mu_{i,j}(t) := \langle \varphi_i^{(n)}(0,\cdot), \psi_j^{(n)}(0,\cdot)\rangle \quad \text{and} \quad \Delta_k := \det(\mu_{i,j})_{1\le i,j\le k}. \tag{4.29}$$

and define

$$\tilde{\varphi}_k^{(n)}(t,x) := \frac{1}{\sqrt{\Delta_k \Delta_{k-1}}} \det \begin{pmatrix} \mu_{1,1} & \mu_{1,2} & \cdots & \mu_{1,k-1} & \varphi_1^{(n)}(t,x) \\ \mu_{2,1} & \mu_{2,2} & \cdots & \mu_{2,k-1} & \varphi_2^{(n)}(t,x) \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{k,1} & \mu_{k,2} & \cdots & \mu_{k,k-1} & \varphi_k^{(n)}(t,x) \end{pmatrix}, \tag{4.30}$$

and

$$\tilde{\psi}_l^{(n)}(t,x) := \frac{1}{\sqrt{\Delta_l \Delta_{l-1}}} \det \begin{pmatrix} \mu_{1,1} & \mu_{1,2} & \cdots & \mu_{1,l} \\ \mu_{2,1} & \mu_{2,2} & \cdots & \mu_{2,l} \\ \vdots & \vdots & & \vdots \\ \mu_{l-1,1} & \mu_{l-1,2} & \cdots & \mu_{l-1,l} \\ \psi_1^{(n)}(t,x) & \psi_2^{(n)}(t,x) & \cdots & \psi_l^{(n)}(t,x) \end{pmatrix}. \tag{4.31}$$

First of all, notice that $\tilde{\varphi}_k^{(n)}$ is linear combination of the $\varphi_\ell^{(n)}$ with $\ell = 1,\dots,k$, with a non-zero coefficient in front of $\varphi_k^{(n)}$ (because $\Delta_k \neq 0$, since both $\{\varphi_\ell^{(n)}, 1\le \ell \le k\}$ and $\{\psi_\ell^{(n)}, 1 \le \ell \le k\}$ form a basis of a k-dimensional vector space). The argument is similar for $\tilde{\psi}_l^{(n)}$. Observe, for $\ell < k$,

$$\langle \tilde{\varphi}_k^{(n)}(t,\cdot), \psi_\ell^{(n)}(t,\cdot)\rangle = \left\{ \begin{array}{l} \text{right hand side of (4.30) with} \\ \text{the last column replaced by} \\ \text{the } \ell\text{th column of (4.30)} \end{array} \right\} = 0. \tag{4.32}$$

Therefore also $\langle \tilde{\varphi}_k^{(n)}(t,\cdot), \tilde{\psi}_\ell^{(n)}(t,\cdot)\rangle = 0$ for $\ell < k$ and thus also for $\ell \neq k$, by merely interchanging the roles of $\tilde{\varphi}$ and $\tilde{\psi}$. The above argument also shows

$$\langle \tilde{\varphi}_k^{(n)}(t,\cdot), \tilde{\psi}_k^{(n)}(t,\cdot)\rangle = \langle \tilde{\varphi}_k^{(n)}(t,\cdot), \psi_k^{(n)}(t,\cdot)\rangle \frac{\Delta_{k-1}}{\sqrt{\Delta_k\Delta_{k-1}}} = \frac{\Delta_k \Delta_{k-1}}{\sqrt{\Delta_k \Delta_{k-1}}^2} = 1. \tag{4.33}$$

The consequence is that now the kernel instead of (4.6) reads

$$K_{n,m}(t_1,x_1;t_2,x_2) = K_{n,0}(t_1,x_1;t_2,x_2) + \sum_{i,j=1}^{m} \tilde{\psi}_i^{(n)}(t_1,x_1)[\tilde{\mu}^{-1}]_{i,j} \tilde{\varphi}_j^{(n)}(t_2,x_2) \tag{4.34}$$

with $\tilde{\mu}_{i,j} = \langle \tilde{\varphi}_i^{(n)}(t,\cdot), \tilde{\psi}_j^{(n)}(t,\cdot)\rangle = \delta_{i,j}$. Therefore, the double sum in (4.34) becomes just

$$\begin{aligned} K_m(t_1,x_1;t_2,x_2) &= \sum_{i,j=1}^{m} \tilde{\psi}_i^{(n)}(t_1,x_1)[\tilde{\mu}^{-1}]_{i,j}\tilde{\varphi}_j^{(n)}(t_2,x_2) \\ &= \sum_{i=1}^{m} \tilde{\psi}_i^{(n)}(t_1,x_1)\tilde{\varphi}_i^{(n)}(t_2,x_2). \end{aligned} \tag{4.35}$$

This last sum is of Darboux-type and can be rewritten as

$$\begin{aligned} &K_m(t_1,x_1;t_2,x_2) \\ &= -\frac{1}{\Delta_m}\det\begin{pmatrix} 0 & \psi_1^{(n)}(t_1,x_1) & \psi_2^{(n)}(t_1,x_1) & \cdots & \psi_m^{(n)}(t_1,x_1) \\ \varphi_1^{(n)}(t_2,x_2) & \mu_{1,1} & \mu_{1,2} & \cdots & \mu_{1,m} \\ \varphi_1^{(n)}(t_2,x_2) & \mu_{2,1} & \mu_{2,2} & \cdots & \mu_{2,m} \\ \vdots & \vdots & \vdots & & \vdots \\ \varphi_m^{(n)}(t_2,x_2) & \mu_{m,1} & \mu_{m,2} & \cdots & \mu_{m,m} \end{pmatrix}, \end{aligned} \tag{4.36}$$

the latter follows from the fact that $K_m(t_1,x_1;t_2,x_2)$ is a bilinear combination of $\psi_i^{(n)}(t_1,x_1)$ and $\varphi_j^{(n)}(t_2,x_2)$, for $1 \le i,j \le m$ and is completely characterized by $\langle K_m(t_1,x_1;t_2,.), \psi_i^{(n)}(t_2,.)\rangle = \psi_i^{(n)}(t_1,x_1)$ for $1 \le i \le m$.

At this point we have to determine the $n \to \infty$ limit of the rescaled kernel, namely

$$\lim_{n\to\infty} \frac{1}{\sqrt{2}\,n^{1/6}} K_m(t_1,x_1;t_2,x_2) \tag{4.37}$$

with x_i, t_i scaled as (4.22). The asymptotics of $\mu_{i,j}$ already appears in Lemma 4.1, and for $\varphi_j^{(n)}$ and $\psi_i^{(n)}$ in Lemma 4.3. Hence Lemma 4.1, together with the fact that m is finite, yields the asymptotics of Δ_m:

$$\begin{aligned} \lim_{n\to\infty} \Delta_m \left(\frac{n}{2e}\right)^{nm} \left(\frac{n^{1/6}}{\sqrt{2}}\right)^{m^2} &= \frac{1}{2^m}\frac{1}{(\tilde{b}-\tilde{a})^{m^2}} \det\binom{k+l-2}{k-1}_{1\le k \le m} \\ &= \frac{1}{2^m}\frac{1}{(\tilde{b}-\tilde{a})^{m^2}}. \end{aligned} \tag{4.38}$$

This result (4.38) and the linearity of the determinant, together with the results of Lemmas 4.1 and 4.3, substituted in (4.36) lead to the limit in

(4.37), namely

$$\frac{1}{(2\pi i)^2}\int_{\Gamma_{\tilde a-\tau_2>}} d\omega \int_{\Gamma_{<\tilde b-\tau_1}} d\widetilde\omega\, \frac{e^{-\omega^3/3+\xi_2\omega}}{e^{-\widetilde\omega^3/3+\xi_1\widetilde\omega}} \frac{-(\tilde b-\tilde a)}{(\omega+\tau_2-\tilde a)(\widetilde\omega+\tau_1-\tilde b)}$$
$$\times\det\begin{pmatrix} 0 & 1 & \cdots & \left(\frac{\tilde b-\tilde a}{\omega+\tau_2-\tilde a}\right)^{m-1} \\ 1 & & & \\ \vdots & & \binom{k+l-2}{k-1}_{1\le k,l\le m} & \\ \left(\frac{-(\tilde b-\tilde a)}{\widetilde\omega+\tau_1-\tilde b}\right)^{m-1} & & & \end{pmatrix} \frac{f(\tau_1,\xi_1)}{f(\tau_2,\xi_2)}. \tag{4.39}$$

