Group factorization, moment matrices and Toda lattices*

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The following differential equations have arisen in the study of two-matrix integrals by the authors [2],

$$\frac{\partial M}{\partial t_n} = \Lambda^n M \qquad \frac{\partial M}{\partial s_n} = -M\Lambda^{\top n}, \quad n = 1, 2, ..., \tag{1}$$

where M is a semi-infinite matrix $M = (M_{ij})_{0 \le i,j < \infty}$; the matrix $\Lambda = (\delta_{i,j-1})_{i,j \ge 0}$ is the customary shift matrix.

A variation of the equations (1) have appeared in the study of Fredholm determinants for Ising correlation functions by Harold Widom [22],

$$\frac{\partial M}{\partial t_n} = [\Lambda^n, M] \qquad \frac{\partial M}{\partial s_n} = [\Lambda^{\top n}, M], \quad n = 1, 2, \ldots. \tag{2}$$

for a certain bi-infinite matrix M and the bi-infinite shift matrix Λ; for background and references, see [18] and [5].

Using different methods, both studies [2] and [22] reach similar conclusions: upon factorizing a moment-like matrix M, equations (1) and (2) lead

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to the two-Toda lattice equations, which are deformations of a couple of matrices (L_1, L_2).

Widom uses (rigorous) functional analysis to connect (2) with the two-Toda lattice, whereas we use the theory of so-called string-orthogonal polynomials. Such ideas go back to the early days, where integrable theory arose naturally in the context of group factorization. A connection with inverse scattering theory was made in [7, 17] and with the sine-Gordon equation in [16].

The main result (section 1) of this paper is to show how a general set-up, involving a group factorization and a set of commuting elements in the corresponding Lie algebra, leads to a set of vector fields on the group, thus unifying equations (1) and (2). These equations lead, upon factorizing an element in the group, to a general two-Toda lattice. For the group $GL(\infty)$, we find the customary two-Toda lattice and the related sine-Gordon like and modified KP equations (section 2).

If the initial condition for equations (1) and (2) satisfies a certain commutation relation, we are lead to two different reductions from 2-Toda to 1-Toda: equation (1) corresponds to the reduction $L_1 = L_2$, implying that $L_1 = L_2$ is tridiagonal; this reduction is explained in Corollary 1.2. Equation (2) corresponds to the reduction $L_1 + L_1^{-1} = L_2 + L_2^{-1}$, implying that $L_1 + L_1^{-1} = L_2 + L_2^{-1}$ is tridiagonal; this reduction was studied in [20].

We discuss, in some detail, the two kinds of equations (1) and (2). In the case (1), discussed in section 3, M is a moment matrix with regard to a certain weight. The Borel decomposition amounts to the study of string-orthogonal polynomials. We give various representations for those polynomials and provide their evolution in time, compatible with equations (1). We also give an example of a string-orthogonal polynomial for a Gaussian weight. Of course, in this case, much more is known: the determinants, obtained from the upper-left corner of M, satisfy partial differential equations, forming a Virasoro algebra; see [2].

In the context of equation (2), we discuss in section 4 a weight, introduced by Widom; this leads to a bi-infinite moment matrix μ. The determinant of the (semi-infinite) upper-left corner of $M = I - \mu$ makes sense and is connected with the Fredholm determinants of certain kernels, as Widom points out in [22]. We also show that this Fredholm determinant can be written in terms of a vertex operator, taylored to the particular reduction from 2- to 1-Toda, mentioned above.

1 Group factorization and 2-Toda lattice

Given a Lie group G and its Lie algebra g, define for $a \in g$ and $M \in G$,

$$\begin{aligned} aM &:= \frac{\partial}{\partial t}(e^{ta}M)\Big|_{t=0} = R_{M^*}\ a \in g, \\ Ma &:= \frac{\partial}{\partial t}(Me^{ta})\Big|_{t=0} = L_{M^*}\ a \in g, \end{aligned} \tag{3}$$

where R_M and L_M denote right and left multiplication in G.

Let $g = g_- \oplus g_+$ be a (direct sum) vector space decomposition into Lie subalgebras g_- and g_+, with the induced (generic) factorization $G = G_-G_+$. Consider the corresponding vector space decomposition of $g \times g$ into Lie subalgebras

$$g \times g = (g \times g)_u \oplus (g \times g)_\ell, \tag{4}$$

where

$$\begin{aligned} (x, y) &= (x, y)_u + (x, y)_\ell \\ &= (x_+ + y_-, x_+ + y_-) + (x_- - y_-, -x_+ + y_+). \end{aligned}$$

For the sake of Theorem 1.1, define two sets of times $t = (t_1, t_2, ...) \in \mathbb{C}^\infty$ and $s = (s_1, s_2, ...) \in \mathbb{C}^\infty$.

Theorem 1.1. *(i) Consider fixed scalars λ and μ, a Lie algebra decomposition $g = g_- \oplus g_+$, and commuting a_i and $b_i \in g$, as follows:*

$$a_i \in g_+, b_i \in g_-, (i = 1, 2, ...) \text{ with } [a_i, a_j] = [b_i, b_j] = \lambda[a_i, b_j] = \mu[a_i, b_j] = 0. \tag{5}$$

The vector fields,

$$\frac{\partial M}{\partial t_n} = a_nM - \lambda Ma_n, \quad \frac{\partial M}{\partial s_n} = -Mb_n + \mu b_nM, \qquad n = 1, 2, ..., \tag{6}$$

acting on the underlying group ($M \in G$), all commute; their integral curves are given by:

$$M(t, s) = e^{\sum_1^\infty (t_na_n + \mu s_nb_n)}M(0, 0)e^{-\sum_1^\infty (s_nb_n + \lambda t_na_n)}. \tag{7}$$

