Universality of the Pearcey process

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Contents

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Abstract

Consider non-intersecting Brownian motions on the line leaving from the origin and forced to two arbitrary points. Letting the number of Brownian particles tend to infinity, and upon rescaling, there is a point of bifurcation, where the support of the density of particles goes from one interval to two intervals. In this paper, we show that at that very point of bifurcation a cusp appears, near which the Brownian paths fluctuate like the Pearcey process. This is a universality result within this class of problems. Tracy and Widom obtained such a result in the symmetric case, when the two target points are symmetric with regard to the origin. This asymmetry enabled us to improve considerably a result concerning the non-linear partial differential equations governing the transition probabilities for the Pearcey process, obtained by Adler and van Moerbeke.

1 Introduction

Consider the probability that n non-intersecting (Dyson) Brownian motions

$$x_1(t) < \ldots < x_n(t)$$

in $\mathbb{R}$ belong to a set $E \in \mathbb{R}$, with all particles leaving from the origin at time $t = 0$ and all forced to end up at $b_1 < b_2 < \ldots < b_p$ at time $t = 1$:

$$\mathbb{P}_n^{(b_1,\dots,b_p)}\left(\text{all } x_j(t)\in E \text{ for } 1\le j\le n \;\middle|\; \begin{array}{c}\text{all } x_j(0)=0\\ n_1 \text{ paths end up at } b_1 \text{ at } t=1\\ \vdots\\ n_p \text{ paths end up at } b_p \text{ at } t=1\end{array}\right)$$

with $\sum_{i=1}^p n_i = n$ and with (local) transition probability

$$p(t;x,y) := \frac{1}{\sqrt{\pi t}} e^{-\frac{(x-y)^2}{t}}. \tag{1.1}$$

A formula by Karlin-McGregor enables one to express this probability as an integral of a product of two determinants involving the transition probability (1.1) above. This further leads to a expression as (i) a GUE-matrix integral with an external potential, (ii) a determinant of a block moment matrix, with p blocks and (iii) a Fredholm determinant of a kernel. Finally it is also the solution of a PDE in the end-points of the interval E and the target points $b_1,\dots,b_p$.

Throughout this paper, we shall be dealing with the case of two target points $p=2$. In this paper, we show that, when $n\to\infty$ and when one looks through a microscope near a certain point of bifurcation, the non-intersecting Brownian motions tend to a new process, the *Pearcey process*, whatever be the location of the target points and whatever be the proportion of particles forced to those points. Tracy and Widom [28] showed this result in the symmetric case; namely when the target points are symmetric with respect to the origin and half of the particles go to either target point. Brézin and Hikami [9, 10, 11, 12] first considered this kernel and Bleher-Kuijlaars [8] obtained strong asymptotics using Riemann-Hilbert techniques.

The *Pearcey process* $\mathcal{P}(t)$ describes a cloud of Brownian particles, evolving in time according to a (matrix) Fredholm determinant,

$$\mathbb{P}^{\mathcal{P}}\left(\text{all } \mathcal{P}(t_j)\in E_j^c, 1\le j\le m\right) = \det\left(I - \left(\chi_{E_i} K^{\mathcal{P}}_{t_i t_j} \chi_{E_j}\right)_{1\le i,j\le m}\right)$$

of the Pearcey kernel

$$\begin{aligned} K^{\mathcal{P}}_{s,t}(x,y) &= -\frac{1}{4\pi^2}\int_X dV \int_{-i\infty}^{i\infty} dU e^{-\frac{U^4}{4}+\frac{tU^2}{2}-Uy} e^{\frac{V^4}{4}-\frac{sV^2}{2}+Vx}\frac{1}{U-V}\\ &\quad -\frac{\mathbb{I}(s<t)}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}} \end{aligned}$$

$$(1.2)$$

The contour X is given by the ingoing rays from $\pm\infty e^{i\pi/4}$ to 0 and the outgoing rays from 0 to $\pm\infty e^{-i\pi/4}$, i.e., X stands for the contour, all rays making an angle of $\pi/4$ with the horizontal axis.

$$\begin{array}{ccc} \nwarrow & & \swarrow \\ & 0 & \\ \nearrow & & \searrow \end{array}$$

For $s = t$, the Pearcey kernel can also be written

$$K^{\mathcal{P}}_{t,t}(x,y) = \frac{p(x)q''(y) - p'(x)q'(y) + p''(x)q(y) - tp(x)q(y)}{x-y} \tag{1.3}$$

with

$$q(y) := \frac{i}{2\pi}\int_{-i\infty}^{i\infty} e^{-\frac{U^4}{4}+\frac{tU^2}{2}-Uy}dU, \qquad p(x) := \frac{1}{2\pi i}\int_X e^{\frac{V^4}{4}-\frac{tV^2}{2}+Vx}dV,$$

satisfying both, the differential equations (using integration by parts)

$$p'''(x) - tp'(x) + xp(x) = 0, \qquad q'''(y) - tq'(y) - yq(y) = 0, \tag{1.4}$$

and the heat equations

$$\frac{\partial p}{\partial t} = -\frac{1}{2}p''(x), \quad \frac{\partial q}{\partial t} = \frac{1}{2}q''(y), \tag{1.5}$$

whereas $K^{\mathcal{P}}_{t,t}$ satisfies the following equation

$$\frac{\partial K^{\mathcal{P}}_{t,t}}{\partial t} = \frac{1}{2}(-p'(x)q(y) + p(x)q'(y)). \tag{1.6}$$

The latter follows from taking $\partial/\partial t$ of the kernel (1.2), which has for effect to multiply the exponentials under the integral (1.2) with $\frac{1}{2}(U^2-V^2)/(U-V) = \frac{1}{2}(U+V)$.

Consider n non-intersecting Brownian motions, with $0 < p < 1$ and $b < a$:

$$\mathbb{P}_n^{(b,a)}\left(\bigcap_{1\le i\le m}\{\text{all } x_j\,(t_i) \text{ for } 1 \le j \le n\} \;\middle|\; \begin{array}{c}\text{all } x_j(0) = 0 \\ pn \text{ paths end up at } a \text{ at } t = 1 \\ \\ (1-p)n \text{ paths end up at } b \text{ at } t = 1\end{array}\right)$$

When $n \to \infty$, the mean density of Brownian particles has its support on one interval for $t \sim 0$ and on two intervals for $t \sim 1$, so that a bifurcation appears for some intermediate time t_0, where one interval splits into two intervals. At this point the boundary of the support of the mean density has a cusp. We show that near this cusp, the same Pearcey process appears, independently of the values of a, b and p, showing "*universality*" of the Pearcey process; see Figure 1. As it turns out, it is convenient to introduce the parametrization

$$p = \frac{1}{1+q^3} \text{ with } 0 < q < \infty \text{ and let } r := \sqrt{q^2-q+1}. \tag{1.7}$$

Theorem 1.1 *For $n \to \infty$, the cloud of Brownian particles lie within a region, having a cusp at location $(x_0\sqrt{n}, t_0)$, with*

$$x_0 = \frac{(2a-b)q+(2b-a)}{q+1} t_0, \qquad \frac{1}{t_0} = 1 + 2\left(\frac{r(a-b)}{q+1}\right)^2. \tag{1.8}$$

Moreover, the following probability tends to the probability for the Pearcey process:

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}_n^{(b\sqrt{n}, a\sqrt{n})} \left(\bigcap_{1\le i\le m} \left\{ all\ x_j \left(t_0 + \left(\frac{c_0\mu}{n^{1/4}}\right)^2 2\tau_i \right) \in x_0 n^{1/2} + c_0 A \tau_i + \frac{c_0\mu}{n^{1/4}} E^c \right\} \right) \\ = \mathbb{P}^{\mathcal{P}} \left(\bigcap_{1\le i\le m} \{\mathcal{P}(\tau_i) \cap E = \emptyset\} \right), \end{aligned} \tag{1.9}$$

using the following constants

$$\mu = \left(\frac{q^2-q+1}{q}\right)^{1/4} > 0, \quad , \quad c_0 := \sqrt{\frac{t_0(1-t_0)}{2}} = t_0 \frac{r(a-b)}{q+1} > 0.$$

$$A = \frac{q^{1/2}(a-x_0) + q^{-1/2}(b-x_0)}{(a-b)} \tag{1.10}$$

In [4], Adler and van Moerbeke showed that the Pearcey transition probability (1.11) satisfies a non-linear PDE, expressible as a Wronskian of the expression (1.12) with some partial. This was obtained from taking a scaling limit, when $n \to \infty$, of the symmetric situation, i.e., where $b = -a$

and $p = 1/2$. It came as a *surprise* to us that considering the asymmetric case leads to a different non-linear PDE, when $n \to \infty$, but nevertheless also expressible as a Wronskian of the same expression (1.12) with some other partial. A separate functional-theoretical argument then enables one to show that the expresssion (1.12) itself vanishes. This was one of the motivations for finding the exact scaling as presented in Theorem 1.1.

To $E = \cup_{i=1}^{r}(y_{2i-1}, y_{2i}) \subset \mathbb{R}$ one associates two operators, a divergence and an Euler operator

$$\partial_{_E} = \sum_{1}^{2r} \frac{\partial}{\partial y_i}, \qquad \varepsilon_{_E} = \sum_{1}^{2r} y_i \frac{\partial}{\partial y_i}.$$

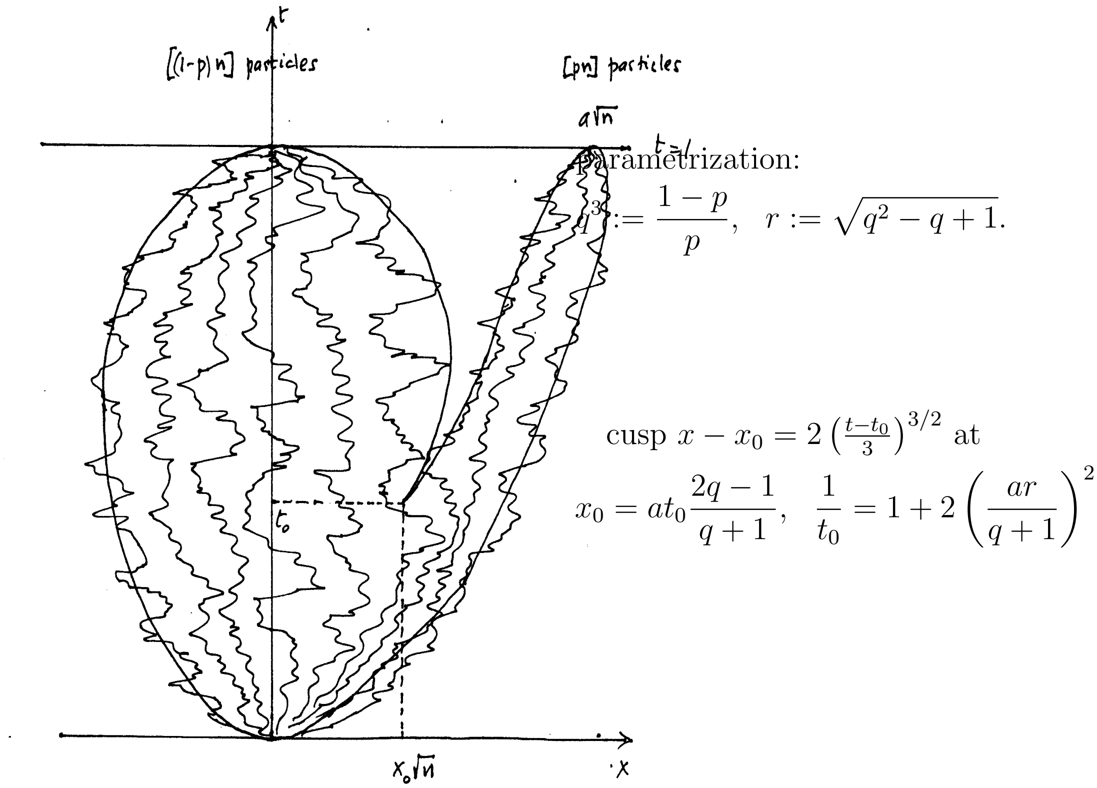


Figure 1: The Pearcey process for $b = 0$.

Theorem 1.2 *The log of the transition probability for the Pearcey process, which is non-stationary,*

$$\mathbb{Q}(t, E) := \log \mathbb{P}^{\mathcal{P}}\left(\mathcal{P}(t) \cap E = \emptyset\right) \tag{1.11}$$

satisfies the following 3rd order non-linear PDE in t and the boundary points of E,

$$\frac{\partial^3 \mathbb{Q}}{\partial t^3} + \frac{1}{8}\left(\varepsilon_E - 2t\frac{\partial}{\partial t} - 2\right)\partial_{_E}^2 \mathbb{Q} - \frac{1}{2}\left\{\partial_{_E}^2 \mathbb{Q}, \partial_{_E} \frac{\partial \mathbb{Q}}{\partial t}\right\}_{\partial_{_E}} = 0, \tag{1.12}$$

with "final condition", given by the Airy process[1] *which is a stationary process, (by moving far out along the cusp* $x = 2\left(\frac{t}{3}\right)^{3/2}$*)*

$$\lim_{t\to\infty} \mathbb{P}^{\mathcal{P}}\left(\frac{\mathcal{P}(t) - 2\left(\frac{t}{3}\right)^{3/2}}{(3t)^{1/6}} \cap (-E) = \emptyset\right) = \det(I - \mathbf{A})_{(-E)}$$

<u>*Remark:*</u> It is interesting to compare the Pearcey PDE with the Airy process PDE; namely for semi-infinite intervals E_1 and E_2, the 3rd order non-linear PDE for the Airy joint probability

$$\mathbb{Q}^{\mathcal{A}}(t;x,y) := \log \mathbb{P}^{\mathcal{A}}\left(\mathcal{A}(t_1) \le \frac{y+x}{2},\ \mathcal{A}(t_2) \le \frac{y-x}{2}\right),\ \text{for } t = t_2 - t_1,$$

reads

$$2t\frac{\partial^3 \mathbb{Q}^{\mathcal{A}}}{\partial t \partial x \partial y} = \left(t^2 \frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 \mathbb{Q}^{\mathcal{A}}}{\partial x^2} - \frac{\partial^2 \mathbb{Q}^{\mathcal{A}}}{\partial y^2}\right) + 8\left\{\frac{\partial^2 \mathbb{Q}^{\mathcal{A}}}{\partial x \partial y}, \frac{\partial^2 \mathbb{Q}^{\mathcal{A}}}{\partial y^2}\right\}_y, \tag{1.13}$$

with "final condition":

$$\lim_{t_2 - t_1 \to \infty} \mathbb{P}^{\mathcal{A}}\left(\mathcal{A}(t_1) \le u_1,\ \mathcal{A}(t_2) \le u_2\right) = \mathcal{F}(u_1)\mathcal{F}(u_2).$$

In the last section (section 7), we develop -in a formal way- the central role played by the spectral curve (or Pastur equation [24]) in the steepest descent analysis used to prove the universal behavior of the kernel as $N \to \infty$ for the different problems of non-intersecting Brownian motions. The spectral curve is precisely the function which appears in the steepest descent analysis. The spectral curve associated to the problem provides the universal limiting kernel obtained after a proper rescaling of the variable around a singularity of the problem.

[1]The Airy process is a stationary process, which describe the statistical fluctuations of the process about the curve appearing in Figure 1, away from the edge and properly rescaled. Its probability given at any time by the Tracy-Widom distribution $\mathcal{F}(x)$. The latter is given by the Fredholm determinant $\det(I - \mathbf{A})$ of the Airy kernel $\mathbf{A}$, restricted to the interval under consideration.

