String-orthogonal polynomials, String Equations and Two-Toda Symmetries*

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Consider a weight $\rho(y,z)dydz = e^{V(y,z)}dydz$ on $\mathbb{R}^2$, with

(0.1) $$V(y,z) := V_1(y) + V_{12}(y,z) + V_2(z) := \sum_1^\infty t_i y^i + \sum_{i,j\geq 1} c_{ij} y^i z^j - \sum_1^\infty s_i z^i,$$

the corresponding inner product

(0.2) $$\langle f,g\rangle = \int_{\mathbb{R}^2} dydz\rho(y,z)f(y)g(z),$$

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and the moment matrix,

$$m_n(t,s,c) =: (\mu_{ij})_{0\le ij\le n-1} = (\langle y^i, z^j\rangle)_{0\le i,j\le n-1}. \tag{0.3}$$

As a function of t, s and c, the moment matrix $m_\infty(t,s,c)$ satisfies the differential equations

$$\frac{\partial m_\infty}{\partial t_n} = \Lambda^n m_\infty, \quad \frac{\partial m_\infty}{\partial s_n} = -m_\infty \Lambda^{\top n}, \quad \frac{\partial m_\infty}{\partial c_{ij}} = \Lambda^i m_\infty \Lambda^{\top j}, \tag{0.3'}$$

where Λ is the semi-infinite shift matrix defined below.

Consider the Borel decomposition of the semi-infinite matrix[1]

$$m_\infty = S_1^{-1} S_2 \text{ with } S_1 \in \mathcal{D}_{-\infty,0}, \quad S_2 \in \mathcal{D}_{0,\infty}$$

with S_1 having 1's on the diagonal, and S_2 having h_i's on the diagonal, with (see section 3)

$$h_i = \frac{\det m_{i+1}}{\det m_i}.$$

The Borel decomposition $m_\infty = S_1^{-1} S_2$ above leads to two strings $(p^{(1)}(y), p^{(2)}(z))$ of monic polynomials in one variable, constructed, in terms of the character $\bar\chi(z) = (z^n)_{n\in \mathbf{Z}, n\ge 0}$, as follows:

$$p^{(1)}(y) =: S_1 \bar\chi(y) \qquad p^{(2)}(z) =: h(S_2^{-1})^\top \bar\chi(z). \tag{0.4}$$

We call these two sequences *string-orthogonal polynomials*; indeed the Borel decomposition of $m_\infty = S_1^{-1} S_2$ above is equivalent to the orthogonality relations:

$$\langle p_n^{(1)}, p_m^{(2)}\rangle = \delta_{n,m} h_n.$$

We show in section 4 the string-orthogonal polynomials have the following expressions in terms of Schur differential polynomials[2]

$$p_n^{(1)}(y) = \sum_{0\le k\le n} \frac{p_{n-k}(-\tilde\partial_t)\det m_n(t,s,c)}{\det m_n(t,s,c)} y^k \ , \ p_n^{(2)}(z) = \sum_{0\le k\le n} \frac{p_{n-k}(\tilde\partial_s)\det m_n(t,s,c)}{\det m_n(t,s,c)} z^k$$

[1] $\mathcal{D}_{k,\ell}$ ($k < \ell \in \mathbf{Z}$) denotes the set of band matrices with zeros outside the strip (k,ℓ).

[2] The Schur polynomials p_k, defined by $e^{\sum_1^\infty t_i z^i} = \sum_0^\infty p_k(t) z^k$ and $p_k(t) = 0$ for $k < 0$, , and not to be confused with the string-orthogonal polynomials $p_i^{(k)}$, $k = 1,2$, lead to differential polynomials

$$p_k(\pm\tilde\partial_t) = p_k\Big(\pm\frac{\partial}{\partial t_1}, \pm\frac{1}{2}\frac{\partial}{\partial t_2}, \pm\frac{1}{3}\frac{\partial}{\partial t_3}, \ldots\Big)$$

To the best of our knowledge, string-orthogonal polynomials were considered for the first time, in the context of symmetric weights $\rho(y,z)dydz$, by Mehta (see [M] and [CMM]).

The *main message* of this work is to show (i) that the expressions $\det m_N$ satisfy the KP-hierarchy in t and s for each $N = 1, 2, ...,$ (section 3) (ii) that the $\det m_N$ hang together in a very specific way (they form the building blocks of the 2-Toda lattice) (section 4), (iii) that $\det m_N$ satisfies an additional Virasoro-like algebra of partial differential equations (section 5), as a result of so-called string equations (section 4). In [AvM2], we have obtained similar results for moment matrices associated with general weights on $\mathbf{R}$ (thus connecting with the standard theory of orthogonal polynomials), rather than on $\mathbf{R}^2$. It would not be difficult to generalize the results of this paper to more general weights $\rho(y,z)dy\,dz$ on $\mathbf{R}^2$, besides those of (0.1).

In terms of the matrix operators

$$\Lambda := \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & 0 & 0 & & \\ & & & & \ddots \end{pmatrix} \text{ and } \varepsilon := \begin{pmatrix} 0 & 0 & & & \\ 1 & 0 & 0 & & \\ & 2 & 0 & & \\ & & & & \ddots \end{pmatrix}$$

acting on $\bar{\chi}$ as

$$\Lambda\bar{\chi}(z) = z\bar{\chi}(z), \ \varepsilon\bar{\chi}(z) = \frac{\partial}{\partial z}\bar{\chi}(z),$$

the matrices

$$L_1 := S_1\Lambda S_1^{-1}, L_2 := S_2\Lambda^\top S_2^{-1}, Q_1 := S_1\varepsilon S_1^{-1}, Q_2 := S_2\varepsilon^\top S_2^{-1}$$

interact with the vector of string-orthogonal polynomials, as follows:

$$L_1 p^{(1)}(y) = yp^{(1)}(y) \qquad Q_1 p^{(1)}(y) = \frac{d}{dy}p^{(1)}(y) \tag{0.5}$$

$$hL_2^\top h^{-1}p^{(2)}(z) = zp^{(2)}(z) \qquad hQ_2^\top h^{-1}p^{(2)}(z) = \frac{d}{dz}p^{(2)}(z).$$

The semi-infinite matrix L_1 (respectively L_2) is lower-triangular (resp. lower-triangular), with one subdiagonal above (resp. below); Q_1 (resp. Q_2) is strictly lower-triangular (resp. strictly upper-triangular). In Theorem 4.1 we prove the matrices L_i and Q_i satisfy so-called *"string equations"*

$$Q_1 + \frac{\partial V}{\partial y}(L_1, L_2) = 0, \qquad Q_2 + \frac{\partial V}{\partial z}(L_1, L_2) = 0. \tag{0.6}$$

When

$$V(y,z)=\sum_1^{\ell_1} t_i y^i + cyz - \sum_1^{\ell_2} s_i z^i,$$

then (0.6) implies that L_1 is a ℓ_2+1-band matrix, and L_2 is a ℓ_1+1-band matrix.

Moreover, as a function of (t,s), the couple $L:=(L_1,L_2)$ satisfies the *" two-Toda lattice equations"*, and as a function of $c_{\alpha,\beta}$, L satisfies another hierarchy of commuting vector fields; so, in terms of an appropriate Lie algebra splitting $(\ )_+$ and $(\ )_-$, to be explained in section 1, we have

$$(0.7)\qquad \frac{\partial L}{\partial t_n}=[(L_1^n,0)_+,L]\quad \frac{\partial L}{\partial s_n}=[(0,L_2^n)_+,L],\quad \frac{\partial L}{\partial c_{\alpha,\beta}}=-[(L_1^\alpha L_2^\beta,0)_-,L],$$

and, what is equivalent, the moment matrix m_∞ satisfies the differential equations (0.3'), with solution (thinking of $t_n=c_{n0}$ and $s_n=-c_{0n}$):

$$m_\infty(t,s,c)=\sum_{\substack{(r_{\alpha\beta})_{\alpha,\beta\geq 0}\in\mathbf{Z}^\infty\\ (\alpha,\beta)\neq(0,0)}}\left(\prod_{(\alpha,\beta)}\frac{c_{\alpha\beta}^{r_{\alpha\beta}}}{r_{\alpha\beta}!}\right)\Lambda^{\sum_{\alpha\geq 1}\alpha r_{\alpha\beta}}m_\infty(0,0,0)\Lambda^{\top\sum_{\beta\geq 1}\beta r_{\alpha\beta}};$$

in particular

$$m_\infty(t,s,0)=e^{\sum_1^\infty t_n\Lambda^n}m_\infty(0,0,0)e^{-\sum_1^\infty s_n\Lambda^{\top n}}.$$

Thus the integrable system under consideration in this paper is a $c_{\alpha\beta}$-deformation of the 2-Toda lattice, which itself is an isospectral deformation of a couple of matrices (L_1,L_2), in general bi-infinite, depending on two sequences of times $t_1,t_2,\dots$ and $s_1,s_2,\dots$.

Moreover, the determinant of the moment matrix has many different expressions; in particular, in terms of the moment matrix at $t=s=0$, using the matrix $E_N(t):=$ (the first N rows of $e^{\sum_1^\infty t_n\Lambda^n}$) of Schur polynomials $p_n(t)$ (see section 3). It can also be expressed as a 2-matrix integral reminiscent of 2-matrix integrals in string theory and in terms of the diagonal elements h_i of the upper-triangular matrix S_2:

(0.8)

$$\begin{aligned}
N!\det m_N(t,s,c) &= N!\det\Big(E_N(t)m_\infty(0,0,c)E_N(-s)^\top\Big)\\
&= \int\int_{u,v\in\mathbb{R}^N} e^{\sum_{k=1}^N V(u_k,v_k)}\prod_{i<j}(u_i-u_j)\prod_{i<j}(v_i-v_j)\,du\,dv\\
&= \prod_0^{N-1} h_i(t,s,c)\\
&=: \tau_N(t,s,c).
\end{aligned}$$

Finally $\tau_N(t,s,c)$ is a *"τ-function"* in the sense of Sato, separately in t and s[3].

The string equations (0.6) play an important role: they have many consequences! In Theorem 5.1, we show $\tau_N = \det m_N(t,s,c)$ satisfies the following set of constraints[4]

$$(0.9)\qquad \left(J_i^{(2)} + (2N+i+1)J_i^{(1)} + N(N+1)J_i^{(0)} + 2\sum_{r,s\geq 1} rc_{rs}\frac{\partial}{\partial c_{i+r,s}}\right)\tau_N = 0$$

$$\left(\tilde{J}_i^{(2)} - (2N+i+1)\tilde{J}_i^{(1)} + N(N+1)\tilde{J}_i^{(0)} + 2\sum_{r,s\geq 1} sc_{rs}\frac{\partial}{\partial c_{r,s+i}}\right)\tau_N = 0,$$

for $i \geq -1$ and $N \geq 0$.

When $V_{12} = cyz$, the relations (0.9) reduce to an inductive system of partial differential equations in t and s, for $\tau_0, \tau_1, \tau_2, \ldots$, (Corollary 5.1.1)

(0.10)

$$\left(J_i^{(2)} + (2N+i+1)J_i^{(1)} + N(N+1)J_i^{(0)}\right)\tau_N + 2c\; p_{i+N}(\tilde{\partial}_t)p_N(-\tilde{\partial}_s)\tau_1 \circ \tau_{N-1} = 0$$

$$\left(\tilde{J}_i^{(2)} - (2N+i+1)\tilde{J}_i^{(1)} + N(N+1)\tilde{J}_i^{(0)}\right)\tau_N + 2c\; p_N(\tilde{\partial}_t)p_{i+N}(-\tilde{\partial}_s)\tau_1 \circ \tau_{N-1} = 0$$

[3]Sato's τ-function $\tau(t)$ in $t \in \mathbf{C}^\infty$ is the determinant of the projection $e^{-\sum t_i z^i} W \to H_+ = \{1, z, z^2, \ldots\}$, where W is a fixed span of functions in z with poles at $z = \infty$ of order $k = 0, 1, 2, \ldots$. Equivalently, a τ-function satisfies the bilinear relations

$$\oint_C \tau(t - [z^{-1}])\tau(t' + [z^{-1}])e^{\sum_1^\infty (t_i - t'_i)z^i} dz = 0,$$

where C is a small contour about $z = \infty$ and where $[\alpha] := \left(\alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, \ldots\right) \in \mathbf{C}^\infty$. The bilinear relations imply that $\tau(t)$ satisfies the KP-hierarchy.

[4]in terms of the customary Virasoro generators in $t_1, t_2, \ldots$:

$$J_n^{(0)} = \delta_{n0}, \qquad J_n^{(1)} = \frac{\partial}{\partial t_n} + (-n)t_{-n}, \qquad J_0^{(1)} = 0$$

$$J_n^{(2)} = \sum_{i+j=n} :J_i^{(1)}J_j^{(1)}: := \sum_{i+j=n}\frac{\partial^2}{\partial t_i \partial t_j} + 2\sum_{-i+j=n} it_i\frac{\partial}{\partial t_j} + \sum_{-i-j=n}(it_i)(jt_j),$$

where ": :" denotes normal ordering, i.e., always pull differentiation to the right, and where a symbol $= 0$, whenever it does not make sense; e.g., $\partial/\partial t_n = 0$ for $n \leq 0$. We also define Virasoro generators in s, namely $\tilde{J}_n^{(k)} := J_n^{(k)}\Big|_{t\to s}$.

for $i \geq -1$ and $N \geq 1$.

involving the Hirota operation[5]; to be precise, the expression $p_n(\tilde{\partial}_t)p_m(-\tilde{\partial}_s)f \circ g$, for $\tilde{\partial}_t$ and $\tilde{\partial}_s$ spelled out in footnote 2, is defined as the coefficient in the (u, v)-Taylor expansion of

$$\begin{aligned} f(t+[u], s-[v])g(t-[u], s+[v]) &= \sum_{n,m\geq 0} u^n v^m p_n(\tilde{\partial}_t)p_m(-\tilde{\partial}_s)f \circ g \\ &= \sum_{n,m\geq 0} u^n v^m \sum_{\substack{i+i'=n \\ j+j'=m \\ i,i',j,j'\geq 0}} p_i(\tilde{\partial}_t)p_j(-\tilde{\partial}_s)f \ .p_{i'}(-\tilde{\partial}_t)p_{j'}(\tilde{\partial}_s)g. \end{aligned}$$

Also, in terms of W-generators, to be defined in (1.36), the matrix integral (0.8) satisfies (Theorem 5.2):

$$(0.11) \qquad \Big(\sum_{k=0}^{i\wedge j} \bar{\alpha}_k^{(i,j)} W_{N,i-j}^{(j-k+1)} + \sum_0^i \bar{\beta}_k^{(i)} \tilde{W}_{N-1,j-i}^{(i-k+1)}\Big)\tau_N = \frac{i!}{(-c)^i}\delta_{ij}\tau_N, \qquad i,j \geq 0,$$

where $\bar{\alpha}_k^{(i,j)}$ and $\bar{\beta}_k^{(i)}$ are numbers defined in (5.10). A relation of this type was conjectured by Morozov [M].

The technique employed here is the one of symmetry vector fields of integrable systems (KP, Toda, two-Toda, etc...), as developped in [ASV1] and [AV2]; they are non-local and time-dependent vector fields transversal to and commuting with the integrable hierarchy, when acting on the (*explicitly*) time-dependent wave functions; however bracketing a symmetry with a Toda vector field (0.7), which acts on the (*implicitly*) time-dependent L, yields another vector field in the hierarchy. In [ASV], we have shown that for generic initial conditions the 2-Toda lattice admits a $w_\infty \times w_\infty$ algebra of symmetries, i.e., an algebra without central extension.

In this study, we consider special initial conditions, preserved by the Toda flows; we show they admit a much wider algebra of symmetries, to be described in Theorem 2.1. This big algebra contains not only the algebra $w_\infty \times w_\infty$ above, but also a Kac-Moody extension $w_\infty \otimes \mathbf{C}(h, h^{-1})$ of w_∞ (in two different ways). Via the Adler-Shiota-van Moerbeke formula, symmetries at the level of the wave functions induce

[5]Given two differential polynomials $p(\partial_t)$ and $q(\partial_s)$, and functions $f(t_1, t_2, ...; s_1, s_2, ...)$, $g(t_1, t_2, ...; s_1, s_2, ...)$, define the Hirota operation for shifts $y = (y_1, y_2, ...)$ and $z = (z_1, z_2, ...)$:

$$p(\partial_t)q(\partial_s)f \circ g(t,s) := p(\frac{\partial}{\partial y})q(\frac{\partial}{\partial z})f(t+y, s+z)g(t-y, s-z)\Big|_{y=z=0}.$$

symmetries at the level of the τ-function; these symmetries form an algebra with central extension. van de Leur [vdL1,2] has a striking representation-theoretical generalization of the ASV-formula; it is an open question to understand whether it includes the wider class of symmetries under consideration here.

The string equations (0.6) are equivalent to the vanishing of a whole algebra of symmetries, viewed as vector fields on the manifold of wave functions; they lead, upon using the ASV-formula, to the algebra of constraints (0.9) and (0.10) (on the 2-matrix integral), viewed as vector fields on the manifold of τ-functions, and this in a straightforward, conceptual and precise fashion.