The final step is to use the following identity, established by observing that both sides are identical upon integrating against x^{i-1}, $1\le i\le m$ from $x=0$ to $x=1$,

$$-\det\begin{pmatrix} 0 & 1 & \cdots & (x+1)^{m-1} \\ 1 & & & \\ \vdots & & \binom{k+l-2}{k-1}_{1\le k,l\le m} & \\ (y+1)^{m-1} & & & \end{pmatrix} = \frac{(xy)^m-1}{xy-1} \tag{4.40}$$

with $y+1=(\tilde a-\tilde b)/(\widetilde\omega+\tau_1-\tilde b)$ and $x+1=(\tilde b-\tilde a)/(\omega+\tau_2-\tilde a)$. Thus, one obtains for (4.37):

$$\begin{aligned}\lim_{n\to\infty}&\frac{1}{\sqrt{2}\,n^{1/6}}K_m(t_1,x_1;t_2,x_2)\\ &=\frac{f(\tau_1,\xi_1)}{f(\tau_2,\xi_2)}\frac{1}{(2\pi i)^2}\int_{\Gamma_{\tilde a-\tau_2>}} d\omega\int_{\Gamma_{<\tilde b-\tau_1}} d\widetilde\omega\,\frac{e^{-\omega^3/3+\xi_2\omega}}{e^{-\widetilde\omega^3/3+\xi_1\widetilde\omega}}\\ &\qquad\times\frac{(\tilde b-\tilde a)}{(\omega+\tau_2-\tilde a)(\widetilde\omega+\tau_1-\tilde b)}\frac{\left(\frac{\widetilde\omega+\tau_1-\tilde a}{\widetilde\omega+\tau_1-\tilde b}\frac{\omega+\tau_2-\tilde b}{\omega+\tau_2-\tilde a}\right)^m-1}{\frac{\widetilde\omega+\tau_1-\tilde a}{\widetilde\omega+\tau_1-\tilde b}\frac{\omega+\tau_2-\tilde b}{\omega+\tau_2-\tilde a}-1},\end{aligned} \tag{4.41}$$

which is equal to the kernel (4.28) obtained previously.

Remark: Using (4.30), (4.31), a 1-border identity analogous to the 2-border identity (4.40), Lemmas 4.1 and 4.3 and (4.38), one finds the limiting biorthogonal functions (which also yield (4.41)):

$$\begin{aligned}\lim_{n\to\infty}2^{1/4}n^{1/12}\tilde\varphi_j^n(t,x) &= \sqrt{\tilde b-\tilde a}\frac{(-1)^{j-1}}{2\pi i}\int_{\Gamma_{\tilde a-\tau>}} d\omega e^{-\frac{\omega^3}{3}+\xi\omega}\frac{(\omega+\tau-\tilde b_j)^{j-1}}{(\omega+\tau-\tilde a_j)^j}\\ \lim_{n\to\infty}2^{1/4}n^{1/12}\tilde\psi_j^n(t,x) &= \sqrt{\tilde b-\tilde a}\frac{(-1)^{j-1}}{2\pi i}\int_{\Gamma_{\tilde b-\tau>}} d\omega e^{\frac{\omega^3}{3}-\xi\omega}\frac{(\omega+\tau-\tilde a)^{j-1}}{(\omega+\tau-\tilde b)^j}.\end{aligned}$$

5 Limit to the Pearcey process

In this section we prove Theorem 1.5. To do this, we must apply the scaling (1.11) to the kernel with m wanderers, where all the $\tilde a_i = \tilde a$ and $\tilde b_i = \tilde b$ and where one uses the shift $\omega \to \omega - \tau_2$ and $\widetilde\omega \to \widetilde\omega - \tau_1$, to yield:

$$\begin{aligned}
K_m^{\tilde a,\tilde b}(\tau_1,\xi_1;\tau_2,\xi_2) &= -\frac{\mathbb{1}(\tau_2>\tau_1)}{\sqrt{4\pi(\tau_2-\tau_1)}} e^{-\frac{(\xi_2-\xi_1)^2}{4(\tau_2-\tau_1)}-\frac{1}{2}(\tau_2-\tau_1)(\xi_2+\xi_1)+\frac{1}{12}(\tau_2-\tau_1)^3} \\
&+\frac{1}{(2\pi i)^2}\int_{\Gamma_{\tilde a >}} d\omega \int_{\Gamma_{<\tilde b}} d\widetilde\omega \, \frac{e^{-(\omega-\tau_2)^3/3+\xi_2(\omega-\tau_2)}}{e^{-(\widetilde\omega-\tau_1)^3/3+\xi_1(\widetilde\omega-\tau_1)}} \frac{1}{\omega-\widetilde\omega}\left(\frac{\omega-\tilde b}{\omega-\tilde a}\right)^m \left(\frac{\widetilde\omega-\tilde a}{\widetilde\omega-\tilde b}\right)^m .
\end{aligned} \tag{5.1}$$

According to Theorem 1.5, the scaling limit we need to take is the following:

$$\begin{aligned}
&\tilde a = \alpha m^{1/3}, \quad \tilde b = \beta m^{1/3}, \\
&\tau_i = T_\pm m^{1/3} + \frac{1}{2}\kappa^2\theta_i m^{-1/6}, \\
&\xi_i = X m^{2/3} - \kappa^2\sigma_\pm\theta_i m^{1/6} - \kappa v_i m^{-1/12}.
\end{aligned} \tag{5.2}$$

Then, we have to compute the large m limit of the rescaled kernel (5.1). We prove the following result, which implies Theorem 1.5.

Proposition 5.1. *Under the above scaling, for any fixed θ_1,θ_2, the limit*

$$\lim_{m\to\infty} \kappa\, m^{-1/12} K_m^{\tilde a,\tilde b}(\tau_1,\xi_1;\tau_2,\xi_2) \equiv K^{\mathcal{P}}(\theta_1,v_1;\theta_2,v_2) \tag{5.3}$$

holds uniformly for v_1,v_2 in a bounded set.

Proof. The first term in (5.1) is a straightforward limit. Indeed, for $\theta_2>\theta_1$,

$$\begin{aligned}
&-\frac{\kappa\, m^{-1/12}}{\sqrt{4\pi(\tau_2-\tau_1)}} e^{-\frac{(\xi_2-\xi_1)^2}{4(\tau_2-\tau_1)}-\frac{1}{2}(\tau_2-\tau_1)(\xi_2+\xi_1)+\frac{1}{12}(\tau_2-\tau_1)^3} \\
&\qquad\qquad = \frac{-1}{\sqrt{2\pi(\theta_2-\theta_1)}} e^{-\frac{(v_2-v_1)^2}{2(\theta_2-\theta_1)}} \frac{Q(1)}{Q(2)}
\end{aligned} \tag{5.4}$$

where the conjugation terms $Q(i)$ are given by

$$Q(i) = \exp\left(\frac{1}{2}\kappa^2(\sigma^2+X)\theta_i m^{1/2} + \kappa\sigma v_i m^{1/4} + \mathcal{O}(m^{-1/4})\right). \tag{5.5}$$