(ii) Then, upon decomposing

$$M(t, s) = S_1^{-1}S_2 \text{ with } S_1 \in G_-, S_2 \in G_+, \tag{8}$$

the elements $L = L^{(a,b)} \in g \times g$, defined for fixed $a = a_n$ and $b = b_n$ by

$$L := L^{(a,b)} := (L_1^{(a)}, L_2^{(b)}) = (S_1 a S_1^{-1}, S_2 b S_2^{-1}) \in g \times g, \tag{9}$$

satisfy the 2-Toda lattice equations associated with the Lie algebra above:

$$\frac{\partial L}{\partial t_n} = [(L_1^{(a_n)}, 0)_u, L] \text{ and } \frac{\partial L}{\partial s_n} = [(0, L_2^{(b_n)})_u, L], \quad n = 1, 2, \dots . \tag{10}$$

<u>Proof</u>: That the vector fields (6) all commute follows from (7) or, alternatively, from the commutation relations (5); e.g.,

$$\left[\frac{\partial}{\partial s_k}, \frac{\partial}{\partial t_n}\right] M = \mu[a_n, b_k]M + M\lambda[b_k, a_n].$$

We have that $S_1 \dot{M} S_2^{-1} \in g$ for any of the vector fields $\partial/\partial t_n$ or $\partial/\partial s_n$ in (6). Moreover, it can be decomposed in two different ways: on the one hand,

$$S_1 \dot{M} S_2^{-1} = S_1 (S_1^{-1} S_2)^{\cdot} S_2^{-1} = -\dot{S}_1 S_1^{-1} + \dot{S}_2 S_2^{-1} \in g_- + g_+ \tag{11}$$

and, on the other hand, one computes for the $\partial/\partial t_n$ and $\partial/\partial s_n$ vector fields separately:

$$\begin{aligned} S_1 \frac{\partial M}{\partial t} S_2^{-1} &= S_1(aM - \lambda M a)S_2^{-1} \\ &= S_1 a S_1^{-1} - \lambda S_2 a S_2^{-1}, \text{ using } M = S_1^{-1} S_2 \\ &= L_1^{(a)} - \lambda S_2 a S_2^{-1}, \text{ by definition (9)} \\ &= (L_1^{(a)})_- + \left((L_1^{(a)})_+ - \lambda S_2 a S_2^{-1}\right) \in g_- + g_+ \ . \end{aligned} \tag{12}$$

Similarly, for the $\partial/\partial s$ vector fields, we compute

$$\begin{aligned} S_1 \frac{\partial M}{\partial s} S_2^{-1} &= S_1(-Mb + \mu b M)S_2^{-1} \\ &= -S_2 b S_2^{-1} + \mu S_1 b S_1^{-1}, \text{ using } M = S_1^{-1} S_2 \\ &= -L_2^{(b)} + \mu S_1 b S_1^{-1}, \text{ by (9)} \\ &= \left(-(L_2^{(b)})_- + \mu S_1 b S_1^{-1}\right) - (L_2^{(b)})_+ \in g_- + g_+. \end{aligned} \tag{13}$$

Comparing formula (11) with (12) and (13) and using the uniqueness of the direct sum decomposition $g = g_- + g_+$, we find

$$\begin{aligned} \frac{\partial S_1}{\partial t} S_1^{-1} &= -(L_1^{(a)})_-, & \frac{\partial S_2}{\partial t} S_2^{-1} &= (L_1^{(a)})_+ - \lambda S_2 a S_2^{-1} \\ \frac{\partial S_1}{\partial s} S_1^{-1} &= (L_2^{(b)})_- - \mu S_1 b S_1^{-1}, & \frac{\partial S_2}{\partial s} S_2^{-1} &= -(L_2^{(b)})_+. \end{aligned} \tag{14}$$

Upon using the representation (9) of $L_1^{(a)}$ and $L_2^{(b)}$ in terms of S_1 and S_2, and the commutation relations $\lambda[a_i, b] = 0$ and $[L_i^{(\alpha)}, L_i^{(\alpha')}] = S_i[\alpha, \alpha']S_i^{-1} = 0$, where $\alpha, \alpha' = a$ for $i = 1$ and $\alpha, \alpha' = b$ for $i = 2$, we find

$$\begin{aligned} \frac{\partial L_1^{(a)}}{\partial t_i} &= [-(L_1^{(a_i)})_-, L_1^{(a)}] = [L_1^{(a_i)} - (L_1^{(a_i)})_-, L_1^{(a)}] = [(L_1^{(a_i)})_+, L_1^{(a)}] \\ \frac{\partial L_2^{(b)}}{\partial t_i} &= [(L_1^{(a_i)})_+, L_2^{(b)}] - S_2\lambda[a_i, b]S_2^{-1} = [(L_1^{(a_i)})_+, L_2^{(b)}], \end{aligned}$$

which establishes the first equation of (10), keeping in mind the decomposition (4). Similarly

$$\begin{aligned} \frac{\partial L_1^{(a)}}{\partial s_i} &= [(L_2^{(b_i)})_-, L_1^{(a)}] - S_1\mu[b_i, a]S_1^{-1} = [(L_2^{(b_i)})_-, L_1^{(a)}] \\ \frac{\partial L_2^{(b)}}{\partial s_i} &= [-(L_2^{(b_i)})_+, L_2^{(b)}] = [L_2^{(b_i)} - (L_2^{(b_i)})_+, L_2^{(b)}] = [(L_2^{(b_i)})_-, L_2^{(b)}], \end{aligned}$$

lead to the second equation (10), ending the proof of Theorem 1.1.