2 Non-intersecting Brownian motions on $\mathbb{R}$, forced to several points

In the expression below, $\mathcal{H}_n(E)$ is the set of all Hermitian matrices with all eigenvalues in E. Note that in general one has the following, using the Karlin-McGregor formula[2] (see [21, 9, 10, 11, 12, 28, 8]):

$$
\begin{aligned}
&\mathbb{P}_n^{(b_1,\dots,b_p)}\left(\text{all } x_j(t)\in E \text{ for } 1\leq j\leq n \;\middle|\; \begin{array}{c}\text{all } x_j(0)=0\\ n_1 \text{ paths end up at } b_1 \text{ at } t=1\\ \vdots\\ n_p \text{ paths end up at } b_p \text{ at } t=1\end{array}\right)\\
&= \lim_{\substack{\text{all }\ \gamma_i\to 0\\ \delta_1,\dots,\delta_{n_1}\to b_1\\ \vdots\\ \delta_{n_1+\dots+n_{p-1}+1},\dots,\delta_n\to b_p}} \int_{E^n}\frac{\prod_1^n dx_i}{Z_n}\det\left(p(t;\gamma_i,x_j)\right)_{1\leq i,j\leq n}\det(p(1-t;x_{i'},\delta_{j'}))_{1\leq i',j'\leq n},\\
&= \frac{1}{Z_n}\int_{\tilde E^n}\Delta_n(x_1,\dots,x_n)\prod_{\ell=1}^{p}\Delta_{n_\ell}(x^{(\ell)})\prod_{j=1}^{n_\ell}e^{-\frac12 {x_j^{(\ell)}}^2+\tilde b_\ell x_j^{(\ell)}}dx_j^{(\ell)}\Bigg|_{\substack{\tilde E=E\sqrt{\frac{2}{t(1-t)}}\\ \tilde b_\ell=\sqrt{\frac{2t}{1-t}}b_\ell}}\\
&= \frac{1}{Z_n}\int_{\mathcal{H}_n\left(E\sqrt{\frac{2}{t(1-t)}}\right)} dM e^{-\frac12\operatorname{Tr}(M^2-2A_tM)}dM\\
&= \frac{1}{Z_n}\det\begin{pmatrix}\left(\displaystyle\int_{\tilde E}x^{i+j}e^{-\frac{x^2}{2}+\tilde b_1 x}dx\right)_{0\leq i\leq n_1-1,\ 0\leq j\leq n-1}\\ \vdots\\ \left(\displaystyle\int_{\tilde E}x^{i+j}e^{-\frac{x^2}{2}+\tilde b_p x}dx\right)_{0\leq i\leq n_p-1,\ 0\leq j\leq n-1}\end{pmatrix}\\
&= \det(I-H_n^{(p)})_{E^c}, \quad (\textbf{Fredholm determinant}) \qquad (2.1)
\end{aligned}
$$

[2] $\Delta_n(x_1,\dots,x_n)$ is the Vandermonde determinant.

where $H_n^{(p)}(x,y)$ is the kernel (setting $t_k = t_\ell = t$)

$$
\begin{aligned}
&H_n^{(p)}(x,y)dy \\
&= -\frac{dy}{2\pi^2\sqrt{(1-t_k)(1-t_\ell)}}\int_{\mathcal{C}} dV \int_{L+i\mathbb{R}} dU\, \frac{e^{-\frac{t_k V^2}{1-t_k}+\frac{2xV}{1-t_k}}}{e^{-\frac{t_\ell U^2}{1-t_\ell}+\frac{2yU}{1-t_\ell}}} \prod_{r=1}^{p}\left(\frac{U-b_r}{V-b_r}\right)^{n_r}\frac{1}{U-V} \\
&\quad - \begin{cases} 0 & \text{for } \ t_k \geq t_\ell \\ \frac{1}{\sqrt{\pi(t_\ell-t_k)}} e^{-\frac{(x-y)^2}{t_\ell-t_k}} e^{\frac{x^2}{1-t_k}-\frac{y^2}{1-t_\ell}}, & \text{for } \ t_k < t_\ell \end{cases} .
\end{aligned}
\tag{2.2}
$$

where X is a contour consisting of the two incoming rays from $\pm\infty e^{i\pi/4}$ to 0 and the two outgoing rays from 0 to $\pm\infty e^{-i\pi/4}$, provided no $b_r = 0$. In the expression above, A_t is the diagonal matrix

$$
A_t := \begin{array}{c} \left(\begin{array}{ccccccccc}
\tilde{b}_1 & & & & & & & & \\
& \ddots & & & \mathbf{O} & & & & \\
& & \tilde{b}_1 & & & & & & \\
& & & \tilde{b}_2 & & & & & \\
& \mathbf{O} & & & \ddots & & & & \\
& & & & & \tilde{b}_2 & & & \\
& & & & & & \ddots & & \\
& & & & & & & \tilde{b}_p & \\
& & & & & & & & \ddots \\
& & & & & & & & & \tilde{b}_p
\end{array}\right) \end{array}
\begin{array}{l} \updownarrow n_1 \\ \\ \updownarrow n_2 \\ \vdots \\ \\ \updownarrow n_p \end{array}
\quad \text{with } \ \tilde{b}_i = b_i\sqrt{\frac{2t}{1-t}},
\tag{2.3}
$$

The main expression appearing in (2.1) contains the matrix integral

$$
\begin{aligned}
\mathbb{P}_n(E; b_1, \ldots, b_p) &= \frac{1}{Z_n}\int_{E^n} \Delta_n(x_1, ..., x_n)\prod_{\ell=1}^{p}\Delta_{n_\ell}(x^{(\ell)})\prod_{j=1}^{n_\ell} e^{-\frac{1}{2}x_j^{(\ell)^2}+b_\ell x_j^{(\ell)}}dx_j^{(\ell)} \\
&= \frac{1}{Z_n}\int_{\mathcal{H}_n(E)} dM e^{-\frac{1}{2}\operatorname{Tr}(M^2-2AM)},
\end{aligned}
\tag{2.4}
$$

which is now viewed as a function of the boundary points of E and the target points b_i, for which one assumes a linear dependence

$$
\sum_1^p c_i b_i = 0 \ \text{ with } \ \sum_1^p c_i = 1.
$$

Introduce the following operators:

$$
\begin{aligned}
\partial_{_E} &:= \left\{ \begin{array}{l} \text{sum of partials in the} \\ \text{boundary points of } E \end{array} \right\} \\
\varepsilon &:= \left\{ \begin{array}{l} \text{Euler operator in the} \\ \text{boundary points of } E \end{array} \right\} - \sum_1^{p-1} b_i \frac{\partial}{\partial b_i} \\
\partial_b^{(\ell)} &:= c_\ell \sum_1^{p-1} \frac{\partial}{\partial b_i} - \frac{\partial}{\partial b_\ell}(1-\delta_{\ell p}), \qquad \text{one checks } \sum_{\ell=1}^{p} \partial_b^{(\ell)} = 0 \qquad (2.5)
\end{aligned}
$$

Proposition: [5] The expression $\log \mathbb{P}_n$ satisfies a non-linear PDE in the boundary points of the interval E and in the target points b_i, given by the (*near-Wronskian*) determinant of a $(p+1)\times(p+1)$ matrix

$$
\det \begin{pmatrix}
F_1 & F_2 & F_3 & \dots & F_p & 0 \\
F_1' & F_2' & F_3' & \dots & F_p' & G_1 \\
F_1'' & F_2'' & F_3'' & \dots & F_p'' & G_2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
F_1^{(p)} & F_2^{(p)} & F_3^{(p)} & \dots & F_p^{(p)} & G_p
\end{pmatrix} = 0, \qquad ' := \partial_{_E} \qquad (2.6)
$$

where the F_ℓ and G_ℓ are given by[3] $(G_0 = 0)$

$$
\begin{aligned}
F_\ell &:= \left(\partial_b^{(\ell)} + c_\ell \partial_{_E}\right) \partial_{_E} \log \mathbb{P}_n + n_\ell \\
G_{\ell+1} &:= \partial_{_E} G_\ell \quad + \sum_{j=1}^{p} (\partial_{_E}^\ell F_j) \left(\partial_{_E} \frac{H_j^{(1)}}{F_j} - \partial_b^{(\ell)} \frac{H_j^{(2)}}{F_j} \right), \\
H_\ell^{(1)} &:= \left(-c_\ell \partial_{_E} \varepsilon + ((\varepsilon - 1)c_\ell + 2)(\partial_b^{(\ell)} + c_\ell \partial_{_E}) \right) \log \mathbb{P}_n + C_\ell \\
H_\ell^{(2)} &:= (1 - \varepsilon + 2 b_\ell \partial_{_E}) \left(\partial_b^{(\ell)} + c_\ell \partial_{_E} \right) \log \mathbb{P}_n. \qquad (2.7)
\end{aligned}
$$

[3] with $C_\ell = -2n_\ell \left((1-c_\ell) b_\ell + \sum_{j\neq \ell} \frac{n_j}{b_\ell - b_j} \right)$.

Example: Setting $H_\ell := \left\{H_\ell^{(1)}, F_\ell\right\}_{\partial_E} - \left\{H_\ell^{(2)}, F_\ell\right\}_{\partial_\ell^{(b)}}$, one has:

$$p = 1 \implies \det\begin{pmatrix} F_1 & 0 \\ F_1' & \frac{H_1}{F_1} \end{pmatrix} = H_1 = 0 \tag{2.8}$$

$$p = 2 \implies F_1 F_2 \det\begin{pmatrix} F_1 & F_2 & 0 \\ F_1' & F_2' & \frac{H_1}{F_1} + \frac{H_2}{F_2} \\ F_1'' & F_2'' & \frac{H_1'}{F_1} + \frac{H_2'}{F_2} \end{pmatrix} = (H_1F_2 + H_2F_1)\{F_1, F_2\}' \\ - (H_1'F_2 + H_2'F_1)\{F_1, F_2\} = 0$$

3 Proof of Theorem 1.1

Proof: It is easily computed by the Pastur-Marcenko method [24], which states that, given a diagonal matrix $A = (a_1, \ldots, a_n)$ and its spectral function $d\sigma(\lambda) := \frac{1}{n}\sum_i \delta(\lambda - a_i)$, the random Hermitian ensemble with probability defined by

$$\frac{1}{Z_n}\int_{\mathcal{H}_n(E)} dM e^{-\frac{n}{2v^2}\operatorname{Tr}(M-A)^2}, \tag{3.1}$$

has, in the limit when $n \to \infty$, a spectral density $d\nu(\lambda)$, whose Stieltjes transform

$$f(z) = \int_{-\infty}^{\infty} \frac{d\nu(\lambda)}{\lambda - z}, \quad \Im m\ z \neq 0,$$

satisfies the integral equation

$$f(z) = \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{\lambda - z - v^2 f(z)}, \tag{3.2}$$

and, in view of the spectral function $d\sigma(\lambda) := \frac{1}{n}\sum_i \delta(\lambda - a_i)$, this becomes,

$$f(z) = \sum_{i=1}^{n} \frac{1/n}{a_i - z - v^2 f(z)}. \tag{3.3}$$

Consider the situation (2.1), where n_i particles are forced to b_i. Then, upon setting the variance $v^2 = 1$ and defining $g(z) := f(z) + z$, the equation (3.3) reads[4]

$$g - z + \sum_{i=1}^{p} \frac{\varepsilon_i}{g - \tilde{b}_i} = 0, \quad \text{with } \varepsilon_i = \frac{n_i}{n}. \tag{3.4}$$

[4] See Section 7, formula (7.18), for further comments on the “fraction numbers” ε_i.

The density of the equilibrium distribution is then given by

$$\frac{d\nu(z)}{dz} = \frac{1}{\pi}|\Im m\ f(z)| = \frac{1}{\pi}|\Im m\ g(z)|\ , \qquad \text{with } z \in \mathbb{R}.$$

Remark: It is precisely this Pastur-Marchenko equation (3.4), which will appear in the argument of steepest descent for the corresponding kernel in section 7. Namely, equation (3.4) is the derivative (7.19) of the S-function, appearing in (7.17).

We will now specialize to two target points $p = 2$, where $n_1 = pn$ and $n_2 = (1-p)n$. From (2.1), it follows that for non-intersecting Brownian motions forced to $b\sqrt{n} < a\sqrt{n}$, one has

$$\mathbb{P}_n^{b\sqrt{n},a\sqrt{n}}(\text{all } x_j(t) \in \sqrt{n}E) = \frac{1}{Z_n}\int_{\mathcal{H}_n\left(E\sqrt{\frac{2}{t(1-t)}}\right)} e^{-\frac{n}{2}\operatorname{Tr}(M-A_t)^2}dM, \qquad (3.5)$$

with

$$A_t := \begin{array}{cc} \left(\begin{array}{cccccc} a\sqrt{\frac{2t}{1-t}} = \alpha & & & & & \\ & \ddots & & & & \mathbf{O} \\ & & a\sqrt{\frac{2t}{1-t}} = \alpha & & & \\ & & & b\sqrt{\frac{2t}{1-t}} = \beta & & \\ & \mathbf{O} & & & \ddots & \\ & & & & & b\sqrt{\frac{2t}{1-t}} = \beta \end{array}\right) & \begin{array}{l} \updownarrow pn \\ \\ \\ \updownarrow (1-p)n \end{array} \end{array}, \qquad (3.6)$$

For A_t as above, the integral equation (3.2) becomes the algebraic equation

$$g - z + \frac{1-p}{g-\beta} + \frac{p}{g-\alpha} = 0$$

which, upon clearing, leads to a cubic equation for g,

$$G(g) := g^3 - (z+\alpha+\beta)\,g^2 + (z(\alpha+\beta)+\alpha\beta+1)\,g - (\alpha\beta z + (1-p)\alpha + p\beta) = 0, \qquad (3.7)$$

with roots given by $g = \tilde{q} + \sqrt[3]{\tilde{r}+\sqrt{\Delta_1}} + \sqrt[3]{\tilde{r}-\sqrt{\Delta_1}}$, with a quartic discriminant in z,

$$\Delta_1(z) = (\alpha-\beta)^2\prod_0^3(z-z_i) = 0;$$

$\tilde{q}, \tilde{r}$ are polynomials of z, α, β. Thus one finds the following

$$\frac{d\nu(z)}{dz} = \frac{1}{\pi}\left|\Im m\ g(z)\right| = \begin{cases} \dfrac{1}{\pi}|\Im m\ g(z)| \text{ for } z \text{ such that } \Delta_1(z) < 0 \\ \\ 0 \text{ for } z \text{ such that } \Delta_1(z) \geq 0 \end{cases}$$

Therefore the support of the equilibrium measure will be given by

either two intervals $[z_2, z_0] \cup [z_1, z_3]$
or two intervals touching $[z_2, z_0] \cup [z_0, z_3]$
or one interval $[z_2, z_3]$

for the real roots of $\Delta_1(z) = (\alpha-\beta)^2 \prod_0^3 (z - z_i) = 0$. Thus depending on the values of the parameters α, β and p, there will be four real roots or two real roots, with a critical situation where two of the four real ones collide, say $z_1 = z_0$. The latter occurs exactly when the discriminant Δ_2 (with regard to z) of Δ_1 vanishes, namely when

$$\Delta_2(\alpha, \beta, p) = 4p(1-p)\rho\Big((\rho-1)^3 - 27p(1-p)\rho\Big)^3\Big|_{\rho=(\alpha-\beta)^2} = 0.$$

This polynomial has one positive root (the others being imaginary), and one checks that, taking into account $\alpha > \beta$, and defining p and r as in (1.7),

$$\alpha - \beta = \sqrt{\rho} = \Big(3p^{1/3}\,(1-p)^{2/3} + 3p^{2/3}(1-p)^{1/3} + 1\Big)^{1/2} = \frac{q+1}{r} > 0 \quad (3.8)$$

For this precise value of the parameter $\alpha - \beta$, two of the four boundary points of the support coincide, namely the roots z_0 and z_1 of $\Delta_1(z)$ coincide:

$$z_0 = z_1 = \beta + \frac{2q-1}{r} = \alpha + \frac{q-2}{r}. \quad (3.9)$$

This double root z_0 is found by stating that, under the condition $\alpha - \beta = (q+1)/r$, the polynomials Δ_1 and Δ_1' have a common root; in other terms for appropriate choices of c_1, c_2, c_3, some linear combination of the polynomials Δ_1 and Δ_1' becomes a linear polynomial $(4z+c_1)\Delta_1(z) - (z^2+c_2z+c_3)\Delta_1'(z) = c_4 z + z_5$ for some c_4 and c_5. Since z_0 is a root of the left hand side, it also must be a root of the right hand side. This is to say $z_0 = -c_5/c_4$, yielding the expression (3.9).