Next, one deals with the double integral, where it is natural to introduce the change of integration variables:

$$\omega = w m^{1/3}, \quad \widetilde\omega = \widetilde w m^{1/3}, \tag{5.6}$$

leading to

$$\frac{\kappa m^{1/4}}{(2\pi i)^2}\int_{\Gamma_{\alpha>}} dw \int_{\Gamma_{<\beta}} d\widetilde{w}\, \frac{1}{w-\widetilde{w}} \frac{e^{mF_0(w)+m^{1/2}F_2(w,\theta_2)+m^{1/4}F_3(w,v_2)+F_4(w,\theta_2)+\mathcal{O}(m^{-1/4})}}{e^{mF_0(\widetilde{w})+m^{1/2}F_2(\widetilde{w},\theta_1)+m^{1/4}F_3(\widetilde{w},v_1)+F_4(\widetilde{w},\theta_1)+\mathcal{O}(m^{-1/4})}}, \tag{5.7}$$

where the functions F_i are given by

$$\begin{aligned} F_0(w) &= -\frac{1}{3}(w-T)^3 + X(w-T) + \ln(w-\beta) - \ln(w-\alpha), \\ F_2(w,\theta) &= \frac{1}{2}((w-T)^2 - X)\kappa^2\theta - (w-T)\kappa^2\sigma\theta, \\ F_3(w,v) &= -(w-T)\kappa v, \\ F_4(w,\theta) &= -\frac{1}{4}(w-T-\sigma)\kappa^4\theta^2. \end{aligned} \tag{5.8}$$

Setting

$$w' := w - T, \quad \alpha' := \alpha - T, \quad \beta' = \beta - T, \tag{5.9}$$

one defines

$$\tilde{F}_0(w') := F_0(w'+T) = -\frac{w'^3}{3} + Xw' + \log(w'-\alpha') - \log(w'-\beta'). \tag{5.10}$$

One now imposes the condition that $\tilde{F}'(w')$ experiences a triple zero at some critical point w'_c; this happens when the following polynomial $P(w')$ is identically zero, with $w'_c \neq w'_0$:

$$\begin{aligned} 0 \equiv P(w') &:= -(w'-\alpha')(w'-\beta')\tilde{F}'_0(w') - (w'-w'_c)^3(w'-w'_1) \\ &= (3w'_c + w'_1 - \alpha' - \beta')w'^3 + (\alpha'\beta' - X - 3{w'_c}^2 - 3w'_c w'_1)w'^2 \\ &\quad + ({w'_c}^3 + 3{w'_c}^2 w'_1 + X(\alpha'+\beta'))w' - (\alpha'\beta' X - \alpha' + \beta' + {w'_c}^3 w'_1). \end{aligned} \tag{5.11}$$

Setting the coefficients of this cubic in w' equal to 0 amounts to 4 equations in 5 unknowns $\alpha',\ \beta',\ w'_c,\ w'_1,\ X$, thus yielding an algebraic curve. At a first stage, let us look at it *purely algebraically*; later we will have to take into account the real character of the parameters, including various inequalities. Close inspection of the four equations suggests the following birational map

$$\alpha' = \frac{2\sigma}{2-x} - r, \quad \beta' = \frac{2\sigma}{2-x} + r, \quad w'_1 = \frac{\sigma(3x-2)}{2-x}, \tag{5.12}$$

with inverse (assuming $\alpha' + \beta' \neq 0$)

$$r = \frac{1}{2}(\beta' - \alpha'), \quad x = \frac{2(2w'_1 + \alpha' + \beta')}{3(\alpha'+\beta')}, \quad \sigma = -\frac{1}{3}(w'_1 - \alpha' - \beta'). \tag{5.13}$$

Substituting this map in $P(w')$, one now solves the 4 equations (5.11) defined by $P(w')$ inductively, beginning with the highest degree in w'. At first, one checks $P(w') = 3(w'_c - \sigma)w'^3 + ...$, leading to $w'_c = \sigma$, together with the value of w'_1, which we already knew from (5.12); thus

$$w'_c = \sigma \quad \text{and} \quad w'_1 = \frac{\sigma(3x-2)}{2-x}. \tag{5.14}$$

Substituting this back into $P(w')$ yields a quadratic polynomial in w'; the vanishing of the coefficient of w'^2 yields

$$X = \frac{2\sigma^2(3x^2-6x+2)}{(x-2)^2} - r^2; \tag{5.15}$$

$P(w')$ becomes thus linear, with vanishing linear and constant terms, yielding

$$r^2 = x^2\sigma^2\frac{(2x-3)}{(x-2)^2} \qquad \text{and} \qquad r = \frac{2x^3\sigma^4}{2-x}. \tag{5.16}$$

The compatibility between the r and r^2 equations (5.16) yields a curve relating x and σ,

$$\mathcal{E}: \quad \sigma^6 = \frac{2x-3}{4x^4}. \tag{5.17}$$

Incidentally, this curve is elliptic; indeed, viewed as a 6-fold cover of the x-plane, the total ramification index equals 12, with a ramification of index 5 above $x = 3/2$, there are two ramification points of index 2 above $x = 0$ and three simple branch points above $x = \infty$; thus the genus $= 1$. Then substituting the value (5.16) of r into (5.12) and (5.15), yields the following expressions for α', β' and X, all defined on the algebraic curve (5.17):

$$\alpha' = \frac{2\sigma}{2-x}(1-x^3\sigma^3)\ , \quad \beta' = \frac{2\sigma}{2-x}(1+x^3\sigma^3), \quad \text{and} \quad X = \sigma^2(1-2x). \tag{5.18}$$

Using these expressions, together with the value of the critical point $w'_c = \sigma$, one checks from (5.10) that

$$\frac{1}{4!}\tilde{F}_0^{iv}(w'_c) = \frac{(x-1)}{2x^2\sigma}, \tag{5.19}$$

and thus

$$\tilde{F}_0(w') = \tilde{F}_0(w'_c) + \frac{(x-1)}{2x^2\sigma}(w'-w'_c)^4 + \mathcal{O}\left((w'-w'_c)^5\right). \tag{5.20}$$

One then *requires the parameters* $\alpha', \beta', X, w'_c, w'_1$ *to be real with* $\alpha' < \beta'$, $w'_c \neq w'_1$, *and* $\alpha' + \beta' \neq 0$. This implies that x, σ and r must be real; the curve relation (5.17) yields two real solutions for σ, namely $\sigma_+ < 0$ and $\sigma_- = -\sigma_+ > 0$. In particular from (5.17) one must have $x > 3/2$, and since $\alpha' < \beta'$, one must have, according to (5.13), that $2r = \beta' - \alpha' > 0$ yielding from (5.16) the inequality $x < 2$. Thus one has $2 > x > 3/2$. Moreover (5.19) will be < 0 for $\sigma = \sigma_+$ and > 0 for $\sigma = \sigma_-$; one then sets $\mp\frac{\kappa^4}{4} := \frac{(x-1)}{2x^2\sigma_\pm}$, with $\kappa > 0$. Since $\frac{2x-3}{4x^4}$, the right hand side of (5.17), is an increasing function of $3/2 < x < 2$, this function has its maximum at $x = 2$, for which $\sigma_\pm = \mp 1/2$, according to the curve relation (5.17). That $\alpha' + \beta' \neq 0$ follows from adding the two first equations in (5.12). Also one has $w_c' = \sigma_+ > \beta'$ and $w_c' = \sigma_- < \alpha'$, since from (5.18) and the curve (5.17), one computes for $\beta' - \sigma_+$ and similarly for $\alpha' - \sigma_-$:

$$\beta' - \sigma_+ = \frac{\sigma_+ x}{2-x}(1 + 2x^2\sigma_+^3) = \frac{\sigma_+ x}{2-x}(1 - \sqrt{2x-3}) < 0 \text{ and } \alpha' - \sigma_- > 0.$$

Finally, from (5.14), it is clear that $w_1' \neq w_c'$, since $\frac{\sigma(3x-2)}{2-x} \neq \sigma$ in the admissible range $x \in (\frac{3}{2}, 2)$, with $w_1' \to w_c'$ for $x \to \frac{3}{2}$.