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1 The 2-Toda lattice and its generic symmetries

Define the column vector $\chi(z) = (z^n)_{n\in\mathbf{Z}}$, and matrix operators Λ, Λ^*, $\varepsilon, \varepsilon^*$ defined as follows:

$$\Lambda\chi(z) = z\chi(z),\ \varepsilon\chi(z) = \frac{\partial}{\partial z}\chi(z),$$

$$\Lambda^*\chi(z) = z^{-1}\chi(z),\ \varepsilon^*\chi(z) = \frac{\partial}{\partial z^{-1}}\chi(z).$$

Note that

$$\Lambda^* = \Lambda^\top = \Lambda^{-1},\ \varepsilon^* = -\varepsilon^\top + \Lambda,$$

and

$$(1.1)\qquad \begin{cases} \Lambda^\top\chi(z^{-1}) = z\chi(z^{-1}),\ \Lambda\chi(z^{-1}) = z^{-1}\chi(z^{-1}), \\ \varepsilon^\top\chi(z^{-1}) = z^{-1}\chi(z^{-1}) - \frac{\partial}{\partial z}\chi(z^{-1}) \\ \varepsilon^{*\top}\chi(z^{-1}) = z\chi(z^{-1}) - \frac{\partial}{\partial z^{-1}}\chi(z^{-1}). \end{cases}$$

Consider the splitting of the algebra $\mathcal{D}$ of pairs (P_1, P_2) of infinite $(\mathbf{Z}\times\mathbf{Z})$ matrices such that $(P_1)_{ij} = 0$ for $j - i \gg 0$ and $(P_2)_{ij} = 0$ for $i - j \gg 0$, used in [ASV2]; to wit:

$$\begin{aligned} \mathcal{D} &= \mathcal{D}_+ + \mathcal{D}_-, \\ \mathcal{D}_+ &= \{(P,P) \mid P_{ij} = 0 \text{ if } |i-j| \gg 0\} = \{(P_1,P_2) \in \mathcal{D} \mid P_1 = P_2\}, \\ \mathcal{D}_- &= \{(P_1,P_2) \mid (P_1)_{ij} = 0 \text{ if } j \geq i,\ (P_2)_{ij} = 0 \text{ if } i > j\}, \end{aligned}$$

with $(P_1, P_2) = (P_1, P_2)_+ + (P_1, P_2)_-$ given by

$$
(1.2) \qquad \begin{aligned} (P_1, P_2)_+ &= (P_{1u} + P_{2\ell}, P_{1u} + P_{2\ell}), \\ (P_1, P_2)_- &= (P_{1\ell} - P_{2\ell}, P_{2u} - P_{1u}); \end{aligned}
$$

P_u and P_ℓ denote the upper (including diagonal) and strictly lower triangular parts of the matrix P, respectively.

The two-dimensional Toda lattice equations

$$
(1.3) \qquad \frac{\partial L}{\partial t_n} = [(L_1^n, 0)_+, L] \quad \text{and} \quad \frac{\partial L}{\partial s_n} = [(0, L_2^n)_+, L] \quad n = 1, 2, \ldots
$$

are deformations of a pair of infinite matrices

$$
(1.4) \qquad L = (L_1, L_2) = \Big(\sum_{-\infty < i \leq 1} a_i^{(1)} \Lambda^i, \sum_{-1 \leq i < \infty} a_i^{(2)} \Lambda^i \Big) \in \mathcal{D},
$$

where $\Lambda = (\delta_{j-i,1})_{i,j \in \mathbf{Z}}$, and $a_i^{(1)}$ and $a_i^{(2)}$ are diagonal matrices depending on $t = (t_1, t_2, \ldots)$ and $s = (s_1, s_2, \ldots)$, such that

$$
a_1^{(1)} = I \quad \text{and} \quad (a_{-1}^{(2)})_{nn} \neq 0 \quad \text{for all } n.
$$

In analogy with Sato's theory, Ueno and Takasaki [U-T] show a solution L of (1.3) has the representation

$$
L_1 = W_1 \Lambda W_1^{-1} = S_1 \Lambda S_1^{-1}, \quad L_2 = W_2 \Lambda^{-1} W_2^{-1} = S_2 \Lambda^{-1} S_2^{-1}
$$

in terms of two pairs of wave operators

$$
\begin{cases} S_1 = \sum_{i \leq 0} c_i(t,s) \Lambda^i, \quad S_2 = \sum_{i \geq 0} c_i'(t,s) \Lambda^i \\ c_i, c_i' : \text{diagonal matrices}, c_0 = I, (c_0')_{ii} \neq 0, \ \text{for all } i \end{cases}
$$

and

$$
(1.5) \qquad W_1 = S_1(t,s) e^{\sum_1^\infty t_k \Lambda^k}, \quad W_2 = S_2(t,s) e^{\sum_1^\infty s_k \Lambda^{-k}}.
$$

One also introduces a pair of wave vectors $\Psi = (\Psi_1, \Psi_2)$, and a pair of adjoint wave vectors, $\Psi^* = (\Psi_1^*, \Psi_2^*)$, instead of a single wave function and a single adjoint wave function:

$$
(1.6) \qquad \begin{cases} \Psi_1(t,s,z) = W_1 \chi(z) = e^{\sum_1^\infty t_k z^k} S_1 \chi(z) \\ \Psi_2(t,s,z) = W_2 \chi(z) = e^{\sum_1^\infty s_k z^{-k}} S_2 \chi(z) \end{cases}
$$

and

$$\text{(1.7)} \qquad \begin{cases} \Psi_1^* = (W_1^\top)^{-1}\chi^*(z) = e^{-\sum_1^\infty t_k z^k}(S_1^\top)^{-1}\chi(z^{-1}) \\ \Psi_2^* = (W_2^\top)^{-1}\chi^*(z) = e^{-\sum_1^\infty s_k z^{-k}}(S_2^\top)^{-1}\chi(z^{-1}). \end{cases}$$

The wave functions Ψ and Ψ^* evolve in t and s according to the following differential equations[6]:

$$\text{(1.8)} \qquad \begin{cases} \frac{\partial \Psi}{\partial t_n} = (L_1^n, 0)_+\Psi = ((L_1^n)_u, (L_1^n)_u)\Psi \\ \frac{\partial \Psi}{\partial s_n} = (0, L_2^n)_+\Psi = ((L_2^n)_\ell, (L_2^n)_\ell)\Psi \end{cases}$$

$$\text{(1.9)} \qquad \begin{cases} \frac{\partial}{\partial t_n}\Psi^* = ((L_1^n, 0)_+)^\top \Psi^* \\ \frac{\partial}{\partial s_n}\Psi^* = -((0, L_2^n)_+)^\top \Psi^*. \end{cases}$$

Besides $L = (L_1, L_2) = (S_1 \Lambda S_1^{-1}, S_2 \Lambda^\top S_2^{-1})$, the operators

$$L^* = (L_1^*, L_2^*) = (L_1^\top, L_2^\top)$$

$$M = (M_1, M_2) := (W_1 \varepsilon W_1^{-1}, W_2 \varepsilon^* W_2^{-1}) = \Big(S_1(\varepsilon + \sum_1^\infty k t_k \Lambda^{k-1}\Big)S_1^{-1}, S_2\Big(\varepsilon^* + \sum_1^\infty k s_k \Lambda^{\top k-1})S_2^{-1}\Big)$$

$$M^* = (M_1^*, M_2^*) = (-M_1^\top + L_1^{\top -1}, -M_2^\top + L_2^{\top -1})$$

satisfy, in view of (1.1):

$$\text{(1.10)} \qquad \begin{aligned} L\Psi = (z, z^{-1})\Psi, \quad & M\Psi = \Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial (z^{-1})}\Big)\Psi, \quad [L, M] = (1, 1), \\ L^*\Psi^* = (z, z^{-1})\Psi^*, \quad & M^*\Psi = \Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial (z^{-1})}\Big)\Psi^*, \quad [L^*, M^*] = (1, 1). \end{aligned}$$

The operators $L, M, L^\top, M^\top$ and $W := (W_1, W_2)$ evolve according to

$$\text{(1.11)} \qquad \begin{cases} \frac{\partial}{\partial t_n}\begin{pmatrix} L \\ M \end{pmatrix} = \left[(L_1^n, 0)_+, \begin{pmatrix} L \\ M \end{pmatrix}\right] \\ \\ \frac{\partial}{\partial s_n}\begin{pmatrix} L \\ M \end{pmatrix} = \left[(0, L_2^n)_+, \begin{pmatrix} L \\ M \end{pmatrix}\right] \end{cases}$$

[6]Here the action is viewed componentwise, e.g., $(A, B)\Psi = (A\Psi_1, B\Psi_2)$ or $(z, z^{-1})\Psi = (z\Psi_1, z^{-1}\Psi_2)$.

(1.12)
$$\begin{cases} \frac{\partial}{\partial t_n}\begin{pmatrix} L^\top \\ M^\top \end{pmatrix} = \left[-\Big((L_1^n,0)_+\Big)^\top, \begin{pmatrix} L^\top \\ M^\top \end{pmatrix}\right] \\ \\ \frac{\partial}{\partial s_n}\begin{pmatrix} L^\top \\ M^\top \end{pmatrix} = \left[-\Big((0,L_2^n)_+\Big)^\top, \begin{pmatrix} L^\top \\ M^\top \end{pmatrix}\right] \end{cases}$$

(1.13)
$$\begin{cases} \frac{\partial W}{\partial t_n} = (L_1^n,0)_+ W, \\ \\ \frac{\partial W}{\partial s_n} = (0,L_2^n)_+ W \end{cases}$$

and consequently

$$\begin{aligned} \frac{\partial W_1^{-1}W_2}{\partial t_n} &= -W_1^{-1}(L_1^n)_u W_1 W_1^{-1} W_2 + W_1^{-1}(L_1^n)_u W_2 = 0 \\ \frac{\partial W_1^{-1}W_2}{\partial s_n} &= 0. \end{aligned}$$

This implies that

$$W_1^{-1}W_2(t,s) = W_1^{-1}W_2(0,0) \tag{1.14}$$

and thus

$$(S_1^{-1}S_2)(t,s) = e^{\Sigma t_n \Lambda^n}(S_1^{-1}S_2)(0,0)e^{-\Sigma s_n (\Lambda^\top)^n}. \tag{1.15}$$

Ueno and Takasaki [U-T] show that the 2-Toda deformations of Ψ, and hence L, can ultimately all be expressed in terms of one sequence of τ-functions

$$\tau(n,t,s) = \tau_n(t_1,t_2,\ldots;s_1,s_2,\ldots) = \det[(S_1^{-1}S_2(t,s))_{i,j}]_{-\infty\le i,j\le n-1}, \quad n\in \mathbf{Z}:$$

to wit:
(1.16)
$$\Psi_1(t,s,z) = \left(\frac{e^{-\eta}\tau_n(t,s)}{\tau_n(t,s)}e^{\sum t_i z^i}z^n\right)_{n\in\mathbf{Z}}, \quad \Psi_2(t,s,z) = \left(\frac{e^{-\tilde\eta}\tau_{n+1}(t,s)}{\tau_n(t,s)}e^{\sum s_i z^{-i}}z^n\right)_{n\in\mathbf{Z}}$$

(1.17)
$$\Psi_1^*(t,s,z) = \left(\frac{e^{\eta}\tau_{n+1}(t,s)}{\tau_{n+1}(t,s)}e^{-\Sigma t_i z^i}z^{-n}\right)_{n\in\mathbf{Z}}, \quad \Psi_2^*(t,s,z) = \left(\frac{e^{\tilde\eta}\tau_n(t,s)}{\tau_{n+1}(t,s)}e^{-\Sigma s_i z^{-i}}z^{-n}\right)_{n\in\mathbf{Z}}.$$

where

$$\eta = \sum_1^\infty \frac{z^{-i}}{i}\frac{\partial}{\partial t_i} \quad \text{and} \quad \tilde\eta = \sum_1^\infty \frac{z^{i}}{i}\frac{\partial}{\partial s_i}, \tag{1.18}$$

so that

$$e^{a\eta+b\tilde\eta} f(t,s) = f(t+a[z^{-1}], s+b[z]) \tag{1.19}$$

with $[\alpha] := (\alpha, \alpha^2/2, \alpha^3/3, \ldots)$. Here the labeling of Ψ_i^* is slightly different from the one of [U-T].

The symmetries of the 2-Toda hierarchy are conveniently expressed in terms of the operators L and M. In view of the relation

$$z^\alpha\Big(\frac{\partial}{\partial z}\Big)^\beta \Psi_1 = M_1^\beta L_1^\alpha \Psi_1, \quad u^\alpha\Big(\frac{\partial}{\partial u}\Big)^\beta\Big|_{u=z^{-1}} \Psi_2 = M_2^\beta L_2^\alpha \Psi_2, \tag{1.20}$$

the Lie algebra

$$w_\infty := \operatorname{span}\Big\{z^\alpha\Big(\frac{\partial}{\partial z}\Big)^\beta \;\Big|\; \alpha,\beta\in\mathbf{Z}, \beta\ge 0\Big\}$$

comes naturally into play. To be precise, denoting by ϕ the algebra antihomomorphism

$$\phi\colon w_\infty\times w_\infty \to \mathcal{D} : \begin{cases} (z^\alpha(\partial/\partial z)^\beta, 0) \mapsto (M_1^\beta L_1^\alpha, 0) \\ \\ (0, u^\alpha(\partial/\partial u)^\beta) \mapsto (0, M_2^\beta L_2^\alpha) \end{cases}$$

we have a Lie algebra antihomomorphism[7]

$$\begin{array}{rcl} \mathbf{Y}\colon w_\infty\times w_\infty & \to & \{\text{symmetries on } \Psi, \Psi^*, L, L^\top, M, M^\top\} \\ p & \longmapsto & \begin{cases} \mathbf{Y}_p\Psi = -\phi(p)_-\Psi, \ \mathbf{Y}_p\Psi^* = (\phi(p)_-)^\top\,\Psi^*, \\ \mathbf{Y}_p L = [-\phi(p)_-, L], \ \mathbf{Y}_p L^\top = [(\phi(p)_-)^\top, L], \\ \mathbf{Y}_p M = [-\phi(p)_-, M], \ \mathbf{Y}_p M^\top = [(\phi(p)_-)^\top, M^\top]. \end{cases} \end{array}$$

So for any admissible[8] polynomial q of L and M, we have

$$\mathbf{Y}_p(q) = [-p_-, q] \quad \text{and} \quad \mathbf{Y}_p\Psi = -p_-\Psi. \tag{1.21}$$

[7] Thus $-\mathbf{Y}$ is a Lie algebra homomorphism; note we shall more often denote $\mathbf{Y}_p$ by $\mathbf{Y}_{\phi(p)}$.

[8] The products $M_i^\beta L_i^\alpha$ are admissible, but, in this generality, the product L_1L_2 obviously does not make sense.

To prove

$$[\mathbf{Y}_{p_1}, \mathbf{Y}_{p_2}] = -\mathbf{Y}_{[p_1,p_2]},$$

one computes

$$[\mathbf{Y}_{p_1}, \mathbf{Y}_{p_2}]\Psi = -Z_1\Psi,$$

where, upon using $\mathcal{D}_-$ and $\mathcal{D}_+$ are Lie subalgebras,

$$\begin{aligned} Z_1 &:= -\Big(\mathbf{Y}_{p_1}(p_2)\Big)_- + \Big(\mathbf{Y}_{p_2}(p_1)\Big)_- - [p_{1-}, p_{2-}] \\ &= [p_{1-}, p_2]_- + [-p_{2-}, p_1]_- - [p_{1-}, p_{2-}] = [p_1, p_2]_-. \end{aligned}$$

Moreover

$$[\mathbf{Y}_p, \frac{\partial}{\partial t_n}]\Psi = [\mathbf{Y}_p, \frac{\partial}{\partial s_n}]\Psi = 0;$$

indeed, from (1.8), the vector field,

$$\mathbf{Z}_q\Psi = q_+\Psi, \quad \mathbf{Z}_q p = [q_+, p] \tag{1.22}$$

represents $\frac{\partial}{\partial t_n}$ for $q = (L_1^n, 0)$ and $\frac{\partial}{\partial s_n}$ for $q = (0, L_2^n)$; now one checks

$$[\mathbf{Y}_p, \mathbf{Z}_q]\Psi = -Z_2\Psi \tag{1.23}$$

where

$$Z_2 = (\mathbf{Z}_q(p))_- + (\mathbf{Y}_p(q))_+ - [q_+, p_-] = [q_+, p]_- + [-p_-, q]_+ - [q_+, p_-] = 0.$$

Remark 1: Some explanation is necessary to understand the equation above $\mathbf{Y}_p\Psi^* = (\phi(p)_-)^\top \Psi^*$: the flows on the transposed matrices $L^\top, M^\top$ are immediately obtained from the original flows by just taking transposes; the flows on the adjoint wave operator Ψ^* are obtained as follows: since

$$\begin{cases} (\Psi_1, \Psi_2) = (W_1, W_2)\chi(z) \\ (\Psi_1^*, \Psi_2^*) = ((W_1^{-1})^\top, (W_2^{-1})^\top)\chi(z^{-1}), \end{cases} \tag{1.24}$$

we have for any deformation $'$, that

$$\Psi_i' = W_i'\chi(z), \quad \Psi_i^{*\prime} = (W_i^{-1})^{\top\prime}\chi(z).$$

Thus, we have the straightforward equivalences:

$$W_i' = A_iW_i \Longleftrightarrow \Psi_i' = A\Psi_i$$

and

$$(W_i^{-1})^{\top\prime} = -A^\top (W_i^{-1})^\top \Longleftrightarrow \Psi_i^{*\prime} = -A^\top \Psi_i^*;$$

so, we conclude

$$\Psi_i' = A\Psi_i \Longleftrightarrow \Psi_i^{*\prime} = -A^\top \Psi_i^*. \tag{1.25}$$

Remark 2: Spelling out the symmetries (1.21) on L and Ψ, one finds

$$\mathbf{Y}_{M_i^\alpha L_i^\beta} L = (-1)^{i-1} \left[\left(-(M_i^\alpha L_i^\beta)_\ell, (M_i^\alpha L_i^\beta)_u\right), L\right] \tag{1.26}$$

$$\mathbf{Y}_{M_i^\alpha L_i^\beta} \Psi = (-1)^{i-1} \left(-(M_i^\alpha L_i^\beta)_\ell, (M_i^\alpha L_i^\beta)_u\right) \Psi, \tag{1.27}$$

and the equivalent equations, upon taking the transpose of the previous ones,

$$\mathbf{Y}_{M_i^\alpha L_i^\beta} L^\top = (-1)^{i-1} \left[\left((M_i^\alpha L_i^\beta)_\ell, -(M_i^\alpha L_i^\beta)_u\right)^\top, L^\top\right]$$

(1.27′)

$$\mathbf{Y}_{M_i^\alpha L_i^\beta} \Psi^* = (-1)^{i-1} \left((M_i^\alpha L_i^\beta)_\ell, -(M_i^\alpha L_i^\beta)_u\right)^\top \Psi^*.$$

The action of a vector field on Ψ, commuting with the Toda flows, induces an action on τ via (1.16), thus leading to the following relation between logarithmic derivatives of Ψ and τ:

$$\left(\frac{\Psi_1'}{\Psi_1}\right)_n = (e^{-\eta} - 1)\frac{\tau_n'}{\tau_n}, \quad \left(\frac{\Psi_2'}{\Psi_2}\right)_n = (e^{-\tilde\eta} - 1)\frac{\tau_{n+1}'}{\tau_{n+1}} + \left(\frac{\tau_{n+1}'}{\tau_{n+1}} - \frac{\tau_n'}{\tau_n}\right). \tag{1.28}$$

Proposition 1.1. *Given the symmetry vector field $\mathbf{Y}_p$, the following relation holds:*

$$\mathbf{Y}_p \log \frac{\tau_{n+1}}{\tau_n} = p_{nn} \tag{1.29}$$

Proof: On the one hand, taking into account the expression (1.16) for Ψ_2, we have, in view of the fact that $e^{-\tilde\eta} f(t,s) = f(t, s-[z])$, the following Taylor expansion about $z=0$:

$$\left(\frac{\mathbf{Y}_p\Psi_2}{\Psi_2}\right)_n = \left(\frac{p_u\Psi_2}{\Psi_2}\right)_n$$

(a) $$= \frac{(p_{nn}+O(z))(\frac{\tau_{n+1}(t,s)}{\tau_n(t,s)}+O(z))z^n}{(\frac{\tau_{n+1}(t,s)}{\tau_n(t,s)}+O(z))z^n}$$

$$= p_{nn}+O(z).$$

On the other hand,

(b) $$(e^{-\tilde\eta}-1)\frac{\tau'_{n+1}}{\tau_{n+1}}+(\frac{\tau'_{n+1}}{\tau_{n+1}}-\frac{\tau'_n}{\tau_n})=\frac{\tau'_{n+1}}{\tau_{n+1}}-\frac{\tau'_n}{\tau_n}+O(z)=(\log\frac{\tau_{n+1}}{\tau_n})'+O(z).$$

According to (1.28), the expressions (a) and (b) are equal, leading to (1.29).