Corollary 1.2. *Considering the data of Theorem 1.1, let $M = S_1^{-1}S_2 \in G$ be such that $a_iM = Mb_i$ for pairs (a_i, b_i) with $i = 1, 2, \ldots$. Then, for each of those pairs (a_i, b_i) and for that M, we have $L_1^{(a)} = L_2^{(b)}$, and this relation is preserved under the vector fields $\partial/\partial t_n$ and $\partial/\partial s_n$. Moreover setting*[1] *$u_n = t_n - s_n$ and $v_n = t_n + s_n$, we have that $L_1^{(a)}$ flows according to the 1-Toda lattice in the u_n and is invariant in the v_n:*

$$\frac{\partial}{\partial u_n} L_1^{(a)} = [-(L_1^{a_n})_-, L_1^{(a)}] \quad \text{and} \quad \frac{\partial}{\partial v_n} L_1^{(a)} = 0.$$

[1] $\frac{\partial}{\partial u_n} = \frac{1}{2}\left(\frac{\partial}{\partial t_n} - \frac{\partial}{\partial s_n}\right)$ and $\frac{\partial}{\partial v_n} = \frac{1}{2}\left(\frac{\partial}{\partial t_n} + \frac{\partial}{\partial s_n}\right)$

Proof: At first, for that specific $M = S_1^{-1}S_2 \in G$ and for each pair $(a,b) := (a_i, b_i)$, we have

$$L_1^{(a)} = S_1 a S_1^{-1} = S_1 a M S_2^{-1} = S_1 M b S_2^{-1} = S_2 b S_2^{-1} = L_2^{(b)},$$

which is preserved under the vector fields $\partial/\partial t_n$ and $\partial/\partial s_n$:

$$\frac{\partial}{\partial t_n}(L_1^{(a)} - L_2^{(b)}) = [(L_1^{(a_n)})_+, L_1^{(a)} - L_2^{(b)}] = 0,$$

$$\frac{\partial}{\partial s_n}(L_1^{(a)} - L_2^{(b)}) = [(L_2^{(b_n)})_-, L_1^{(a)} - L_2^{(b)}] = 0.$$

Moreover,

$$\begin{aligned}\left(\frac{\partial}{\partial t_n} + \frac{\partial}{\partial s_n}\right) L_1^{(a)} &= [(L_1^{(a_n)})_+, L_1^{(a)}] + [(L_2^{(b_n)})_-, L_1^{(a)}] \\ &= [(L_1^{(a_n)})_+ + (L_1^{(a_n)})_-, L_1^{(a)}] \\ &= [L_1^{(a_n)}, L_1^{(a)}] = 0\end{aligned}$$

and

$$\left(\frac{\partial}{\partial t_n} - \frac{\partial}{\partial s_n}\right) L_1^{(a)} = [(L_1^{(a_n)})_+ - (L_1^{(a_n)})_-, L_1^{(a)}] = -2[(L_1^{(a_n)})_-, L_1^{(a)}],$$

ending the proof of Corollary 1.2.

2 Application to $GL(\infty)$ and its Borel decomposition

Here we apply Theorem 1.1 and Corollary 1.2 to an appropriate closure of $GL(\infty)$ and $g\ell(\infty)$; the a_i and b_i will be represented by powers of the shift matrix $\Lambda = (\delta_{i,j-1})_{i,j\in\mathbb{Z}} \in \overline{g\ell(\infty)}$ and its transpose $\Lambda^\top$ respectively. We now consider matrices M such that

$$\tau_n(M) = \det(M_n), \quad \text{with} \quad M_n = (M_{ij})_{-\infty<i,j\leq n-1}$$

makes sense.

Theorem 2.1. *Consider the equations on* $\overline{GL(\infty)}$

$$\frac{\partial M}{\partial t_n} = \Lambda^n M \qquad \frac{\partial M}{\partial s_n} = -M\Lambda^{\top^n}, \quad n = 1, 2, ..., \tag{15}$$

or

$$\frac{\partial M}{\partial t_n} = [\Lambda^n, M] \qquad \frac{\partial M}{\partial s_n} = [\Lambda^{\top^n}, M], \quad n = 1, 2, \dots; \tag{16}$$

then the Borel decomposition $M = S_1^{-1} S_2$, *with*

$$\begin{aligned} S_1 \in G_- &= \{\text{lower-triangular invertible matrices, with 1's on the diagonal}\} \\ S_2 \in G_+ &= \{\text{upper-triangular invertible matrices}\} \end{aligned} \tag{17}$$

leads to matrices

$$L_1 = S_1 \Lambda S_1^{-1} \text{ and } L_2 = S_2 \Lambda^\top S_2^{-1} \tag{18}$$

satisfying the (standard) 2-Toda equations

$$\frac{\partial L_i}{\partial t_n} = [(L_1^n)_+, L_i] \text{ and } \frac{\partial L_i}{\partial s_n} = [(L_2^n)_-, L_i], \quad n = 1, 2, \dots, \tag{19}$$

with

$$(L_1^k)_{nn} = \frac{\partial}{\partial t_k} \log \frac{\tau_{n+1}(M)}{\tau_n(M)} \quad (L_2^k)_{nn} = -\frac{\partial}{\partial s_k} \log \frac{\tau_{n+1}(M)}{\tau_n(M)}, \tag{20}$$

expressible in terms of $\tau_n = \tau_n(M)$. *In particular*

$$L_1 = ... + \frac{\partial}{\partial t_1} \log \frac{\tau_{n+1}}{\tau_n} + \Lambda \quad \text{and} \quad L_2 = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2} \Lambda^{-1} - \frac{\partial}{\partial s_1} \log \frac{\tau_{n+1}}{\tau_n} + ... \tag{21}$$

Proof: The statement follows at once from Theorem 1, by picking $a_n = \Lambda^n$ and $b_n = \Lambda^{\top^n}$ for $n = 1, 2, ...$, for which assumption (5) holds. Note, the first set (15) of equations corresponds to (6) with $\lambda = 0$ and $\mu = 0$, whereas the second set (16) corresponds to $\lambda = \mu = 1$.