The critical time t_0 is then obtained from setting $t = t_0$ in (3.6),

$$\alpha = a\sqrt{\frac{2t_0}{1-t_0}} \quad \text{and} \quad \beta = b\sqrt{\frac{2t_0}{1-t_0}} \tag{3.10}$$

from which one computes t_0, by taking the difference and by using (3.8),

$$t_0 = \frac{(q+1)^2}{(q+1)^2 + 2(a-b)^2(q^2-q+1)},$$

and from which one further computes

$$c_0 := \sqrt{\frac{t_0(1-t_0)}{2}} = t_0\frac{r(a-b)}{q+1} \quad \text{and} \quad \frac{t_0}{c_0} = \sqrt{\frac{2t_0}{1-t_0}} = \frac{q+1}{(a-b)r}, \tag{3.11}$$

confirming the expression for c_0 in (1.10). Then from (3.10) and (3.11) one deduces

$$\alpha = a\frac{t_0}{c_0} = \frac{a(q+1)}{(a-b)r} \qquad \beta = b\frac{t_0}{c_0} = \frac{b(q+1)}{(a-b)r}. \tag{3.12}$$

Defining $x_0 := z_0 c_0$, one computes from (3.9),

$$z_0 = \frac{x_0}{c_0} = \frac{(2a-b)q + (2b-a)}{(a-b)r}. \tag{3.13}$$

Next, one computes the double root of the G-equation (3.7) for the value $z = z_0$. Indeed, using (3.9) and (3.12), one checks that for some root $g = g_0$,

$$G = \frac{\partial G}{\partial g} = \frac{\partial^2 G}{\partial g^2} = 0,$$

where

$$g_0 := \frac{1}{3}(z_0 + \alpha + \beta) = \frac{1}{3}(2a\frac{t_0}{c_0} + b\frac{t_0}{c_0} + \frac{q-2}{r}) = \frac{aq+b}{(a-b)r}, \tag{3.14}$$

and from (3.12) and $b < a$ that

$$\beta < g_0 < \alpha. \tag{3.15}$$

The point z_0 in (3.13) refers to the matrix integral variables on the right hand side of (3.5), which can then be transformed into the Brownian motion variables, according to $E=\tilde{E}\sqrt{\frac{t(1-t)}{2}}$ at $t=t_0$, which gives the transformation from the matrix integral variables to the Brownian motion variables. Hence the critical point in the Brownian picture takes place at (using (3.11) and (3.13))

$$(x_0\sqrt{n},t_0)=(z_0c_0\sqrt{n},t_0)=\left(\frac{(2a-b)q+(2b-a)}{q+1}t_0\sqrt{n},\ t_0\right).$$

To prove the main statement (1.9) in Theorem 1.1, start with the kernel (2.2) of the non-intersecting Brownian motion:

$$\begin{aligned}
&H_n(x,y;t_k,t_\ell)dy\\
&= -\frac{dy}{2\pi^2\sqrt{(1-t_k)(1-t_\ell)}}\int_{\mathcal{C}}dV\int_{L+i\mathbb{R}}dU\ \frac{e^{-\frac{tV^2}{1-t_k}+\frac{2xV}{1-t_k}}}{e^{-\frac{tU^2}{1-t_\ell}+\frac{2yU}{1-t_\ell}}}\\
&\qquad\times\left(\frac{U-b}{V-b}\right)^{n_2}\left(\frac{U-a}{V-a}\right)^{n_1}\frac{1}{U-V}. \qquad (3.16)
\end{aligned}$$

One first needs to prove that for some $\varphi_n(\lambda,\tau)$:

$$\begin{aligned}
&\lim_{n\to\infty}\varphi_n(x,t_k)H_n(x,y;t_k,t_\ell)\varphi_n(y,t_\ell)^{-1}\Big|_{\text{rescaling}}\\
&= K^{\mathcal{P}}(\xi,\eta;\tau_k,\tau_\ell)\\
&:= -\frac{1}{4\pi^2}\int_X d\omega_v\int_{-i\infty}^{i\infty}d\omega_u\ \frac{e^{-\frac{\omega_u^4}{4}+\frac{\tau_\ell\omega_u^2}{2}-\omega_u\eta}}{e^{-\frac{\omega_v^4}{4}+\frac{\tau_k\omega_v^2}{2}-\omega_v\xi}}\ \frac{1}{\omega_u-\omega_v}
\end{aligned}$$

with the rescaling $\left\{\begin{matrix}x\\y\end{matrix}\right\}$

$$t_i=t_0+(c_0\mu)^2\frac{2\tau_i}{n^{1/2}},\quad \left\{\begin{matrix}x\\y\end{matrix}\right\}=c_0\left(z_0n^{1/2}+A\left\{\begin{matrix}\tau_k\\\tau_\ell\end{matrix}\right\}+\mu\frac{\left\{\begin{matrix}\xi\\\eta\end{matrix}\right\}}{n^{1/4}}\right), \qquad (3.17)$$

with constants $A,\ \mu$, given by (1.10).

Consider the change of variables $U:=\frac{c_0u\sqrt{n}}{t_0}$. The form of the Brownian motion kernel (2.2) suggests, by putting the two U-factors of the integrand in the exponential, the function $F(u)$, which one observes, at leading order, is closely related to the function $G(u)$, defined in (3.7); namely[5]

$$F(u) \quad := \quad \frac{u^2}{2} - uz + p\log(u-\alpha) + (1-p)\log(u-\beta)\Big|_{\alpha=\frac{at_0}{c_0},\quad \beta=\frac{bt_0}{c_0},\quad z=\frac{x_0}{c_0}} \tag{3.18}$$

with

$$F'(u) = \left.\frac{G(u)}{(u-\alpha)(u-\beta)}\right|_{\alpha=\frac{at_0}{c_0},\quad \beta=\frac{bt_0}{c_0},\quad z=\frac{x_0}{c_0}}.$$

Remember from (3.14) that

$$u_0 := g_0 = \frac{aq+b}{(a-b)r} \tag{3.19}$$

is a root of $G(u) = 0$ and two of its derivatives,

$$G(u_0) = G'(u_0) = G''(u_0) = 0 \text{ and } G'''(u_0) = 6$$

and then one computes, since[6] $(u_0-\alpha)(u_0-\beta) = -q(q^2-q+1)^{-1} = -\mu^{-4}$,

$$F'(u_0) = F''(u_0) = F'''(u_0) = 0 \text{ and } \frac{1}{4!}F^{(iv)}(u_0) = -\frac{(q^2-q+1)}{4q} \tag{3.20}$$

and so

$$F(u) \quad = \quad F(u_0) - \frac{\mu^4}{4}(u-u_0)^4 + \mathbf{O}(u-u_0)^5.$$

In the calculation below, the first equality $\stackrel{*}{=}$ is obtained by doing all the substitions below, except for the last one $u \mapsto \omega_u$, whereas the second equality $\stackrel{**}{=}$ is obtained by the substitution to the new integration variable $u \mapsto \omega_u$; the expression after $\stackrel{**}{=}$ contains a term $n^{1/4}\omega_u$, which contains the new integration variable and which blows up as $n^{1/4}$. Hence this coefficient must be put $= 0$,

[5]The function $F(u)$ should contain the term $\log \frac{\sqrt{n}c_0}{t_0}$; however as the same rescaling is made in the v-variables, this same term will appear with a different sign and therefore they will cancel. Consequently, this term will be omitted.

[6]upon using (3.9), $\alpha - \beta = (q+1)/r$ and the root $u_0 = \frac{1}{3}(z_0+\alpha+\beta)$ as in (3.14).

yielding the value of A as in (1.10). In the next equality one uses this value, thus yielding in the end,

$$
\left.\frac{tU^2}{1-t}-\frac{2Uy}{1-t}+n_2\log(U-b)+n_1\log(U-a)\right|_{\substack{n_1=np,\quad n_2=(1-p)n\\ U=\frac{c_0u\sqrt{n}}{t_0}\\ t=t_0+(c_0\mu)^2\frac{2\tau_\ell}{n^{1/2}}\\ a\mapsto a\sqrt{n},\quad b\mapsto b\sqrt{n}\\ y=c_0\left(z_0n^{1/2}+A\tau_\ell+\mu\frac{\eta}{n^{1/4}}\right)\\ u=u_0+\frac{\omega_u}{\mu n^{1/4}}}}
$$

$$
\begin{aligned}
&\overset{*}{=}\; nF(u)+n^{1/2}\;\tau_\ell u\mu^2\left(\frac{u}{2}-\frac{t_0}{c_0}x_0-\frac{A}{\mu^2}\right)-n^{1/4}u\eta\mu\\
&\qquad\qquad +t_0u\mu^4\tau_\ell^2\left(\frac{u}{2}-\frac{t_0}{c_0}x_0-\frac{A}{\mu^2}\right)+O(n^{-1/4})\\
&\overset{**}{=}\; nF(u_0)+n^{1/2}\;\tau_\ell u_0\mu^2\left(\frac{u_0}{2}-\frac{t_0}{c_0}x_0-\frac{A}{\mu^2}\right)\\
&\qquad\qquad +n^{1/4}\big(\tau_\ell\omega_u(u_0-\frac{t_0}{c_0}x_0-\frac{A}{\mu^2})-\eta u_0\big)\mu+O(1)\\
&=\; nF(u_0)+n^{1/2}\frac{\tau_\ell u_0^2\mu^2}{2}-n^{1/4}u_0\mu\eta-\left(\frac{\omega_u^4}{4}-\frac{\tau_\ell\omega_u^2}{2}+\eta\omega_u\right)-\frac{t_0u_0^2\mu^4}{2}\tau_\ell^2\\
&\quad +O(n^{-1/4}),
\end{aligned}
$$

using the value (1.10) of

$$
A=\sqrt{q}+\frac{t_0}{c_0}\mu^2\,(b-x_0)=\frac{q^{1/2}(a-x_0)+q^{-1/2}(b-x_0)}{(a-b)}.
$$

Similarly,

$$
\left.\frac{tV^2}{1-t}-\frac{2Vx}{1-t}+n_2\log(V-b)+n_1\log(V-a)\right|_{\substack{n_1=np,\quad n_2=(1-p)n\\ V=\frac{c_0v\sqrt{n}}{t_0}\\ t=t_0+(c_0\mu)^2\frac{2\tau_k}{n^{1/2}}\\ a\mapsto a\sqrt{n},\quad b\mapsto b\sqrt{n}\\ x=c_0\left(z_0n^{1/2}+A\tau_k+\mu\frac{\xi}{n^{1/4}}\right)\\ v=v_0+\frac{\omega_v}{\mu n^{1/4}}}}
$$

$$
\begin{aligned}
= \quad & nF(u_0)+n^{1/2}\frac{\tau_k u_0^2\mu^2}{2}-n^{1/4}u_0\mu\xi-\left(\frac{\omega_v^4}{4}-\frac{\tau_k\omega_v^2}{2}+\xi\omega_v\right)-\frac{t_0u_0^2\mu^4}{2}\tau_k^2\\
& +O(n^{-1/4})
\end{aligned}
$$

Moreover, using $\frac{2c_0^2}{(t_0-1)t_0}=-1$, together with the rescalings above, one finds

$$
\frac{dy}{2\pi^2\sqrt{(1-t_k)(1-t_\ell)}}\frac{dUdV}{U-V}=-\frac{d\eta}{4\pi^2}\frac{d\omega_u d\omega_v}{(\omega_u-\omega_v)}+O(n^{-1/2})
$$

Summarizing, one obtains:

$$
\begin{aligned}
&\frac{-dy}{2\pi^2\sqrt{(1-t_k)(1-t_\ell)}}\int_{\mathcal{C}}dV\int_{L+i\mathbb{R}}dU\ \frac{e^{-\frac{t_kV^2}{1-t_k}+\frac{2xV}{1-t_k}}}{e^{-\frac{t_\ell U^2}{1-t_\ell}+\frac{2yU}{1-t_\ell}}}\\
&\qquad\qquad\qquad\qquad\times\left(\frac{U-b}{V-b}\right)^{n_2}\left(\frac{U-a}{V-a}\right)^{n_1}\frac{1}{U-V}\\
&= \quad e^{u_0\mu(\xi-\eta)n^{1/4}}e^{\frac12\sqrt{n}u_0^2\mu^2(\tau_k-\tau_\ell)}e^{\frac12 t_0u_0^2\mu^4(\tau_k^2-\tau_\ell^2)}\\
&\quad\ \frac{d\eta}{4\pi^2}\int_X d\omega_v\int_{-i\infty}^{i\infty}d\omega_u e^{\frac{\omega_v^4}{4}-\frac{\tau_k\omega_v^2}{2}+\xi\omega_v-\frac{\omega_u^4}{4}+\frac{\tau_\ell\omega_u^2}{2}-\eta\omega_u}\frac{1}{\omega_u-\omega_v}+O(n^{-1/4}),
\end{aligned}
$$

where we deformed the u,v contours, by translating them by u_0, so they are no longer emanating from 0, but from u_0. Moreover, taking into account the extra-piece appearing in (2.2) for $\tau_k<\tau_\ell$, one computes

$$
\begin{aligned}
&\frac{dy}{\sqrt{\pi(t_\ell-t_k)}}e^{-\frac{(x-y)^2}{t_\ell-t_k}}e^{\frac{x^2}{1-t_k}-\frac{y^2}{1-t_\ell}}\Bigg|_{\substack{t_i=t_0+(c_0\mu)^2\frac{2\tau_i}{n^{1/2}}\\ x=c_0\left(z_0n^{1/2}+A\tau_k+\mu\frac{\xi}{n^{1/4}}\right)\\ y=c_0\left(z_0n^{1/2}+A\tau_\ell+\mu\frac{\eta}{n^{1/4}}\right)}}\\
&= \quad e^{u_0\mu n^{1/4}(\xi-\eta)}e^{\frac12\sqrt{n}u_0^2\mu^2(\tau_k-\tau_\ell)}e^{\frac12 t_0u_0^2\mu^4(\tau_k^2-\tau_\ell^2)}\\
&\qquad\qquad\frac{d\eta}{\sqrt{2\pi(\tau_\ell-\tau_k)}}e^{-\frac{(\xi-\eta)^2}{2(\tau_\ell-\tau_k)}}\left(1+O(n^{-1/4})\right). \qquad (3.21)
\end{aligned}
$$

In other terms, setting

$$
D(\xi,\tau):=\operatorname{diag}\left(\ldots,e^{-u_0\mu\xi n^{1/4}}e^{-\frac12\sqrt{n}\tau_ku_0^2\mu^2}e^{-\frac12 t_0u_0^2\mu^4\tau_k^2},\ldots\right)_{1\le k\le m},
$$

one has the following limit, upon using the (t_i, x, y)-rescaling in (3.21) and upon conjugation of the matrix kernel,

$$\lim_{n\to\infty} D(\xi,\tau)(K_{t_k,t_\ell}(x,y))_{1\le k,\ell\le m} D(\eta,\tau)^{-1} = \left(K^{\mathcal{P}}_{\tau_k,\tau_\ell}(\xi,\eta)\right)_{1\le k,\ell\le m},$$

which leads to the desired kernel (1.2), upon replacing the integration variables $\omega_u \to U,\ \omega_v \to V$.

Since the above argument is obviously formal, one needs to make a rigorous steepest descent analysis on the conjugated kernel above. In the following section, steepest descent contours will be found, depending on whether $q < 1,\ q = 1\ q > 1$, or what is the same $p > \frac{1}{2},\ p = \frac{1}{2},\ p < \frac{1}{2}$. Notice the duality $q < 1 \leftrightarrow q > 1$.