To summarize, using the change of variables (5.9), the relations (5.18) imply, for a given $\alpha < \beta$, two values $T_\pm$ of T below, and thus

$$\begin{gathered} \alpha = T_\pm + \frac{2\sigma_\pm}{2-x}(1 - x^3\sigma_\pm^3), \quad \beta = T_\pm + \frac{2\sigma_\pm}{2-x}(1 + x^3\sigma_\pm^3), \\ X = \sigma_\pm^2(1-2x), \quad T_\pm = \frac{\alpha+\beta}{2} - \frac{2\sigma_\pm}{2-x}, \end{gathered} \tag{5.21}$$

from which (1.14) in the statement of Theorem 1.5 follows, with inequalities

$$3/2 < x < 2, \qquad 0 < |\sigma_\pm| < \frac{1}{2}, \qquad X < 0, \qquad T_- < \frac{\alpha+\beta}{2} < T_+. \tag{5.22}$$

Also the critical point w_c of $F_0(w)$ and the extra-root w_1 of $F_0'(w)$ occur at

$$w_c := \sigma_\pm + T_\pm, \text{ with } \begin{array}{c} \alpha < \beta < w_c \\ w_c < \alpha < \beta \end{array} \text{ and } w_1 = T_\pm + \frac{\sigma_\pm(3x-2)}{2-x} \neq w_c. \tag{5.23}$$

The statement about the *uniqueness of the solution* (x, σ) to the equations (5.21) and (5.17), given arbitrary $\alpha < \beta$, remains to be shown. Indeed, upon using the identities (5.21) obtained for α and β, together with the curve equation (5.17), it is easy to see that the right hand side of the equation,

$$0 < \beta - \alpha = \frac{4\sigma_\pm^4 x^3}{2-x} = \frac{4x^{1/3}\,(2x-3)^{2/3}}{2-x}, \tag{5.24}$$

is a monotonically increasing function in the range $3/2 < x < 2$; therefore the right hand side of that equation takes on every value in $(0, \infty)$ exactly once and thus given arbitrary $\alpha < \beta$, there is a unique value $x \in (3/2, 2)$ satisfying the first equation in (5.24). Substituting this value of x in the $T_\pm$-equation of (5.21), the value of $T_\pm$ is specified unambiguously and thus $T_\pm$ can take on any value in $\mathbb{R}$. Therefore, only when $\alpha,\ \beta \to 0$, do

$$x \to 3/2,\ \sigma_\pm \to 0 \text{ and } T_\pm \to 0, \text{ and thus, by (5.23)},\ w_c, w_1 \to 0, \tag{5.25}$$

which proves the remark at the end of Theorem 1.5.

Then, the series expansions about the critical point $\sigma_\pm$ give

$$\begin{aligned} F_0(w) &= F_0(w_c) \mp \frac{1}{4}\kappa^4(w-w_c)^4 + \mathcal{O}((w-w_c)^5), \\ F_2(w,\theta) &= F_2(w_c,\theta) + \frac{1}{2}\kappa^2\theta(w-w_c)^2, \\ F_3(w,v) &= F_3(w_c,v) - \kappa v(w-w_c), \\ F_4(w,\theta) &= \mathcal{O}(w-w_c). \end{aligned} \tag{5.26}$$

We now apply the steepest descent method, which we spell out for the opening cusp; i.e., for T_+ and $\sigma = \sigma_+ < 0$. By Cauchy's Residue theorem, one can deform the paths as indicated in Figure 9. The contribution of the last contour is zero. Indeed, the integration over w is trivial, since the only pole is simple at $w = \widetilde{w}$. All the factors involving α and β cancel exactly. Thus, we remain with a contour in $\widetilde{w}$ around β of an analytic function (no pole at β anymore) which is zero. The deformation also involves contributions which vanish at infinity.

The final and most important step is to deal with the previous last contours of Figure 9. First a remark on the integration paths in (5.7). For large w and $\widetilde{w}$, the leading term is the cubic in F_0, which means that without any error, we can let the the directions of the path w go to infinity in the cones of angles in $(\pi/2, 5\pi/6)$ and $(-5\pi/6, -\pi/2)$ instead of $2\pi/3$ and $-2\pi/3$. Similarly for $\widetilde{w}$ we can let it go to infinity in the cones with angles in $(\pi/6, \pi/2)$ and $(-\pi/2, -\pi/6)$ instead of $\pi/3$ and $-\pi/3$. Finally, the small contour around β can be also deformed to go to infinity as soon as it does in directions in $(5\pi/6, 7\pi/6)$. Therefore without errors we can deform the contours to become as in Figure 10. Let us verify that these paths satisfy the steepest descent property. This will be done for the case $\sigma = \sigma_+$; the case $\sigma = \sigma_-$ is essentially the same.

Slope of the function $F_0(w)$ starting from $w_c = \sigma_+$. Consider the curve given by

$$w = w_c + \zeta\frac{-\sigma x}{2-x}e^{\pm i(\pi/2+\delta)},\ \text{for } 0 < \delta < \pi/3 \text{ and } \zeta \geq 0. \tag{5.27}$$

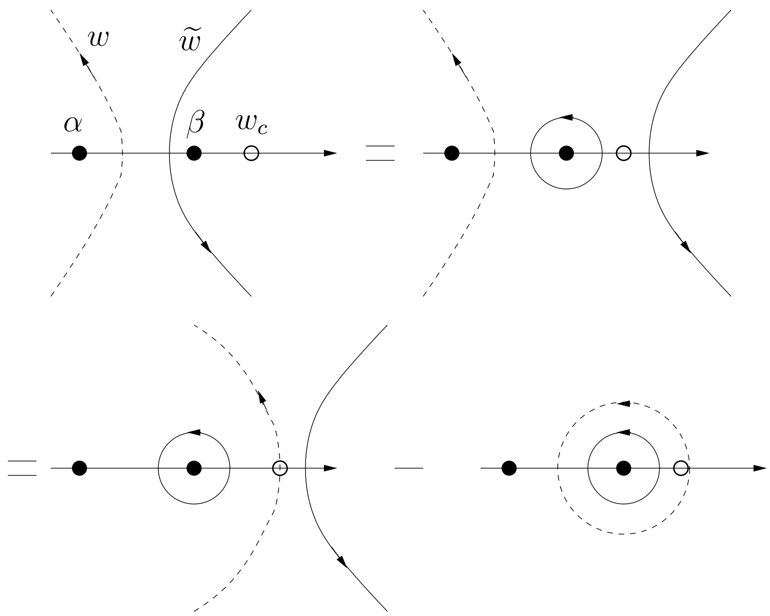

Figure 9: First deformation of the paths which then pass close to the critical point $w_c = \sigma_+ + T$. The solid contours are for $\widetilde{w}$, while the dashed ones for w. (For the case of σ_-, one has $w_c < \alpha$ and the figures is essentially reflected.) The circles in the first and second figures on the right hand side become dashed and in the second figure it sits on the right hand side of the full curve. In the third figure, the inner circle is dashed and the outer is full. In effect, the roles of w and w' are interchanged.

Remember that $\sigma < 0$. Then, at first one verifies

$$\frac{\partial}{\partial \zeta}\mathrm{Re} F_0(w) = \frac{x^2\sigma^3\zeta^3 P_3(\zeta;x,\delta)}{(2-x)^3 P_2(\zeta;x,-\sigma,\delta)P_2(\zeta;x,\sigma,\delta)} < 0 \text{ for } \zeta > 0, \qquad (5.28)$$

with

$$\begin{aligned} P_2(\zeta;x,-\sigma) &= \zeta^2 - 2\zeta(1+2x^2\sigma^3)\sin\delta + (1+2x^2\sigma^3)^2 \\ P_3(\zeta;x,\delta) &= \delta(3x+\mathcal{O}(\delta^2))\zeta^3 + (2(2-x)+\mathcal{O}(\delta^2))\zeta^2 \\ &\quad + \delta((10x^2+4x-24)+\mathcal{O}(\delta^2))\zeta + (8(x-1)(2-x)+\mathcal{O}(\delta^2)). \end{aligned} \qquad (5.29)$$

Indeed, $P_2(u;x,\pm\sigma) > 0$, since its discriminant, as a quadric in u, is < 0. Moreover $P_3(u;x,\delta) > 0$, for $0 < \delta < \varepsilon(x)$ with $\varepsilon(x)$ sufficiently small, since in that case the coefficients of $u^0, ..., u^3$ are positive (since $x \in (3/2, 2)$) and $\zeta \geq 0$. Thus the chosen path for w is a path of steepest descent.