Finally the matrix M admits the following Borel decomposition

$$M = S_1^{-1} S_2 = S^{-1}(M) h(M) (S^{-1}(M^\top))^\top, \tag{22}$$

with $S_1 := S(M)$ and $S_2 := h(M)(S^{-1}(M^\top))^\top$, where

$$S(M) = \begin{pmatrix} \ddots & & & & \\ \ddots & 1 & 0 & 0 & \\ \ddots & \frac{\tau_{01}}{\tau_1} & 1 & 0 & \\ & \frac{\tau_{02}}{\tau_2} & \frac{\tau_{12}}{\tau_2} & 1 & \\ & & & & \ddots \end{pmatrix} \quad \text{and} \quad h(M) = \begin{pmatrix} \ddots & & & & \\ & \frac{\tau_1}{\tau_0} & & O & \\ & & \frac{\tau_2}{\tau_1} & & \\ & O & & \frac{\tau_3}{\tau_2} & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{23}$$

with $\tau_n(M)$ defined in the beginning of this section and

$$\tau_{in}(M) := (-1)^{i+n} \det(\text{minor of } (i,n)^{\text{th}} \text{ entry in the last column of } M_n).$$

From (22), the diagonal of S_2 is given by the entries of the diagonal matrix $h(M)$; therefore (20) follows at once from the equations (14) for $\frac{\partial S_2}{\partial t_k} S_2^{-1}$ and $\frac{\partial S_2}{\partial s_k} S_2^{-1}$, ending the proof of Theorem 2.1.

Remark: It is important to note that equations (15) make sense also for truncated M's:

$$\bar{M} := (M_{ij})_{i,j \geq 0},$$

but equations (16) cease to make sense for truncated M's. Indeed, in the truncated situation and for $\lambda, \mu \neq 0$, the conditions $\lambda[a_i, b_j] = \mu[a_i, b_j] = 0$ are not met, since $[\bar{\Lambda}^i, \bar{\Lambda}^{\top j}] \neq 0$ for $i, j \geq 1$.

In the next statement, we show that the sinh-Gordon-like and modified KP equations appear very naturally in the 2-Toda lattice context; For the sake of completeness, we reproduce the proof of these facts; see, for instance, [20] and [11].

Corollary 2.2. *The functions* $q_n = \log \frac{\tau_{n+1}}{\tau_n}$ *satisfy the sinh-Gordon-like equation,*

$$\frac{\partial^2}{\partial s_1 \partial t_1} q_n = e^{q_n - q_{n-1}} - e^{q_{n+1} - q_n}, \tag{24}$$

and the modified KP equation,

$$3\frac{\partial^2 q_n}{\partial t_2^2} + 6\frac{\partial q_n}{\partial t_2}\frac{\partial^2 q_n}{\partial t_1^2} - 4\frac{\partial^2 q_n}{\partial t_1 \partial t_3} + \frac{\partial^4 q_n}{\partial t_1^4} - 6\left(\frac{\partial q_n}{\partial t_1}\right)^2 \frac{\partial^2 q_n}{\partial t_1^2} = 0. \tag{25}$$

Proof: Since the t- and s-vector fields all commute, we have

$$\frac{\partial}{\partial s_1}(L_1)_+ - \frac{\partial}{\partial t_1}(L_2)_- + [(L_1)_+, (L_2)_-] = 0,$$

which upon using (21), keeping the diagonal terms only and upon setting $q_n = \log \frac{\tau_{n+1}}{\tau_n}$, leads to (24).

To prove the modified KP equation for u_n, one uses the fact that the τ_n's satisfy a bilinear equation

$$\oint_{z=\infty} \tau_n(t - [z^{-1}], s)\tau_{m+1}(t' + [z^{-1}], s')e^{\sum_1^\infty (t_i - t_i')z^i} z^{n-m-1}dz$$
$$= \oint_{z=0} \tau_{n+1}(t, s - [z])\tau_m(t', s' + [z])e^{\sum_1^\infty (s_i - s_i')z^{-i}} z^{n-m-1}dz.$$

Setting $t \mapsto t - a, t' \mapsto t + a,\ s \mapsto s - b,\ s' \mapsto s + b$, and using the standard Hirota symbol[2], one finds

$$\sum_{k=0}^{\infty} p_{m+k}(-2a)p_k(\tilde{\partial}_t)e^{\sum_0^\infty (a_i \frac{\partial}{\partial t_i} + b_i \frac{\partial}{\partial s_i})} \tau_{n+m+1} \circ \tau_n$$
$$= \sum_{k=0}^{\infty} p_{-m+k}(-2b)p_k(\tilde{\partial}_s)e^{\sum_0^\infty (a_i \frac{\partial}{\partial t_i} + b_i \frac{\partial}{\partial s_i})} \tau_{n+m} \circ \tau_{n+1}.$$

Setting $m = 0$, the coefficient of the a_1^3-term in the (a, b)-Taylor expansion of this expression leads to

$$\left(\left(\frac{\partial}{\partial t_1} \right)^3 + 3\frac{\partial^2}{\partial t_1 \partial t_2} - 4\frac{\partial}{\partial t_3} \right) \tau_{n+1} \circ \tau_n = 0,$$

which amounts to equation (25) for $q_n = \log \frac{\tau_{n+1}}{\tau_n}$, ending the proof of Corollary 2.2.