4 Steepest descent analysis

In this section, we will deform the contours for both the u and v integration into steepest decent contours. They will be as depicted in the picture below; all lines are at an angle of $0, \pi/4$ or $\pi/2$ with the horizontal line.

After having set $U := \frac{c_0 u\sqrt{n}}{t_0}$ and $V := \frac{c_0 v\sqrt{n}}{t_0}$ in the exponential appearing in the kernel, one was led to a function

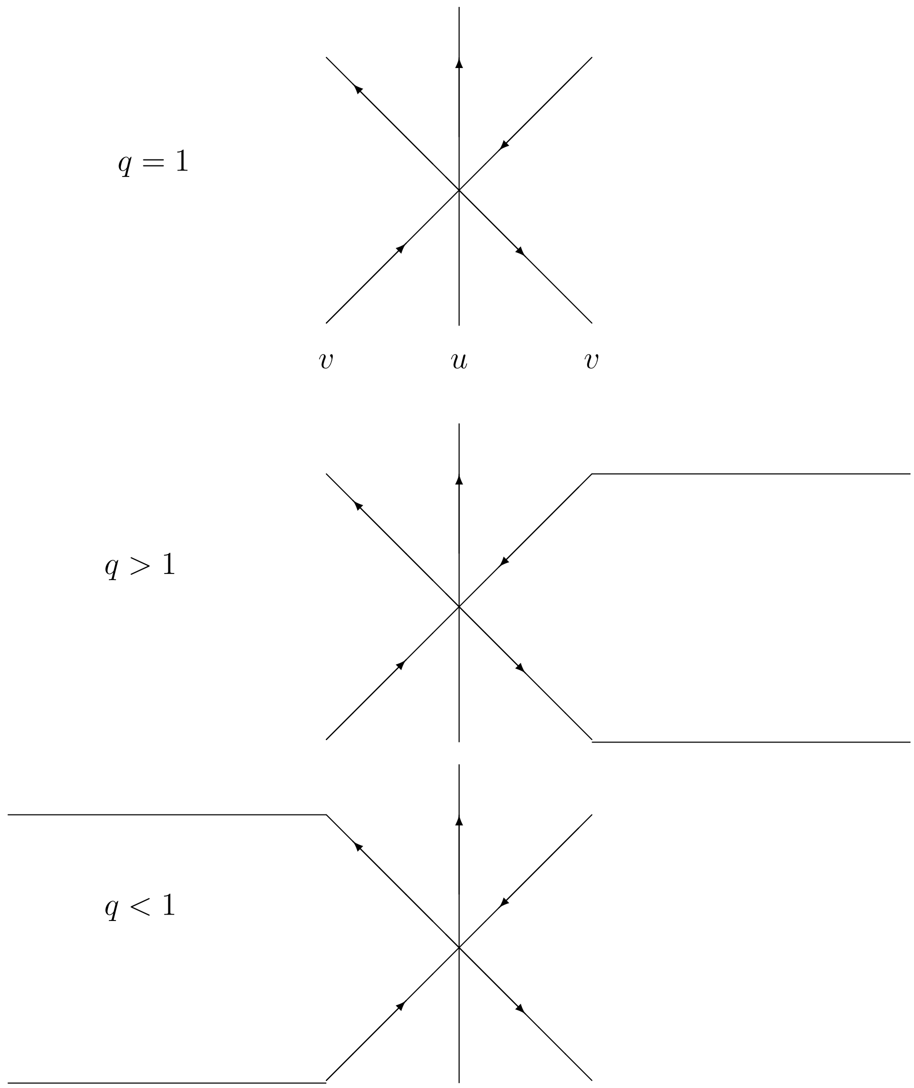
$$F(u) = \frac{u^2}{2} - uz_0 + p\log(u-\alpha) + (1-p)\log(u-\beta),$$

where (upon using (3.9), (3.12) and (3.19)),

$$z_0 = \frac{x_0}{c_0} = \beta + \frac{2q-1}{r} = \alpha + \frac{q-2}{r},\ \alpha = \frac{at_0}{c_0},\ \beta = \frac{bt_0}{c_0},\ u_0 = \beta + \frac{q}{r}.$$

Then one checks (with $p = (1+q^3)^{-1}$ and $r = \sqrt{q^2-q+1}$),

$$z_0 - u_0 = \frac{q-1}{r}, \qquad u_0 - \alpha = -\frac{1}{r}, \qquad u_0 - \beta = \frac{q}{r}. \tag{4.1}$$



(i) First we derive the steepest descent contour for the q-independent u-**integration**, which is the vertical line through u_0. Indeed, one checks that for

$$\Re e F(u_0 + iy) = \frac{1}{2}(u_0^2 - y^2) - u_0 z_0 + \frac{p}{2} \log((u_0 - \alpha)^2 + y^2) + (1-p) \log((u_0 - \beta)^2 + y^2)$$

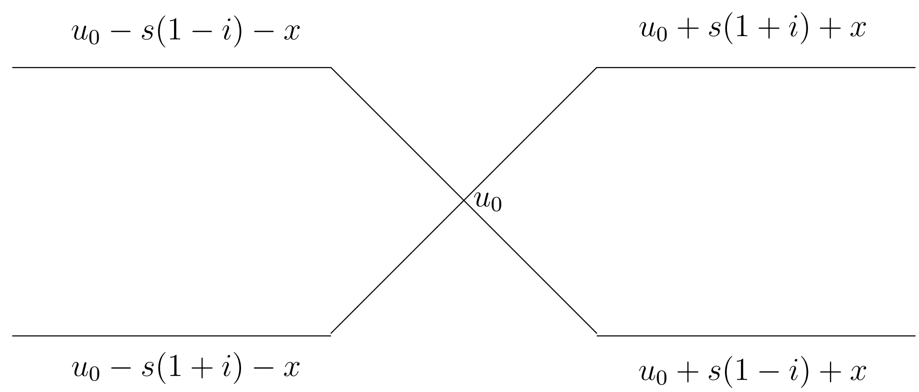
and using (4.1), the derivative equals

$$\frac{\partial}{\partial y}\Re e F(u_0+iy) \quad = \quad -y^3\frac{y^2+\frac{q}{r^2}}{(y^2+\frac{1}{r^2})(y^2+\frac{q^2}{r^2})} = -y^3\left\{\begin{array}{c}\text{a positive}\\ \text{function}\end{array}\right\},$$

showing $\Re e F(u_0+iy)$ has a maximum at $y=0$, which takes care of the u-contour.

(ii) The next point is to deal with the v**-integration**. Returning to the non-intersecting Brownian motion kernel (3.16), the F-function goes with a negative sign on the v-contour and thus (with $\varepsilon=\pm 1$)

$$\begin{aligned}-\Re e F(u_0+x(\varepsilon+i)) \quad = \quad & -\frac{1}{2}((u_0+x\varepsilon)^2-x^2)+(u_0+x\varepsilon)z_0 \\ & -\frac{p}{2}\log((u_0-\alpha+x\varepsilon)^2+x^2) \\ & -\frac{1-p}{2}\log((u_0-\beta+x\varepsilon)^2+x^2)\end{aligned}$$



$0<q<1$ $\qquad\qquad$ $1<q$

$$s=\frac{q}{r|q-1|}$$

Using (4.1),one checks for $\varepsilon = \pm 1$

$$\begin{aligned}
-\left(\left(u_0-\alpha+x\varepsilon\right)^2+x^2\right)\left(\left(u_0-\beta+x\varepsilon\right)^2+x^2\right)\frac{\partial}{\partial x}\Re eF(u_0+x(\varepsilon+i)) \\
= \quad -\frac{4x^3}{r}\left(-\left(q-1\right)\varepsilon x+\frac{q}{r}\right) \\
= \quad -x^3\times\left\{\begin{array}{c}\text{a positive}\\ \text{function}\end{array}\right\}
\end{aligned}$$

for

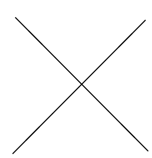
$$\begin{array}{ll}
\left\{\begin{array}{ll} -\infty < x < \frac{q}{r(q-1)}, & \varepsilon = 1 \\ -\frac{q}{r(q-1)} < x < \infty, & \varepsilon = -1 \end{array}\right\} & \text{if } q>1, \\
\left\{\begin{array}{ll} -\infty < x < \infty, & \varepsilon = 1 \\ -\infty < x < \infty, & \varepsilon = -1 \end{array}\right\} & \text{if } q=1, \\
\left\{\begin{array}{ll} -\frac{q}{r(1-q)} < x < \infty, & \varepsilon = 1 \\ -\infty < x < \frac{q}{r(1-q)}, & \varepsilon = -1 \end{array}\right\} & \text{if } q<1,
\end{array} \tag{4.2}$$

Therefore the function $-\Re eF(u_0+x(\varepsilon+i))$, restricted to the segments specified by (4.2), has its maximum at u_0. One then completes those segments by horizontal lines starting from the end of those segments as in the figure above. Along those horizontal half lines, one must check that the maximum is attained at the points $u_0 \pm s(1-i)$ and $u_0 \pm s(1+i)$. To carry out this computation, the four horizontal segments can readily be represented by

$$u_0+\delta s(\varepsilon+i)+\delta\varepsilon x$$

with

$\delta = -\varepsilon = 1$ $\qquad$ $\delta = \varepsilon = 1$



$-\delta = \varepsilon = 1$ $\qquad$ $-\delta = -\varepsilon = 1$

in the four corresponding regions of the figure above.

To deal with the horizontal segment, setting $s = \frac{q}{r|q-1|}$, one finds

$$\begin{aligned}
-\Re e F(u_0 + \delta s(\varepsilon + i) + \delta\varepsilon x) &= -\frac{1}{2}((u_0 + \epsilon\delta(s+x))^2 - s^2) + (u_0 + \epsilon\delta(s+x))z_0 \\
&\quad -\frac{p}{2}\log\left((u_0 - \alpha + \epsilon\delta(s+x))^2 + s^2\right) \\
&\quad -\frac{1-p}{2}\log\left((u_0 - \beta + \epsilon\delta(s+x))^2 + s^2\right).
\end{aligned}$$

One computes,

$$\begin{aligned}
&-\left(\left(u_0 + (s+x)\epsilon\delta - \alpha\right)^2 + s^2\right)\left(\left(u_0 + (s+x)\epsilon\delta - \beta\right)^2 + s^2\right) \\
&\qquad\qquad \frac{\partial}{\partial x}\Re e\ F(u_0 + \delta s(\varepsilon + i) + \delta\varepsilon x)\Big|_{s = \frac{\varepsilon\delta q}{r(q-1)}}
\end{aligned}$$

$$= \frac{q^4}{r^5(q-1)^5}\left(\begin{array}{l} qz^5 + \varepsilon\delta(q^2+3q+1)z^4 + 3(q+1)^2z^3 + \varepsilon\delta(q+3)z^2 \\ +4(q^2+q+1)z + \varepsilon\delta(1+q^2) \end{array}\right)\Bigg|_{z = \frac{r(q-1)}{q}x} \tag{4.3}$$

Since $x > 0$ is increasing as one moves away from u_0 along the horizontal lines; one has simultaneously,

$$\begin{array}{llll} s = \frac{q}{r(q-1)}, & \varepsilon\delta = +1, & q-1>0, & z>0 \\ s = -\frac{q}{r(q-1)}, & \varepsilon\delta = -1, & q-1<0, & z<0 \end{array}$$

and thus, since $q > 0$, the right hand side of (4.3) is > 0. Thus the above derivative is negative on the four lines and so, when one moves away from u_0 along the horizontal paths in all four directions (as in the picture above) the function $-\Re e\ F(z)$ goes down so that combining both calculations, the maximum will be attained at u_0.

In order to show that in the limit the Pearcey kernel is obtained, one picks τ_i's, and ξ, η in a compact set of $\mathbb{R}$, one integrates the u and v variables along the contour in a neighborhood of radius $n^{-1/4}n^{\frac{1}{20}-\varepsilon}$. Then $\left|\frac{\omega_u}{\mu}\right|, \left|\frac{\omega_u}{\mu}\right| \leq \delta \leq n^{\frac{1}{20}-\varepsilon}$, as $n \to \infty$. In this range lemma 4.1 will apply, whereas outside this neighborhood, the rest of the contour makes no contribution, because of the steepest descent estimates. So, one needs the following estimate:

Lemma 4.1 *Given the function $F(u)$ as in (3.18), one has the following estimate,*

$$n\left|F(u_0 + \frac{\delta}{n^{1/4}}) - F(u_0) - F^{(iv)}(u_0)\frac{1}{4!}\left(\frac{\delta}{n^{1/4}}\right)^4\right| \leq \frac{64\delta^5}{5n^{1/4}}\left(q + \frac{1}{q}\right)^5.$$

Proof: By Taylor's Theorem and using $F'(u_0) = F''(u_0) = F'''(u_0) = 0$ and $\frac{1}{4!}F^{(iv)}(u_0) = -\frac{r^2}{4q}$, as in (3.20), one has

$$\left| F(u_0 + \frac{\delta}{n^{1/4}}) - F(u_0) - \frac{1}{4!}\left(\frac{\delta}{n^{1/4}}\right)^4 F^{(iv)}(u_0) \right| \leq \left(\frac{\delta}{n^{1/4}}\right)^5 \max_{|u-u_0| \leq \frac{\delta}{n^{1/4}}} \frac{\left|F^{(v)}(u)\right|}{5!}$$

From the explicit expression (3.18) for F, from the fact that $u_0 - \beta = q/r$ and $a - u_0 = 1/r$ and that $\beta < u_0 < \alpha$, one deduces[7]

$$\begin{aligned}
\max_{|u-u_0| \leq \frac{\delta}{n^{1/4}}} \left| \frac{F^{(v)}(u)}{5!} \right| &= \sup_{|u-u_0| \leq \frac{\delta}{n^{1/4}}} \frac{1}{5} \left| \frac{1-p}{(u-\beta)^5} + \frac{p}{(u-\alpha)^5} \right| \\
&\leq \frac{2}{5 \min(|\alpha - u_0 - \frac{\delta}{n^{1/4}}|, |u_0 - \beta - \frac{\delta}{n^{1/4}}|)^5} \\
&\leq \frac{2}{5}\left(\frac{1}{r}\min(1,q) - \frac{\delta}{n^{1/4}}\right)^{-5} \\
&\leq \frac{64}{5}\left(\frac{r}{(\min(1,q))}\right)^5 \left(\begin{array}{c} \text{by picking } n \text{ large enough} \\ \text{such that } \frac{\delta}{n^{1/4}} \leq \frac{1}{2r}\min(1,q) \\ \text{since } \delta \leq n^{\frac{1}{20}-\varepsilon} \end{array} \right) \\
&\leq \frac{64}{5}\left(q + \frac{1}{q}\right)^5
\end{aligned}$$

ending the proof of Lemma 4.1. ■

5 Proof of Theorem 1.2

In this section, we denote by a and b the Brownian motions target points, where we put $b = 0$. We denote the old time in the Brownian motion formula by $\bar{t}$ and the new rescaled time and space in the Brownian motion formula (Theorem 1.1) by $\bar{\tau}$ and $\bar{\eta}$. Set $E^c = \cup_{i=1}^r (y_{2i-1}, y_{2i}) \subset \mathbb{R}$. Let x be the spatial variable for the matrix integral and α the variable appearing in the diagonal matrix, the other one being $= 0$.

