The solution to the q-KdV equation*

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Abstract: Let KdV stand for the Nth Gelfand-Dickey reduction of the KP hierarchy. The purpose of this paper is to show that any KdV solution leads effectively to a solution of the q-approximation of KdV. Two different q-KdV approximations were proposed, by E. Frenkel [7] and Khesin, Lyubashenko and Roger [12]. We show there is a dictionary between the solutions of q-KP and the 1-Toda lattice equations, obeying some special requirement; this is based on an algebra isomorphism between difference operators and D-operators, where $Df(x) = f(qx)$. Therefore every notion about the 1-Toda lattice can be transcribed into q-language. So, q-KdV is yet another Toda discretization of KdV.

Consider the q-difference operators D and D_q, defined by

$$Df(y) = f(qy) \quad \text{and} \quad D_q f(y) := \frac{f(qy) - f(y)}{(q-1)y},$$

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and the q-pseudo-differential operators

$$Q = D + u_0(x)D^0 + u_{-1}D^{-1} + ... \text{ and } \quad Q_q = D_q + v_0(x)D_q^0 + v_{-1}(x)D_q^{-1} + ...$$

The following q-versions of KP were proposed by E. Frenkel [7] and by Khesin, Lyubashenko and Roger [12], for $n = 1, 2, ...$:

$$\frac{\partial Q}{\partial t_n} = [(Q^n)_+ , Q] \qquad \textit{(Frenkel system)} \tag{0.1}$$

$$\frac{\partial Q_q}{\partial t_n} = [\left(Q_q^n\right)_+ , Q_q], \qquad \textit{(KLR system)} \tag{0.2}$$

where $(\ )_+$ and $(\ )_-$ refer to the q-differential and strictly q-pseudo-differential part of $(\ )$. The two systems are identical, after a (constant) upper-triangular linear transformation from the u_i's to the v_i's, as will become clear from the isomorphism between q-operators and difference operators, explained below. The purpose of this paper is to give a large class of solutions to both systems.

The δ-function $\delta(z) := \sum_{i\in\mathbf{Z}} z^i$; enjoys the property $f(\lambda,\mu)\delta(\lambda/\mu) = f(\lambda,\lambda)\delta(\lambda/\mu)$. Consider an appropriate space of functions $f(y)$ representable by "Fourier" series in the basis $\varphi_n(y) := \delta(q^{-n}x^{-1}y)$ for fixed $q \neq 1$,

$$f(y) = \sum_{-\infty}^{\infty} f_n \varphi_n(y);$$

the operators D, defined by $Df(y) = f(qy)$, and multiplication by a function $a(y)$ act on the basis elements, as follows:

$$D\varphi_n(y) = \varphi_{n-1}(y) \text{ and } a(y)\varphi_n(y) = a(xq^n)\varphi_n(y).$$

Therefore, the Fourier transform,

$$f \longmapsto \hat{f} = (..., f_n, ...)_{n\in\mathbf{Z}},$$

induces an algebra isomorphism, mapping D-operators onto a special class of Λ-operators in the shift $\Lambda := \left(\delta_{i,j-1}\right)_{i,j\in\mathbf{Z}}$, as follows:

$$\sum_i a_i(y)D^i \longmapsto \sum_i \ \mathrm{diag}\,(..., a_i(xq^n), ...)_{n\in\mathbf{Z}}\,\Lambda^i; \tag{0.3}$$

conversely, any difference operator, depending on x, of the type (0.3) i.e., annihilated by $D - Ad_\Lambda$, where $(Ad_\Lambda)a = \Lambda a \Lambda^{-1}$, maps into a D-operator. This is the crucial basic isomorphism used throughout this paper.

To the shift Λ and to a fixed diagonal matrix $\lambda = \mathrm{diag}(\lambda_{n+1})_{n\in\mathbf{Z}}$, we associate new operators

$$\tilde{\Lambda} = -\lambda\Lambda \quad \text{and} \quad \tilde{\tilde{\Lambda}} = \tilde{\Lambda} + \lambda = -\lambda(\Lambda - 1).$$

Observe that, under the isomorphism (0.3),

$$D \longmapsto \Lambda\ , \quad \frac{1}{(q-1)x} D \longmapsto \tilde{\Lambda} \quad \text{and} \quad D_q \longmapsto \tilde{\tilde{\Lambda}},$$

upon setting $\lambda_n^{-1} = (1-q)xq^{n-1}$.