In order to spell out the precise relationship between the symmetry vector fields on Ψ, and those acting on τ, we need to discuss generating functions for the symmetries.

A generating function of the **symmetries on the Ψ-manifold** is given by

$$\mathbf{Y}_N\Psi = -N_-\Psi \tag{1.30}$$

which naturally leads to the symmetry on the L-manifold expressed in the Lax form

$$\mathbf{Y}_N L = [-N_-, L], \tag{1.31}$$

with

$$N=(N_1,0) \quad \text{or} \quad (0,N_2), \tag{1.32}$$

$$N_i := (\mu-\lambda)e^{(\mu-\lambda)M_i}\delta(\lambda,L_i)=\sum_{k=1}^{\infty}\frac{(\mu-\lambda)^k}{k!}\sum_{\ell=-\infty}^{\infty}\lambda^{-\ell-k}k(M_i^{k-1}L_i^{k-1+\ell}),$$

where $\delta(\lambda,z):=\sum_{-\infty}^{\infty}\lambda^{-n}z^{n-1}$.

A generating function of the **symmetries on the τ-manifold** is given by a vector of the vertex operators, based on the one of Date, Jimbo, Kashiwara and Miwa, acting on a single τ-function,

$$X(t,\lambda,\mu) := \exp\Big(\sum_1^\infty t_i(\mu^i-\lambda^i)\Big)\exp\Big(\sum_1^\infty(\lambda^{-i}-\mu^{-i})\tfrac{1}{i}\tfrac{\partial}{\partial t_i}\Big),$$

(1.33)

$$= \sum_{k=0}^\infty \tfrac{(\mu-\lambda)^k}{k!}\sum_{\ell=-\infty}^\infty \lambda^{-\ell-k}W_\ell^{(k)}, \quad \text{with } W_\ell^{(0)}=\delta_{\ell 0}.$$

So, the τ-manifold symmetries for the 2-Toda lattice can be expressed as a vector of vertex operators $X(t,\lambda,\mu)$ acting on the vector of τ-functions[9]:

$$\mathbf{X}(t,\lambda,\mu) := \Big(\Big(\frac{\mu}{\lambda}\Big)^n X(t,\lambda,\mu)\Big)_{n\in\mathbf{Z}} = \left(\sum_{k=0}^\infty \frac{(\mu-\lambda)^k}{k!}\sum_{\ell=-\infty}^\infty \lambda^{-\ell-k}W_{n,\ell}^{(k)}\right)_{n\in\mathbf{Z}} \tag{1.34}$$

$$\tilde{\mathbf{X}}(s,\lambda,\mu) := \Big(\Big(\frac{\lambda}{\mu}\Big)^n X(s,\lambda,\mu)\Big)_{n\in\mathbf{Z}} = \left(\sum_{k=0}^\infty \frac{(\mu-\lambda)^k}{k!}\sum_{\ell=-\infty}^\infty \lambda^{-\ell-k}\tilde{W}_{n,\ell}^{(k)}\right)_{n\in\mathbf{Z}} \tag{1.35}$$

with

$$W_{n,\ell}^{(k)} = \sum_{j=0}^k \binom{n}{j}(k)_j W_\ell^{(k-j)} \quad \text{and} \quad \tilde{W}_{n,\ell}^{(k)} = W_{-n,\ell}^{(k)}\Big|_{t\to s}. \tag{1.36}$$

One easily computes (remember the definition of the customary Virasoro generators J's in footnote 2)

$$W_n^{(0)}=\delta_{n,0}, \quad W_n^{(1)}=J_n^{(1)} \quad \text{and} \quad W_n^{(2)} = J_n^{(2)}-(n+1)J_n^{(1)}, \qquad n\in\mathbf{Z} \tag{1.37}$$

and

(1.38)

$$\begin{aligned} W_{m,i}^{(1)} &= W_i^{(1)}+mW_i^{(0)} & W_{m,i}^{(2)} &= W_i^{(2)}+2mW_i^{(1)}+m(m-1)W_i^{(0)} \\ &= J_i^{(1)}+m\delta_{i0} & &= J_i^{(2)}+(2m-i-1)J_i^{(1)}+m(m-1)\delta_{i0}. \end{aligned}$$

For future use, record the formulas

(1.39)

$$\begin{aligned} \frac{1}{2}W_{m,i}^{(2)}+(i+1)W_{m,i}^{(1)} &= \frac{1}{2}J_i^{(2)}+\Big(m+\frac{i+1}{2}\Big)J_i^{(1)}+\frac{m(m+1)}{2}\delta_{i0} \\ \frac{1}{2}W_{m,i}^{(2)}-W_{m,i}^{(1)} &= \frac{1}{2}J_i^{(2)}+\Big(m-\frac{i+3}{2}\Big)J_i^{(1)}+\frac{m(m-3)}{2}\delta_{i0}. \end{aligned}$$

[9]using $\left(\frac{\mu}{\lambda}\right)^\alpha = \sum_{k\ge0}\binom{\alpha}{k}\left(\frac{\mu-\lambda}{\lambda}\right)^k$ and $\binom{\alpha}{k} = \frac{(\alpha)_k}{k!}$

The corresponding expression $\tilde{W}^{(k)}_{m,i}$ can be read off from the above, using (1.36), wich $J^{(k)}_n$ replaced by $\tilde{J}^{(k)}_n = J^{(k)}_n \mid_{t\to s}$.

The symmetry $\mathbf{Y}_N$ acting on Ψ and L as in (1.30) and (1.31) lifts to the vertex operator $\mathbf{X}$ acting on τ, as in (1.34) and (1.35), according to the Adler-Shiota-van Moerbeke formula [ASV2]:

Theorem 1.2. *The vector fields $\mathbf{Y}_{N_1}$ and $\mathbf{Y}_{N_2}$ acting on the Ψ-manifold and the vertex operators of type $\mathbf{X}(t,\lambda,\mu)$ and $\frac{\mu}{\lambda}\tilde{\mathbf{X}}(s,\lambda,\mu)$ acting on the τ-manifold are related as follows:*

$$\frac{\mathbf{Y}_{(N_1,0)}\Psi}{\Psi} = \left((e^{-\eta}-1)\frac{\mathbf{X}(t,\lambda,\mu)\tau}{\tau}, (\Lambda e^{-\tilde{\eta}}-1)\frac{\mathbf{X}(t,\lambda,\mu)\tau}{\tau}\right),$$

$$\frac{\mathbf{Y}_{(0,N_2)}\Psi}{\Psi} = \frac{\mu}{\lambda}\left((e^{-\eta}-1)\frac{\tilde{\mathbf{X}}(s,\lambda,\mu)\tau}{\tau}, (\Lambda e^{-\tilde{\eta}}-1)\frac{\tilde{\mathbf{X}}(s,\lambda,\mu)\tau}{\tau}\right).$$

Expanding $\mathbf{X}$ and $\tilde{\mathbf{X}}$ in terms of W-generators, and N_i in terms of $M_i^\alpha L_i^\beta$, we obtain:

Corollary 1.2.1. *For $n,k \in \mathbf{Z}$, $n \geq 0$, the symmetry vector fields $\mathbf{Y}_{M_i^n L_i^{n+k}}$, $(i=1,2)$ acting on Ψ lead to the correspondences*

$$-\frac{((M_1^n L_1^{n+k})_\ell \Psi_1)_m}{\Psi_{1,m}z^m} = \frac{1}{n+1}(e^{-\eta}-1)\frac{W^{(n+1)}_{m,k}(\tau_m)}{\tau_m}, \tag{1.40}$$

$$\frac{((M_1^n L_1^{n+k})_u \Psi_2)_m}{\Psi_{2,m}z^m} = \frac{1}{n+1}\left(e^{-\tilde{\eta}}\frac{W^{(n+1)}_{m+1,k}(\tau_{m+1})}{\tau_{m+1}} - \frac{W^{(n+1)}_{m,k}(\tau_m)}{\tau_m}\right), \tag{1.41}$$

$$\frac{((M_2^n L_2^{n+k})_\ell \Psi_1)_m}{\Psi_{1,m}z^m} = \frac{1}{n+1}(e^{-\eta}-1)\frac{\tilde{W}^{(n+1)}_{m-1,k}(\tau_m)}{\tau_m}, \tag{1.42}$$

$$-\frac{((M_2^n L_2^{n+k})_u \Psi_2)_m}{\Psi_{2,m}z^m} = \frac{1}{n+1}\left(e^{-\tilde{\eta}}\frac{\tilde{W}^{(n+1)}_{m,k}(\tau_{m+1})}{\tau_{m+1}} - \frac{\tilde{W}^{(n+1)}_{m-1,k}(\tau_m)}{\tau_m}\right), \tag{1.43}$$

and acting on Ψ^*

$$\frac{(((M_1^n L_1^{n+k})_\ell)^\top \Psi_1^*)_m}{\Psi^*_{1,m}z^{-m}} = \frac{1}{n+1}(e^{\eta}-1)\frac{W^{(n+1)}_{m+1,k}(\tau_{m+1})}{\tau_{m+1}} \tag{1.44}$$

$$(1.45)\qquad -\frac{(((M_1^n L_1^{n+k})_u)^\top \Psi_2^*)_m}{\Psi_{2,m}^* z^{-m}} = \frac{1}{n+1}\left(e^{\tilde\eta}\frac{W_{m,k}^{(n+1)}(\tau_m)}{\tau_m} - \frac{W_{m+1,k}^{(n+1)}(\tau_{m+1})}{\tau_{m+1}}\right)$$

$$(1.46)\qquad -\frac{(((M_2^n L_2^{n+k})_\ell)^\top \Psi_1^*)_m}{\Psi_{1,m}^* z^{-m}} = \frac{1}{n+1}(e^{\eta}-1)\frac{\tilde W_{m,k}^{(n+1)}(\tau_{m+1})}{\tau_{m+1}}$$

$$(1.47)\qquad \frac{(((M_2^n L_2^{n+k})_u)^\top \Psi_2^*)_m}{\Psi_{2,m}^* z^{-m}} = \frac{1}{n+1}\left(e^{\tilde\eta}\frac{\tilde W_{m-1,k}^{(n+1)}(\tau_m)}{\tau_m} - \frac{\tilde W_{m,k}^{(n+1)}(\tau_{m+1})}{\tau_{m+1}}\right).$$

Remark: The expansion of the vertex operator (1.33) is obtained by multiplying the series

$$e^{\sum_1^\infty t_i(\mu^i-\lambda^i)} = \sum_{k=0}^\infty (\mu-\lambda)^k \sum_{i=1}^\infty f_i^{(k)}(t)\lambda^{i-1},$$

and

$$e^{\sum_1^\infty (\lambda^{-i}-\mu^{-i})\frac{1}{i}\frac{\partial}{\partial t_i}} = \sum_{k=0}^\infty (\mu-\lambda)^k \sum_{i=0}^\infty g_i^{(k)}(\frac{\partial}{\partial t})\lambda^{-k-i},$$

where $f_i^{(k)}(u)$ and $g_i^{(k)}(u)$ are polynomials in u, vanishing at $u=0$, except $f_1^{(0)} = g_0^{(0)} = 1$. Therefore

$$X(t,\lambda,\mu) = \sum_{k=0}^\infty \frac{(\mu-\lambda)^k}{k!}\Big(\sum_{\ell=-k}^{-\infty} \lambda^{-k-\ell} k! f_{-k-\ell+1}^{(k)} + \sum_{\ell=-\infty}^{\infty} \lambda^{-k-\ell}(\text{differential operator})\Big),$$

showing, for all $\ell \in \mathbf{Z}$,

(1.48)

$$W_\ell^{(k)} = (\text{polynomial in } t, \text{without constant term}) + \left\{\begin{array}{c}\text{pure differential}\\ \text{operator}\end{array}\right\} + \delta_{\ell,0}\delta_{k,0},$$

with

$$W_\ell^{(k)} = (\text{pure differential operator}) \text{ if and only if } \ell \geq -k+1 \text{ with } \ell, k \neq 0.$$

Observe, in view of (1.36), that for $n, \ell \in \mathbf{Z}$,

$$(1.49)\qquad \begin{aligned} W_{n,\ell}^{(k)} &= (\text{differential operator}) + (\text{polynomial in } t, \text{without}\\ &\qquad \text{constant term}) + (n)_k\delta_{\ell,0}\\ \tilde W_{n,\ell}^{(k)} &= (\text{differential operator}) + (\text{polynomial in } s, \text{without}\\ &\qquad \text{constant term}) + (-n)_k\delta_{\ell,0}. \end{aligned}$$

2 A larger class of symmetries for special initial conditions

In this section we consider the submanifold of L, M and Ψ such that polynomials $p(L_1, M_1, L_2, M_2)$ in L_1, M_1, L_2, M_2 are legitimate objects; e.g., $L_1 L_2$ makes sense. This point of view leads to a much wider algebra of symmetries containing the ones of section 1. Henceforth, we change the point of view; namely, p refers to a polynomial and not to a vector of polynomials as in previous section.

From (1.11), it follows that, for such polynomials p, the Toda vector fields can be rewritten as:

$$\frac{\partial p}{\partial t_n} = [(L_1^n)_u\,, p] \quad \text{and} \quad \frac{\partial p}{\partial s_n} = [(L_2^n)_\ell\,, p] \quad n = 1, 2, \ldots. \tag{2.1}$$

Letting $p = p(L_1, M_1, L_2, M_2)$ be a polynomial, we redefine symmetries as follows

$$\mathbf{Y}_p \Psi = (-p_\ell \Psi_1, p_u \Psi_2) = \begin{cases} -(p, 0)_- \Psi \\ (0, p)_- \Psi \end{cases} \tag{2.2}$$

and thus

$$\mathbf{Y}_p(L_1, L_2) = ([-p_\ell, L_1]\,, [p_u, L_2]) = \begin{cases} [-(p, 0)_-, L] \\ [(0, p)_-, L] \end{cases} \tag{2.3}$$

and similarly for L_1, L_2 replaced by M_1, M_2. Then

$$\begin{aligned} [\mathbf{Y}_p, \mathbf{Y}_{\bar p}]\Psi &= \Big((\mathbf{Y}_p(\bar p, 0))_- - (\mathbf{Y}_{\bar p}(p, 0))_- + [(p, 0)_-, (\bar p, 0)_-] \Big)\Psi \\ &= -\Big((\mathbf{Y}_p(0, \bar p))_- - (\mathbf{Y}_{\bar p}(0, p))_- - [(0, p)_-, (0, \bar p)_-] \Big)\Psi \\ &= \left(\Big(((\mathbf{Y}_p(\bar p))_\ell - (\mathbf{Y}_{\bar p}(p))_\ell + [p_\ell, \bar p_\ell] \Big)\Psi_1, \right. \\ &\qquad \left. -((\mathbf{Y}_p(\bar p))_u - (\mathbf{Y}_{\bar p}(p))_u - [p_u, \bar p_u] \Big)\Psi_2 \right) = \mathbf{Y}_{\{p, \bar p\}}\Psi \end{aligned} \tag{2.4}$$

where

$$\{p, \bar p\} := -\mathbf{Y}_p(\bar p) + \mathbf{Y}_{\bar p}(p) - [p_\ell, \bar p_\ell] + [p_u, \bar p_u]. \tag{2.5}$$

We verify that these more general vector fields are indeed symmetries, i.e.

$$(i) \qquad [\mathbf{Y}_p, \frac{\partial}{\partial t_n}] = 0, \qquad\qquad (ii) \qquad [\mathbf{Y}_p, \frac{\partial}{\partial s_n}] = 0,$$

using the argument of (1.23). For the case (i), set

$$Q = (q,0) = (L_1^n, 0) \quad P = (p,0),$$

and, using (2.1) and (2.2), we define

$$\text{(a)} \qquad \tilde{\mathbf{Z}}_Q(P) = \frac{\partial}{\partial t_n}(p,0) = [Q_+, P] \text{ and } \tilde{\mathbf{Z}}_Q\Psi = Q_+\Psi$$

$$\text{(b)} \quad \tilde{\mathbf{Y}}_P(Q) = (-[p_\ell, q], 0) = [-(p_\ell, -p_u), (q,0)] = [-P_-, Q], \text{ and } \tilde{\mathbf{Y}}_P\Psi = -P_-\Psi;$$

we are exactly in the situation (1.21) and (1.23), leading at once to $[\mathbf{Y}_P, \mathbf{Z}_Q] = 0$, which is (i). A similar argument leads to (ii)

Let $\mathcal{A}$ be the Lie algebra of generalized symmetries of the 2-Toda vector fields, with the standard commutator [,]. Also, the ring of polynomials $\mathbf{C}[L_1, L_2, M_1, M_2]$ in the (noncommutative) variables L_1, L_2, M_1, M_2, forms a Lie algebra, not only with regard to the standard bracket [,], but also with regard to the new bracket $\{p, \bar{p}\}$ defined in (2.5), as wil be seen in the next theorem; denote by

$A_1, B_1, C_1, ...$ = any monomial (L_1, M_1, I) and $A_2, B_2, C_2, ...$ = any monomial (L_2, M_2, I).