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[2]For an arbitrary polynomial p,

$$p(\partial)f \circ g(t) \equiv p(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots)f(t + y)g(t - y)|_{y=0}.$$

For the Schur polynomials p_n, defined by $e^{\sum_0^\infty t_n z^n} = \sum_0^\infty z^n p_n(t)$, introduce the following standard notation:

$$p(\tilde{\partial}_t) := p\left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \ldots \right).$$

Corollary 2.3. *If M satisfies any of the two equations (15) or (16), and if M satisfies $\Lambda M = M\Lambda^{\top}$, then, in particular, M is symmetric and the tridiagonal matrices $L := L_1 = L_2$ satisfy the 1-Toda equations*

$$\frac{\partial L}{\partial u_n} = [-(L^n)_-, L],$$

for the u_n-variables introduced in Corollary 1.2.

Proof: The statement follows at once from Corollary 1.2, except for the symmetry; indeed $M_{k\ell} = M_{\ell k}$ with $k \geq \ell$, by setting $i = j = \ell$ and $n = k - \ell$ in

$$M_{i+n,j} = (\Lambda^n M)_{ij} = (M\Lambda^{\top^n})_{ij} = M_{i,j+n},$$

establishing Corollary 2.3.

Moment matrices, viewed in the context of Hermitean matrix integrals or viewed as solutions to the 1-Toda lattice, have been considered in the random matrix literature, for instance in [13, 14], but also in the Grassmannian context in [9] and [15].

3 A solution to the semi-infinite Toda lattice and string-orthogonal polynomials

An example of a semi-infinite matrix M, satisfying the equation (15), is constructed as follows. Consider a weight on $\mathbb{R}^{\not\models}$,

$$\rho(y, z) := \rho_{t,s}(y, z) := \rho_0(y, z) e^{\sum_1^{\infty}(t_i y^i - s_i z^i)} \tag{26}$$

and the corresponding inner product between functions of one real variable:

$$\langle f, g \rangle = \int_{\mathbb{R}^{\not\models}} dydz \rho(y, z) f(y) g(z).$$

Define also the semi-infinite moment matrix

$$M = (\mu_{ij})_{0 \leq i,j < \infty} = (\langle y^i, z^j \rangle)_{0 \leq i,j < \infty} \tag{27}$$

and the finite matrix $M_n := (\mu_{ij})_{0 \leq i,j \leq n-1}$.

Theorem 3.1. *The moment matrix M with regard to the weight (26) satisfies the equations (15), i.e.,*

$$\frac{\partial M}{\partial t_n} = \Lambda^n M \quad \text{and} \quad \frac{\partial M}{\partial s_n} = -M\Lambda^{\top n}; \tag{28}$$

for generic s and t, the matrix M admits a decomposition $M = S_1^{-1}S_2$, leading to semi-infinite matrices L_1 and L_2, as in (18), satisfying the two-Toda lattice equations (19). Also $q_n = \log(\det M_{n+1}/\det M_n)$ satisfies the sinh-Gordon-like equation (24) and the modified KP equation (25).

Proof: The proof follows at once from Theorem 2.1 and Corollary 2.2, upon noting that

$$\left(\frac{\partial M}{\partial t_n}\right)_{ij} = \frac{\partial}{\partial t_n}\langle y^i, z^j\rangle = \langle y^{n+i}, z^j\rangle = (\Lambda^n M)_{ij}$$

$$\left(\frac{\partial M}{\partial s_n}\right)_{ij} = \frac{\partial}{\partial s_n}\langle y^i, z^j\rangle = -\langle y^i, z^{j+n}\rangle = (-M\Lambda^{\top n})_{ij}.$$

The finite matrix M_n admits generically a Borel decomposition and its determinant τ_n satisfies (24) and (25), ending the proof of theorem 3.1.

In the next theorem, we show that performing the Borel decomposition of a semi-infinite matrix M is tantamount to the process of constructing *string-orthogonal polynomials* with regard to M.

They are defined as two sets of monic polynomials of degree i, each depending on one variable (y and $z \in \mathbb{R}$)

$$\{p_i^{(1)}(y)\}_{i=0}^{\infty} \quad \text{and} \quad \{p_i^{(2)}(z)\}_{i=0}^{\infty},$$

orthogonal in the following sense

$$\langle p_i^{(1)}, p_j^{(2)}\rangle = h_i\delta_{ij}.$$

Note, since $\mu_{ij} = \langle y^i, z^j\rangle$, the string-orthogonality of polynomials or entire functions can purely be expressed in terms of the entries of the matrix M, regardless of the weight $\rho dy\, dz$. To the best of our knowledge, string-orthogonal polynomials were considered for the first time, in the context of symmetric weights $\rho(y,z)dydz$, by Mehta (see [13] and [6]). The ideas in Theorem 3.2 are mainly due to [1].

Theorem 3.2. *(i) For $\overline{g\ell(\infty)}$, the matrices S_1 and S_2 in*[3] *the Borel decomposition $M = S_1^{-1}S_2$ of the semi-infinite matrix M, lead to vectors*[4]

$$p^{(1)}(y) = S_1\chi(y) \text{ and } h^{-1}p^{(2)}(z) = (S_2^\top)^{-1}\chi(z) \tag{29}$$

of monic string-orthogonal polynomials with regard to the matrix M, i.e.

$$\langle p_i^{(1)}, p_j^{(2)}\rangle = h_i\delta_{ij}. \tag{30}$$

and conversely.