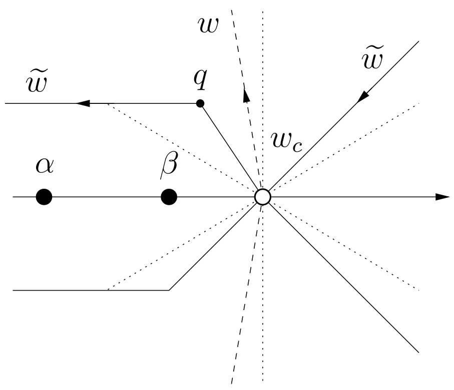

Figure 10: Second deformation of the previous last contours in Figure 9 for $\sigma = \sigma_+$. At the critical point w_c, the path of $\widetilde{w}$ has angles $\pm\pi/4, \pm 3\pi/4$, while the path of w leaves it with any angle in $(\pi/2, 3\pi/4)$. The value of q will be chosen during the analysis. For $\sigma = \sigma_-$, one picks the mirror image of the figure above about the vertical line through w_c, with w and $\tilde{w}$ also flipped; i.e., the dashed and solid lines are interchanged.

Slope of the function $-F_0(\widetilde{w})$ from $w_c = \sigma_+$ to q and w_c to $e^{i\pi/4}\infty$. Consider the curves parameterized by

$$\widetilde{w} = w_c + \frac{-\sigma x}{2-x}(\varepsilon \pm i)\zeta, \text{ for } \varepsilon = \pm 1 \text{ and } \zeta \geq 0. \tag{5.30}$$

One verifies

$$\frac{\partial}{\partial \zeta}\mathrm{Re}(-F_0(\widetilde{w})) = \frac{2x^3\sigma^3\zeta^3 P_3^\varepsilon(\zeta;x)}{(2-x)^3 P_2^\varepsilon(\zeta;x,-\sigma)P_2^\varepsilon(\zeta;x,\sigma)}, \tag{5.31}$$

where, using the curve relation (5.17),

$$P_3^\varepsilon(\zeta;x) = \varepsilon\zeta^3 + 2\zeta^2 + \varepsilon\zeta\frac{-x^2+6x-4}{x} + \frac{4}{x}(x-1)(2-x) \tag{5.32}$$

and

$$\begin{aligned} |\widetilde{w}-\alpha|^2 &= 2\left(\frac{x\sigma}{x-2}\right)^2 P_2^\varepsilon(\zeta;x,-\sigma) > 0, \\ |\widetilde{w}-\beta|^2 &= 2\left(\frac{x\sigma}{x-2}\right)^2 P_2^\varepsilon(\zeta;x,\sigma) > 0 \end{aligned} \tag{5.33}$$

with

$$P_2^\varepsilon(\zeta;x,-\sigma) := \zeta^2 + \varepsilon\zeta(1-2x^2\sigma^3) - 2x^2\sigma^3 + x - 1, \tag{5.34}$$

showing at once the denominator of (5.31) is > 0.

For $\varepsilon = 1$, the polynomial P_3^ε in the numerator of (5.31) is > 0 for $\zeta > 0$, because its coefficients are all > 0 in the range $2 > x > 3/2$ and for $\zeta > 0$. Therefore the derivative (5.31) is < 0 for $\zeta > 0$.

For $\varepsilon = -1$, the polynomial $P_3^\varepsilon(\zeta; x)$ will be < 0 for large enough $\zeta > 0$. However in the range $2 > x > 3/2$,

$$-x^2 + 6x - 4 > 0 \quad \text{and} \quad 0 < \frac{4(x-1)(2-x)}{-x^2+6x-4} < \frac{4}{11}, \tag{5.35}$$

and thus for $0 < \zeta \leq \zeta_0$ with

$$\zeta_0 := \frac{4(x-1)(2-x)}{-x^2+6x-4}, \tag{5.36}$$

the cubic above is strictly positive:

$$P_3^\varepsilon(\zeta; x)\Big|_{\varepsilon=-1} = \zeta^2(2-\zeta) - \frac{1}{x}(-x^2+6x-4)\left(\zeta - \frac{4(x-1)(2-x)}{-x^2+6x-4}\right) > 0. \tag{5.37}$$

This is the reason why for $\varepsilon = -1$ we bend the path at q to be horizontal, with q set to be equal to

$$q := w_c + \zeta_0 \frac{-\sigma x}{2-x}(\pm i - 1). \tag{5.38}$$

Slope of the function $-F_0(\widetilde{w})$ from q to $q - \infty$. Consider the horizontal line given by

$$\widetilde{w} = q - \zeta \frac{-\sigma x}{2-x}, \quad \zeta \geq 0. \tag{5.39}$$

Then, in the range $-\sigma > 0$ and $x \in (3/2, 2)$,

$$\frac{\partial}{\partial \zeta}\mathrm{Re}(-F(\widetilde{w})) = \frac{\sigma x}{2-x}\left[\frac{-\sigma x}{(2-x)^2(x^2-6x+4)}\right]^2 \frac{x\sigma^2 P_6(\zeta; x)}{|\widetilde{w}-\alpha|^2|\widetilde{w}-\beta|^2} < 0 \tag{5.40}$$

with

$$P_6(\zeta; x) = (2-x)^2 Q_2(\zeta; x) + (2-x)\zeta^3 P_7(x) + \zeta^4 \tilde{Q}_2(\zeta; x). \tag{5.41}$$

Indeed, $Q_2(\zeta; x)$ and $\tilde{Q}_2(\zeta; x)$ are quadratic polynomials in ζ, with coefficients polynomial in x and $P_7(x)$ is a seventh degree polynomial in x. All three coefficients of $Q_2(\zeta; x)$ are > 0 for $3/2 \leq x \leq 2$, while $P_7(x) > 0$ also as long as $2 \geq x \geq 3/2$. The coefficients of ζ^0 and ζ^2 of $\tilde{Q}_2(\zeta; x)$ are > 0 for $2 \geq x \geq 1.70$, which moreover has a positive minimum in ζ for $2 \geq x \geq 1.70$, thus proving the assertion for $2 > x > 1.7$. A little numerics in fact shows we can remove the restriction $2 > x > 1.7$ and deduce the inequality for $2 > x > 3/2$.

Thus we have also shown that the chosen path for $\widetilde{w}$ is of steep descent. Thus the steep descent method can be applied along these curves where the maximum of $\mathrm{Re}F_0(w)$, $-\mathrm{Re}F_0(\widetilde{w})$ occur at the saddle point w_c. The main contribution comes from the integration over a δ-neighborhood of the critical point w_c for both w and $\widetilde{w}$. For small δ, the error made is of order $e^{-\mu m}$ with $\mu \sim \delta^4$. Let us therefore choose $\delta = \kappa^{-1}m^{-1/4}m^{\gamma}$ with any $\gamma \in (0, 1/20)$ fixed (i.e., $m^{-1/5} \gg \delta \gg m^{-1/4}$). Then, the only non-vanishing contribution in the $m \to \infty$ limit is given by the integrations with $|w - w_c| \leq \delta$, $|\widetilde{w} - w_c| \leq \delta$. In these small neighborhoods, we can apply series expansions (5.26). After the change of variables

$$z := \kappa m^{1/4}(w - w_c), \quad \tilde{z} := \kappa m^{1/4}(\widetilde{w} - w_c) \tag{5.42}$$

we finally get for $\sigma = \sigma_{\pm}$,

$$\frac{Q(2)}{Q(1)}(5.7) = \begin{cases} \dfrac{1}{(2\pi i)^2} \displaystyle\int dz \int d\tilde{z} \frac{1}{z - \tilde{z}} \frac{e^{-z^4/4+\theta_2 z^2/2 - v_2 z + R_2}}{e^{-\tilde{z}^4/4+\theta_1 \tilde{z}^2/2 - v_1 \tilde{z} + R_1}} \\[2ex] \dfrac{1}{(2\pi i)^2} \displaystyle\int dz \int d\tilde{z} \frac{1}{z - \tilde{z}} \frac{e^{z^4/4+\theta_2 z^2/2 - v_2 z + R_2}}{e^{\tilde{z}^4/4+\theta_1 \tilde{z}^2/2 - v_1 \tilde{z} + R_1}} \end{cases} \tag{5.43}$$

where $Q(i) = \exp\big(F_2(w_c, \theta_i)m^{1/2} + F_3(w_c, v_i)m^{1/4} + \mathcal{O}(m^{-1/4})\big)$ is the conjugation given in (5.5), and where the R_i are error terms, to be discussed later. Note the involution $\theta_1 \leftrightarrow -\theta_2, v_1 \leftrightarrow v_2,\ z \leftrightarrow -\tilde{z}$ between the two integrals on the right hand side of (5.43), which also respect the integration paths. The error terms R_i include the following local contributions:

(a) $\mathcal{O}(m^{-1/4})$ of (5.7) (uniform for v_i in a bounded set),

(b) $\mathcal{O}(\delta) = \mathcal{O}(m^{-1/5})$ from $F_4(w)$ in (5.26) (uniform for θ_i in a bounded set),

(c) $\mathcal{O}(m\delta^5) = \mathcal{O}(m^{5(\gamma-1/20)})$, which is the corrections in the series expansions of $F_0(w)$ of order higher than 4, see (5.26).

Indeed, (b) is immediate since F_4 is linear (see (5.8)). To see that (c) holds, we need to control the fifth derivative of F_0 at w_c. We have

$$\max_{|w-w_c|\leq\delta} \left| F_0^{(v)}(w) \right| = 4 \max_{|w-w_c|\leq\delta} \left| \frac{1}{(w-\alpha)^5} - \frac{1}{(w-\beta)^5} \right| \leq \frac{16}{(w_c-\beta)^5} \tag{5.44}$$

for m large enough.

Finally, taking the $m \to \infty$ limit to (5.43) the error terms vanishes and at the same time the integrals extend to infinity. The fact that z is not exactly $i\mathbb{R}$ is irrelevant, since the result is identical as soon as the direction has an angle strictly smaller than $\pi/4$ to the imaginary axis. Similarly one can deform the $\tilde{z}$-path as depicted in Figure 4. This ends the proof of Proposition 5.1 and thus also of Theorem 1.5. $\square$

Remark 5.2. For future use, we point out that the elliptic curve $\mathcal{E}$ has three points above $x = \infty$, only one of which is real, namely

$$\sigma = \frac{2^{-1/6}}{x^{1/2}} + \dots \quad \text{for} \quad x \to \infty. \tag{5.45}$$

At this point at infinity, one has, using the estimate (5.45), $\beta - \alpha = \lim_{x\to\infty} \frac{4\sigma^4 x^3}{2-x} = -2^{4/3}$, and assuming $T = \frac{\alpha+\beta}{2} - \frac{4\sigma}{2-x} = 0$, also $\alpha + \beta = 0$. This implies that $\beta = -\alpha = -2^{1/3}$. Note how this contrasts with $(\sigma, x) = (0, 3/2)$, $(X, T) = (0, 0)$, in which case $\alpha = \beta = 0$. To summarize, near the real points on $\mathcal{E}$, namely near $x = 3/2$ and $x = \infty$, one has the following leading terms (set $\gamma := \frac{2^{1/2}}{3^{2/3}}$):

$$\begin{aligned}
&(\sigma, x) \sim \left(\gamma\Big(x - \frac{3}{2}\Big)^{\frac{1}{6}}, \frac{3}{2}\right),\ (\alpha,\beta) \sim 0,\ (X,T) \sim 0,\ \frac{w_c - w_1}{-4\gamma} \sim \Big(x - \frac{3}{2}\Big)^{\frac{1}{6}} \\
&(\sigma, x) \sim \left(\frac{2^{-\frac{1}{6}}}{\sqrt{x}}, \infty\right),\ (\alpha,\beta) \sim (2^{\frac{1}{3}}, -2^{\frac{1}{3}}),\ (X,T) \sim (-2^{\frac{2}{3}}, 0),\ \frac{w_c - w_1}{2^{-1/6}} \sim \frac{4}{\sqrt{x}}.
\end{aligned} \tag{5.46}$$

6 Limit to the quintic kernel

In this section we present a conjecture concerning the process that will occur in the situation illustrated in Figure 5. In Theorem 1.5 and, in particular, in formula (5.25), it was observed that when $\alpha, \beta \to 0$ (and only then), the tips of the cusps $(\tau, \xi) \sim (\pm T m^{1/3}, X m^{1/3})$ tend to the same point and that $w_c - w_1 = \frac{4\sigma \pm (1-x)}{2-x} \to 0$, i.e., the cube root of $F_0'(w)$ turns into a quartic root. This also means that the starting and end points a and b for the wanderers tend to coincide and that the line connecting both points becomes vertical and tangent to the ellipse, as described in Figure 1. This corresponds to the first situation in (5.46). We now pick the second situation in (5.46), for which the cube root of $F_0'(w)$ also turns into a quartic root. However, this forces the points a and b to be a bit beyond $\sqrt{2n}$; this means in particular that $\tilde{a} > \tilde{b}$, which actually violates the condition $\tilde{a} < \tilde{b}$ in Theorem 1.1. One can think of the passage from $x = 3/2$ to $x = \infty$ as a *transition process.*

The most natural strategy would be to set $a = b = \sqrt{2n} + \sqrt{2m}$ and take $m, n \to \infty$ together. Then, under an appropriate scaling limit, we expect to get a process with a *quintic kernel.* Of course, there will be a parameter tuning regulating how close the two Airy fields come together. For example, if $m = n$, then we have to choose $a = 2\sqrt{2n} + \mathcal{O}(n^{-1/6})$ since the fluctuations of the first n Brownian bridges alone live on the $n^{-1/6}$ scale.

Evidence in favor of Conjecture 1.7. To give some evidence to this conjecture, we present two pieces of rigorous mathematics, concerning the (one-time) kernel, with $\tilde{a} < \tilde{b}$, with time τ absorbed into $\tilde{a}$,

$$K_m^{\tilde{a},\tilde{b}}(\xi_1;\xi_2) = \frac{1}{(2\pi i)^2} \int_{\Gamma_{\tilde{a}>}} d\omega \int_{\Gamma_{<\tilde{b}}} d\widetilde{\omega} \, \frac{e^{-\omega^3/3+\xi_2\omega}}{e^{-\widetilde{\omega}^3/3+\xi_1\widetilde{\omega}}} \frac{\left(\frac{\widetilde{\omega}-\tilde{a}}{\omega-\tilde{a}}\right)^m \left(\frac{\omega-\tilde{b}}{\widetilde{\omega}-\tilde{b}}\right)^m}{\omega - \widetilde{\omega}}. \tag{6.1}$$

Proposition 6.1. *The kernel $K_m^{\tilde{a},\tilde{b}}$, as in (6.1), can be continued analytically to a new kernel $\widetilde{K}_m^{\tilde{a},\tilde{b}}$, as in (6.2), with same integrand as kernel (6.1), by moving $\tilde{a}$ and $\tilde{b}$ in the complex plane from their original position $\tilde{a} < \tilde{b}$ to a new position $\tilde{b} < \tilde{a}$ on the real line:*

$$\widetilde{K}_m^{\tilde{a},\tilde{b}}(\xi_1;\xi_2) = \frac{1}{(2\pi i)^2} \int_{\Omega} d\omega \int_{\widetilde{\Omega}} d\widetilde{\omega} \, \frac{e^{-\omega^3/3+\xi_2\omega}}{e^{-\widetilde{\omega}^3/3+\xi_1\widetilde{\omega}}} \frac{\left(\frac{\widetilde{\omega}-\tilde{a}}{\omega-\tilde{a}}\right)^m \left(\frac{\omega-\tilde{b}}{\widetilde{\omega}-\tilde{b}}\right)^m}{\omega - \widetilde{\omega}}. \tag{6.2}$$

integrated over contours Ω and $\widetilde{\Omega}$ as in Figure 11.