Lemma 2.1. *The two brackets [,] and { , } of the Lie algebra $\mathbf{C}[L_1, L_2, M_1, M_2]$ interact as follows: for monomials $p = A_1A_2B_1B_2C_1C_2\ldots$ and $\bar{p} = \bar{A}_1\bar{A}_2\bar{B}_1\bar{B}_2\bar{C}_1\bar{C}_2\ldots$ in the algebra, two different expressions hold,*

$$(2.6) \qquad \begin{aligned} \{p, \bar{p}\} &= [p, \bar{p}] - \bar{A}_1[p_1, \bar{A}_2]\bar{B}_1\bar{B}_2\bar{C}_1\bar{C}_2\ldots - \bar{A}_1\bar{A}_2\bar{B}_1[p, \bar{B}_2]\bar{C}_1\bar{C}_2\ldots - \ldots \\ &\quad + A_1[\bar{p}, A_2]B_1B_2C_1C_2\ldots + A_1A_2B_1[\bar{p}, B_2]C_1C_2\ldots + \ldots \\ &= [p, \bar{A}_1]\bar{A}_2\bar{B}_1\bar{B}_2\bar{C}_1\bar{C}_2\ldots + \bar{A}_1\bar{A}_2[p, \bar{B}_1]\bar{B}_2\bar{C}_1\bar{C}_2\ldots + \ldots \\ &\quad + A_1[\bar{p}, A_2]B_1B_2C_1C_2\ldots + A_1A_2B_1[\bar{p}, B_2]C_1C_2\ldots + \ldots \end{aligned}$$

Theorem 2.2. *There is a Lie algebra homomorphism*

$$\mathbf{C}[L_1, L_2, M_1, M_2],\ \{\ \} \longmapsto \mathcal{A},\ [\ ]$$

$$p \to \mathbf{Y}_p : \mathbf{Y}_p\Psi = -(p,0)_-\Psi = (0,p)_-\Psi$$

i.e.,

$$[\mathbf{Y}_p, \mathbf{Y}_{\bar{p}}]\Psi = \mathbf{Y}_{\{p,\bar{p}\}}\Psi.$$

Moreover, we have the following inclusion (Lie algebra (anti)-homomorphism)

$$\mathcal{A} \supset w_\infty \oplus w_\infty \text{ with } \{A_1, A_2\} = 0,$$

and

$$\{A_1, B_1\} = [A_1, B_1] \quad \{A_2, B_2\} = -[A_2, B_2];$$

Also

$$\mathcal{A} \supset w_\infty \otimes \mathbf{C}(h, h^{-1}), \qquad \textit{(Loop algebra over } w_\infty), \tag{2.7}$$

in two different ways:

$$\{A_1 L_2^k, B_1 L_2^\ell\} = [A_1, B_1] L_2^{k+\ell}, \{L_1^k A_2, L_1^\ell B_2\} = -L_1^{k+\ell}[A_2, B_2]. \tag{2.8}$$

Corollary 2.2.1. *The vector fields*

$$\mathbf{Y}_p \Psi = -(p, 0)_- \Psi \text{ with } p = M_1^n L_1^{n+k}, \ M_2^n L_2^{n+k}, \ L_1^\alpha L_2^\beta$$

all commute with the Toda vector fields $\partial/\partial t_n$ and $\partial/\partial s_n$. The vector fields $\partial/\partial c_{\alpha,\beta}$, corresponding to $p = L_1^\alpha L_2^\beta$, all commute among themselves.

Remark 2.2.2: The τ-function can then be considered as a function of t, s and c, as follows:

$$\tau(c_{10} + t_1, c_{20} + t_2, ..., ; c_{01} + s_1, c_{02} + s_2, ...; c_{11}, c_{12}, c_{21}, ...).$$

Proof of Lemma 2.1, Theorem 2.2 and Corollary 2.2.1.: Since $\mathbf{Y}_p(.)$ and $[x, .]$ are derivations, one computes for $p = A_1 A_2 B_1 B_2$ and $\bar{p} = \bar{A}_1 \bar{A}_2 \bar{B}_1 \bar{B}_2$, that

$$\begin{aligned}
\mathbf{Y}_p(\bar{p}) &= \mathbf{Y}_p(\bar{A}_1)\bar{A}_2\bar{B}_1\bar{B}_2 + \bar{A}_1\mathbf{Y}_p(\bar{A}_2)\bar{B}_1\bar{B}_2 + \bar{A}_1\bar{A}_2\mathbf{Y}_p(\bar{B}_1)\bar{B}_2 + \bar{A}_1\bar{A}_2 B_1\mathbf{Y}_p(\bar{B}_2) \\
&= -[p_\ell, \bar{A}_1]\bar{A}_2\bar{B}_1\bar{B}_2 + \bar{A}_1[p_u, \bar{A}_2]\bar{B}_1\bar{B}_2 - \bar{A}_1\bar{A}_2[p_\ell, \bar{B}_1]\bar{B}_2 + \bar{A}_1\bar{A}_2\bar{B}_1[p_u, \bar{B}_2] \\
&\text{or} \left\{ \begin{array}{l} = -[p_\ell, \bar{A}_1\bar{A}_2\bar{B}_1\bar{B}_2] + \bar{A}_1[p_\ell + p_u, \bar{A}_2]\bar{B}_1\bar{B}_2 + \bar{A}_1\bar{A}_2\bar{B}_1[p_\ell + p_u, \bar{B}_2] \\ = [p_u, \bar{A}_1\bar{A}_2\bar{B}_1\bar{B}_2] - [p_\ell + p_u, \bar{A}_1]\bar{A}_2\bar{B}_1\bar{B}_2 - \bar{A}_1\bar{A}_2[p_\ell + p_u, \bar{B}_1]\bar{B}_2 \end{array} \right. \\
&\text{or} \left\{ \begin{array}{l} = -[\bar{p}_\ell, \bar{p}] + \bar{A}_1[p, \bar{A}_2]\bar{B}_1\bar{B}_2 + \bar{A}_1\bar{A}_2\bar{B}_1[p, \bar{B}_2] \\ = [p_u, \bar{p}] - [p, \bar{A}_1]\bar{A}_2\bar{B}_1\bar{B}_2 - \bar{A}_1\bar{A}_2[p, \bar{B}_1]\bar{B}_2 \end{array} \right.
\end{aligned}$$

and by symmetry, using the first expression, one finds

$$\mathbf{Y}_{\bar{p}}(p) = -[\bar{p}_\ell, p] + A_1[\bar{p}, A_2]B_1B_2 + A_1A_2B_1[\bar{p}, B_2].$$

Using the definition (2.5) of $\{p,\bar{p}\}$, one computes

$$
\begin{aligned}
\{p,\bar{p}\} \quad &= \quad -\mathbf{Y}_p(\bar{p}) + \mathbf{Y}_{\bar{p}}(p) - [p_\ell,\bar{p}_\ell] + [p_u,\bar{p}_u] \\
\text{or} \quad &\left\{\begin{array}{l}
= [p_\ell,\bar{p}] + [p,\bar{p}_\ell] - [p_\ell,\bar{p}_\ell] + [p_u,\bar{p}_u] \\
\quad -\bar{A}_1[p,\bar{A}_2]\bar{B}_1\bar{B}_2 - \bar{A}_1\bar{A}_2\bar{B}_1[p,\bar{B}_2] + A_1[\bar{p},A_2]B_1B_2 + A_1A_2B_1[\bar{p},B_2] \\
= -[p_u,\bar{p}] + [p,\bar{p}_\ell] - [p_\ell,\bar{p}_\ell] + [p_u,\bar{p}_u] \\
\quad +[p,\bar{A}_1]\bar{A}_2\bar{B}_1\bar{B}_2 + \bar{A}_1\bar{A}_2[p,\bar{B}_1]\bar{B}_2 + A_1[\bar{p},A_2]B_1B_2 + A_1A_2B_1[\bar{p},B_2]
\end{array}\right. \\
\text{or} \quad &\left\{\begin{array}{l}
= [p,\bar{p}] - \bar{A}_1[p,\bar{A}_2]\bar{B}_1\bar{B}_2 + A_1[\bar{p},A_2]B_1B_2 - \bar{A}_1\bar{A}_2\bar{B}_1[p,\bar{B}_2] + A_1A_2B_1[\bar{p},B_2] \\
= [p,\bar{A}_1]\bar{A}_2\bar{B}_1\bar{B}_2 + \bar{A}_1\bar{A}_2[p,\bar{B}_1]\bar{B}_2 + A_1[\bar{p},A_2]B_1B_2 + A_1A_2B_1[\bar{p},B_2],
\end{array}\right.
\end{aligned}
$$

using

$$[p_\ell,\bar{p}] + [p,\bar{p}_\ell] - [p_\ell,\bar{p}_\ell] + [p_u,\bar{p}_u] = [p_\ell,\bar{p}] + [p_u,\bar{p}_\ell] + [p_u,\bar{p}_u] = [p_\ell,\bar{p}] + [p_u,\bar{p}] = [p,\bar{p}]$$

and

$$[p_u,\bar{p}] - [p,\bar{p}_\ell] + [p_\ell,\bar{p}_\ell] - [p_u,\bar{p}_u] = 0,$$

establishing (2.6).

In particular we have the following:

$$
\begin{aligned}
(2.9) \qquad \{A_1A_2, B_1B_2\} &= [A_1,B_1]A_2B_2 - A_1B_1[A_2,B_2] \\
\{A_2A_1, B_2B_1\} &= A_2B_2[A_1,B_1] - [A_2,B_2]A_1B_1 \\
\{A_1A_2, B_2B_1\} &= [A_1,B_2][B_1,A_2] - A_1[A_2,B_2]B_1 + B_2[A_1,B_1]A_2,
\end{aligned}
$$

from which it follows at once that

$$(2.10) \qquad \{A_1,A_2\} = 0, \qquad \{A_1,B_1\} = [A_1,B_1], \qquad \{A_2,B_2\} = -[A_2,B_2];$$

this implies that $w_\infty \oplus w_\infty \subset \mathcal{A}$. Moreover, the first relation of (2.9) leads to (2.8), implying (2.7) with $h = L_1$ or $h = L_2$, ending the proof of theorem 2.2. From (2.9), it follows that

(2.11)

$$
\begin{aligned}
\{L_1^iL_2^j, L_1^{i'}L_2^{j'}\} \quad &= \quad \{L_2^jL_1^i, L_2^{j'}L_1^{i'}\} = 0 \\
\{L_1^iL_2^j, L_2^{i'}L_1^{j'}\} \quad &= \quad [L_1^i, L_2^{i'}][L_1^{j'}, L_2^j],
\end{aligned}
$$

establishing corollary 2.2.1.

Theorem 2.3. *Given functions*

$$f(L_1, L_2) = \sum a_{ij} L_1^i L_2^j \text{ and } g(L_1, L_2) = \sum b_{ij} L_1^i L_2^j,$$

then the following matrices form a Virasoro algebra (without central extension) for the $\{\ ,\ \}$*-bracket*[10]*:*

$$\{L_1^{i+1}(M_1 + f), L_1^{j+1}(M_1 + f)\} = (i - j) L_1^{i+j+1}(M_1 + f) \tag{2.12}$$

$$\{(M_2 - g)L_2^{i+1}, (M_2 - g)L_2^{j+1})\} = (j - i)(M_2 - g)L_2^{i+j+1}. \tag{2.13}$$

If $\frac{\partial g}{\partial z_1} = \frac{\partial f}{\partial z_2}$*, then the two representations decouple*

$$\{L_1^{i+1}(M_1 + f), (M_2 - g)L_2^{j+1}\} = 0. \tag{2.14}$$

Proof: At first, we need to prove

(2.15)

$$\begin{aligned}
\{L_1^k M_1, L_1^\alpha f L_2^\beta\} &= -\alpha L_1^{k+\alpha-1} f L_2^\beta - L_1^{k+\alpha} \frac{\partial f}{\partial z_1} L_2^\beta \\
\{M_2 L_2^k, L_1^\alpha f L_2^\beta\} &= \beta L_1^\alpha f L_2^{k+\beta-1} + L_1^\alpha \frac{\partial f}{\partial z_2} L_2^{k+\beta}.
\end{aligned}$$

Indeed, it suffices to do so for a monomial $f = L_1^i L_2^j$; e.g., using (2.9), we have

$$\begin{aligned}
\{L_1^k M_1, L_1^{\alpha+i} L_2^{\beta+j}\} &= [L_1^k M_1, L_1^{\alpha+i}] L_2^{\beta+j} = -(\alpha + i) L_1^{\alpha+i+k-1} L_2^{\beta+j} \\
\{M_2 L_2^k, L_1^{\alpha+i} L_2^{\beta+j}\} &= -L_1^{\alpha+i}[M_2 L_2^k, L_2^{\beta+j}] = (\beta + j) L_1^{\alpha+i} L_2^{\beta+j+k-1}.
\end{aligned}$$

Using the relation (2.10), (2.11) and (2.15) above, we find

$$\begin{aligned}
&\{L_1^{i+1}(M_1 + f), L_1^{j+1}(M_1 + f)\} \\
&= \{L_1^{i+1} M_1, L_1^{j+1} M_1\} + \{L_1^{i+1} f, L_1^{j+1} M_1\} + \{L_1^{i+1} M_1, L_1^{j+1} f\} + \{L_1^{i+1} f, L_1^{j+1} f\} \\
&= (i - j) L_1^{i+j+1} M_1 + (i + 1) L_1^{i+j+1} f + L_1^{i+j+2} \frac{\partial f}{\partial z_1} - (j + 1) L_1^{i+j+1} f - L_1^{i+j+2} \frac{\partial f}{\partial z_1} \\
&= (i - j) L_1^{i+j+1}(M_1 + f),
\end{aligned}$$

and the same relations yield

[10] $f = f(L_1, L_2) = \sum_{i,j} a_{ij} L_1^i L_2^j$ and $g = g(L_1, L_2) = \sum_{i,j} b_{ij} L_1^i L_2^j$

$$\begin{aligned}
&\{(M_2-g)L_2^{i+1},(M_2-g)L_2^{j+1}\}\\
&= \{M_2L_2^{i+1},M_2L_2^{j+1}\}-\{gL_2^{i+1},M_2L_2^{j+1}\}-\{M_2L_2^{i+1},gL_2^{j+1}\}+\{gL_2^{i+1},gL_2^{j+1}\}\\
&= (j-i)M_2L_2^{i+j+1}+(i+1)gL_2^{i+j+1}+\frac{\partial g}{\partial z_2}L_2^{i+j+2}-(j+1)gL_2^{i+j+1}-\frac{\partial g}{\partial z_2}L_2^{i+j+2}\\
&= (j-i)(M_2-g)L_2^{i+j+1},
\end{aligned}$$

confirming (2.12) and (2.13). Finally, one checks (2.14),

$$\begin{aligned}
&\{L_1^{i+1}(M_1+f),(M_2-g)L_2^{j+1}\}\\
&= \{L_1^{i+1}M_1,M_2L_2^{j+1}\}+\{L_1^{i+1}f,M_2L_2^{j+1}\}-\{L_1^{i+1}M_1,gL_2^{j+1}\}-\{L_1^{i+1}f,gL_2^{j+1}\}\\
&= 0-L_1^{i+1}\frac{\partial f}{\partial z_2}L_2^{j+1}+L_1^{i+1}\frac{\partial g}{\partial z_1}L_2^{j+1}+0=-L_1^{i+1}\Big(\frac{\partial f}{\partial z_2}-\frac{\partial g}{\partial z_1}\Big)L_2^{j+1},
\end{aligned}$$

ending the proof of Theorem 2.3.