(ii) The string-orthogonal polynomials $p_n^{(1)}(y)$ and $p_n^{(2)}(z)$ are explicitly given either in terms of $\tau_n = \det M_n$ or in terms of the measure $\rho(u,v)du\,dv$ defining the moments, as follows[5]

$$\begin{aligned}
p_n^{(1)}(y) &= \frac{1}{\det M_n}\det\left(\begin{array}{ccc|c} & & & 1\\ & M_n & & y\\ & & & \vdots\\ \hline \mu_{n0} & \dots & \mu_{n,n-1} & y^n\end{array}\right) && (31)\\
&= \frac{1}{\det M_n}\iint_{(\mathbb{R}^n)^2}\prod_{k=1}^n(y-u_k)\Delta(u)\Delta(v)\prod_1^n\rho(u_k,v_k)du\,dv && (32)\\
&= y^n\frac{\tau_n(t-[y^{-1}],s)}{\tau_n(t,s)} && (33)
\end{aligned}$$

and

$$\begin{aligned}
p_n^{(2)}(z) &= \frac{1}{\det M_n}\det\left(\begin{array}{ccc|c} & M_n^\top & & 1\\ & & & z\\ & & & \vdots\\ \hline \mu_{0,n} & \dots & \mu_{n-1,n} & z^n\end{array}\right) && (34)\\
&= \frac{1}{\det M_n}\iint_{(\mathbb{R}^n)^2}\prod_{k=1}^n(z-v_k)\Delta(u)\Delta(v)\prod_1^n\rho(u_k,v_k)du\,dv && (35)\\
&= z^n\frac{\tau_n(t,s+[z^{-1}])}{\tau_n(t,s)}, && (36)
\end{aligned}$$

[3]with S_1 and S_2 as in Theorem 2.1; the diagonal matrix h is defined, such that $h^{-1}S_2$ has 1's on the diagonal.

[4]$\chi(z) := (1, z, z^2, ...)$

[5]$\Delta(a) := \prod_{1\le i<j\le n}(a_i - a_j)$ and $[\alpha] := (\alpha, \alpha^2/2, \alpha^3/3, ...)$

where τ_n has the integral representation

$$\det \tau_n = M_n = \int\int_{(\mathbb{R}^n)^2} \Delta(u)\Delta(v) \prod_{k=1}^{n} \rho_{t,s}(u_k, v_k) du\, dv. \tag{37}$$

<u>Proof</u>: Given string orthogonal polynomials

$$p_i^{(1)}(y) = \sum_{k=0}^{i} P_{ik} y^k \qquad p_j^{(2)}(z) = \sum_{\ell=0}^{j} Q_{j\ell} z^\ell,$$

thus satisfying $\langle p_i^{(1)}, p_j^{(2)} \rangle = h_i \delta_{ij}$, we have upon setting $h := \mathrm{diag}(h_0, h_1, ...)$,

$$\begin{aligned} h &= \left(\langle p_i^{(1)}(y), p_j^{(2)}(z) \rangle \right)_{0 \le i,j < \infty} \\ &= \left(\sum_{\substack{0 \le k \le i \\ 0 \le \ell \le j}} P_{ik} \langle y^k, z^\ell \rangle (Q^\top)_{\ell j} \right)_{0 \le i,j < \infty} \\ &= PMQ^\top \end{aligned}$$

with P and Q being lower-triangular matrices, yielding the Borel decomposition of M. The converse is true as well: given the Borel decomposition $M = P^{-1} h Q^{\top^{-1}}$ of the matrix M of moments $\mu_{ij} = \langle y^i, z^j \rangle$, we find polynomials $p_i^{(1)}$ and $p_j^{(2)}$, defined as above, such that $h_i \delta_{ij} = \langle p_i^{(1)}(y), p_j^{(2)}(z) \rangle$.

To show the expressions (31) and (34) for $p^{(1)}$ and $p^{(2)}$, it suffices to compute

$$\begin{aligned} \langle p_n^{(1)}(y), z^r \rangle &= \frac{1}{\det M_n} \det \begin{pmatrix} & & & \langle 1, z^r \rangle \\ & M_n & & \vdots \\ & & & \langle y^{n-1}, z^r \rangle \\ \mu_{n0} & \dots & \mu_{n,n-1} & \langle y^n, z^r \rangle \end{pmatrix} \\ &= \frac{1}{\det M_n} \det \begin{pmatrix} & & & \mu_{0,r} \\ & M_n & & \vdots \\ & & & \mu_{n-1,r} \\ \mu_{n0} & \dots & \mu_{n,n-1} & \mu_{n,r} \end{pmatrix} \\ &= \begin{cases} 0 & \text{if } r \le n-1, \text{ (matrix with two equal columns)} \\ \dfrac{\det M_{n+1}}{\det M_n} & \text{if } r = n, \end{cases} \end{aligned}$$

and thus

$$\langle p_n^{(1)}, p_m^{(2)} \rangle = \delta_{m,n} \frac{\det M_{n+1}}{\det M_n}.$$

The proof of (32) and (35) is quite similar, whereas (33) and (36) follows from replacing t_i by $t_i - y^{-i}/i$ and s_i by $s_i + z^{-i}/i$ in (37), and using

$$e^{-\sum_1^\infty \frac{a^i}{i}} = 1 - a,$$

ending the proof of Theorem 3.2.

Theorem 3.3. *The string-orthogonal polynomials are eigenvectors of the matrices L_1 and $L_2^\top$:*

$$yp^{(1)}(y) = L_1 p^{(1)}(y) \qquad zh^{-1}p^{(2)}(z) = L_2^\top h^{-1} p^{(2)}(z) \tag{38}$$

and flow in t and s according to the equations

$$\begin{aligned} \frac{\partial p^{(1)}}{\partial t_k} &= -(L_1^k)_- p^{(1)} & \frac{\partial}{\partial t_k} h^{-1} p^{(2)} &= -\left((L_1^k)_+\right)^\top h^{-1} p^{(2)} \\ \frac{\partial p^{(1)}}{\partial s_k} &= (L_2^k)_- p^{(1)} & \frac{\partial}{\partial s_k} h^{-1} p^{(2)} &= (L_2^k)_+ h^{-1} p^{(2)}. \end{aligned} \tag{39}$$