Defining the simple vertex operators

$$X(t,z) := e^{\sum_1^\infty t_i z^i} e^{-\sum_1^\infty \frac{z^{-i}}{i} \frac{\partial}{\partial t_i}}, \tag{0.4}$$

we now make a statement concerning the so-called one-Toda lattice; the latter describes deformations of of a bi-infinite matrix L, which is lower-triangular, except for 1's just above the main diagonal. The *first formula* (0.6) below gives a solution to the *Frenkel system* (Theorem 0.1), upon replacing $\tilde{\Lambda}$ by Λ, which amounts to conjugating L by a constant diagonal matrix ε; see (2.2). The *second formula* (0.6) gives, via the isomorphism (0.3), a solution to the *KLR system* (Theorem 0.2). Thus in the L-representation the two systems are related by a trivial diagonal conjugation. Note, Theorem 0.1 is given for arbitrary $\lambda = (..., \lambda_{-1}, \lambda_0, \lambda_1, ...)$.

We shall need the well-known Hirota symbol for a polynomial p,

$$p(\pm\tilde{\partial}) f \circ g := p\left(\pm\frac{\partial}{\partial y_1}, \pm\frac{1}{2}\frac{\partial}{\partial y_2}, ...\right) f(t+y)g(t-y)\Big|_{y=0}.$$

Note A_+ refers to the upper-triangular part of a matrix A, including the diagonal, and for $\alpha \in \mathbf{C}$, set $[\alpha] := (\alpha, \frac{1}{2}\alpha^2, \frac{1}{3}\alpha^3, ...) \in \mathbf{C}^\infty$.

Theorem 0.1. *Given an integer $N \geq 2$, consider an arbitrary τ-function for the KP equation such that $\partial\tau/\partial t_{iN} = 0$ for $i = 1, 2, 3, ...$ (N-KdV hierarchy).*

For a fixed $\lambda,\ \nu,\ c \in \mathbf{C}^\infty$, the infinite sequence of τ-functions [1]

$$\tau_n := X(t,\lambda_n)...X(t,\lambda_1)\tau(c+t), \quad \tau_0 = \tau(c+t), \quad \textit{for } n \geq 0;$$

satisfies the 1-Toda bilinear identity for all $t, t' \in \mathbf{C}^\infty$ and all $n > m$:

$$\oint_{z=\infty} \tau_n(t-[z^{-1}])\tau_{m+1}(t'+[z^{-1}])e^{\sum_1^\infty (t_i - t_i')z^i} z^{n-m-1} dz = 0.$$

The bi-infinite matrix (a full matrix below the main diagonal), where p_ℓ are the elementary Schur polynomials,

$$L = \sum_{\ell=0}^{\infty} \operatorname{diag}\left(\frac{p_\ell(\tilde\partial)\tau_{n+2-\ell}\circ\tau_n}{\tau_{n+2-\ell}\tau_n}\right)_{n\in\mathbf{Z}} \tilde\Lambda^{1-\ell} \tag{0.5}$$

has the following properties:

(i) L^N *satisfies the* 1*-Toda lattice*

$$\frac{\partial L^N}{\partial t_n} = [(L^n)_+, L^N], \quad n = 1, 2, ..,$$

(ii) L^N *is <u>upper triangular</u> and admits the following expression in terms*[2] *of $\tilde\Lambda$ and $\tilde{\tilde\Lambda}$:*

$$\begin{aligned}
L^N &= \tilde\Lambda^N + \sum_1^N (\lambda_j + b_j)\tilde\Lambda^{p-1} + \left(\sum_0^{N-1} a_j + \sum_{1\leq i\leq j\leq N-1} (\lambda_i + b_i)(\lambda_j + b_j)\right)\tilde\Lambda^{N-2} \\
&\qquad + ... + \lambda_1^N \tilde\Lambda^0 \\
&= \tilde{\tilde\Lambda}^N + \left(\sum_1^N b_j\right)\tilde{\tilde\Lambda}^{N-1} \\
&\qquad + \left(\sum_0^{N-1} a_j - \sum_1^{N-1}(b_N - b_i)\lambda_i + \sum_{1\leq i\leq j\leq N-1} b_i b_j\right)\tilde{\tilde\Lambda}^{N-2} + ...
\end{aligned} \tag{0.6}$$

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[1] τ_n for $n < 0$ is defined later in (3.3).