In the next theorem, we provide the explicit solution to a large class of symmetries and, in particular, to the system of differential equations (0.3'). Since the S_1 and S_2 of the Borel decomposition of the moment matrix m_∞ will turn out to be the S_1 and S_2 of this section, it will suffice to prove theorem 2.4. Note that symmetry vector fields of the form

(2.16)
$$\frac{\partial\Psi}{\partial c}:=\mathbf{Y}_p\Psi=-(p,0)_-\Psi=(-p_\ell,p_u)(\Psi_1,\Psi_2)\text{ with }p=p_1(L_1,M_1)p_2(L_2,M_2)$$

induce vector fields on $S_1^{-1}S_2$:

(2.17)
$$\frac{\partial S_1^{-1}S_2}{\partial c}=p_1(\Lambda,\varepsilon+\sum_{k=1}^\infty kt_k\Lambda^{k-1})S_1^{-1}S_2p_2(\Lambda^\top,\varepsilon^*+\sum_{k=1}^\infty ks_k\Lambda^{\top k-1}).$$

Theorem 2.4. *For symmetries of the form (2.16), we find the solution*

(2.18) $(S_1^{-1}S_2)(t,s,c)=$
$$\sum_{r=0}^\infty\frac{c^r}{r!}p_1^r\Big(\Lambda,\varepsilon+\sum_1^\infty kt_k\Lambda^{k-1})e^{\sum_1^\infty t_k\Lambda^k}(S_1^{-1}S_2)(0,0,0)e^{-\sum_1^\infty s_k\Lambda^{\top k}}p_2^r(\Lambda^\top,\varepsilon^*+\sum_1^\infty kt_k\Lambda^{\top k-1})$$

with

(2.19)
$$\tau_N(t,s,c)=\det\Big(S_1^{-1}S_2(t,s,c)_{ij}\Big)_{0\le i,j\le N-1}.$$

Corollary 2.4.1. *In particular, symmetries of the form*

$$\frac{\partial \Psi}{\partial c_{\alpha\beta}} := \mathbf{Y}_{L_1^\alpha L_2^\beta} \Psi,$$

have the following solution

$$S_1^{-1} S_2(t, s, c_{\alpha\beta}) = \sum_{r=0}^{\infty} \frac{c_{\alpha\beta}^r}{r!} \Lambda^{r\alpha} e^{\sum_1^\infty t_k \Lambda^k} S_1^{-1} S_2(0,0,0) e^{-\sum_1^\infty s_k \Lambda^{\top k}} (\Lambda^\top)^{r\beta}$$

Corollary 2.4.2. *In general, setting* $t = c_{10}$ *and* $s = -c_{0,1}$*, we have*

$S_1^{-1} S_2(t, s, c_{11}, c_{12}, \ldots) =$

$$\sum_{\substack{(r_{\alpha\beta})_{\alpha,\beta\geq 0} \in \mathbf{Z}^\infty \\ (\alpha,\beta)\neq(0,0)}} \left(\prod_{(\alpha,\beta)} \frac{c_{\alpha\beta}^{r_{\alpha\beta}}}{r_{\alpha\beta}!} \right) \Lambda^{\sum_{\alpha\geq 1} \alpha r_{\alpha\beta}} S_1^{-1} S_2(0,0,0,\ldots) \Lambda^{\top \sum_{\beta \geq 1} \beta r_{\alpha\beta}};$$

Proof of theorem 2.4: Note that the vector field (2.16) induces the vector field

$$\mathbf{Y}_p(S_1) = -p_\ell S_1 \text{ and } \mathbf{Y}_p(S_2) = p_u S_2$$

and thus for $p = p_1(L_1, M_1) p_2(L_2, M_2)$ as in (2.16), we find

$$\begin{aligned} \mathbf{Y}_p(S_1^{-1} S_2) &= -S_1^{-1}(-p_\ell S_1) S_1^{-1} S_2 + S_1^{-1} p_u S_2 = S_1^{-1} p S_2 \\ &= S_1^{-1} p S_2 \\ &= p_1(\Lambda, \varepsilon + \sum_1^\infty k t_k \Lambda^{k-1}) S_1^{-1} S_2 p_2(\Lambda^\top, \varepsilon^* + \sum_1^\infty k s_k (\Lambda^\top)^{k-1}), \end{aligned}$$

a linear differential equation for $S_1^{-1} S_2$. Since for any linear differential equation

$$\frac{\partial x}{\partial c} = L(x) \quad \Longleftrightarrow x(c) = e^{cL} x(0) = \sum \frac{c^r}{r!} L^r x(0),$$

the conclusion (2.18) follows. Corollary 2.4.1. is merely a special case of (2.18), while corollary 2.4.2 takes into account the t-, s- and $c_{\alpha\beta}$-flows all at once.

3 Borel decomposition of moment matrices, τ-functions and string-orthogonal polynomials

Consider the weight $\rho(y,z)dydz = e^{V(y,z)}dydz$ on $\mathbb{R}^2$, the corresponding inner product $\langle\ ,\ \rangle$ and the moment matrix[11] $m_n(t,s,c)$, as in (0.1),(0.2) and (0.3). In the introduction, *string-orthogonal polynomials* were defined as two sequences of monic polynomials of degree i, each depending on one variable (y and $z \in \mathbb{R}$)

$$\{p_i^{(1)}(y)\}_{i=0}^{\infty} \quad \text{and} \quad \{p_i^{(2)}(z)\}_{i=0}^{\infty},$$

orthogonal in the following sense

$$\langle p_i^{(1)}, p_j^{(2)}\rangle = h_i \delta_{ij}.$$

Also consider the $N \times \infty$ matrix of Schur polynomials[12] $p_i(t)$:

$$\begin{aligned} E_N(t) &= \left(\begin{array}{ccccc|cc} 1 & p_1(t) & p_2(t) & \dots & p_{N-1}(t) & p_N(t) & \dots \\ 0 & 1 & p_1(t) & \dots & p_{N-2}(t) & p_{N-1}(t) & \dots \\ \vdots & \vdots & & & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & p_1(t) & p_2(t) & \dots \\ 0 & 0 & 0 & \dots & 1 & p_1(t) & \dots \end{array}\right) \\ &= (p_{j-i}(t))_{\substack{0\le i\le N-1 \\ 0\le j<\infty}} \end{aligned}$$

The polynomials $p_n^{(1)}(y)$ and $p_n^{(2)}(z)$ have explicit expressions in terms of the moments

$$p_n^{(1)}(y) = \frac{1}{\det m_n(t,s,c)} \det \left(\begin{array}{ccc|c} & & & 1 \\ & m_n(t,s,c) & & y \\ & & & \vdots \\ \hline \mu_{n,0}(t,s,c) & \dots & \mu_{n,n-1}(t,s,c) & y^n \end{array}\right) \tag{3.1}$$

---

[11]In this and subsequent sections

$$(t,s,c) = \Big((t_n)_{n\ge1}, (s_n)_{n\ge1}, (c_{\alpha\beta})_{\alpha,\beta\ge1}\Big) =: (c_{\alpha\beta})_{\substack{\alpha,\beta\ge0 \\ (\alpha,\beta)\neq(0,0)}}$$

with the understanding that $t_n = c_{n0}$, $s_n = -c_{0n}$.

[12]The Schur polynomials p_i, defined by $e^{\sum_1^\infty t_i z^i} = \sum_0^\infty p_k(t) z^k$ and $p_k(t) = 0$ for $k<0$, are not to be confused with the string-orthogonal polynomials $p_i^{(k)}$, $k=1,2$.

and

$$p_n^{(2)}(z) = \frac{1}{\det m_n(t,s,c)} \det \left(\begin{array}{ccc|c} & & & 1 \\ & & & z \\ & m_n^\top(t,s,c) & & \vdots \\ \hline \mu_{0,n}(t,s,c) & \dots & \mu_{n-1,n}(t,s,c) & z^n \end{array} \right),$$

from which

$$h_i = \langle p_i^{(1)}, p_i^{(2)} \rangle = \langle p_i^{(1)}, z^i \rangle = \frac{\det m_{i+1}}{\det m_i}.$$

For future use, we need some notation: for an arbitrary matrix

$$A = (a_{ij})_{0 \le i,j \le N-1}$$

with N finite or infinite, denote by A_i the upper-left $i \times i$ minor and $\Delta_i := \det A_i$, with $i \ge 1$ and $\Delta_0 = 1$; assuming all $\Delta_i \neq 0$, define the following monic polynomials of degree i $(0 \le i \le N-1)$

(3.2) $$P_i(z,A) := \frac{1}{\Delta_i} \det \begin{pmatrix} & & & 1 \\ & A_i & & \vdots \\ & & & z^{i-1} \\ a_{i0} & \dots & a_{i,i-1} & z^i \end{pmatrix} =: \sum_{0 \le j \le i} \frac{\Delta_{ji}^{(i+1)}}{\Delta_i} z^j = z^i + \dots,$$

with

$$\Delta_{ii}^{(i+1)} = \Delta_i \text{ and } \Delta_{ji}^{(i+1)} = 0, \quad i < j.$$

Given the character $\bar{\chi}(z) = (1, z, z^2, \dots)^\top$, the lower triangular matrix $S(A)$, with 1's along the diagonal, is defined such that

(3.3) $$S(A)\chi(z) := (P_0(z,A), P_1(z,A), \dots)^\top.$$

Also define the $N \times N$ diagonal matrix $h = h(A)$

(3.4) $$h := diag(h_0, h_1, \dots, h_{N-1}) := diag(\frac{\Delta_1}{\Delta_0}, \dots, \frac{\Delta_N}{\Delta_{N-1}}),$$

and so $\Delta_N = \prod_0^{N-1} h_i$. Theorem 3.1 shows that performing the Borel decomposition of the matrix m_∞ is tantamount to the construction of the string-orthogonal polynomials . Given m_∞, define $S(m_\infty)$, $S(m_\infty^\top)$ and $h(m_\infty)$ by means of (3.3) and (3.4). We now state Theorem 3.1, whose proof can be found in [AvM3].

Theorem 3.1. *The vectors of string-orthogonal polynomials are given by*

$$p^{(1)}(y) = L^{-1}\chi(y) \text{ and } p^{(2)}(z) = (U^{-1})^\top \chi(z) \tag{3.5}$$

where L and U are the lower- and upper-triangular matrices respectively, with 1's on the diagonal, appearing in the Borel decomposition of the matrix m_∞:

$$m_\infty = L\ h\ U := S^{-1}(m_\infty)h(m_\infty)(S^{-1}(m_\infty^\top))^\top. \tag{3.6}$$

Corollary 3.1.1. *The string-orthogonality relations are equivalent to the Borel decomposition of the moment matrix m_∞:*

$$\langle p_i^{(1)}, p_j^{(2)}\rangle = h_i\delta_{ij} \Longleftrightarrow S(m_\infty)m_\infty(S(m_\infty^\top))^\top = h(m_\infty). \tag{3.7}$$

Remark 3.1.1: Any matrix Δ (finite or infinite), with all $\Delta_i \neq 0$, can be realized as a moment matrix m_∞ and the recipe (3.6) provides its Borel decomposition.

The determinant of the moment matrix m_N can be expressed in terms of a double matrix integral, as stated in the following theorem:

Theorem 3.2. *The following integral*

(3.8)

$$\begin{aligned}\tau_N(t,s,c) &:= \int\int_{\vec{u},\vec{v}\in\mathbb{R}^N} e^{\sum_{k=1}^N V(u_k,v_k)} \prod_{i<j}(u_i-u_j)\prod(v_i-v_j)d\vec{u}d\vec{v} \\ &= N!\det m_N(t,s,c) \\ &= N!\det\Big(E_N(t)m_\infty(0,0,c)E_N(-s)^\top\Big)\end{aligned}$$

is a τ-function of t and $-s$; in particular τ_N satisfies the KP-hierarchy in t and s, for each $N = 0, 1, \ldots$; moreover the $h_n = \langle p_n^{(1)}, p_n^{(2)}\rangle$ are given by

$$h_n(t,s,c) = \frac{\tau_{n+1}(t,s,c)}{\tau_n(t,s,c)} \qquad \tau_0(t,s,c) = 1. \tag{3.9}$$

Remark: For $V(y,z) = \sum_1^\infty t_i y^i + cyz - \sum_1^\infty s_i z^i$, the integral (3.8) is well known [AS1] to be the two matrix integral, integrated over the space of Hermitean matrices of size N:

$$\Big(\frac{\pi}{\sqrt{c}}\Big)^{-N^2+N} \cdot \prod_{k=1}^N k! \int\int_{\mathcal{H}_N\times\mathcal{H}_N} dM_1 dM_2 e^{\operatorname{tr} V(M_1,M_2)}.$$

Before giving the proof of this theorem, we shall need several Lemmas:

Lemma 3.2.1.
$$m_N(t,s) = E_N(t) m_\infty(0,0) E_N(-s)^\top.$$
and
$$m_\infty(t,s) = e^{\sum_1^\infty t_n \Lambda^n} m_\infty(0,0) e^{-\sum_1^\infty s_n \Lambda^{\top n}}.$$

Proof:

$$\begin{aligned}
m_N(t,s) &= \Big(\int\int_{\mathbb{R}^2} u^{\ell-1} v^{k-1} e^{\Sigma t_i u^i} e^{V_{12}(u,v)} e^{-\Sigma s_j v^j} du\, dv\Big)_{1\le \ell,k\le N}\\
&= \left(\sum_{i,j=0}^{\infty} p_i(t) \Big(\int\int_{\mathbb{R}^2} e^{V_{12}(u,v)} u^{i+\ell-1} v^{j+k-1} du\, dv\Big) p_j(-s)\right)_{1\le \ell,k\le N}\\
&= \Big(\sum_{i,j=0}^{\infty} p_i(t) \mu_{i+\ell-1,j+k-1}(0,0) p_j(-s)\Big)_{1\le \ell,k\le N}\\
&= E_N(t) m_\infty(0,0) E_N(-s)^\top.
\end{aligned}$$

The second line follows from the first one, since $E_\infty(t) = e^{\sum t_n \Lambda^n}$, establishing Lemma 3.2.1.

Lemma 3.2.2. *For arbitrary sequences of monic polynomials $p_k^{(1)}$ and $p_k^{(2)}$, we have*

$$\begin{aligned}
&\det(u_k^{\ell-1})_{1\le \ell,k\le N} \det(v_k^{\ell-1})_{1\le \ell,k\le N}\\
&\qquad = \sum_{\sigma\in S_N} \det\Big(u_{\sigma(k)}^{\ell-1} v_{\sigma(k)}^{k-1}\Big)_{1\le \ell,k\le N}\\
&\qquad = \det\Big(p_{\ell-1}^{(1)}(u_k)\Big)_{1\le \ell,k\le N} \det\Big(p_{\ell-1}^{(2)}(v_k)\Big)_{1\le \ell,k\le N}.
\end{aligned}$$

Proof: This is a standard result about Vandermonde determinants.

Consider a linear space W^0 spanned by formal (Laurent) series in z (for large z):

$$W^0 = \operatorname{span}\Big\{\sum_{j=-\infty}^{N-1} a_{jk} z^j, k = 0, \ldots, N-1\Big\} \oplus z^N H_+.$$

Lemma 3.2.3. *Letting*
$$W^t = e^{\Sigma t_i z^i} W^0,$$
the τ-function
$$\tau(t) = \det(\mathrm{Proj} : W^t \to H_+)$$

has the form

$$\tau(t) = \det(E_N(t)a_N),$$

where a_N *is the* (∞, N)*-matrix of coefficients*

$$a_N = \left(\begin{array}{ccc} a_{N-1,0} & \dots & a_{N-1,N-1} \\ \vdots & & \vdots \\ a_{10} & \dots & a_{1,N-1} \\ a_{00} & \dots & a_{0,N-1} \\ \hline a_{-10} & \dots & a_{-1,N-1} \\ a_{-20} & \dots & a_{-2,N-1} \\ \vdots & & \vdots \end{array}\right) \tag{3.10}$$

Proof:

$$\begin{aligned} W^t &= \operatorname{span}\Big\{\sum_{i=0}^{\infty} p_i(t) z^i \sum_{j=-\infty}^{N-1} a_{jn} z^j, n = 0, 1, \dots, N-1\Big\} \oplus \Big\{\sum_0^{\infty} p_i(t) z^{i+N+k}, k = 0, 1, 2, \dots\Big\} \\ &= \operatorname{span}\Big\{\sum_{k\in\mathbf{Z}} z^k \sum_{j\geq\max(-k,-N+1)} p_{k+j}(t) a_{-jn}, n = 0, 1, \dots, N-1\Big\} \oplus \\ &\qquad \Big\{\sum_0^{\infty} p_i(t) z^{i+N+k}, k = 0, 1, 2, \dots\Big\}. \end{aligned}$$

Therefore, recording the coefficients of z^k in the kth row starting from the bottom $(k = 0, 1, 2, \dots)$, the projection $W^t \to H_+$ has the following matrix representation:

$$\begin{aligned} \tau(t) &= \det \ \operatorname{Proj}(W^t \to H_+) \\ &= \det \left(\begin{array}{ccc|cccc} & & & \vdots & \vdots & \vdots & \\ & & & p_2 & p_1 & 1 & \dots \\ \vdots & & \vdots & p_1 & 1 & 0 & \dots \\ \sum_{j\geq -N+1} a_{-j0} p_{j+N} & \dots & \sum_{j\geq -N+1} a_{-j,N-1} p_{j+N} & 1 & 0 & 0 & \dots \\ \hline \sum_{j\geq -N+1} a_{-j0} p_{j+N-1} & \dots & \sum_{j\geq -N+1} a_{-j,N-1} p_{j+N-1} & & & & \\ \vdots & & \vdots & & O & & \\ \sum_{j\geq -1} a_{-j0} p_{j+1} & \dots & \sum_{j\geq -1} a_{-j,N-1} p_{j+1} & & & & \\ \sum_{j\geq 0} a_{-j0} p_j & \dots & \sum_{j\geq 0} a_{-j,N-1} p_j & & & & \end{array}\right) \end{aligned}$$

$$
\begin{aligned}
&= \det\begin{pmatrix} \sum_{j\geq -N+1} a_{-j0}p_{j+N-1} & \dots & \sum a_{-j,N+1}p_{j+N-1} \\ \vdots & & \vdots \\ \sum_{j\geq 0} a_{-j0}p_j & & \sum_{j\geq 0} a_{-j,N-1}p_j \end{pmatrix} \\
&= \det(E_N a_N),
\end{aligned}
$$

ending the proof of Lemma 3.3.