Proof: one computes

$$yp^{(1)}(y) = yS_1\chi(y) = S_1\Lambda\chi(y) = S_1\Lambda S_1^{-1}p^{(1)}(y) = L_1p^{(1)}(y)$$

$$zh^{-1}p^{(2)}(z) = z(S_2^\top)^{-1}\chi(z) = (S_2^\top)^{-1}\Lambda S_2^\top h^{-1}p^{(2)}(z) = L_2^\top h^{-1}p^{(2)}(z),$$

establishing (38). Finally, using (14) and the definition (29) of $p^{(1)}$ and $h^{-1}p^{(2)}$, we have (set $(L_i^k)_+^\top := \left((L_i^k)_+\right)^\top$)

$$\begin{aligned} \frac{\partial p^{(1)}(z)}{\partial t_k} &= \frac{\partial S_1}{\partial t_k}\chi(z) = -(L_1^k)_- S_1\chi(z) = -(L_1^k)_- p^{(1)}(z) \\ \frac{\partial p^{(1)}(z)}{\partial s_k} &= \frac{\partial S_1}{\partial s_k}\chi(z) = (L_2^k)_- S_1\chi(z) = (L_2^k)_- p^{(1)}(z) \\ \frac{\partial h^{-1}p^{(2)}(z)}{\partial t_k} &= \frac{\partial (S_2^\top)^{-1}}{\partial t_k}\chi(z) = -(L_1^k)_+^\top (S_2^\top)^{-1}\chi(z) = -(L_1^k)_+^\top h^{-1}p^{(2)}(z) \\ \frac{\partial h^{-1}p^{(2)}(z)}{\partial s_k} &= \frac{\partial (S_2^\top)^{-1}}{\partial s_k}\chi(z) = (L_2^k)_+^\top (S_2^\top)^{-1}\chi(z) = (L_2^k)_+^\top h^{-1}p^{(2)}(z), \end{aligned}$$

establishing Theorem 3.3.

Example: string-orthogonal polynomials for a Gaussian weight.

Monic string-orthogonal polynomials with respect to the Gaussian weight

$$\rho(x,y) = e^{-\frac{1}{2}(x^2+y^2-2cxy)} = e^{-\frac{1}{2}(x,Cx)}, \text{ with } C = \begin{pmatrix} 1 & -c \\ -c & 1 \end{pmatrix}$$

are given by

$$\tilde{p}_n^{(1)}(x) = \tilde{p}_n^{(2)}(x) = \left(\frac{1}{2\alpha}\right)^n H_n(\alpha x), \text{ with } h_n = \frac{2\pi n! c^n}{(1-c^2)^{n+1/2}} \tag{40}$$

where

$$\alpha = \left(\frac{1-c^2}{2}\right)^{1/2} \text{ and } H_n(y) = 2^n y^n + ...$$

are the standard Hermite polynomials, with generating function

$$e^{2yz-z^2} = \sum_0^\infty \frac{H_n(y)}{n!} z^n. \tag{41}$$

The proof of this fact relies on the moment generating function (see [8])

$$\begin{aligned}\langle e^{a_1 x}, a^{a_2 y}\rangle_\rho &= \sum_{j\geq 0} \frac{a_1^j a_2^j}{i!j!} M_{ij}(0,0) \\ &= \int\int_{\mathbb{R}^2} e^{a_1 x + a_1 y} \rho(x,y) dx\, dy = 2\pi(\det C)^{-1/2} e^{\frac{1}{2}(a, C^{-1}a)} \end{aligned} \tag{42}$$

We compute

$$\begin{aligned}\sum_{n,m\geq 0} \frac{u^n v^m}{n!m!} \langle H_n(\alpha x), H_m(\alpha y)\rangle_\rho &= \langle \sum_{n\geq 0} \frac{u^n}{n!} H_n(\alpha x), \sum \frac{v^m}{m!} H_m(\alpha y)\rangle_\rho \\ &= e^{-u^2-v^2} \langle e^{2\alpha x u}, e^{2\alpha y v}\rangle \quad \text{using (41)} \\ &= \frac{2\pi}{\sqrt{1-c^2}} e^{-u^2-v^2} e^{u^2+v^2+2cuv} \quad \text{using (42)} \\ &= \frac{2\pi}{\sqrt{1-c^2}} \sum_{n\geq 0} (2c)^n n! \frac{(uv)^n}{(n!)^2}.\end{aligned}$$

Therefore

$$\langle H_n(\alpha x), H_m(\alpha y)\rangle = \frac{2\pi n!}{\sqrt{1-c^2}}(2c)^n\delta_{nm},$$

and thus

$$\langle (\frac{1}{2\alpha})^n H_n(\alpha x), (\frac{1}{2\alpha})^m H_m(\alpha y)\rangle = \frac{2\pi n! c^n}{(1-c^2)^{n+1/2}}\delta_{nm},$$

establishing formula (40) for the polynomials $\tilde{p}_n^{(1)}(x) = \tilde{p}_n^{(2)}(x)$.

4 A solution to the bi-infinite Toda lattice

In contrast with (26), consider now the weight, introduced by Widom [22]

$$\rho(y,z) := \rho_{t,s}(y,z) := \rho_0(y,z)e^{\sum_1^\infty \left(t_n(y^n - z^{-n}) + s_n(y^{-n} - z^n)\right)} \tag{43}$$

and the corresponding inner product

$$\langle f, g\rangle = \int_{\mathbb{R}^{\not 2}} dy\, dz\, \rho(y,z) f(y) g(z).$$

Consider the bi-infinite moment matrix

$$M = I - \mu := (\delta_{ij} - \langle y^i, z^j\rangle)_{-\infty<i,j<\infty} \tag{44}$$

and the matrix $M_n := (M_{ij})_{-\infty<i,j\leq n-1}$. The Fredholm determinant of M makes sense for good weights $\rho_0(y,z)$; see [22].