[7] Remember $r = \sqrt{q^2 - q + 1}$

The reader is reminded of the different players in the argument below, in accordance with (2.1),

$$\mathbb{P}_{\rm Br}(\bar t, y, a\sqrt{n})$$
$$:= \mathbb{P}_n^{(0,a\sqrt{n})}\left(\text{all } x_j(\bar t)\in E \text{ for } 1\le j\le n \;\middle|\; \begin{array}{l} \text{all } x_j(0)=0 \\ n_1 \text{ paths end up at } a\sqrt{n} \text{ at } \bar t = 1 \\ n_2 \text{ paths end up at } 0 \text{ at } \bar t = 1 \end{array} \right)$$

and, setting $\tilde b_1 = \alpha,\ \tilde b_2 = 0$,

$$\mathbb{P}_n(\alpha, x_i) := \frac{1}{Z_n}\int_{\tilde E^n} \Delta_n(x_1,...,x_n)\prod_{\ell=1}^{2}\Delta_{n_\ell}(x^{(\ell)})\prod_{j=1}^{n_\ell} e^{-\frac12 x_j^{(\ell)2}+\tilde b_\ell x_j^{(\ell)}}dx_j^{(\ell)}$$

with $\mathbb{P}_n(\alpha,x_i)$ satisfying the PDE[8], as in (2.6) and (2.8),

$$\det\begin{pmatrix} F_1 & F_2 & 0 \\ F_1' & F_2' & F_1F_2\left(\frac{H_1}{F_1}+\frac{H_2}{F_2}\right) \\ F_1'' & F_2'' & F_1F_2\left(\frac{H_1'}{F_1}+\frac{H_2'}{F_2}\right)\end{pmatrix} = 0,$$

with F_i and H_i given in (2.7). From (2.1), one has the following relationship

$$\mathbb{P}_{Br}\left(\bar t, y, a\sqrt{n}\right) = \mathbb{P}_n\left(a\sqrt{n}\sqrt{\frac{2\bar t}{1-\bar t}}, y_i\sqrt{\frac{2}{\bar t(1-\bar t)}}\right) = \mathbb{P}_n(\alpha, x_i)$$

and also from Theorem 1.1,

$$\mathbb{P}_{Br}(\bar t, y, a\sqrt{n})\Big|_{\bar t = t_0+(c_0\mu)^2\frac{2\bar\tau}{n^{1/2}},\quad y = x_0 n^{1/2}+c_0A\bar\tau + c_0\mu\frac{\bar\eta}{n^{1/4}}}$$
$$= \mathbb{P}^{\mathcal{P}}\left(\mathcal{P}(\bar\tau)\cap\cup_{i-1}^{r}(\bar\eta_{2i-1},\bar\eta_{2i}) = \emptyset\right) + O(n^{-1/4}) \tag{5.1}$$

Setting $b=0$, the formulae (1.7), (1.8) and (1.10) in Theorem 1.1 simplify:

$$\sqrt{\frac{2t_0}{1-t_0}} = \frac{q+1}{ar},\quad x_0 = \frac{2q-1}{q+1}at_0\ ,\quad c_0 = \sqrt{\frac{t_0(1-t_0)}{2}} = \frac{at_0r}{q+1}$$
$$A = \frac{q-(2q-1)t_0}{\sqrt{q}},\qquad \mu = \frac{\sqrt{r}}{q^{1/4}}. \tag{5.2}$$

---

[8] Given $\tilde E^c = \cup_{i=1}^r(x_{2i-1},x_{2i})\subset\mathbb{R}$, the prime in the formula below denotes $' := \sum\frac{\partial}{\partial x_i}$.

For convenience, we set

$$\tau := \frac{\bar{\tau}}{\sqrt{q}}, \quad \eta := \frac{\bar{\eta}}{q^{1/4}}, \quad z := \left(\frac{q^2-q+1}{n}\right)^{1/4}.$$

One finds, using $z^2 = r/\sqrt{n}$ and formulae (5.2),

$$\begin{aligned}
\bar{t} = t_0 + (c_0\mu)^2 \frac{2\bar{\tau}}{\sqrt{n}} &= t_0 \left(1 + (1-t_0)\frac{\bar{\tau}}{\sqrt{q}} \left(\frac{q^2-q+1}{n}\right)^{1/2}\right) \\
&= t_0 \left(1 + (1-t_0)\tau z^2\right)
\end{aligned}$$

Moreover, one computes

$$\begin{aligned}
\alpha = a\sqrt{n}\sqrt{\frac{2\bar{t}}{1-\bar{t}}} &= \sqrt{n}a\sqrt{\frac{2t_0}{1-t_0}}\sqrt{\frac{1+(1-t_0)\tau z^2}{1-t_0\tau z^2}} \\
&= \frac{q+1}{z^2}\sqrt{\frac{1+(1-t_0)\tau z^2}{1-t_0\tau z^2}}.
\end{aligned}$$

We also have, using the formulae (5.2) above,

$$\begin{aligned}
x &= y\sqrt{\frac{2}{\bar{t}(1-\bar{t})}} \\
&= \sqrt{\frac{2}{\bar{t}(1-\bar{t})}} c_0 \left(\frac{x_0}{c_0}\sqrt{n} + A\bar{\tau} + \mu\frac{\bar{\eta}}{n^{1/4}}\right) \\
&= \sqrt{\frac{2}{\bar{t}(1-\bar{t})}} c_0 \left((2q-1)\frac{\sqrt{n}}{r} + (q-(2q-1)t_0)\frac{\bar{\tau}}{q^{1/2}} + \frac{\bar{\eta}}{q^{1/4}}\frac{\sqrt{r}}{n^{1/4}}\right) \\
&= \frac{1}{\sqrt{(1+(1-t_0)\tau z^2)}\sqrt{(1-t_0\tau z^2)}} \left(\frac{2q-1}{z^2} + (q-t_0(2q-1))\tau + \eta z\right)
\end{aligned}$$

One also checks

$$n_1 = np = \frac{n}{q^3+1} = \frac{n}{(q+1)(q^2-q+1)} = \frac{z^{-4}}{(q+1)}, \quad n_2 = n(1-p) = \frac{q^3 z^{-4}}{(q+1)}.$$

Consider the map $T_z : (\tau, \eta_i) \mapsto (\alpha, x_i)$,

$$\begin{aligned}
(\alpha, x_i) &:= T_z(\tau, \eta_i) \\
&= \left(\frac{(q+1)\sqrt{1+(1-t_0)\tau z^2}}{z^2\sqrt{(1-t_0\tau z^2)}}, \frac{\left(\frac{2q-1}{z^2} + (q-(2q-1)t_0)\,\tau + \eta_i z\right)}{\sqrt{(1+(1-t_0)\tau z^2)\,(1-t_0\tau z^2)}}\right)
\end{aligned}$$

and its inverse $T_z^{-1} : (\alpha, x_i) \mapsto (\tau, \eta_i)$,

$$\begin{aligned}(\tau,\eta_i) &= T_z^{-1}(\alpha,x_i)\\ &= \left(\frac{\alpha^2 z^4-(q+1)^2}{z^2(t_0z^4\alpha^2+(q+1)^2(1-t_0))},\ \frac{\alpha(x_iq-\alpha q+x_i)z^4-(q-1)(q+1)^2}{z^3(t_0z^4\alpha^2+(q+1)^2(1-t_0))}\right)\end{aligned}$$

Then setting

$$\log\mathbb{P}_n(\alpha,x_i) = F(\tau,\eta_i) = F(T_z^{-1}(\alpha,x_i)),$$

setting $B = \cup_{i-1}^r(x_{2i-1},x_{2i}) \subset \mathbb{R}$, taking the derivatives $\partial_x := \sum \frac{\partial}{\partial x_i}$, $\varepsilon_x := \sum x_i \frac{\partial}{\partial x_i}$ and then taking a series in z, the functions F_i and H_i in (2.7) have the following form,

$$\begin{aligned}
F_1 &= -\frac{\partial}{\partial\alpha}\partial_x\log\mathbb{P}_n+n_1 = \frac{1}{q+1}\left(z^{-4}+z^{-2}F''-2z^{-1}\frac{\partial F'}{\partial\tau}+O(1)\right)\\
F_2 &= (\frac{\partial}{\partial\alpha}+\partial_x)\partial_x\log\mathbb{P}_n+n_2 = \frac{1}{q+1}\left(q^3z^{-4}+qz^{-2}F''+2z^{-1}\frac{\partial F'}{\partial\tau}+O(1)\right)\\
H_1 &= \left\{H_1^{(1)},F_1\right\}_{\partial_x}+\left\{H_1^{(2)},F_1\right\}_{\alpha} = \frac{2}{(q+1)^4}\begin{pmatrix} q(2q^2+3q+2)F'''z^{-9}\\ -(4q^3-1+6q^2+3q)\frac{\partial F''}{\partial\tau}z^{-8}\\ +O(z^{-7})\end{pmatrix}\\
H_2 &= \left\{H_2^{(1)},F_2\right\}_{\partial_x}-\left\{H_2^{(2)},F_2\right\}_{\alpha} = \frac{2q^3}{(q+1)^4}\begin{pmatrix} -q(2q^2+3q+2)F'''z^{-9}\\ +(4q^3-1+6q^2+3q)\frac{\partial F''}{\partial\tau}z^{-8}\\ +O(z^{-7})\end{pmatrix}
\end{aligned}$$

with

$$\begin{aligned}
H_1^{(2)} &= (1-\varepsilon_x+\alpha\frac{\partial}{\partial\alpha}+2\alpha\partial_x)(-\frac{\partial}{\partial\alpha})\log\mathbb{P}_n\\
H_2^{(2)} &= (1-\varepsilon_x+\alpha\frac{\partial}{\partial\alpha})(\frac{\partial}{\partial\alpha}+\partial_x)\log\mathbb{P}_n\\
H_1^{(1)} &= -2\frac{\partial}{\partial\alpha}\log\mathbb{P}_n-2n_1(\alpha+\frac{n_2}{\alpha})\\
H_2^{(1)} &= (1+\varepsilon_x-\alpha\frac{\partial}{\partial\alpha})\frac{\partial}{\partial\alpha}\log\mathbb{P}_n+2\frac{n_1n_2}{\alpha}
\end{aligned}$$

Then one computes (2.8), setting $' = \partial_x$ and setting $\partial_E = \sum\frac{\partial}{\partial y_i}$ for $E =$

(y_1, y_2),

$$\det\left(\begin{array}{ccc} F_1 & F_2 & 0 \\ F_1' & F_2' & F_1F_2\left(\frac{H_1}{F_1}+\frac{H_2}{F_2}\right) \\ F_1'' & F_2'' & F_1F_2\left(\frac{H_1'}{F_1}+\frac{H_2'}{F_2}\right) \end{array}\right)\Bigg|_{(\alpha,x_i)=T_z(\tau,\eta_i),\text{ with } \tau=\frac{t}{\sqrt{q}},\ \eta_i=\frac{y_i}{q^{1/4}}}$$

$$\begin{aligned} &= -2q^{6+1/2}\frac{q-1}{(q+1)^5}\left\{\partial_{_E}^3\log\mathbb{P}^{\mathcal{P}},\mathbb{X}\right\}_{\partial_E} z^{-18} \\ &\quad+\frac{1}{8}\left(\left\{\frac{\partial}{\partial t}\partial_{_E}^2\log\mathbb{P}^{\mathcal{P}},\mathbb{X}\right\}_{\partial_E}+O(q-1)\right)z^{-17}+O(z^{-16}), \end{aligned}$$

where, setting $\mathbb{Q}(t,E) := \log\mathbb{P}^{\mathcal{P}}\left(\mathcal{P}(t)\cap E=\emptyset\right)$,

$$\mathbb{X} := 8\frac{\partial^3\mathbb{Q}}{\partial t^3}+\left(\varepsilon_{_E}-2t\frac{\partial}{\partial t}-2\right)\partial_{_E}^2\mathbb{Q}-4\left\{\partial_{_E}^2\mathbb{Q},\partial_{_E}\frac{\partial\mathbb{Q}}{\partial t}\right\},$$

where we made use of (5.1), which states that $\log\mathbb{P}_n(\alpha,x_i)=\mathbb{Q}(t,E)+O(z)$.

• For $q\neq 1$, the function $\log\mathbb{P}^{\mathcal{P}}$, which is independent of q by the universality result, satisfies the differential equation, given by the leading term z^{-18},

$$\left\{\partial_E^3\log\mathbb{P}^{\mathcal{P}},\mathbb{X}\right\}=0. \tag{5.3}$$

• For $q=1$, the z^{-18}-term vanishes and thus $\log\mathbb{P}$ satisfies another equation, namely the one appearing in the z^{-17}-term,

$$\left\{\partial_E^2\frac{\partial}{\partial t}\log\mathbb{P}^{\mathcal{P}},\mathbb{X}\right\}=0. \tag{5.4}$$

This means that $\log\mathbb{P}^{\mathcal{P}}$ satisfies the two equations (5.3) and (5.4).

Thus for $E=(x,y)\subset\mathbb{R}$, setting $u_\pm=\frac{1}{2}(y\pm x)$, and $\log\mathbb{P}^{\mathcal{P}}=H(t;\frac{1}{2}(y+x),\frac{1}{2}(y-x))$,

$$\begin{aligned} \partial_{_E}\log\mathbb{P}^{\mathcal{P}} &= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\log\mathbb{P}^{\mathcal{P}}=\frac{\partial}{\partial u_+}H \\ \varepsilon_{_E}\log\mathbb{P}^{\mathcal{P}} &= \left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\log\mathbb{P}^{\mathcal{P}}=\left(u_+\frac{\partial}{\partial u_+}+u_-\frac{\partial}{\partial u_-}\right)H. \end{aligned}$$

Since the wronskian of two functions equals the derivative of the ratio, modulo a non-zero multiplicative term, one concludes from equation (5.3) that $\mathbb{X} = c(t,u_-)\frac{\partial^3 H}{\partial u_+^3}$, with $c(t,u_-)$ a function depending on all variables except u_+; putting this equation in equation (5.3), one finds

$$c(t,u_-)\left\{\frac{\partial^3 H}{\partial u_+^3}, \frac{\partial^3 H}{\partial t \partial u_+^2}\right\}_{u_+} = c(t,u_-)\left\{\partial_E^3 \log \mathbb{P}^{\mathcal{P}}, \frac{\partial}{\partial t}\partial_E^2 \log \mathbb{P}^{\mathcal{P}}\right\}_{\partial_E} = 0,$$

implying $c(t,u_-) = 0$ for all t, u_- and thus $H(t;u_+,u_-) = \log \mathbb{P}^{\mathcal{P}}$ satisfies the equation $\mathbb{X} = 0$, provided the Wronskian $\left\{\partial_E^3 \log \mathbb{P}^{\mathcal{P}}, \frac{\partial}{\partial t}\partial_E^2 \log \mathbb{P}^{\mathcal{P}}\right\}_{\partial_E} \neq 0$. This will be shown in the next section, using functional theoretical arguments. This ends the proof of Theorem 1.2, except for the "final condition", which will be shown in [1]. ∎

6 An estimate for the Wronskian

Proposition 6.1 *The Wronskian*

$$\left\{\frac{\partial}{\partial t}\partial_E^2 \log \mathbb{P}^{\mathcal{P}}, \partial_E^3 \log \mathbb{P}^{\mathcal{P}}\right\}_{\partial_E}$$

is a non-zero function.