Proof of Theorem 3.2: Indeed, using Lemma 3.2.2 and Lemma 3.2.1, we have

(3.11)

$$
\begin{aligned}
&\int\int_{u,v\in \mathbb{R}^N} e^{\sum_{k=1}^N V(u_k,v_k)} \prod_{i<j}(u_i-u_j)\prod_{i<j}(v_i-v_j)du_1\dots du_N\, dv_1\dots dv_N \\
&= \int\int_{u,v\in \mathbb{R}^N} \prod_{k=1}^N e^{V(u_k,v_k)} \sum_{\sigma\in S_N} \det\Big(u_{\sigma(k)}^{\ell-1}v_{\sigma(k)}^{k-1}\Big)_{1\leq \ell,k\leq N} du_1\dots du_N\, dv_1\dots dv_N \\
&= \sum_{\sigma\in S_N} \det\Big(\int\int_{\mathbb{R}^2} e^{V(u,v)}u^{\ell-1}v^{k-1}du\, dv\Big)_{1\leq \ell,k\leq N} \\
&= N!\det(m_N(t,s)) \\
&= N!\det(E_N(t)m_\infty(0,0)E_N(-s)^\top).
\end{aligned}
$$

Lemma 3.2.3 shows the latter expression is a τ-function in t and in $-s$, where alternately $a_N^{(1)}(-s) = m_\infty(0,0)E_N^\top(-s)$ yields an initial plane in the Grassmannian parametrized by s, or $a_N^{(2)}(t) = m_\infty^\top(0,0)E_N^\top(t)$ yields an initial plane parametrized by t. Finally (3.9) follows at once from (3.1'), ending the proof of Theorem 3.2.

4 From string-orthogonal polynomials to the two-Toda lattice and the string equation

Given the (t,s,c)-dependent weight $e^{V(y,z)}$ as in (0.1), its moment matrix m_∞ admits, according to Theorem 3.1, the Borel decomposition

(4.1) $$m_\infty = S_1^{-1}h(\bar{S}_2^{-1})^\top,$$

upon setting (in the notation (3.3)),

(4.2) $$S_1 := S(m_\infty), \qquad \bar{S}_2 := S(m_\infty^\top).$$

Also, according to Theorem 3.1, the semi-infinite matrices $S_1 = S_1(t,s,c)$ and $\bar S_2 = \bar S_2(t,s,c) \in \mathcal{D}_{-\infty,0}$ lead to string-orthogonal (monic) polynomials $(p^{(1)}, p^{(2)})$ with $\langle p_n^{(1)}, p_m^{(2)} \rangle = \delta_{n,m} h_n$; we also define new matrices $\bar S_1, S_2 \in \mathcal{D}_{0,\infty}$ and vectors Ψ_1 and Ψ_2^*, as follows:

$$(4.3)\qquad \begin{aligned} & p^{(1)}(z) =: S_1 \bar\chi(z) && \text{and} \quad p^{(2)}(z) =: \bar S_2 \bar\chi(z) \\ & \bar S_1 := h(S_1^{-1})^\top && \text{and} \quad S_2 := h(\bar S_2^{-1})^\top \\ & \Psi_1 := e^{\Sigma t_k z^k} p^{(1)}(z) && \text{and} \quad \Psi_2^* := e^{-\Sigma s_k z^{-k}} h^{-1} p^{(2)}(z^{-1}) \\ & \quad\; = e^{\Sigma t_k z^k} S_1 \bar\chi(z) && \qquad\qquad = e^{-\Sigma s_k z^{-k}} (S_2^{-1})^\top \bar\chi(z^{-1}). \end{aligned}$$

Also define matrices $L_1, \bar L_2 \in \mathcal{D}_{-\infty,1}$, $L_2, \bar L_1 \in \mathcal{D}_{-1,\infty}$, and $Q_1, \bar Q_2 \in \mathcal{D}_{-\infty,-1}$ so that

$$(4.4)\qquad \begin{aligned} &\text{(i)} && z p_n^{(1)}(z) = \textstyle\sum_{\ell \le n+1} (L_1)_{n\ell} p_\ell^{(1)}(z) && z p_n^{(2)}(z) = \textstyle\sum_{\ell \le n+1} (\bar L_2)_{n\ell} p_\ell^{(2)}(z) \\ &\text{(ii)} && \tfrac{\partial}{\partial z} p_n^{(1)}(z) = \textstyle\sum_{\ell \le n-1} (Q_1)_{n\ell} p_\ell^{(1)}(z) && \tfrac{\partial}{\partial z} p_n^{(2)}(z) = \textstyle\sum_{\ell \le n-1} (\bar Q_2)_{n\ell} p_\ell^{(2)}(z) \\ &\text{(iii)} && \bar L_1 = h L_1^\top h^{-1}, && \bar L_2 = h L_2^\top h^{-1}. \end{aligned}$$

Using the latter, we check,

$$z\Psi_1 = e^{\Sigma t_k z^k} z p^{(1)}(z) = e^{\Sigma t_k z^k} L_1 p^{(1)} = L_1 \Psi_1,$$

and

$$z^{-1}\Psi_2^*(z) = e^{-\Sigma s_k z^{-k}} h^{-1} z^{-1} p^{(2)}(z^{-1}) = e^{-\Sigma s_k z^{-k}} h^{-1} \bar L_2 p^{(2)}(z^{-1})) = L_2^\top \Psi_2^*,$$

leading to the formula (4.5) below upon setting $L_2^* = L_2^\top$; at the same time, we also define the matrices M_1 and M_2^*:

$$(4.5)\qquad \begin{aligned} (L_1, L_2^*)(\Psi_1, \Psi_2^*) &= (z, z^{-1})(\Psi_1, \Psi_2^*) \\ (M_1, M_2^*)(\Psi_1, \Psi_2^*) &:= \Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial z^{-1}}\Big)(\Psi_1, \Psi_2^*). \end{aligned}$$

Theorem 4.1 ("String equations"). *The matrices L and Q satisfy the constraints*[13],

$$(4.6)\qquad \text{(i)}\quad Q_1 + \frac{\partial V}{\partial y}(L_1, L_2) = 0, \qquad \text{(ii)}\quad \bar Q_2 + \left(\frac{\partial V}{\partial z}(\bar L_1^\top, \bar L_2^\top)\right)^\top = 0,$$

[13] If $f(y,z) = \Sigma c_{ij} y^i z^j$, then $f(L_1, L_2) \equiv \Sigma c_{ij} L_1^i L_2^j$.

and the matrices L and M,

$$(4.7)\qquad \text{(i)}\quad M_1+\frac{\partial V_{12}}{\partial y}(L_1,L_2)=0,\qquad \text{(ii)}\quad M_2^*+\Big(\frac{\partial V_{12}}{\partial z}(L_1,L_2)\Big)^\top=0.$$

Corollary 4.1.1. *If*

$$V(y,z)=\sum_1^{\ell_1}t_iy^i+cyz-\sum_1^{\ell_2}s_iz^i,$$

then L_1 is a ℓ_2+1-band matrix, having thus ℓ_2-1 subdiagonals below the main diagonal; also, L_2 is a ℓ_1+1-band matrix, with ℓ_1-1 subdiagonals above the main diagonal.

Theorem 4.2 ("Toda equations"). *The Borel decomposition of the moment matrix for the weight e^V*

$$(4.8)\qquad m_\infty=S_1^{-1}h(\bar S_2^{-1})^\top=S_1^{-1}S_2$$

provides matrices S_i such that $L=(L_1,L_2)=(S_1\Lambda S_1^{-1},S_2\Lambda^\top S_2^{-1})$. With regard to the parameters (t,s,c) appearing in the weight e^V, the matrices m_∞ and L satisfy the two-Toda equations and symmetry equations [14]

$$\frac{\partial m_\infty}{\partial t_n}=\Lambda^n m_\infty,\quad \frac{\partial m_\infty}{\partial s_n}=-m_\infty\Lambda^{\top n},\quad \frac{\partial m_\infty}{\partial c_{\alpha\beta}}=\Lambda^\alpha m_\infty\Lambda^{\top\beta},$$

$$(4.9)\qquad \frac{\partial L}{\partial t_n}=[(L_1^n,0)_+,L]\quad \frac{\partial L}{\partial s_n}=[(0,L_2^n)_+,L],\quad \frac{\partial L}{\partial c_{\alpha,\beta}}=-[(L_1^\alpha L_2^\beta,0)_-,L].$$

Corollary 4.2.1. *The moment matrix $m_\infty(t,s,c)$ can be expressed in terms of its initial value $m_\infty(0,0,0)$ as follows:*

$$m_\infty(t,s,c)=\sum_{\substack{(r_{\alpha\beta})_{\alpha,\beta\ge0}\in\mathbf{Z}^\infty\\(\alpha,\beta)\neq(0,0)}}\Big(\prod_{(\alpha,\beta)}\frac{c_{\alpha\beta}^{r_{\alpha\beta}}}{r_{\alpha\beta}!}\Big)\Lambda^{\sum_{\alpha\ge1}\alpha r_{\alpha\beta}}m_\infty(0,0,0)\Lambda^{\top\sum_{\beta\ge1}\beta r_{\alpha\beta}}.$$

[14]Note $\frac{\partial L}{\partial c_{\alpha\beta}}=-[(L_1^\alpha L_2^\beta,0)_-,L]=[(0,L_1^\alpha L_2^\beta)_-,L]$.

Corollary 4.2.2. *The string-orthogonal polynomials, defined in (3.1) or (4.3), have the following expression in terms of the τ-function $\tau_n = \det m_n$ (see (0.3) and footnote 2):*

$$p_n^{(1)}(y) = y^n \frac{\tau_n(t-[y^{-1}],s,c)}{\tau_n(t,s,c)} = \sum_{0\le k\le n} \frac{p_{n-k}(-\tilde\partial_t)\det m_n(t,s,c)}{\det m_n(t,s,c)} y^k \qquad p_n^{(2)}(z) = z^n \frac{\tau_n(t,s+[z^{-1}],c)}{\tau_n(t,s,c)} = \sum_{0\le k\le n} \frac{p_{n-k}(\tilde\partial_s)\det m_n(t,s,c)}{\det m_n(t,s,c)} z^k.$$

Theorem 4.3 ("Trace formula"). *([ASV4]) The following holds for $n,m\ge -1$:*

$$\sum_{0\le i\le N-1} (L_1^{n+1}\ L_2^{m+1})_{ii} = \frac{1}{\tau_N(t,s)} p_{n+N}(\tilde\partial_t) p_{m+N}(-\tilde\partial_s)\tau_1\circ\tau_{N-1}.$$

Remark 1: Alternatively, also according to [ASV4], the above formula can be written in terms of derivatives of τ_{N-1} with coefficients involving the moments $\mu_{ij} = \langle y^i, z^j\rangle$, defined in (0.3):

$$\sum_{0\le i\le N-1} (L_1^{n+1}\ L_2^{m+1})_{ii} = \frac{1}{\tau_N(t,s)} \sum_{\substack{i+i'=n+N\\ j+j'=m+N\\ i,i',j,j'\ge 0}} \mu_{ij} p_{i'}(-\tilde\partial_t) p_{j'}(\tilde\partial_s)\tau_{N-1}(t,s).$$

The reader will find the proof of Theorem 4.3 in [ASV4]. Before proving Theorems 4.1 and 4.2, we first need a few Lemmas:

Lemma 4.4. *The matrices $L_i, \bar L_i, M_1, M_2^*, Q_1, \bar Q_2$ satisfy:*

$$\text{(4.10)} \qquad \begin{cases} L_1 = S_1\Lambda S_1^{-1}, & \bar L_2 = \bar S_2 \Lambda \bar S_2^{-1} \\ \bar L_1 = \bar S_1 \Lambda^\top \bar S_1^{-1}, & L_2 = S_2\Lambda^\top S_2^{-1} \end{cases}$$

and

$$\text{(4.11)} \qquad M_1 = Q_1 + \frac{\partial V_1}{\partial y}(L_1), \quad M_2^* = h^{-1}\bar Q_2 h + \frac{\partial V_2}{\partial z}(L_2^\top).$$

Proof: From (4.3) and (4.4),

$$L_1 S_1 \bar\chi(z) = L_1 p_1 = z p_1 = z S_1\bar\chi(z) = S_1 z\bar\chi(z) = S_1\Lambda\bar\chi(z), \qquad \text{for all } z,$$

together with $\bar{L}_1 = hL_1^\top h^{-1}$, implies the formulas (4.10) for L_1 and $\bar{L}_1$; a similar argument implies those for L_2 and $\bar{L}_2$.

To prove (4.11), observe, from (4.3) and (4.4), that for $\Psi_1 = \Psi_1(t,s,c;z)$

$$\begin{aligned} M_1\Psi_1 = \frac{\partial \Psi_1}{\partial z} &= \frac{\partial}{\partial z}\left(e^{\Sigma t_k z^k} p^{(1)}(z)\right) \\ &= \sum_{k\geq 1} k t_k z^{k-1}\Psi_1 + e^{\Sigma t_i z^k}\frac{\partial}{\partial z}p^{(1)}(z) \\ &= (\Sigma k t_k L_1^{k-1} + Q_1)\Psi_1, \\ &= \Big(\frac{\partial V_1}{\partial y}(L_1) + Q_1\Big)\Psi_1, \end{aligned}$$

and similarly that for $\Psi_2^* = \Psi_2^*(t,s,c;z)$,

$$\begin{aligned} M_2^*\Psi_2^* = \frac{\partial \Psi_2^*}{\partial z^{-1}} &= \frac{\partial}{\partial z^{-1}}\left(e^{-\Sigma s_k z^{-k}} h^{-1} p^{(2)}(z^{-1})\right) \\ &= -\Sigma k s_k z^{-k+1}\Psi_2^* + e^{-\Sigma s_k z^{-k}} h^{-1}\frac{\partial}{\partial z^{-1}}p^{(2)}(z^{-1}) \\ &= -\Sigma k s_k (L_2^\top)^{k-1}\Psi_2^* + e^{-\Sigma s_k z^{-k}} h^{-1}\bar{Q}_2 p^{(2)}(z^{-1}) \\ &= \Big(\frac{\partial V_2(L_2^\top)}{\partial z} + h^{-1}\bar{Q}_2 h\Big)\Psi_2^*, \end{aligned}$$

concluding the proof of Lemma 4.4.

Proof of Theorem 4.1: The following calculation, must be done in the following spirit: for arbitrarily large m, set all $t_i = 0$, for $i > 2m$ and $t_{2m} \neq 0$. Using in the fifth equality the fact that $c_{i0} = t_i$ and $c_{0i} = -s_i$ (see (4.1)), compute, for all integer n and $m \geq 0$,

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \frac{\partial}{\partial y}\Big\{e^{V_1(y)} p_n^{(1)}(y)\Big(\int_{\mathbb{R}} p_m^{(2)}(z) e^{V(y,z)-V_1(y)} dz\Big)\Big\} dy \\ &= \int_{\mathbb{R}} e^{V_1(y)}\Big(\frac{\partial V_1}{\partial y}(y) p_n^{(1)}(y) + \frac{\partial p_n^{(1)}}{\partial y}(y)\Big)\Big(\int_{\mathbb{R}} p_m^{(2)}(z) e^{V-V_1} dz\Big) dy \\ &\qquad + \int_{\mathbb{R}} e^{V_1(y)} p_n^{(1)}(y)\Big(\int_{\mathbb{R}} p_m^{(2)}(z) e^{V-V_1}\frac{\partial}{\partial y}(V - V_1) dz\Big) dy \\ &= \int_{\mathbb{R}} \sum_\ell (Q_1)_{n\ell} p_\ell^{(1)}(y)\left(\int_{\mathbb{R}} p_m^{(2)}(z) e^V dz\right) dy + \int_{\mathbb{R}} p_n^{(1)}(y)\left(\int_{\mathbb{R}} p_m^{(2)}(z) e^V \frac{\partial V}{\partial y} dz\right) dy \end{aligned}$$

$$
\begin{aligned}
&= \sum_\ell (Q_1)_{n\ell} \int\int_{\mathbb{R}^2} p_\ell^{(1)}(y) p_m^{(2)}(z) e^{V(y,z)} dydz \\
&\qquad\qquad + \int\int_{\mathbb{R}^2} p_n^{(1)}(y) p_m^{(2)}(z) \sum_{\substack{i\geq 0\\ j\geq 0}} i c_{ij} y^{i-1} z^j e^V dydz \\
&= (Q_1)_{nm} h_m + \sum_{i,j} i c_{ij} \int\int_{\mathbb{R}^2} (y^{i-1} p_n^{(1)}(y))(z^j p_m^{(2)}(z)) dydz \\
&= (Q_1)_{nm} h_m + \sum_{i,j} i c_{ij} \int\int_{\mathbb{R}^2} \Big(\sum_\alpha (L_1^{i-1})_{n\alpha} p_\alpha^{(1)}(y)\Big)\Big(\sum_\beta (\bar{L}_2^j)_{m\beta} p_\beta^{(2)}(z)\Big) e^V dydz \\
&= (Q_1)_{nm} h_m + \sum_{i,j} i c_{ij} \sum_{\alpha,\beta} (L_1^{i-1})_{n\alpha} (\bar{L}_2^j)_{m\beta} \int\int_{\mathbb{R}^2} p_\alpha^{(1)}(y) p_\beta^{(2)}(z) e^V dydz \\
&= (Q_1)_{nm} h_m + \sum_{i,j,\alpha\geq 0} i c_{ij} (L_1^{i-1})_{n\alpha} (\bar{L}_2^j)_{m\alpha} h_\alpha \\
&\overset{(*)}{=} (Q_1)_{nm} h_m + \sum_{i,j,\alpha\geq 0} i c_{ij} (L_1^{i-1})_{n\alpha} (L_2^j)_{\alpha m} h_m \\
&= \left((Q_1)_{nm} + \sum_{i,j\geq 0} i c_{ij} (L_1^{i-1} L_2^j)_{nm} \right) h_m \\
&= \left(Q_1 + \frac{\partial V}{\partial y}(L_1, L_2) \right)_{nm} h_m,
\end{aligned}
$$

upon using, in the equality $\overset{(*)}{=}$, the property $h(\bar{L}_2^j)^\top = L_2^j h$ (see (4.4, iii)); since $h_m \neq 0$, relation (4.6, i) holds.