Also, consider one of the four 2-Toda vector vertex operators and its expansion in W-generators, defined by us in [1][6],

$$\begin{aligned}
\mathbb{X}(t,y,z): &= (\mathbb{X}_n(t,y,z))_{n\in\mathbb{Z}} := \chi(z^{-1})\chi(y)X(-t,z)X(t,y) \\
&= -\frac{1}{y-z}\sum_{k=0}^{\infty}\frac{(y-z)^k}{k!}\sum_{\ell=-\infty}^{\infty} z^{-\ell-k} W_\ell^{(k)} \left(\frac{y^n}{z^{n-1}}\right)_{n\in\mathbb{Z}}.
\end{aligned}$$

Theorem 4.1. *The moment matrix M for the weight (43) satisfies the equations (16); the decomposition of $M = S_1^{-1}S_2$ leads to bi-infinite matrices L_1 and L_2 satisfying the 2-Toda-equations.*

[6]where $\chi(z) = \text{diagonal}(..., z^{-1}, z^0, z^1, ...)$ and $X(t,y) := e^{\sum_1^\infty t_i y^i} e^{\sum_1^\infty \frac{y^{-i}}{i}\frac{\partial}{\partial t_i}}$

Proof: The simple computation

$$\frac{\partial \mu_{ij}}{\partial t_n} = \langle y^{i+n}, z^j \rangle - \langle y^i, z^{j-n} \rangle = (\Lambda^n \mu - \mu \Lambda^n)_{ij}$$

$$\frac{\partial \mu_{ij}}{\partial s_n} = \langle y^{i-n}, z^j \rangle - \langle y^i, z^{j+n} \rangle = (\Lambda^{\top n} \mu - \mu \Lambda^{\top n})_{ij}, \tag{45}$$

together with Theorem 2.1, establishes Theorem 4.1.

Replacing, with Widom, $\rho_0(y, z)$ by $\rho_0(z)\delta(y - z)$ in (43) leads to a measure which has its support on the diagonal $y = z$:

$$\rho(z) := \rho_{t-s}(z) := e^{\sum_1^\infty (t_n - s_n)(z^n - z^{-n})} \rho_0(z). \tag{46}$$

Corollary 4.2. *Given the weight (46), the Borel decomposition of* $M = I - \mu$ *leads to the 1-Toda lattice and* $q_n = \log(\tau_{n+1}/\tau_n)$*, satisfies equations (24) and (25). Moreover* τ_n *has three alternative expressions: at first*

$$\tau_n = \det M_n = \det(I - \mu_n);$$

secondly, τ_n *can be expressed in terms of a Fredholm determinant*

$$\tau_n = \det(I - K_n)$$

of an integral operator associated with the kernel

$$K_n(y, z) = -\frac{y^n \rho^{\frac{1}{2}}(y) \rho^{\frac{1}{2}}(z) z^n}{1 - yz}. \tag{47}$$

Finally, it can also be expressed as a kind of continuous "soliton" formula, in terms of the vertex operator above, acting on the function 1*:*

$$\tau_n = e^{-\int \mathbb{X}_n(t, z, z^{-1}) \rho_0(z) dz} 1, \tag{48}$$

where $\mathbb{X}_n(t, z, z^{-1})$ *is the vertex operator realization of a Virasoro-type central extension:*

$$\frac{\partial}{\partial z}(z^{k+1} - z^{-k+1}) \mathbb{X}_n(t, z, z^{-1}) = [\frac{1}{2}(W_k^{(2)} - W_{-k}^{(2)}) + (2n+1) W_k^{(1)}, \mathbb{X}_n(t, z, z^{-1})]. \tag{49}$$

Proof: The Borel decomposition of $M = I - \mu = S_1^{-1} S_2$, together with equations (16), lead to the 1-Toda lattice, since $\rho_{t-s}(z)$ defined in (46) only depends on $t-s$; hence, so does M, S_1 and S_2. Note, in this case, the reduction from 2-Toda to 1-Toda is given by the requirement that $L := L_1 + L_1^{-1} = L_2 + L_2^{-1}$ be tridiagonal.

Moreover, the kernel

$$
\begin{aligned}
K_N(y,z) &= -\frac{y^N \rho^{\frac{1}{2}}(y)\rho^{\frac{1}{2}}(z) z^N}{1-yz} \\
&= \frac{y^{N-1} \rho^{\frac{1}{2}}(y)\rho^{\frac{1}{2}}(z) z^{N-1}}{1-y^{-1}z^{-1}} \\
&= y^{N-1} \rho^{\frac{1}{2}}(y)\rho^{\frac{1}{2}}(z) z^{N-1} \sum_{i\geq 0} \frac{1}{y^i z^i} \\
&= \sum_{\ell \leq N-1} y^{\ell} \rho^{\frac{1}{2}}(y)\rho^{\frac{1}{2}}(z) z^{\ell}
\end{aligned}
$$

defines an integral operator acting on the space

$$
\mathcal{H} = \text{span } \{\varphi_i(z), -\infty < i < \infty\}, \quad \varphi_i(z) = \rho^{\frac{1}{2}}(z) z^i,
$$

as follows

$$
\varphi \mapsto K_N \varphi(y) = \int K_N(y,z)\varphi(z)\, dz.
$$

Then

$$
\begin{aligned}
(K_N \varphi_i)(y) &= \int dz \sum_{\ell \leq N-1} y^{\ell} \rho^{\frac{1}{2}}(y)\rho^{\frac{1}{2}}(z) z^{\ell} \rho^{\frac{1}{2}}(z) z^i \\
&= \sum_{\ell \leq N-1} \mu_{i\ell}\varphi_{\ell}(y),
\end{aligned}
$$

with $\mu_{i\ell}$ as in (44). Therefore we have $\det(I - \mu_N) = \det(I - K_N)$. The proof of (48) and (49) is a special case of a theorem relating Fredholm determinants and vertex operators; see [4]. This ends the proof of corollary 4.2.

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