In order to prove this proposition, we need a number of lemmas; the proofs will be functional-theoretical and rely on the techniques and on some of the formulae in [26]. Given the Pearcey kernel $K^{\mathcal{P}}$, one defines, for a given set $E = \cup_{k=1}^r [a_{2k-1}, a_{2k}] \subset \mathbb{R}$, the following[9]:

$$K_E^{\mathcal{P}} := K^{\mathcal{P}} \chi_E, \qquad I + R := (I - K_E^{\mathcal{P}})^{-1} \doteq: \rho(x,y) \tag{6.1}$$

and thus one has

$$(I - K_E^{\mathcal{P}})^{-1} K_E^{\mathcal{P}} = K_E^{\mathcal{P}} + (E_E^{\mathcal{P}})^2 + ... = R. \tag{6.2}$$

Also, from (1.2) and the differential equations (1.4) for $p(x)$ and $q(y)$, it follows that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) K^{\mathcal{P}}(x,y) = p(x)q(y). \tag{6.3}$$

[9]Given a kernel, viewed as an operator, the equality $\doteq$ refers to the corresponding kernel

Note that for a general kernel L, one has

$$[D, L] \doteq \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) L(x, y). \tag{6.4}$$

Define the functions

$$\hat{p} := (I - K_E^{\mathcal{P}})^{-1} p, \quad \hat{q} := (I - K_E^{\mathcal{P}^\top})^{-1} q \tag{6.5}$$

and[10]

$$u := \int_E (I - K_E^{\mathcal{P}})^{-1} p(\alpha) q(\alpha) d\alpha = \langle \hat{p}(\alpha), q(\alpha) \chi_E(\alpha) \rangle. \tag{6.6}$$

Lemma 6.2 *For a disjoint union $E = \bigcup_{k=1}^r [a_{2k-1}, a_{2k}]$, one has the following identity*[11]*:*

$$\partial_E^2 \log \mathbb{P}^{\mathcal{P}} = \partial_E^2 \log \det(I - K_E^{\mathcal{P}}) = \partial_E u = \sum_k (-1)^k \hat{p}(a_k) \hat{q}(a_k).$$

Proof: At first notice that

$$\begin{aligned} [D, K_E^{\mathcal{P}}] = [D, K^{\mathcal{P}} \chi_E] &\doteq \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) (K^{\mathcal{P}} \chi_E) \\ &= p(x) q(y) \chi_E(y) - \sum_k (-1)^k K^{\mathcal{P}}(x, a_k) \delta(y - a_k), \end{aligned} \tag{6.7}$$

from which one deduces, using notations (6.1) and (6.2),

$$\begin{aligned} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) R(x, y) &\doteq [D, (I - K_E^{\mathcal{P}})^{-1}] \\ &\doteq (I - K_E^{\mathcal{P}})^{-1} p(x) q(y) \chi_E(y) (I - K_E^{\mathcal{P}})^{-1} \\ &\quad - \sum_k (-1)^k (I - K_E^{\mathcal{P}})^{-1} K^{\mathcal{P}}(x, a_k) \delta(y - a_k) (I - K_E^{\mathcal{P}})^{-1} \\ &= \hat{p}(x) \hat{q}(y) \chi_E(y) - \sum_k (-1)^k R(x, a_k) \rho(a_k, y). \end{aligned} \tag{6.8}$$

[10] $\langle f, g \rangle := \int_{\mathbb{R}} f(x) g(x) dx$.
[11] Remembering $\partial_E = \sum_1^r \frac{\partial}{\partial a_j}$.

and

$$\frac{\partial}{\partial a_k}R(x,y)=\frac{\partial}{\partial a_k}(I+R) \;=\; \frac{\partial}{\partial a_k}(I-K_E^{\mathcal{P}})^{-1}$$

$$= (I-K^{\mathcal{P}}\chi_E)^{-1}K^{\mathcal{P}}\frac{\partial\chi_E}{\partial a_k}(I-K^{\mathcal{P}}\chi_E)^{-1}$$

$$= R(x,z)(\delta(z-a_k)(-1)^k)\rho(z,y)$$

$$= (-1)^k R(x,a_k)\rho(a_k,y). \tag{6.9}$$

Then combining (6.8) and (6.9), one finds

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\sum_k\frac{\partial}{\partial a_k}\right)R(x,y) \;=\; [D,(I-K_E^{\mathcal{P}})^{-1}]+\sum_k\frac{\partial}{\partial a_k}R(x,y)$$

$$\doteq \hat{p}(x)\hat{q}(y)\chi_E(y), \tag{6.10}$$

and hence, setting $x=y=a_j$, the total derivative becomes

$$\sum_k\frac{d}{da_k}R(a_j,a_j)\doteq\hat{p}(a_j)\hat{q}(a_j). \tag{6.11}$$

One then computes the derivative of u, as defined in (6.6), with respect to a_k: (of course, the functions p and q do not involve the interval E)

$$\frac{\partial u}{\partial a_k} \;=\; \frac{\partial}{\partial a_k}\left\langle(I-K_E^{\mathcal{P}})^{-1}p,q\chi_E\right\rangle$$

$$= \left\langle\left(\frac{\partial}{\partial a_k}(I-K_E^{\mathcal{P}})^{-1}\right)p,q\chi_E\right\rangle+\left\langle\hat{p},q\frac{\partial\chi_E}{\partial a_k}\right\rangle$$

$$= (-1)^k\langle R(x,a_k)\hat{p}(a_k),q\chi_E\rangle+\langle\hat{p},q\delta(y-a_k)(-1)^k\rangle,\quad\text{using (6.9)},$$

$$= (-1)^k\left(\hat{p}(a_k)\langle R(x,a_k),q\chi_E\rangle+\hat{p}(a_k)q(a_k)\right)$$

$$= (-1)^k\hat{p}(a_k)((I+R^\top)q(a_k)-q(a_k)+q(a_k))$$
$$= (-1)^k\hat{p}(a_k)\hat{q}(a_k). \tag{6.12}$$

Combining (6.11) and (6.12) yields

$$\frac{\partial u}{\partial a_j} = \sum_k \frac{d}{da_k}(-1)^j R(a_j, a_j)$$

and then summing with respect to j,[12]

$$\begin{aligned}
\sum_k \frac{\partial u}{\partial a_k} &= \sum_k \frac{d}{da_k}\left(\sum_j (-1)^j R(a_j, a_j)\right) \\
&= -\left(\sum_k \frac{d}{da_k}\right)^2 \log\det(I - K_E^{\mathcal{P}})^{-1} \\
&= \partial_E^2 \log\det(I - K_E^{\mathcal{P}}) = \partial_E^2 \log \mathbb{P}^{\mathcal{P}},
\end{aligned}$$

which, together with (1.2), establishes Lemma 6.2. ■

Lemma 6.3 *Given $E = [x, y]$, the following estimates hold*

$$\begin{aligned}
\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u &= (y - x)(p(x)q(x))' + \mathbf{O}(y - x)^2 \\
\frac{\partial u}{\partial t} &= \frac{1}{2}(y - x)(p''q - pq'')(x) + \mathbf{O}(y - x)^2.
\end{aligned}$$

Proof: Using the fact that, for a small interval E, the integral $\int_x^y$ has order $x - y$, using $R(\alpha, y) - R(\alpha, x) = \mathbf{O}(y - x)$, one deduces from the formula of Lemma 6.2, (remember the definitions (6.5) of $\hat{p}$ and $\hat{q}$)

$$\begin{aligned}
\partial_E u &= \hat{p}(y)\hat{q}(y) - \hat{p}(x)\hat{q}(x) \\
&= \left(p(y) + \int_x^y R(y, \alpha)p(\alpha)d\alpha\right)\left(q(y) + \int_x^y R(\alpha, y)q(\alpha)d\alpha\right) \\
&\quad - \left(p(x) + \int_x^y R(x, \alpha)p(\alpha)d\alpha\right)\left(q(x) + \int_x^y R(\alpha, x)q(\alpha)d\alpha\right) \\
&= p(y)q(y) - p(x)q(x) + \mathbf{O}(y - x)^2 \\
&= (y - x)(p(x)q(x))' + \mathbf{O}(y - x)^2.
\end{aligned}$$

[12] Here one uses identity (1.1) in [26],

$$\frac{d}{da_j}\log\det(I - K_E^{\mathcal{P}})^{-1} = (-1)^{j+1}R(a_j, a_j).$$

Using the heat equations (1.5) satisfied by p, q and the PDE (1.6) for $K^{\mathcal{P}}$, one checks

$$\begin{aligned}
2\frac{\partial u}{\partial t} &= 2\frac{\partial}{\partial t}\langle (I-K_E^{\mathcal{P}})^{-1}p, q\rangle \\
&= \Big\langle (I-K_E^{\mathcal{P}})^{-1} 2\frac{\partial K_E^{\mathcal{P}}}{\partial t}(I-K_E^{\mathcal{P}})^{-1}p, q\Big\rangle \\
&\quad + \Big\langle (I-K_E^{\mathcal{P}})^{-1} 2\frac{\partial p}{\partial t}, q\Big\rangle + \Big\langle (I-K_E^{\mathcal{P}})^{-1}p, 2\frac{\partial q}{\partial t}\Big\rangle \\
&= \Big\langle \int_E dy\,(-p'(x)q(y)+p(x)q'(y))\,\hat{p}(y), \Big((I-K_E^{\mathcal{P}^\top})^{-1}q\Big)(x)\Big\rangle \\
&\quad - \Big\langle p''(x), \Big((I-K_E^{\mathcal{P}^\top})^{-1}q\Big)(x)\Big\rangle + \Big\langle (I-K_E^{\mathcal{P}})^{-1}p(x), q''(x)\Big\rangle \\
&= u(x)(-\langle p',\hat{q}\rangle + \langle q',\hat{p}\rangle) - \langle p'',\hat{q}\rangle + \langle \hat{p}, q''\rangle,\ \text{using } \langle p,\hat{q}\rangle = \langle \hat{p},q\rangle = u, \\
&= \mathbf{O}(y-x)^2 - \langle p'',q\rangle + \langle p,q''\rangle \\
&= -(y-x)(p''(x)q(x)-p(x)q''(x)) + \mathbf{O}(y-x)^2,
\end{aligned}$$

thus ending the proof of Lemma 6.3. ■

Proof of Proposition 6.1: One computes the following Wronskian and expand for small $y-x$, given $E=(x,y)$. Indeed, from $\partial_E u = \partial_E^2 \log \mathbb{P}^{\mathcal{P}}$(see Lemma 6.2), the estimates of Lemma 6.3 and the the differential equations (1.4) for p and q, one computes

$$\begin{aligned}
&\left\{\frac{\partial}{\partial t}\partial_E^2 \log \mathbb{P}^{\mathcal{P}}, \partial_E^3 \log \mathbb{P}^{\mathcal{P}}\right\}_{\partial_E} \\
&= \left\{\partial_E \frac{\partial u}{\partial t}, \partial_E^2 u\right\}_{\partial_E} \\
&= -\frac{(y-x)^2}{2}\left(\{(p''q-pq'')', (pq)''\}_x + \mathbf{O}(y-x)\right) \\
&= \frac{(y-x)^2}{2}\left(\begin{array}{c} (-t(p''q-pq'')+3x(pq)'+2pq)(pq)'' + \mathbf{O}(y-x) \\ +(t(p'q-pq')-2xpq+p''q'-p'q'')(t(pq)'+3(p'q')') \end{array}\right)
\end{aligned}$$

with the coefficient of $(y-x)^2/2$, for $x = t = 0$ being equal to

$$2pq(pq)'' - 3(p'q')'(p'q'' - p''q') \neq 0$$

which is nonzero, ending the proof of Proposition 6.1. ∎

7 Steepest descent analysis and replica duality

In this section, we emphasize the central role played by the spectral curve (or Pastur equation [24]) in the steepest descent analysis used to prove the universal behavior of the kernel as $N \to \infty$. More precisely, we point out that the kernel used for the steepest descent analysis takes a very universal form in the different problems of non-intersecting Brownian motions studied so far. We further give show that the study of the spectral curve associated to the considered problem gives the universal limiting kernel obtained after a proper rescaling of the variable around a singularity of the problem: we give a "physical meaning" to the computations performed in the preceding sections as well as a way to generalize it to more complicated problems.

In a first part, we show how this spectral curve arises in an integral representation of the kernel in the case of the matrix model in an external field. We then show how the expression of this kernel in terms of the spectral curves exhibits a universal behavior in the large matrix limit depending on the local properties of this curve. We finally apply this procedure to prove the appearance of the Pearcy and Airy kernels in the context described in the previous sections.

7.1 From Hermitian matrix integrals to double contour integrals

In this section, we derive the a double integral representation of the kernel by using the replica formulation introduced by Brézin and Hikami [9]. In order to make this paper self-contained, we show this duality by simply performing gaussian integrals.

Let us consider the partition function

$$Z(A) := \int_{H_N} dM e^{-N \operatorname{Tr}\left(\frac{M^2}{2} - AM\right)} \tag{7.1}$$

where one integrates over hermitian matrices M of size $N \times N$ and A is a deterministic diagonal matrix, with arbitrary k[13], of the form

$$A := \mathrm{diag}(\overbrace{a_1, \dots, a_1}^{n_1}, \overbrace{a_2, \dots, a_2}^{n_2}, \dots, \overbrace{a_k, \dots, a_k}^{n_k}). \tag{7.2}$$

Diagonalizing the matrix M and using the HCIZ formula [16, 17], one is left with the integration over the eigenvalues $(x_1, x_2, \dots, x_N)$ of M:

$$Z(A) = \int_{\mathbb{R}^N} \prod_{i=1}^{N} dx_i \frac{\Delta(x)}{\Delta(a)} e^{-N \sum_i \left(\frac{x_i^2}{2} - x_i a_i\right)}. \tag{7.3}$$

In order to compute the density of state $R_1(\lambda)$ and the ℓ-point correlation functions R_ℓ defined by

$$R_k(\lambda_1, \lambda_2, \dots, \lambda_\ell) = \frac{1}{N^\ell} \left\langle \prod_{i=1}^{\ell} \mathrm{Tr}\, \delta(\lambda_i \mathbb{I} - M) \right\rangle, \tag{7.4}$$

where the average is taken with respect to the probability measure

$$\frac{1}{Z(A)} \prod_{i=1}^{N} dx_i \frac{\Delta(x)}{\Delta(a)} e^{-N \sum_i \left(\frac{x_i^2}{2} - x_i a_i\right)}, \tag{7.5}$$

we consider their "Fourier" transforms

$$U_l(t_1, t_2, \dots, t_l) = \left\langle \prod_{i=1}^{l} \mathrm{Tr}\, e^{iNt_iM} \right\rangle \tag{7.6}$$

and, in particular, one gets the Fourier transform of the two points correlation function:

$$U_2(t_1, t_2) = \frac{1}{Z(A)N^2} \sum_{\alpha_1, \alpha_2 = 1}^{N} \int \left(\prod_{j=1}^{N} dx_j \right) \frac{\Delta(x)}{\Delta(a)} e^{-N \sum_{j=1}^{N} \left[\frac{x_j^2}{2} - x_j \left(a_j + it_1 \delta_{j,\alpha_1} + it_2 \delta_{j,\alpha_2} \right) \right]}. \tag{7.7}$$

[13]The previous section considers the particular case $k = 2$.

One can now integrate the variables x_j by noting that

$$\int \left(\prod_{j=1}^{N} dx_j\right) \Delta(x) e^{-N\sum_{j=1}^{N}\left[\frac{x_j^2}{2}+x_j b_j\right]} = \Delta(b) e^{\frac{N}{2}\sum_{j=1}^{N} b_j^2} \tag{7.8}$$

and expanding $\Delta(x) = \prod_{i\neq j}(x_i - x_j)$:

$$\begin{aligned} & U_2(t_1,t_2) \\ = & \sum_{\alpha_1,\alpha_2=1}^{N} e^{N\left(it_1 a_{\alpha_1} + it_2 a_{\alpha_2} - \frac{t_1^2+t_2^2}{2} - t_1 t_2 \delta_{\alpha_1,\alpha_2}\right)} \times \\ & \times \frac{\prod_{1\le l<m\le N} (a_l - a_m + it_1(\delta_{l,\alpha_1} - \delta_{m,\alpha_1}) + it_2(\delta_{l,\alpha_2} - \delta_{m,\alpha_2}))}{\prod_{1\le l<m\le N} (a_l - a_m)}. \end{aligned} \tag{7.9}$$

One can see that this can be written as a double contour integral

$$\begin{aligned} & U_2(t_1,t_2) \\ = & \frac{e^{-N\frac{t_1^2+t_2^2}{2}}}{t_1 t_2} \oint\oint \frac{dudv}{(2i\pi)^2} e^{Ni(t_1 u + t_2 v)} \frac{(u-v+it_1-it_2)(u-v)}{(u-v+it_1)(u-v-it_2)} \times \\ & \times \prod_k \left(1+\frac{it_1}{u-a_k}\right)\left(1+\frac{it_2}{v-a_k}\right) \\ = & \frac{e^{-N\frac{t_1^2+t_2^2}{2}}}{t_1 t_2} \oint\oint \frac{dudv}{(2i\pi)^2} e^{Ni(t_1 u + t_2 v)} \left(1 - \frac{t_1 t_2}{(u-v+it_1)(u-v-it_2)}\right) \times \\ & \times \prod_k \left(1+\frac{it_1}{u-a_k}\right)\left(1+\frac{it_2}{v-a_k}\right), \end{aligned} \tag{7.10}$$

where the integration contours encircle all the eigenvalues a_k and the pole $v = u - it_1$[14].