The proof of (4.6, ii) is similar to the previous proof, upon interchanging $1 \longleftrightarrow 2$ and $m \longleftrightarrow n$; indeed, compute for all m and n,

$$
\begin{aligned}
0 &= \int_{\mathbb{R}} \frac{\partial}{\partial z} e^{V_2(z)} p_m^{(2)}(z) \Big(\int_{\mathbb{R}} p_n^{(1)}(y) e^{V(y,z) - V_2(z)} dy \Big) dz \\
&= (\bar{Q}_2)_{mn} h_n + \sum_{i,j\geq 0} j c_{ij} \int\int_{\mathbb{R}^2} (z^{j-1} p_m^{(2)}(z))(y^i p_n^{(1)}(y) e^V dydz \\
&= (\bar{Q}_2)_{mn} h_n + \sum_{i,j} j c_{ij} \sum_{\alpha,\beta} (\bar{L}_2^{j-1})_{m\alpha} (L_1^i)_{n\beta} \int\int p_\alpha^2(z) p_\beta^1(y) e^V dydz \\
&= (\bar{Q}_2)_{mn} h_n + \sum_{i,j,\alpha\geq 0} j c_{ij} (\bar{L}_2^{j-1})_{m\alpha} (L_1^i)_{n\alpha} h_\alpha \\
&\overset{(*)}{=} \Big((\bar{Q}_2)_{mn} + \sum_{i,j,\alpha\geq 0} j c_{ij} (\bar{L}_2^{j-1})_{m\alpha} (\bar{L}_1^i)_{\alpha n} \Big) h_n
\end{aligned}
$$

$$= \left(\bar{Q}_2 + \left(\frac{\partial V}{\partial z}(\bar{L}_1^\top, \bar{L}_2^\top)\right)^\top\right)_{m,n} h_n,$$

upon using in $\overset{(*)}{=}$ that $h(\bar{L}_1^i)^\top = L_1^i h$ (see (4.5, iii)).

Now observe from (4.6)(i) and (4.11) that

$$\begin{aligned} 0 = Q_1 + \frac{\partial V}{\partial y}(L_1, L_2) &= Q_1 + \frac{\partial V_1}{\partial y}(L_1) + \frac{\partial V_{12}}{\partial y}(L_1, L_2) \\ &= M_1 + \frac{\partial V_{12}}{\partial y}(L_1, L_2), \end{aligned}$$

yielding (4.7)(i). To prove (4.7)(ii), compute from (4.6)(ii) and (4.11) that

$$\begin{aligned} 0 &= h^{-1}\bar{Q}_2 h + h^{-1}\left(\frac{\partial V}{\partial z}(\bar{L}_1^\top, \bar{L}_2^\top)\right)^\top h \\ &= h^{-1}\bar{Q}_2 h + \left(\frac{\partial V}{\partial z}(h\bar{L}_1^\top h^{-1}, h\bar{L}_2^\top h^{-1})\right)^\top \\ &= h^{-1}\bar{Q}_2 h + \left(\frac{\partial V}{\partial z}(L_1, L_2)\right)^\top, \\ &= h^{-1}\bar{Q}_2 h + \frac{\partial V_2}{\partial z}(L_2^\top) + \left(\frac{\partial V_{12}}{\partial z}(L_1, L_2)\right)^\top \\ &= M_2^* + \left(\frac{\partial V_{12}}{\partial z}(L_1, L_2)\right)^\top, \end{aligned}$$

concluding the proof of Theorem 4.1.

Proof of Corollary 4.1.1: From (4.6(i)) and (4.6(ii)), it follows that

$$Q_1 + \sum_1^{\ell_1} it_i L_1^{i-1} + cL_2 = 0 \quad \text{and} \quad h\bar{Q}_2^\top h^{-1} + cL_1 - \sum_1^{\ell_2} is_i L_2^{i-1} = 0.$$

But Q_1 is zero on and above the diagonal; $\sum_1^{\ell_1} it_i L_1^{i-1}$ has $\ell_1 - 1$ subdiagonals above the main diagonal and zeroes beyond, and thus also L_2. Similarly $\bar{Q}_2^\top$ has zeroes on and below the diagonal; $\sum_1^{\ell_2} is_i L_2^{i-1}$ has $\ell_2 - 1$ subdiagonals below the main diagonal and zeroes beyond, and thus also L_1, establishing the corollary.

Lemma 4.5. *The wave operators $S=(S_1,S_2)$ satisfy for $n=1,2,...,$ and $\alpha,\beta=0,1,...,$*

$$
(4.12)\qquad
\begin{array}{lll}
\dfrac{\partial S_1}{\partial t_n}=-(L_1^n)_\ell S_1, & \dfrac{\partial S_1}{\partial s_n} = (L_2^n)_\ell S_1, & \dfrac{\partial S_1}{\partial c_{\alpha\beta}}=-(L_1^\alpha L_2^\beta)_\ell S_1\\
\dfrac{\partial S_2}{\partial t_n}=(L_1^n)_u S_2, & \dfrac{\partial S_2}{\partial s_n}=-(L_2^n)_u S_2, & \dfrac{\partial S_2}{\partial c_{\alpha\beta}} = (L_1^\alpha L_2^\beta)_u S_2
\end{array}
$$

and the wave vectors Ψ_1 and Ψ_2^*

$$
(4.13)\qquad
\begin{cases}
(i) & \dfrac{\partial(\Psi_1,\Psi_2^*)}{\partial t_n}=\left((L_1^n)_{u,}-((L_1^n)_u)^\top\right)(\Psi_1,\Psi_2^*)\quad n=1,2,\dots\\
(ii) & \dfrac{\partial(\Psi_1,\Psi_2^*)}{\partial s_n}=\left((L_2^n)_{\ell,}-((L_2^n)_\ell)^\top\right)(\Psi_1,\Psi_2^*)\\
(iii) & \dfrac{\partial(\Psi_1,\Psi_2^*)}{\partial c_{\alpha\beta}}=\left(-(L_1^\alpha L_2^\beta)_{\ell,}-((L_1^\alpha L_2^\beta)_u)^\top\right)(\Psi_1,\Psi_2^*)\quad \alpha,\beta\geq 0.
\end{cases}
$$

Proof: Since $p_k^{(1)}$ and $p_k^{(2)}$ are monic polynomials,

$$
(4.14)\qquad \frac{\partial p_k^{(i)}}{\partial c_{\alpha\beta}}=\sum_{m<k}A_{km}^{(i)}p_m^{(i)}\quad i=1,2
$$

with $A^{(1)}$ and $A^{(2)}\in\mathcal{D}_{-\infty,-1}$. So, for arbitrary ℓ and $k\geq 0$,

$$
\begin{aligned}
&\frac{\partial}{\partial c_{ij}}\langle p_k^{(1)},p_\ell^{(2)}\rangle\\
=\ &\langle\frac{\partial p_k^{(1)}}{\partial c_{ij}},p_\ell^{(2)}\rangle+\langle p_k^{(1)},\frac{\partial p_\ell^{(2)}}{\partial c_{ij}}\rangle+\langle y^i p_k^{(1)},z^j p_\ell^{(2)}\rangle\\
=\ &\sum_{m<k}A_{km}^{(1)}\langle p_m^{(1)},p_\ell^{(2)}\rangle+\sum_{m<\ell}A_{\ell m}^{(2)}\langle p_k^{(1)},p_m^{(2)}\rangle+\sum_{m,n}(L_1^i)_{km}(\bar L_2^j)_{\ell n}\langle p_m^{(1)},p_n^{(2)}\rangle\\
=\ &A_{k\ell}^{(1)}h_\ell+A_{\ell k}^{(2)}h_k+\sum_r(L_1^i)_{kr}(\bar L_2^j)_{\ell r}h_r
\end{aligned}
$$

$$
\text{or}\quad\begin{cases}
=A_{k\ell}^{(1)}h_\ell+A_{\ell k}^{(2)}h_k+\sum\limits_r (L_1^i)_{kr}h_\ell(L_2^j)_{r\ell}\\
=A_{k\ell}^{(1)}h_\ell+A_{\ell k}^{(2)}h_k+\sum\limits_r (\bar L_2^j)_{\ell r}(\bar L_1^i)_{rk}h_k
\end{cases}\quad\text{using (4.4) (iii)}
$$

leading to

$$
(4.15) \qquad \begin{cases} \ell < k & 0 = \dfrac{\partial}{\partial c_{ij}} \Big\langle p_k^{(1)}, p_\ell^{(2)} \Big\rangle = \Big(A_{k\ell}^{(1)} + (L_1^i L_2^j)_{k\ell} \Big) h_\ell \\ \ell = k & \dfrac{\partial}{\partial c_{ij}} h = (L_1^i L_2^j)_0 h \\ \ell > k & 0 = \dfrac{\partial}{\partial c_{ij}} \langle p_k^{(1)}, p_\ell^{(2)} \rangle = \Big(A_{\ell k}^{(2)} + (\bar{L}_2^j \bar{L}_1^i)_{\ell k} \Big) h_k. \end{cases}
$$

Setting these expressions for the matrices $A^{(i)} \in \mathcal{D}_{-\infty,-1}$ in (4.14), leads to

$$
(4.16) \qquad \frac{\partial p^{(1)}}{\partial c_{\alpha\beta}} = -(L_1^\alpha L_2^\beta)_\ell p^{(1)} \quad \text{and} \quad \frac{\partial p^{(2)}}{\partial c_{\alpha\beta}} = -(\bar{L}_2^\beta \bar{L}_1^\alpha)_\ell p^{(2)}.
$$

Since $p^{(1)} = S_1 \bar{\chi}$ and $p^{(2)} = \bar{S}_2 \bar{\chi}$, (4.16) leads to

$$
(4.17) \qquad \frac{\partial S_1}{\partial c_{\alpha\beta}} = -(L_1^\alpha L_2^\beta)_\ell S_1 \quad \text{and} \quad \frac{\partial \bar{S}_2}{\partial c_{\alpha\beta}} = -(\bar{L}_2^\beta \bar{L}_1^\alpha)_\ell \bar{S}_2,
$$

for integer $\alpha, \beta \geq 0$, and since $S_2 = h(\bar{S}_2^{-1})^\top$,

$$
\begin{aligned}
\frac{\partial S_2}{\partial c_{\alpha\beta}} &= \frac{\partial h}{\partial c_{\alpha\beta}} \left(\bar{S}_2^{-1} \right)^\top - h \left(\bar{S}_2^{-1} \frac{\partial \bar{S}_2}{\partial c_{\alpha\beta}} \bar{S}_2^{-1} \right)^\top \\
&= \Big((L_1^\alpha L_2^\beta)_0 + h((\bar{L}_2^\beta \bar{L}_1^\alpha)_\ell)^\top h^{-1} \Big) S_2, \quad \text{using (4.15), (4.17),} \\
&= \left((L_1^\alpha L_2^\beta)_0 + \left(((L_1^\alpha L_2^\beta)^\top)_\ell \right)^\top \right) S_2, \quad \text{using (4.4(iii))} \\
&= (L_1^\alpha L_2^\beta)_u S_2,
\end{aligned}
$$

from which it follows that

$$
(4.18) \qquad \frac{\partial (S_2^{-1})^\top}{\partial c_{\alpha\beta}} = -((L_1^\alpha L_2^\beta)_u)^\top (S_2^{-1})^\top.
$$

In particular, setting $c_{n0} = t_n$ or $-c_{0n} = s_n$ leads to (4.12).

Since, according to the definition (4.3),

$$
\Psi_1 = e^{\Sigma t_k z^k} S_1 \bar{\chi}(z) \quad \text{and} \quad \Psi_2^* = e^{-\Sigma s_k z^{-k}} (S_2^{-1})^\top \bar{\chi}(z^{-1})
$$

depends on $c_{\alpha\beta}$, $\alpha, \beta \geq 1$, through S_1 and S_2 only, the expressions for $\partial S_1/\partial c_{\alpha\beta}$ and $\partial S_2/\partial c_{\alpha\beta}$ yield the corresponding derivatives (4.13, iii). Equations (4.13, i) and (4.13, ii) follow from (4.13, iii), taking into account the fact that t_n and s_n also appear in the exponents of (Ψ_1, Ψ_2^*). This ends the proof of Lemma 4.5.

Proof of Theorem 4.2: The formulas

$$L_1 = S_1 \Lambda S_1^{-1}, \qquad L_2 = S_2 \Lambda^\top S_2^{-1},$$

and the expression (4.8) for m_∞ combined with the differential equations (4.12) for S_i lead to

$$\begin{aligned}
\frac{\partial m_\infty}{\partial c_{\alpha\beta}} &= \frac{\partial S_1^{-1} S_2}{\partial c_{\alpha\beta}} \\
&= S_1^{-1}(L_1^\alpha L_2^\beta)_\ell S_1^{-1} S_1 S_2 + S_1^{-1}(L_1^\alpha L_2^\beta)_u S_2 \\
&= S_1^{-1} L_1^\alpha L_2^\beta S_2 \\
&= \Lambda^\alpha S_1^{-1} S_2 \Lambda^{\top\beta} = \Lambda^\alpha m_\infty \Lambda^{\top\beta},
\end{aligned}$$

and thus the conclusion of Theorem 4.2. Corollary 4.2.1 is an immediate consequence of Theorem 4.2, while Corollary 4.2.2 follows from (1.16), (1.17) and (4.3).

5 Virasoro constraints on two-matrix integrals

For the general 2-Toda lattice, define the bi-infinite matrices for all integer $i \geq -1$,

(5.1) $$v_i^{(1)} = L_1^{i+1}(M_1 + \frac{\partial V_{12}}{\partial y}(L_1, L_2)), \quad v_i^{(2)} = -(M_2^{*\top} + \frac{\partial V_{12}}{\partial z}(L_1, L_2))L_2^{i+1},$$

(bi-infinite 2-Toda lattice).

Note that, according to the formula preceeding (1.10), we have

(5.2) $$M_2^{*\top} = L_2^{-1} - M_2.$$

In view of formulas (1.39), define differential operators $\mathcal{K}_{m,i}^{(\alpha)}$ and $\mathcal{L}_{m,i}^{(\alpha)}$ for $m \geq 0$, $i \geq -1$, $\alpha = 1, 2$,

(5.3)

$$\begin{aligned}
\mathcal{K}_{m,i}^{(1)} &:= \frac{1}{2} W_{m,i}^{(2)} + (i+1) W_{n,i}^{(1)} + \sum_{r,s \geq 1} r c_{rs} \frac{\partial}{\partial c_{i+r,s}} \\
&= \frac{1}{2} J_i^{(2)} + (m + \frac{i+1}{2}) J_i^{(1)} + \sum_{r,s \geq 1} r c_{rs} \frac{\partial}{\partial c_{i+r,s}} + \frac{m(m+1)}{2} \delta_{i0} \\
&=: \mathcal{L}_{m,i}^{(1)} + \frac{m(m+1)}{2} \delta_{i0}
\end{aligned}$$

and

(5.4)

$$\begin{aligned}\mathcal{K}^{(2)}_{m,i} &:= \frac{1}{2}\tilde W^{(2)}_{m-1,i} - \tilde W^{(1)}_{m-1,i} + \sum_{r,s\geq 1} sc_{rs}\frac{\partial}{\partial c_{r,s+i}}\\ &= \frac{1}{2}\tilde J^{(2)}_i - (m+\frac{i+1}{2})\tilde J^{(1)}_i + \sum_{r,s\geq 1} sc_{rs}\frac{\partial}{\partial c_{r,s+i}} + \frac{(m-1)(m+2)}{2}\delta_{i0}\\ &=: \mathcal{L}^{(2)}_{m,i} + \frac{(m-1)(m+2)}{2}\delta_{i0}.\end{aligned}$$

Consider the usual weight e^V, the corresponding string-orthogonal polynomials, the semi-infinite wave vectors Ψ_1 and Ψ_2^* derived from them in (4.3) and the semi-infinite matrices L_1, L_2, M_1, M_2^* defined in (4.4) and (4.5). Note that in this context neither L_2^{-1} nor M_2 appearing in (5.2) make sense although the combination $M_2^{*\top} = L_2^{-1} - M_2$ makes perfectly good sense.

Recall the bracket $\{\, ,\,\}$ introduced in (2.5) and made explicit in (2.6); we now state

Theorem 5.1. *For the general 2-Toda lattice, the bi-infinite matrices $v_i^{(\alpha)}$ ($i \geq -1$, $\alpha = 1, 2$) form a (decoupled) algebra $Vir^+ \otimes Vir^+$ for the bracket $\{\, ,\,\}$:*

$$\{v_i^{(\alpha)}, v_j^{(\alpha)}\} = (-1)^{\alpha+1}(i-j)v^{(\alpha)}_{i+j} \qquad \{v_i^{(1)}, v_j^{(2)}\} = 0. \tag{5.5}$$

For string-orthogonal polynomials, we have

$$v_i^{(\alpha)} = 0 \quad \text{and} \quad \tau_0 = 1;$$

they imply for

$$\tau_m = \frac{1}{m!}\int\int_{u,v\in\mathbb{R}^m} e^{\sum_1^m V(u_k,v_k)}\prod_{i<j}(u_i-u_j)\prod_{i<j}(v_i-v_j)du\,dv$$

the algebra $Vir^+ \otimes Vir^+$ of constraints

$$\Big(\mathcal{L}^{(\alpha)}_{m,i} + \frac{m(m+1)}{2}\delta_{i,0}\Big)\tau_m = 0, \quad \text{for } m \geq 0, i \geq -1, \alpha = 1, 2. \tag{5.6}$$

Lemma 5.2. *For the general 2-Toda lattice, we have the following correspondence for the vector field $\mathbf{Y}_{v_i^{(\alpha)}}$ acting on the wave functions (Ψ_1, Ψ_2^*):*

$$\begin{aligned} -\frac{((v_i^{(\alpha)})_\ell\Psi_1)_m}{\Psi_{1,m}z^m} &= (e^{-\eta}-1)\frac{\mathcal{L}^{(\alpha)}_{m,i}\tau_m}{\tau_m}\\ -\frac{((v_i^{(\alpha)})_u)^\top\Psi_2^*)_m}{\psi^*_{2,m}z^{-m}} &= (e^{\tilde\eta}-1)\frac{\mathcal{L}^{(\alpha)}_{m,i}\tau_m}{\tau_m} - \frac{\mathcal{L}^{(\alpha)}_{m+1,i}\tau_{m+1}}{\tau_{m+1}} + \frac{\mathcal{L}^{(\alpha)}_{m,i}\tau_m}{\tau_m} - (m+1)\delta_{i0} \end{aligned} \tag{5.7}$$

for $i, m \in \mathbf{Z}$, $\alpha = 1, 2$.