Proof. $\tilde{b}$ corresponds to the black dot and $\tilde{a}$ to the white dot in Figures 11 and 12; the dashed line refers to the ω-integration and the solid line to the $\tilde{\omega}$-integration.

We noticed in Remark 5.2 that the elliptic curve $\mathcal{E}$, introduced in (5.17), contains another real point, namely one covering $x = \infty$ for which $(\alpha, \beta) = (2^{1/3}, -2^{-1/3})$. This clearly violates the inequality $\tilde{a} = \alpha m^{1/3} < \tilde{b} = \beta m^{1/3}$, crucial for the derivation of the kernel (6.1).

• Keeping $\tilde{\omega}$ fixed but arbitrary on the solid line, one sees that the dashed line of Figure 12 (a) can be deformed into the dashed lines of Figure 12 ((b) + (c)). Then one notices that the (c)-contribution vanishes. Indeed, (i) if $\tilde{\omega}$ belongs to the solid line, outside the dashed circle, the ω-integral vanishes, the integrand being holomorphic; (ii) if $\tilde{\omega}$ belongs to the solid line, inside the dashed circle, one picks up a residue and thus the ω-integral equals $\frac{1}{2\pi i} e^{(\xi_2-\xi_1)\tilde{\omega}}$; further integrated with regard to $\widetilde{\omega}$, one obtains

$$\frac{1}{2\pi i} \int_{\text{solid circle of (c)}} d\tilde{\omega} \; e^{(\xi_2-\xi_1)\tilde{\omega}} = 0,$$

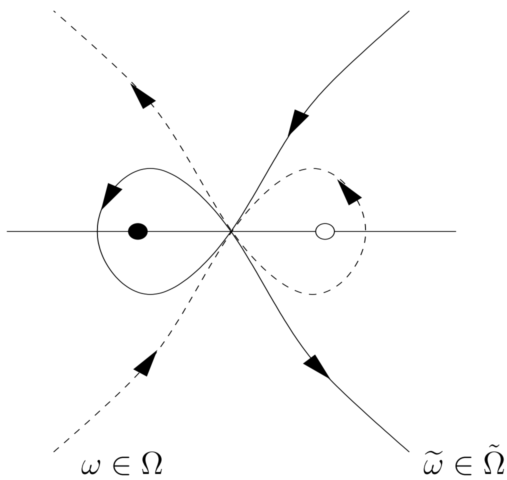

Figure 11: New contour Ω and $\tilde{\Omega}$, with the *black dot* $= \tilde{b}$ and the *white dot* $= \tilde{a}$. The solid line $\widetilde{\Omega}$ refers to the integration of the $\widetilde{\omega}$-variable, while the dashed line Ω refers to the ω-integration.

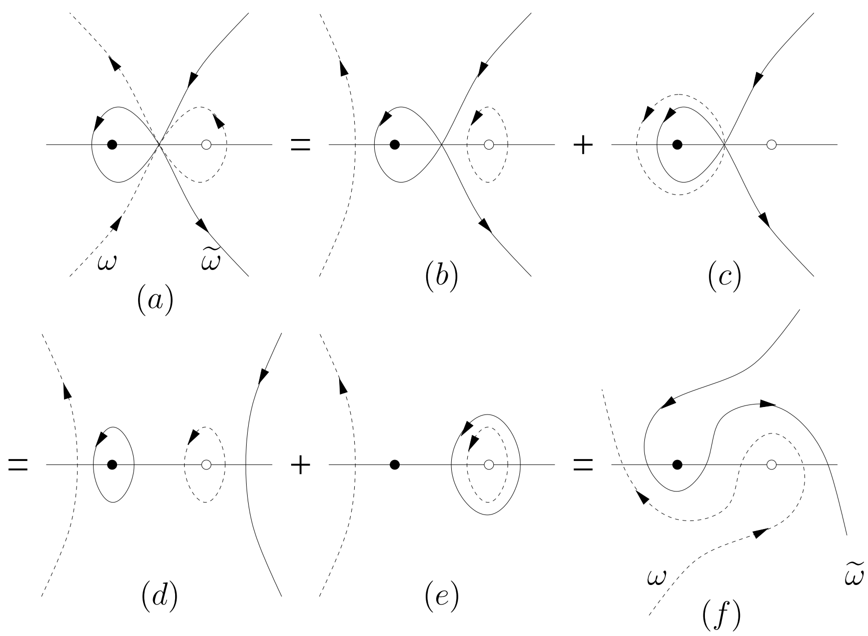

Figure 12: Representation of the deformation of the integration variables for the case $\tilde{b} < \tilde{a}$. All the contours are clockwise oriented, the black dot is $\tilde{b}$, the white dot is $\tilde{a}$. The solid line refers to the integration of the $\widetilde{\omega}$-variable, while the dashed line for ω-integration. The contributions of (c) and (e) are exactly zero.

and thus the only contribution comes from (b).

• At the next stage, picking an arbitrary $\omega \in$ dashed contour (b), one deforms the solid contour (b) into the solid contours (d) + (e). In the same way, if $\omega \notin$ dashed circle, the $\tilde{\omega}$-integration contributes nothing, the integrand being holomorphic; if $\omega \in$ dashed circle, the $\tilde{\omega}$-integration contributes $\frac{1}{2\pi i}e^{(\xi_2-\xi_1)\omega}$; further integrated with regard to ω, one obtains

$$-\frac{1}{2\pi i}\int_{\text{dashed circle of (e)}} d\omega\ e^{(\xi_2-\xi_1)\omega} = 0,$$

and thus the integration over the (d)-contour is the only contribution. Finally, the solid and dashed contours of (d) can further be deformed into contours (f), thus leading to the contours of Figure 2, as the black dot $\tilde{b}$ migrates to the right of the white dot $\tilde{a}$ through the $\mathbb{C}$-plane; this ends the proof of Proposition 6.1. $\square$

Proposition 6.2. *Consider the kernel* $\widetilde{K}_m^{\tilde{a},\tilde{b}}(\xi_1;\xi_2)$, *as in (6.2) with* $\tilde{a} > \tilde{b}$. *Then, defining the scaling*

$$\begin{aligned}
\tilde{a} &= (2m)^{1/3}\left(1+\tfrac{1}{6}\theta m^{-2/5}+\tfrac{1}{2}\eta m^{-3/5}\right)\\
\tilde{b} &= (2m)^{1/3}\left(-1-\tfrac{1}{6}\theta m^{-2/5}+\tfrac{1}{2}\eta m^{-3/5}\right)\\
\xi_i &= -(2m)^{2/3}\left(1-\tfrac{1}{6}\theta m^{-2/5}-\tfrac{1}{2}\left(v_i-\tfrac{1}{18}\theta^2\right)m^{-4/5}\right),
\end{aligned} \tag{6.3}$$

one obtains, in the $m \to \infty$ *limit, the quintic kernel* $K^{\mathcal{Q}}(\xi_1,\xi_2)$,

$$\begin{aligned}
\lim_{m\to\infty} & \frac{2^{-1/3}m^{-2/15}}{(2\pi i)^2}\int_{\Omega} d\omega\int_{\widetilde{\Omega}} d\widetilde{\omega}\ \frac{e^{-\omega^3/3+\xi_2\omega}}{e^{-\widetilde{\omega}^3/3+\xi_1\widetilde{\omega}}}\ \frac{1}{\omega-\widetilde{\omega}}\left(\frac{\omega-\tilde{b}}{\omega-\tilde{a}}\right)^m\left(\frac{\widetilde{\omega}-\tilde{a}}{\widetilde{\omega}-\tilde{b}}\right)^m\\
&= \frac{1}{(2\pi i)^2}\int_{\mathcal{C}} dz\int_{\widetilde{\mathcal{C}}} d\tilde{z}\frac{1}{z-\tilde{z}}\frac{e^{\frac{2}{5}z^5-\frac{1}{3}\theta z^3-\eta z^2+v_2 z}}{e^{\frac{2}{5}\tilde{z}^5-\frac{1}{3}\theta\tilde{z}^3-\eta\tilde{z}^2+v_1\tilde{z}}} =: K^{\mathcal{Q}}(\theta,\eta;v_1,v_2),
\end{aligned} \tag{6.4}$$

where $\mathcal{C}$ *and* $\widetilde{\mathcal{C}}$ *are the paths defined in Figure 6. The limit is uniform for* θ,η,v_1,v_2 *in a bounded set.*