We can now go back to the correlation function

$$R_2(\lambda,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dt_1 dt_2}{4\pi^2} e^{-iN(t_1\lambda + t_2\mu)} U(t_1,t_2). \tag{7.11}$$

[14] see for instance [9] for more details around eq.(2-20) and eq.(4-40).

By first integrating on t_1 and t_2 with the shifts $t_1 \to t_1 - iu$ and $t_2 \to t_2 - iu$, this latter equation reads:

$$R_2(\lambda, \mu) = K_N(\lambda, \lambda)K_N(\mu, \mu) - K_N(\mu, \lambda)K_N(\lambda, \mu) \tag{7.12}$$

where the kernel is defined by

$$K_N(\lambda, \mu) = \int \frac{dt}{2\pi} \oint \frac{dv}{2i\pi} \prod_{k=1}^{N} \left(\frac{it - a_k}{v - a_k} \right) \frac{1}{v - it} e^{-N\left(\frac{v^2+t^2}{2} + it\lambda - v\mu\right)} \tag{7.13}$$

where the integration contour for v goes around all the points a_k and the integration for t is parallel to the real axis and avoids the v contour. Moreover, it can be derived in a very similar way that any k-point function can be written as the Fredholm determinant:

$$R_k(x_1, \ldots, x_k) = \det \left[K_N(x_i, x_j)\right]_{i,j=1}^{k}. \tag{7.14}$$

By Wick rotating the integration variable $t \to it$, one gets

$$K_N(\lambda, \mu) = \int \frac{dt}{2i\pi} \oint \frac{dv}{2i\pi} \prod_{k=1}^{N} \left(\frac{t - a_k}{v - a_k} \right) \frac{1}{v - t} e^{-N\left(\frac{v^2-t^2}{2} + t\lambda - v\mu\right)} \tag{7.15}$$

where the integration contour for t is now parallel to the imaginary axis. One can then rewrite it under a more factorized form:

$$K_N(\lambda, \mu) = \int \frac{dt}{2i\pi} \oint \frac{dv}{2i\pi} e^{-N(S(\mu,v) - S(\lambda,t))} \frac{1}{v - t} \tag{7.16}$$

where

$$S(x, y) = \frac{y^2}{2} - xy + \sum_{i=1}^{k} \epsilon_i \ln(y - a_i) \tag{7.17}$$

with the "fraction numbers" given by

$$\epsilon_i := n_i/N. \tag{7.18}$$

The first step in the steepest descent analysis of this kernel is to look for the stationary points of the exponent in the integrand, i.e. we look for y as a function of x solution of the equation

$$\partial_y S(x, y) = y - x + \sum_{i=1}^{k} \frac{\epsilon_i}{y - a_i} = 0 \tag{7.19}$$

which is nothing but the equation of the classical spectral curve introduced in the preceding sections (see 3.18) and in the general study of the one matrix model in an external field.

Remark: If the external matrix A is highly degenerated, the ϵ_i are fixed and do not depend in N. Thus, the action $S(x, y)$ does not depend on N in this case.

Remark: The density of states is given by $\rho(\lambda) = K_N(\lambda, \lambda)$ and its derivative wrt λ can be factorized:

$$\frac{1}{N}\frac{\partial\rho(\lambda)}{\partial\lambda} \;=\; \int \frac{dt}{2i\pi} \oint \frac{dv}{2i\pi} e^{-N(S_(\lambda,v)-S(\lambda,t))} = \phi(\lambda)\psi(\lambda) \tag{7.20}$$

where

$$\phi(\lambda) = \int_{i\mathbb{R}} \frac{idt}{2\pi} e^{NS(\lambda,t)} \quad \text{and} \quad \psi(\lambda) = \oint \frac{dv}{2i\pi} e^{-NS(\lambda,v)}. \tag{7.21}$$

7.2 Saddle points, spectral curve and universality

Let us now forget about the matrix model and consider a general kernel of the form

$$K_N(\lambda, \mu) = (-1)^N \int \frac{idt}{2\pi} \oint \frac{dv}{2i\pi} e^{-N(S(\mu,v)-S(\lambda,t))} \frac{1}{v-t} \tag{7.22}$$

where the derivative of the action $S(x, y)$:

$$\partial_y S(x, y) = \frac{\mathcal{E}(x, y)}{D(x, y)} \tag{7.23}$$

is a rational function which can be written as the ratio of two polynomials in both variables. Then the locus of stationary points of S is given by an algebraic equation

$$\mathcal{E}(x, y) = 0, \tag{7.24}$$

referred to as the *spectral curve* in the sequel. Let us study some of its properties necessary to classify the different universal behavior of $K_N(\lambda, \mu)$ as $N \to \infty$.

Generically, the equation $\mathcal{E}(x, y) = 0$ has d_y distinct solutions in y for a given value of x. Let us denote them $Y_i(x)$ as functions of x:

$$\mathcal{E}(x, y) = g_{d_y}(x) \prod_{i=1}^{d_y} (y - Y_i(x)) \tag{7.25}$$

where $g_{d_y}(x)$ is the leading coefficient of $\mathcal{E}(x,y)$ as a polynomial in y. However, there exists finitely many branch points x_i such that $\mathcal{E}(x,y)=0$ has a double zero, i.e. two solutions $Y_j(x_i)=Y_l(x_i)$ coincide. One can also characterize them by the property:

$$\partial_y \, \mathcal{E}(x_i,y)\big|_{y:=Y_j(x_i)} = 0. \tag{7.26}$$

More generally, a ℓth order branch point $x_i^{(\ell)}$ is defined by

$$\partial_y^m \, \mathcal{E}(x_i^{(\ell)},y)\Big|_{y:=Y_j(x_i^{(l\ell)})} = 0, \text{for} \;\; m \leq \ell, \quad \text{ with } \; \partial_y^{\ell+1} \, \mathcal{E}(x_i^{(\ell)},y)\Big|_{y:=Y_j(x_i^{(\ell)})} \neq 0. \tag{7.27}$$

Other types of singularities might occur, but we will not refer to them in this paper.

Remark: A $(\ell+1)$th order branch point can be obtained when a ℓth order branch point and a simple branch point merge.

In the next sections, we show that, as $N \to \infty$, the kernel $K_N(\lambda,\mu)$ has a universal behavior for λ and μ approaching the same point with an appropriate scaling (depending on N and the singular behavior of the spectral curve at this point). We study the first singularities and show this universal behavior by a *local analysis* of the spectral curve whereas the usual Riemann-Hilbert study [**?**] involves a global analysis of the latter.

7.3 Steepest descent analysis

The asymptotic of this kernel when $N \to \infty$ exhibits different regimes depending on the value of its argument. More precisely, we get a universal kernel associated to the neighborhood of any point x_0 of the spectral curve: if x_0 is a generic point of the spectral curve, we get the sine kernel; if x_0 is a simple branch point, we get the Airy kernel; if x_0 is a higher order singularity of the curve, we get a universal kernel associated to this particular singularity.

In any case, we proceed with the same steps: we choose a point x_0. We then focus on its neighborhood by a change of variable consistent with the singularity of the curve at x_0. We expand the exponent in the kernel around this point by simply writing a Taylor series expansion.

7.3.1 Simple branch point: the Airy kernel

Consider a one-parameter family of algebraic functions $S(x,y|t)$, parametrized by t, such that there exists a critical point (x_c,y_c,t_c) satisfying

$$S_y(x_c,y_c|t_c) = S_{yy}(x_c,y_c|t_c) = 0, \tag{7.28}$$

$$S_{yyy}(x_c,y_c|t_c) \neq 0, \quad S_{yx}(x_c,y_c|t_c) \neq 0, \quad S_{yt}(x_c,y_c|t_c) \neq 0. \tag{7.29}$$

It means that the spectral curve $\mathcal{E}(x,y|t_c)$ has a simple branch point at (x_c,y_c). One shows that in the neighborhood of this branch point, the kernel can be rescaled in such a way that it converges to the Airy kernel as $N\to\infty$. Consider the changes of variables allowing to focus on the neighborhood of the critical point:

$$\begin{cases} t := t_c + \frac{\alpha_t T}{N^{\frac{1}{3}}} \\ x := x_c + \frac{\alpha_x}{N^{\frac{1}{3}}} + \frac{\beta_x X}{N^{\frac{2}{3}}} \\ y := y_c + \frac{\alpha_y Y}{N^{\frac{1}{3}}} \end{cases}, \tag{7.30}$$

and expand $S(x,y|t)$ around this critical point using (7.30) as $N\to\infty$. One then expands $S(x,y|t)$ and $S(\tilde{x},\tilde{y}|t)$ in aTaylor series in t, x and y, thus obtaining for the expression in the integrand of the kernel:

$$\begin{aligned}
&N\left(S(x,y;t) - S(\tilde{x},\tilde{y};t)\right) \\
&= N^{\frac{1}{3}}\Big(\alpha_y\left[\alpha_x S_{xy}(x_c,y_c;t_c) + \alpha_t T S_{ty}(x_c,y_c;t_c)\right](Y-\widetilde{Y}) + \beta_x X S_x(x_c,y_c;t_c)\Big) \\
&\quad +(Y^3-\widetilde{Y}^3)\frac{\alpha_y^3}{6}S_{yyy}(x_c,y_c;t_c) \\
&\quad +(Y^2-\widetilde{Y}^2)\frac{\alpha_y^2}{2}\left[\alpha_x S_{xyy}(x_c,y_c;t_c) + \alpha_t T S_{tyy}(x_c,y_c;t_c)\right] \\
&\quad +(XY-\widetilde{X}\widetilde{Y})\beta_x\alpha_y S_{xy}(x_c,y_c;t_c) \\
&\quad +(X-\widetilde{X})\beta_x\left[\alpha_x S_{xx}(x_c,y_c;t_c) + \alpha_t T S_{tx}(x_c,y_c;t_c)\right] \\
&\quad +(Y-\widetilde{Y})\alpha_y\left[\frac{\alpha_x^2}{2}S_{xxy}(x_c,y_c;t_c) + \alpha_x\alpha_t T S_{txy}(x_c,y_c;t_c) + \frac{\alpha_t^2}{2}T^2 S_{tty}(x_c,y_c;t_c)\right] \\
&\quad +O(N^{-\frac{1}{3}});
\end{aligned} \tag{7.31}$$

which we write for brevity as

$$\begin{aligned}
&N\left(S(x,y;t) - S(\tilde{x},\tilde{y};t)\right) = \\
&= N^{\frac{1}{3}}\left[\alpha_1\left(Y-\widetilde{Y}\right) + \alpha_2\left(X-\widetilde{X}\right)\right] + \alpha_3\left(Y^3-\widetilde{Y}^3\right) + \alpha_4\left(Y^2-\widetilde{Y}^2\right) \\
&\qquad +\alpha_5\left(XY-\widetilde{X}\widetilde{Y}\right) + \alpha_6\left(Y-\widetilde{Y}\right) + \alpha_7\left(X-\widetilde{X}\right)
\end{aligned} \tag{7.32}$$

for coefficients $\{\alpha_i\}_{i=1}^7$ functions of the scaling parameters α_t, α_x, β_x and α_y.

The coefficients α_2 and α_7 can be eliminated by conjugation of the kernel leaving the Fredholm determinants invariant. In order to recover the Airy kernel, one has to fix the remaining coefficients by

$$\alpha_1 = 0 \quad , \quad \alpha_3 = 1/3 \quad , \quad \alpha_5 = -1 \quad \text{and} \quad \alpha_6 - \alpha_4^2 = 0. \tag{7.33}$$

α_y and β_x are determined by the constraints on α_3 and α_5 respectively, while the first equation gives $\frac{\alpha_t T}{\alpha_x}$ and the last one determines α_x^2.

7.3.2 Double branch point: the Pearcey kernel

Let us now consider an algebraic a function $S(x, y|t)$ such that there exists a critical point (x_c, y_c, t_c) satisfying

$$S_y(x_c, y_c|t_c) = S_{yy}(x_c, y_c|t_c) = S_{yyy}(x_c, y_c|t_c) = 0, \tag{7.34}$$

$$S_{yyyy}(x_c, y_c|t_c) \neq 0, \quad S_{yx}(x_c, y_c|t_c) \neq 0, \quad S_{yt}(x_c, y_c|t_c) \neq 0. \tag{7.35}$$

It means that the spectral curve $\mathcal{E}(x, y|t_c)$ has a double branch point at (x_c, y_c). In the neighborhood of this critical point, one can rescale the kernel in such a way that it converges to the Pearcey kernel as $N \to \infty$. To this effect, consider the changes of variables in the neighborhood of the critical point:

$$\begin{cases} t := t_c + \frac{\alpha_t T}{N^{\frac{1}{2}}} \\ x := x_c + \frac{\alpha_x}{N^{\frac{1}{2}}} + \frac{\beta_x X}{N^{\frac{3}{4}}} \\ y := y_c + \frac{\alpha_y Y}{N^{\frac{1}{4}}} \end{cases} \tag{7.36}$$

and expand $S(x, y|t)$ around the critical point using (7.36) as $N \to \infty$.

$$\begin{aligned} & N\left(S(x, y; t) - S(\tilde{x}, \tilde{y}; t)\right) = \\ & = N^{\frac{1}{4}}\left[\beta_x S_x(x_c, y_c; t_c)(X - \widetilde{X}) + \alpha_y (Y - \widetilde{Y})\left(\alpha_x S_{xy}(x_c, y_c; t_c) + \alpha_t T S_{ty}(x_c, y_c; t_c)\right)\right] \\ & + \frac{\alpha_y^4}{24} S_{yyyy}(x_c, y_c; t_c)(Y^4 - \widetilde{Y}^4) \\ & + \frac{\alpha_y^2}{2}\left[\alpha_t T S_{tyy}(x_c, y_c; t_c) + \alpha_x S_{xyy}(x_c, y_c; t_c)\right](Y^2 - \widetilde{Y}^2) \\ & + \beta_x \alpha_y S_{xy}(x_c, y_c; t_c)(YX - \widetilde{Y}\widetilde{X}) + O(N^{-\frac{1}{4}}). \end{aligned} \tag{7.37}$$

<u>*Remark:*</u> One can see that (as in the previous case) all the terms in this expression, except the scaling factor $\beta_x S_x(x_c, y_c; t_c)(X - \widetilde{X})$, depend only on derivatives of S

wrt y. This shows that the kernel depends only on the spectral curve $\mathcal{E}(x,y|t) := S_y(x,y|t)$.

One can see that the variables to be integrated Y and $\widetilde{Y}$ appear only in terms which do not blow up as $N \to \infty$ except one which is proportional to $N^{\frac{1}{4}}$. One can get rid of this term by fine-tuning the coefficients of the change of variable. Indeed, imposing the constraint on $\frac{\alpha_t T}{\alpha_x}$,

$$\alpha_x S_{xy}(x_c, y_c; t_c) + \alpha_t T S_{ty}(x_c, y_c; t_c) = 0 \tag{7.38}$$

eliminates this term. One can finally normalize the scaling coefficients α_y, α_x and β_x respectively by setting

$$\frac{\alpha_y^4}{24} S_{yyyy}(x_c, y_c; t_c) = -\frac{1}{4}, \qquad \beta_x \alpha_y S_{xy}(x_c, y_c; t_c) = -1 \tag{7.39}$$

$$\frac{\alpha_x \alpha_y^2}{2}\left[\frac{\alpha_t T}{\alpha_x} S_{tyy}(x_c, y_c; t_c) + S_{xyy}(x_c, y_c; t_c)\right] = \frac{\tau}{2}, \tag{7.40}$$

yielding the quartic exponent appearing in the Pearcey kernel:

$$\begin{aligned} &N\left(S(x,y;t) - S(\tilde{x},\tilde{y};t)\right) \\ &= N^{\frac{1}{4}} \beta_x S_x(x_c, y_c; t_c)(X - \widetilde{X}) \\ &\quad + \frac{1}{4}(Y^4 - \widetilde{Y}^4) - \frac{\tau}{2}(Y^2 - \widetilde{Y}^2) + (YX - \widetilde{Y}\widetilde{X}) + O(N^{-\frac{1}{4}}), \end{aligned} \tag{7.41}$$

where the term in $(X - \widetilde{X})$ can be eliminated by conjugation of the kernel.