Proof of Lemma 5.2: At first observe that $v_i^{(1)}$ and $v_i^{(2)}$ defined in (5.1) equals,

$$\begin{aligned} v_i^{(1)} &= M_1 L_1^{i+1} + (i+1) L_1^i + \sum_{r,s\geq 1} r c_{rs} L_1^{i+r} L_2^s \\ v_i^{(2)} &= M_2 L_2^{i+1} - L_2^i - \sum_{r,s\geq 1} s c_{rs} L_1^r L_2^{i+s}. \end{aligned}$$

Then using (1.27) and (1.27'), (4.13,(iii)), Corollary 1.2.1 and the notations (5.3) and (5.4), we have the following:

$$\begin{aligned} \frac{(\mathbf{Y}_{v_i(\alpha)}\Psi_1)_m}{\Psi_{1,m} z^m} &= (-1)^\alpha \frac{((v_i^{(\alpha)})_\ell \Psi_1)_m}{\Psi_{1,m} z^m} = (e^{-\eta} - 1)\frac{\mathcal{K}_{m,i}^{(\alpha)}\tau_m}{\tau_m} = (e^{-\eta} - 1)\frac{\mathcal{L}_{m,i}^{(\alpha)}\tau_m}{\tau_m} \\ \frac{(\mathbf{Y}_{v_i(\alpha)}\Psi_2^*)_m}{\Psi_{2,m}^* z^{-m}} &= (-1)^\alpha \frac{(((v_i^{(\alpha)})_u)^\top)\Psi_2^*)_m}{\Psi_{2,m}^* z^{-m}} \\ &= (e^{\tilde\eta} - 1)\frac{\mathcal{K}_{m,i}^{(\alpha)}\tau_m}{\tau_m} - \frac{\mathcal{K}_{m+1,i}^{(\alpha)}\tau_{m+1}}{\tau_{m+1}} + \frac{\mathcal{K}_{m,i}^{(\alpha)}\tau_m}{\tau_m}, \\ &= (e^{\tilde\eta} - 1)\frac{\mathcal{L}_{m,i}^{(\alpha)}\tau_m}{\tau_m} - \frac{\mathcal{L}_{m+1,i}^{(\alpha)}\tau_{m+1}}{\tau_{m+1}} + \frac{\mathcal{L}_{m,i}^{(\alpha)}\tau_m}{\tau_m} - (m+1)\delta_{i0}, \end{aligned}$$

where the term $(m+1)\delta_{i0}$ is caused by the constant which distinguishes $\mathcal{L}_{m,i}^{(\alpha)}$ from $\mathcal{K}_{m,i}^{(\alpha)}$, ending the proof of Lemma 5.2.

Proof of Theorem 5.1: That the matrices $v_i^{(\alpha)}$ form an algebra $Vir^+ \otimes Vir^+$ for $\{\,,\,\}$ with structure relations (5.5) follows immediately from Theorem 2.3. Since the maps

$$v, \{\,,\,\} \mapsto \mathbf{Y}_v, [\,,\,]$$

and

$$\mathbf{Y}_v \text{ acting on } L \mapsto \mathbf{Y}_v \text{ acting on } \tau$$

are Lie algebra homomorphisms, the vector fields $\mathbf{Y}_{v_i^{(\alpha)}}$ induce on τ an algebra $Vir^+ \otimes Vir^+$ of constraints, as well.

For string-orthogonal polynomials, the matrices L_1, M_1, L_2 and M_2^* satisfy the string relations (4.7), implying, upon multiplication by L_1^{i+1} and L_2^{i+1} respectively, that both $v_i^{(1)}$ and $v_i^{(2} = 0$ for $i \geq -1$; thus the right hand side of (5.7) vanishes. But the terms on the right hand side of (5.7), containing $e^{-\eta} - 1$ and $e^{\tilde\eta} - 1$, are Taylor series in z^{-1} and z respectively, without independent term, whereas the remaining part is independent of z; therefore we have

$$\frac{\mathcal{L}^{(\alpha)}_{m,i}\tau_m}{\tau_m} = a^{(\alpha)}(m,i,c), \qquad m \geq 1 \text{ and } \alpha = 1,2,$$

is a function of $m, i, c = (c_{ij}$, independent of t and s. Substituting the latter into the right hand side of (5.7), yields the difference relation

(5.8)

$$\frac{\mathcal{L}^{(\alpha)}_{m+1,i}\tau_{m+1}}{\tau_{m+1}} - \frac{\mathcal{L}^{(\alpha)}_{m,i}\tau_m}{\tau_m} + (m+1)\delta_{i0} = a^{(\alpha)}(m+1,i,c) - a^{(\alpha)}(m,i,c) + (m+1)\delta_{i0} = 0,$$

for $m \geq 0$, $i \geq -1$ and $\alpha = 1,2$. Moreover, since $\mathcal{L}^{(\alpha)}_{0,i}$ (for $i \geq -1$) is a differentiation operator, and since $\tau_0 = 1$, we have the boundary condition

$$a^{(\alpha)}(0,i,c) = \frac{\mathcal{L}^{(\alpha)}_{0,i}\tau_0}{\tau_0} = 0, \qquad \text{for } i \geq -1. \tag{5.9}$$

So, the difference relation (5.8) together with the boundary condition (5.9) implies

$$a^{\alpha}(m,i,c) = -\frac{m(m+1)}{2}\delta_{i0}, \qquad m \geq 0, i \geq -1,$$

establishing Theorem 5.1.

Corollary 5.1.1. *For the two-matrix integral τ_N, relations (5.6) are equivalent to*

$$\left(J^{(2)}_i + (2N+i+1)J^{(1)}_i + N(N+1)J^{(0)}_i + 2\sum_{r,s\geq 1} rc_{rs}\sum_{\alpha=0}^{N-1}(L_1^{i+r}L_2^s)_{\alpha\alpha}\right)\tau_N = 0$$

$$\left(\tilde{J}^{(2)}_i - (2N+i+1)\tilde{J}^{(1)}_i + N(N+1)\tilde{J}^{(0)}_i + 2\sum_{r,s\geq 1} sc_{rs}\sum_{\alpha=0}^{N-1}(L_1^{r}L_2^{s+i})_{\alpha\alpha}\right)\tau_N = 0,$$

for $i \geq -1$ and $N \geq 0$.

In particular, when $V_{12} = cyz$, we have

$$\left(J^{(2)}_i + (2N+i+1)J^{(1)}_i + N(N+1)J^{(0)}_i\right)\tau_N + 2c\; p_{i+N}(\tilde{\partial}_t)p_N(-\tilde{\partial}_s)\tau_1 \circ \tau_{N-1} = 0$$

$$\left(\tilde{J}^{(2)}_i - (2N+i+1)\tilde{J}^{(1)}_i + N(N+1)\tilde{J}^{(0)}_i\right)\tau_N + 2c\; p_N(\tilde{\partial}_t)p_{i+N}(-\tilde{\partial}_s)\tau_1 \circ \tau_{N-1} = 0$$

or, in terms of the moments μ_{ij},

$$
\begin{aligned}
\left(J_i^{(2)}+(2N+i+1)J_i^{(1)}+N(N+1)J_i^{(0)}\right)\tau_N+2c\sum_{\substack{k+k'=N+i\\ \ell+\ell'=N\\ k,k',\ell,\ell'\geq 0}}\mu_{k\ell}p_{k'}(-\tilde\partial_t)p_{\ell'}(\tilde\partial_s)\tau_{N-1} &= 0\\
\left(\tilde J_i^{(2)}-(2N+i+1)\tilde J_i^{(1)}+N(N+1)\tilde J_i^{(0)}\right)\tau_N+2c\sum_{\substack{k+k'=N\\ \ell+\ell'=N+i\\ k,k',\ell,\ell'\geq 0}}\mu_{k\ell}p_{k'}(-\tilde\partial_t)p_{\ell'}(\tilde\partial_s)\tau_{N-1} &= 0
\end{aligned}
$$

for $i\geq -1$ *and* $N\geq 1$.

Proof: It suffices to replace the partial derivatives[15]

$$
\begin{aligned}
\frac{\partial\tau_N}{\partial c_{\alpha\beta}} &= N!\sum_{i=0}^{N-1}\frac{\partial h_i}{\partial c_{\alpha\beta}}h_0\dots\widehat{h_i}\dots h_{N-1}, \quad \text{using Theorem 3.2,}\\
&= N!\sum_{i=0}^{N-1}(L_1^\alpha L_2^\beta)_{ii}h_ih_0\dots\widehat{h_i}\dots h_{N-1}, \quad \text{using Proposition 1.1,}\\
&= N!\sum_{i=0}^{N-1}(L_1^\alpha L_2^\beta)_{ii}\prod_0^{N-1}h_k\\
(*)\qquad &= \tau_N\sum_{i=0}^{N-1}(L_1^\alpha L_2^\beta)_{ii}, \quad \text{using (3.9),}\\
(**)\qquad &= p_{\alpha+N-1}(\tilde\partial_t)p_{\beta+N-1}(\tilde\partial_s)\tau_1\circ\tau_{N-1}, \quad \text{using theorem 4.3,}
\end{aligned}
$$

in the expression $\mathcal{L}_{m,i}^{(\alpha)}$ (see (5.3) and (5.4)) of (5.6) by the multiplication operator (*). For $V_{12}=cyz$, one replaces $\frac{\partial\tau_N}{\partial c_{\alpha\beta}}$ by the Hirota-type expression (**) and then one sets all $c_{ij}=0$, for all $i,j\geq 1$, except $c_{11}=c$; the second set of equations uses the formula in Remark 1, following Theorem 4.3, thus ending the proof of Corollary 5.1.1.

Note that, since

$$
\left(\frac{\partial}{\partial z}\right)^p z^n=\sum_{k=0}^{p\wedge n}\binom{p}{k}(n)_k\,z^{n-k}\left(\frac{\partial}{\partial z}\right)^{p-k}=:\sum_{k=0}^{p\wedge n}\alpha_k^{(n,p)}z^{n-k}\left(\frac{\partial}{\partial z}\right)^{p-k}, \quad p,n\geq 0
$$

[15] $\widehat{h_i}$ means with h_i omitted

we have using (1.20), that

$$
\begin{aligned}
[L_1^n, M_1^p] &= \sum_{k=1}^{p\wedge n} \alpha_k^{(n,p)} M_1^{p-k} L_1^{n-k}, \qquad \text{with } \alpha_k^{(n,p)} := \begin{pmatrix} p \\ k \end{pmatrix} (n)_k \\
(M_2 - L_2^{-1})^n &= \sum_{k=0}^{n} \beta_k^{(n)} M_2^{n-k} L_2^{-k}, \qquad \text{with } \beta_k^{(n)} := \begin{pmatrix} n \\ k \end{pmatrix} (-1)^k, \quad \beta_0^{(n)} = 1.
\end{aligned}
$$

Defining

$$
\bar{\alpha}_k^{(i,j)} := \frac{\alpha_k^{(i,j)}}{(-c)^j (j-k+1)} \text{ and } \bar{\beta}_k^{(i)} = \frac{\beta_k^{(i)}}{(i-k+1)c^i}, \tag{5.10}
$$

we now state

Theorem 5.2. *For $V_{12} = cyz$, we have*

$$
\Big(\sum_{k=0}^{i\wedge j} \bar{\alpha}_k^{(i,j)} W_{m,i-j}^{(j-k+1)} + \sum_{0}^{i} \bar{\beta}_k^{(i)} \tilde{W}_{m-1,j-i}^{(i-k+1)}\Big)\tau_m = \frac{i!}{(-c)^i} \delta_{ij} \tau_m, \qquad i,j \geq 0.
$$

<u>Proof</u>: For $V_{12} = cyz$, the string equations $v_{-1}^{(\alpha)} = 0$ (see (5.1)) reduce to

$$
M_1 + cL_2 = 0 \quad \text{ and } \quad M_2^{*\top} + cL_1 = 0, \tag{5.11}
$$

implying

$$
(-\frac{1}{c})^j L_1^i M_1^j - L_1^i L_2^j = 0 \quad \text{ and } \quad (\frac{1}{c})^i (-M_2^{*\top})^i L_2^j - L_1^i L_2^j = 0.
$$

Subtracting we find, upon replacing $M_2^{*\top}$ by the expression (5.2),

$$
\begin{aligned}
0 &= (-\frac{1}{c})^j L_1^i M_1^j - (\frac{1}{c})^i (-M_2^{*\top})^i L_2^j \\
&= (-\frac{1}{c})^j L_1^i M_1^j - (\frac{1}{c})^i (M_2 - L_2^{-1})^i L_2^j \\
&= \sum_{k=0}^{i\wedge j} \frac{\alpha_k^{(i,j)}}{(-c)^j} M_1^{j-k} L_1^{i-k} - \sum_{k=0}^{i} \frac{\beta_k^{(i)}}{c^i} M_2^{i-k} L_2^{j-k}.
\end{aligned}
$$

Applying Corollary 1.2.1, we have, using the notation (5.10),

$$
\begin{aligned}
0 &= -\frac{\left(\left(\sum_{k=0}^{i\wedge j} \frac{\alpha_k^{(i,j)}}{(-c)^j} M_1^{j-k} L_1^{i-k} - \sum_{k=0}^{i} \frac{\beta_k^{(i)}}{c^i} M_2^{i-k} L_2^{j-k}\right)_\ell \Psi_1\right)_m}{\Psi_{1,m} z^m} \\
&= (e^{-\eta} - 1) \frac{\left(\sum_0^{i\wedge j} \bar{\alpha}_k^{(i,j)} W_{m,i-j}^{(j-k+1)} + \sum_0^i \bar{\beta}_k^{(i)} \tilde{W}_{m-1,j-i}^{(i-k+1)}\right)\tau_m}{\tau_m}
\end{aligned} \tag{5.12}
$$

and

$$
\begin{aligned}
0 &= \frac{-\left(\left(\left(\sum_{k=0}^{i\wedge j}\frac{\alpha_k^{(i,j)}}{(-c)^j}M_1^{j-k}L_1^{i-k}-\sum_{k=0}^{i}\frac{\beta_k^{(i)}}{c^i}M_2^{i-k}L_2^{j-k}\right)_u\right)^\top\Psi_2^*\right)_m}{\Psi_{2,m}^* z^{-m}} \\
&= e^{\tilde\eta}\frac{\left(\sum_0^{i\wedge j}\bar\alpha_k^{(i,j)}W_{m,i-j}^{(j-k+1)}+\sum_0^i\bar\beta_k^{(i)}\tilde W_{m-1,j-i}^{(i-k+1)}\right)\tau_m}{\tau_m} \\
&\quad -\frac{\left(\sum_{k=0}^{i\wedge j}\bar\alpha_k^{(i,j)}W_{m+1,i-j}^{(j-k+1)}+\sum_{k=0}^{i}\bar\beta_k^{(i)}\tilde W_{m,j-i}^{(i-k+1)}\right)\tau_{m+1}}{\tau_{m+1}}.
\end{aligned} \tag{5.13}
$$

Upon using the same argument on (5.12) and (5.13) as in the proof of Theorem 5.1, one shows the ratio is independent of t and s; so

$$
\frac{\left(\sum_0^{i\wedge j}\bar\alpha_k^{(i,j)}W_{m,i-j}^{(j-k+1)}+\sum_0^i\bar\beta_k^{(i)}\tilde W_{m-1,j-i}^{(i-k+1)}\right)\tau_m}{\tau_m} =: a^{ij}(m,c), \tag{5.14}
$$

and according to (5.13),

$$
a^{ij}(m+1,c)=a^{ij}(m,c), \qquad m=0,1,2,\dots
$$

Then, we compute (5.14) for $m=0$:

$a^{ij}(0,c)$

$$
\begin{aligned}
a^{ij}(0,c) &= \frac{\left(\sum_{k=0}^{i\wedge j}\bar\alpha_k^{(ij)}W_{0,i-j}^{(j-k+1)}+\sum_{k=0}^{i}\bar\beta_k^{i}\tilde W_{-1,j-i}^{(i-k+1)}\right)\tau_0}{\tau_0} \\
&= \frac{\left(\sum_{k=0}^{i\wedge j}\bar\alpha_k^{(ij)}(0)_{j-k+1}\delta_{i,j}+\sum_{k=0}^{i}\bar\beta_k^{i}(1)_{i-k+1}\delta_{i,j}\right)\tau_0}{\tau_0} \\
&\qquad +\frac{\left((\text{differential operator})+(\text{polynomial in}\,t)\right)\tau_0}{\tau_0}, \quad \text{using (1.49)}, \\
&= \bar\beta_i^i(1)_1\delta_{i,j}+(\text{polynomial in } t) \qquad \text{since } \tau_0=1,\,(0)_{j-k+1}=0 \\
&= \frac{(i)_i(-1)^i}{c^i}\delta_{ij}, \qquad \text{since } a^{ij}(0,c) \text{ is independent of } t \\
&= \frac{i!}{(-c)^i}\delta_{ij}.
\end{aligned}
$$

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