Proof. We shall give the proof in the case of $\eta = 0$; the case $\eta \neq 0$ is easy to implement. As in the case of the Pearcey process (see Theorem 1.5), consider the scaling $\xi_i = Xm^{2/3}$, $\tilde{a} = \alpha m^{1/3}$, $\tilde{b} = \beta m^{1/3}$ and the change of integration variables $\omega = wm^{1/3}$, $\widetilde{\omega} = \widetilde{w}m^{1/3}$. Then, the kernel (6.2) becomes

$$(6.2) = \frac{m^{1/3}}{(2\pi i)^2}\int_{\Omega} dw\int_{\widetilde{\Omega}} d\widetilde{w}\,\frac{e^{mF(w)-mF(\widetilde{w})}}{w-\widetilde{w}} \tag{6.5}$$

with

$$F(w) := -w^3/3 + Xw + \ln(w-\beta) - \ln(w-\alpha). \tag{6.6}$$

where Ω and $\widetilde{\Omega}$ are the contours of Figure 11, with the black dot being β and the white dot α. Here, one imposes the property that $F'(w)$ experiences a 4-fold zero at some point w_c, with α, β, X real and $\alpha \neq w_c$, $\beta \neq w_c$; i.e. one requires all $A_i = 0$:

$$\begin{aligned}-(w-\alpha)(w-\beta)F'(w) - (w-w_c)^4 &=: A_0 w^3 + A_1 w^2 + A_2 w + A_3 \\ &= (4w_c - \alpha - \beta)w^3 + (\alpha\beta - 6w_c^2 - X)w^2 \\ &\quad + (4w_c^3 + X(\alpha+\beta))w - X\alpha\beta + \alpha - \beta - w_c^4.\end{aligned} \tag{6.7}$$

The coefficients $A_0 = A_1 = 0$ imply $w_c = \frac{1}{4}(\alpha+\beta)$ and $X = \alpha\beta - 6w_c^2$ and consequently $A_2 = -\frac{1}{16}(\alpha+\beta)(5\alpha^2 - 6\alpha\beta + 5\beta^2) = 0$, whose only real solution is given by $\alpha = -\beta$ and thus $w_c = 0$ and $X = \alpha\beta = -\alpha^2$. For these values, one has $A_3 = -\alpha(\alpha^3 - 2) = 0$, implying $\alpha = -\beta = 2^{1/3}$, $X = -2^{2/3}$ and $w_c = 0$. To summarize

$$\beta = -2^{1/3} < w_c = 0 < \alpha = 2^{1/3} \quad \text{and} \quad X = -2^{2/3}.$$

Note this solution corresponds precisely to the real point on the elliptic curve $\mathcal{E}$, covering $x = \infty$, as obtained on the second line of (5.46) (see Remark 5.2). Since we have a quintic leading term $\sim mw^5$, we make the change of variables $w = m^{-1/5} z\alpha$ and $\widetilde{w} = m^{-1/5}\tilde{z}\alpha$. The precise coefficients are chosen in order to simplify the final formula. Indeed, with (6.3) we obtain

$$mF(w) = mF(0) + v_2 z - \theta z^3/3 + 2z^5/5 + \mathcal{O}(z^7 m^{-2/5}, zm^{-2/5}), \tag{6.8}$$

with the error uniform for θ, v_i in a bounded set. The prefactor in (6.5), after the changes of variables, becomes $m^{1/3} m^{-1/5} 2^{1/3}(1 + \mathcal{O}(m^{-2/5}))$, which cancels with the $2^{-1/3} m^{-2/15}$ in front of the l.h.s. of (6.4) (as $m \to \infty$). Except for the error terms, the result of the theorem would follow.

What remains to be seen is that the higher order expansions in the series do not contribute. We do it by the steepest descent method as for the Pearcey case. Consider the curve parametrized by $w = e^{\pm 3\pi/5} x$. Then, for the function F, as in (6.6), , with α, β and X substituted,

$$F(w) = -\frac{1}{3}w^3 - 2^{2/3} w + \ln\left(w + 2^{1/3}\right) - \ln\left(w - 2^{1/3}\right) \tag{6.9}$$

one checks

$$\frac{\partial}{\partial x}\mathrm{Re}F(w) = -2\frac{x^4\,(x^2\cos(\pi/5) + 1)}{x^4 + 2x^2\cos(\pi/5) + 1} < 0 \quad \text{for all } x > 0, \tag{6.10}$$

One then checks that along the dotted loop in Figure 11, $\mathrm{Re}F(w) - \mathrm{Re}F(0) < 0$ for $w \neq 0$. It is at once visible by superimposing the dashed contour of

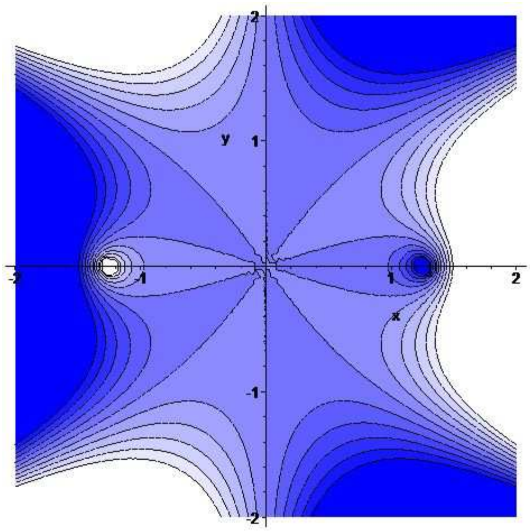

Figure 13: Contourplot of the function $\mathrm{Re}(F(x+iy)-F(0))$. The value is high in dark regions and low in light regions.

Figure 11 onto the contour plot, as in Figure 13. Then along the curve given by $\widetilde{w}=e^{\pm 2\pi/5}x$,

$$\frac{\partial}{\partial x}(-\mathrm{Re}F(\widetilde{w})) = -2\frac{x^4\,(x^2\cos(\pi/5)+1)}{x^4+2x^2\cos(\pi/5)+1} < 0 \quad \text{for all } x>0, \tag{6.11}$$

and along the solid loop in Figure 11, $-\mathrm{Re}F(\tilde{w})+\mathrm{Re}F(0)<0$ for $\tilde{w}\neq 0$. This shows that the curves have the steepest descent property. Thus, if we integrate (in w) around a δ-neighborhood of the origin, the error term will be only of order $\mathcal{O}(e^{-\mu m})$ with $\mu\sim\delta^5$. We choose $\delta=m^{-1/5}m^{\gamma}$ for any $\gamma\in(0,2/35)$. Then, uniformly for θ, v_1, v_2 in a bounded set, the error term $\mathcal{O}(z^7 m^{-2/5})=\mathcal{O}(m^{7\gamma-2/5})\to 0$ as $m\to\infty$, as $z<\delta m^{1/5}/\alpha<m^{\gamma}/\alpha$. In the limit, the only part of the contour in Figure 11, which contributes in the end are the 8 rays emanating from the origin, which one deforms so as to form consecutive angles $\pi/5$. This then yields the quintic kernel $K^{\mathcal{Q}}(\theta,\eta;v_1,v_2)$ with the integration paths $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ of Figure 6, thus establishing Proposition 6.2. $\square$

Remark 6.3. It is an open problem to know whether the Fredholm determinant of the quintic kernel defines a probability.

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