7.3.3 kth order branch point

More generally, consider a point $(x_c, y_c; t_c)$ of the spectral curve $\mathcal{E}(x_c, y_c; t_c)$ where[15]

$$\partial_y^m \mathcal{E}(x_c, y_c; t_c) = 0, \quad \text{for all } m \leq l, \quad \text{with } \partial_y^{l+1}\mathcal{E}(x_c, y_c; t_c) \neq 0, \tag{7.42}$$

while still assuming that $\partial_t \mathcal{E}$ and $\partial_x \mathcal{E}$ do not vanish at this critical point.

Let us use this example to explain how one can guess the rescaling to obtain the desired universal kernel. Generically, one looks for a rescaling of the form:

$$\left\{ \begin{array}{l} t := t_c + \frac{\alpha_t T}{N^{\gamma_t}} \\ x := x_c + \frac{\alpha_x X}{N^{\gamma_x}} \\ y := y_c + \frac{\alpha_y Y}{N^{\gamma_y}} \end{array} \right. \tag{7.43}$$

[15]Remark that the Airy case corresponds to $l = 1$ and the Pearcey case to $l = 2$.

for some critical exponents γ_t, γ_x and γ_y to be determined. One then writes the Taylor expansion of the action S around the critical point using this rescaling: this expansion can be seen as a series in $N^{-\gamma}$ for some exponent γ depending on the rescaling. Since the action is multiplied by N in the kernel and integrating over the variable y, one needs that all the terms depending on Y in this expansion are of order $\frac{1}{N}$ at most as $N \to \infty$. In particular, considering the Taylor series with respect to y:

$$S(x_c, y; t_c) = S(x_c, y_c; t_c) + \cdots + \frac{(\alpha_y Y)^{l+2}}{(l+2)!} \partial_y^{l+1} \mathcal{E}(x_c, y_c; t_c) N^{-(l+2)\gamma_y} + o\left(N^{-(l+2)\gamma_y}\right),$$

one must impose $(l+2)\gamma_y = 1$ for this contribution to appear, thus fixing the critical exponent for y:

$$\gamma_y = \frac{1}{l+2}. \tag{7.44}$$

On the other hand, one expects x and y to couple in the exponent of the limiting integrand (otherwise, it would only give a simple multiplicative factor). Thus, the first contribution of a mixed derivative wrt to x and y of the action must be of order $\frac{1}{N}$ which imposes:

$$\gamma_x = 1 - \gamma_y = \frac{l+1}{l+2}. \tag{7.45}$$

One can remark that we could stop the procedure here and not rescale the time which could remain decoupled from x and y. But one could also try to couple it with y by taking $\alpha_t \neq 0$. If one wants this coupling to be different from the one between x and y, one must impose $\gamma_t < \gamma_x$ and one gets, in the simplest case[16]

$$\gamma_t = 1 - 2\gamma_y = \frac{l}{l+2}. \tag{7.46}$$

By doing so, one has coupled t to y. But one has also introduced a divergent term coming from the coefficient of S_{ty}. One can compensate this term by completing the change of variable for x and writing:

$$\begin{cases} t := t_c + \frac{\alpha_t T}{N^{\frac{l}{l+2}}} \\ x := x_c + \frac{\beta_x}{N^{\frac{l}{l+2}}} + \frac{\alpha_x X}{N^{\frac{l+1}{l+2}}} \\ y := y_c + \frac{\alpha_y Y}{N^{\frac{1}{l+2}}} \end{cases}. \tag{7.47}$$

[16]This means that one wants to have a coupling TY^2, but one could also obtain in a more complicated way higher order times by looking for couplings of the form TY^k with $k > 1$.

Doing so, one finally gets the associated kernel in terms of the rescaled variables:

$$K_k(X,X') = \int dY \int dY' e^{\frac{Y^l}{l!} - \frac{TY^2}{2} + XY} e^{-\frac{Y'^l}{l!} - \frac{TY'^2}{2} + X'Y'} \frac{1}{Y-Y'}. \qquad (7.48)$$

<u>*Remark:*</u> One omits a detailed proof here, although the details can be filled in. Indeed, in order to do so, one carefully studies the integration contours and the normalization, which implies a rescaling of the kernel as well as conjugation. It is done in the preceding sections for the Airy and Pearcey cases for two ending points.

7.4 General case

In a more general case, i.e. the two arguments x and y of K_N approaching any singular or non-singular point of the algebraic curve, one can use the same study leading to a different universal limiting kernel associated to each type of singularity. Let us summarize the procedure one has to follow to obtain the universal kernel associated to a given singularity:

- First compute the multiple derivatives of the action $S(x,y;t)$ with respect to x, y and t at the considered critical point and fix which are the first non-vanishing derivatives;
- Fix the critical exponent in the rescaling by studying the large N behavior of the Taylor expansion of the action around the critical point, under consideration;
- Normalize the change of variables and compute the rescaling of the kernel through the Taylor expansion of the action of the action around the critical point: one then gets the universal limiting kernel;
- Finally, carefully study the integration contours to check that they give a right path for the steepest descent analysis.

7.5 Example: back to the matrix model and non-intersecting Brownian motions

Let us now apply the preceding procedure to the case of the random matrix model in an external field studied in the preceding sections: one considers N Brownian motions starting from 0 and going to two ending points at $t = 1$.

As reminded in section 2 of this paper, the kernel, given in Eq. (2.2), can easily be taken to the form of Eq. (7.22) :

$$H_n(\bar{\mu},\bar{\lambda};t)=\int\frac{idu}{2\pi}\oint\frac{dv}{2i\pi}e^{-n(S(\bar{\mu},v;t)-S(\bar{\lambda},u;t))}\frac{1}{v-u} \tag{7.49}$$

by using the rescaling

$$v:=\frac{Vt}{c\sqrt{n}}\,,\ u:=\frac{Ut}{c\sqrt{n}}\,,\ \tilde{a}:=\frac{at}{c\sqrt{n}}\,,\ \tilde{b}:=\frac{bt}{c\sqrt{n}}\,,\ \mu:=\frac{x}{\sqrt{n}}\text{ and }\lambda:=\frac{y}{\sqrt{n}} \tag{7.50}$$

with $\frac{t}{c}=\sqrt{\frac{2t}{1-t}}$ and the action

$$S(x,u;t):=\frac{u^2}{2}-x\frac{u}{c}+p\log(u-\tilde{a})+(1-p)\log(u-\tilde{b}) \tag{7.51}$$

which is nothing but the function $S(u,x;t)=F(u)$ of Eq. (3.18) studied in section 3. The study of this section, and particularly Eq. (3.20), states that

$$S_u(x_0,u_0;t_0)=S_{uu}(x_0,u_0;t_0)=S_{uuu}(x_0,u_0;t_0)=0 \tag{7.52}$$

at the critical branch point (x_0,u_0) of the spectral curve

$$S_u(x,u;t_0)=u-\frac{x}{c_0}+\frac{p}{u-\tilde{a}_0}+\frac{1-p}{u-\tilde{b}_0}=0 \tag{7.53}$$

where one notes $c_0=c(t_0)$, $\tilde{a}_0=\frac{at_0}{c_0}$ and $\tilde{b}_0=\frac{bt_0}{c_0}$. Moreover, using the notations and computations of section 3, one has:

$$S_{ux}(x_0,u_0;t_0)=-\frac{1}{c_0},\qquad\frac{S_{uuuu}(x_0,u_0;t_0)}{4!}=-\frac{q^2-q+1}{4q} \tag{7.54}$$

and

$$S_{ut}(x_0,u_0;t_0)=\frac{x_0(1-2t_0)}{2c_0t_0(1-t_0)}+\frac{p\tilde{a}_0c_0^2}{t_0^2(1-t_0)^2\,(u-\tilde{a}_0)^2}+\frac{(1-p)\tilde{b}_0c_0^2}{t_0^2(1-t_0)^2(u-\tilde{b}_0)^2}. \tag{7.55}$$

This implies we are in the case of a double branch point studied in section 7.3.2 and thus, with the right rescaling, the kernel converges to the Pearcey kernel as $n\to\infty$. Let us now check the conditions to obtain precisely the Pearcey kernel. From the study of section 7.3.2, one must consider the rescaling

$$\left\{\begin{array}{l} t := t_0 + \frac{\alpha_t \tau}{N^{\frac{1}{2}}} \\ x := x_0 + \frac{\alpha_x}{N^{\frac{1}{2}}} + \frac{\beta_x X}{N^{\frac{3}{4}}} \\ u := u_0 + \frac{\alpha_u Y}{N^{\frac{1}{4}}} \end{array}\right. \tag{7.56}$$

with the conditions

$$\alpha_x S_{xu}(x_0, u_0; t_0) + \alpha_t \tau S_{tu}(x_0, u_0; t_0) = 0 \tag{7.57}$$

$$\frac{\alpha_u^4}{24} S_{uuuu}(x_0, u_0; t_0) = -\frac{1}{4}, \tag{7.58}$$

$$\frac{\alpha_u^2}{2}\left[\alpha_t \tau S_{tuu}(x_0, u_0; t_0) + \alpha_x S_{uux}(x_0, u_0; t_0)\right] = \frac{\tau}{2} \tag{7.59}$$

and

$$\beta_x \alpha_u S_{ux}(x_0, u_0; t_0) = -1. \tag{7.60}$$

Let us check that these conditions agree with the rescalling Eq. (3.17). From conditions (7.58) and (7.60), one gets (μ was defined in Eq. (1.10))

$$\alpha_u = \frac{1}{\mu}, \quad \beta_x = c_0 \mu, \tag{7.61}$$

whereas Eq. (7.59) and Eq. (7.57) give

$$\alpha_t = \frac{(1-t_0)^2 c_0^2}{2} \frac{1}{\frac{p\tilde{a}_0}{(u_0-\tilde{a}_0)^3} + \frac{(1-p)\tilde{b}_0}{(u_0-\tilde{b}_0)^3}} \left(\frac{q^2-q+1}{q}\right)^{\frac{1}{2}} = 2c_0^2 \mu^2 \tag{7.62}$$

and Eq. (7.57) states that

$$\begin{aligned} \alpha_x \quad &= \quad \tau \left[\frac{x_0(1-t_0)(1-2t_0)}{2c_0} + \frac{p\tilde{a}_0 c_0^2}{t_0(u_0-\tilde{a}_0)^2} + \frac{(1-p)\tilde{b}_0 c_0^2}{t_0(u_0-\tilde{b}_0)^2}\right] \\ &\quad \times \frac{\mu^2}{\frac{p\tilde{a}_0}{(u_0-\tilde{a}_0)^3} + \frac{(1-p)\tilde{b}_0}{(u_0-\tilde{b}_0)^3}}. \end{aligned}$$

It is then easily checked by plugging in the values of x_0, t_0 and u_0 in terms of q (with Maple for example) that this rescaling coincides with Eq. (3.17), that is to say, the rescaling considered in the preceding part.

7.6 Application: Brownian bridges from one point to k points

We can also use this analysis to study more general statistical systems. Let us now consider N non-intersecting Brownian motions stating from 0 at time $t=0$ and ending at k points a_i by groups of n_i particles where $n_i = \epsilon_i N$. It is a classical result that their probability measure at a given time t is given by the probability measure of the eigenvalues of an Hermitian random matrix in an external field with external matrix $A(t)$ whose eigenvalues are given by $a_i(t) = \sqrt{\frac{2t}{1-t}} a_i$.

Then, the spectral curve is given by:

$$\mathcal{E}(x,y) = y - x + \sum_{i=1}^{k} \frac{\epsilon_i}{y - a_i(t)} = 0 \tag{7.63}$$

and has genus zero. One can straightforwardly (just solving the equation in x) find a rational parametrization under the form:

$$x(z) = z + \sum_{i=1}^{k} \frac{\epsilon_i}{(z - a_i(t))}, \qquad y(z) = z \tag{7.64}$$

and the branch points are given by

$$x'(z) = 1 - \sum_{i=1}^{k} \frac{\epsilon_i}{(z - a_i(t))^2} = 0. \tag{7.65}$$

For a generic time t, one has $2k$ branch points, $2l$ of them $\{z_{2i-1}, z_{2i}\}_{i=1}^{l}$ being real and lying in the so-called physical sheet of the spectral curve (see [6, 8, 23] for an extensive study of the spectral curve). Thus, by he study of section 3 and 7.3, one can conclude that the kernel converges to the Airy kernel in the neighborhood of these real branch points z_i recovering the results of [23].

Now, as the time decreases from 1 to 0, some real branch points $z_{2i}(t)$ and $z_{2i+1}(t)$ merge for some critical value of the time t_c. In the Brownian motion setting, it correspond to the times when one big group of particles splits into two smaller. In terms of the family of spectral curves parameterized by the time t, it corresponds to the merging of two cuts. In the neighborhood of this double branch point, thanks to the study of section 7.3, one can rescale the kernel in such a way that it converges to the Pearcey kernel as $N \to \infty$. This generalizes the result of th. 1.1 to any cusp in this kind of processes and recovers the former results of [23].

It is then natural to ask the question: Can we have higher order singularity in these processes? When the ϵ_i's are independent of N and the a_i's real, the answer is no.

Indeed, it amounts to knowing what is the highest order possible for a real root of Eq. (7.65). This problem can be rephrased as knowing the higher order possible for a real root of the equation

$$T = \sum_{i=1}^{k} \frac{\epsilon_i}{(z - a_i)^2}, \tag{7.66}$$

with the constraint $\sum_i \epsilon_i = 1$ and the rescaled time evolution $T = \frac{2t}{1-t} \in [0, \infty]$.

One can prove that the real roots of this equation are at most double. For this purpose, let us follow the evlution of the roots as t decreases from 1 to 0, i.e. for T going from ∞ to 0.

For T large, this equation has obviously $2k$ simple real roots z_i located around their $T \to \infty$ value, i.e. $|z_{2i-1} - a_i| \ll 1$ and $|z_{2i} - a_i| \ll 1$ with $z_{2i-1} < a_i < z_{2i}$. Now, for any real solution of this equation, one can compute

$$\frac{dz_j}{dT} = -\frac{1}{2\sum_i \frac{\epsilon_i}{(z_j - a_i)^3}}. \tag{7.67}$$

This derivative does not change sign as long as z_j does not cross any a_i. It means that z_{2i} (resp. z_{2i+1}) keeps on going from a_i to a_{i+1} (resp. from a_{i+1} to a_i) as T decreases, unless it reaches another real root. Following this process one sees that the two real roots z_{2i} and z_{2i+1} meet for some critical time T_c giving birth to a double real root of this equation. In this first part of the process, one can thus only encounter double real roots.

Let us keep on decreasing time. For $T < T_c$ and close to it, the double root gives rise to two simple complex conjugated roots. Let us thus now consider a simple complex root $z = r + i\theta$. For θ close to 0, one can compute

$$\frac{d\theta}{dT} = -\frac{1}{6\sum_i \frac{\epsilon_i \theta}{(r - a_i)^4}} \tag{7.68}$$

to first order in θ. The complex roots are thus repelled by the real axis and cannot thus give birth to real roots. In this second part of the process, one do not have real roots anymore. The only multiple real roots are thus obtained when $a_i < z_{2i} \to z_{2i+1} < a_{i+1}$.

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