[2] in the expressions below, the coefficients of the $\tilde{\tilde\Lambda}$'s are diagonal matrices, whose 0th component is given by the expression appearing below; e.g., $\sum_1^N b_j$ stands for $\operatorname{diag}(\sum_1^N b_{j+n})_{n\in\mathbf{Z}}$ and λ_1^N stands for $\operatorname{diag}(\lambda_{1+n}^N)_{n\in\mathbf{Z}}$.

with

$$b_k = \frac{\partial}{\partial t_1} \log \frac{\tau(c+t-\sum\limits_1^k [\lambda_i^{-1}])}{\tau(c+t-\sum\limits_1^{k-1} [\lambda_i^{-1}])}, \qquad a_k = \left(\frac{\partial}{\partial t_1}\right)^2 \log \tau\left(c+t-\sum_1^k [\lambda_i^{-1}]\right), \tag{0.7}$$

for $k \geq 1$*. These expressions for* $k \leq 0$ *will be given in (3.4) and (3.5).*

In view of (0.7), the shift

$$\Lambda:\ b_k \longmapsto \Lambda b_k = b_{k+1} \quad \text{and} \quad a_k \longmapsto \Lambda a_k = a_{k+1}$$

corresponds to the following transformation,

$$\Lambda:\ c \mapsto c - [\lambda_1^{-1}] \quad \text{and} \quad \lambda_i \mapsto \lambda_{i+1}. \tag{0.8}$$

Therefore, in order that L^N satisfies the form of the right hand side of (0.3), we must make c and λ_i depend on x and q, such that the map Λ on $a,\ b,\ \lambda$ corresponds to D, in addition to the fact that all λ_i must tend to ∞ simultaneously and c to $(x, 0, 0, ...)$, when q goes to 1. So, $c(x)$ and $\lambda(x)$ must satisfy:

$$\begin{cases} Dc(x) = c(x) - [\lambda_1^{-1}(x)] \\ D\lambda_n(x) = \lambda_{n+1}(x) \\ \lim_{q\to 1} \lambda_i = \infty \\ \lim\limits_{q\to 1} c(x) = \bar{x} := (x, 0, 0, ...); \end{cases} \tag{0.9}$$

its only solution is given by:

$$\lambda_n^{-1} = (1-q)xq^{n-1} \quad \text{and} \quad c(x) = \left(\frac{(1-q)x}{1-q}, \frac{(1-q)^2x^2}{2(1-q^2)}, \frac{(1-q)^3x^3}{3(1-q^3)}, ...\right), \tag{0.10}$$

and thus $D^n c(x) = c(x) - \sum_1^n [\lambda_i^{-1}]$. With this choice of λ_n,

$$\frac{1}{(q-1)x} D \longmapsto \tilde{\Lambda} \quad \text{and} \quad D_q := \frac{D-1}{(q-1)x} \longmapsto \tilde{\tilde{\Lambda}}. \tag{0.11}$$

In analogy with (0.4), we define the simple q-vertex operators:

$$X_q(x,t,z) := e_q^{xz} X(t,z) \ \text{ and } \ \tilde{X}_q(x,t,z) := \left(e_q^{xz}\right)^{-1} X(-t,z). \tag{0.12}$$

in terms of (0.4) and the q-exponential $e_q^x := e^{\sum_1^\infty \frac{(1-q)^k x^k}{k(1-q^k)}}$. Therefore under the isomorphism (0.3), Theorem 0.1 can be translated into q-language, to read:

Theorem 0.2. *Any KdV τ-function leads to a q-KdV τ-function $\tau(c(x)+t)$; the latter satisfies the bilinear relations below, for all $x \in \mathbf{R},\ t, t' \in \mathbf{C}^\infty$, and all $n > m$, which tends to the standard KP-bilinear identity, when q goes to 1:*

$$\begin{aligned}
&\oint_{z=\infty} D^n \left(X_q(x,t,z)\tau(c(x)+t)\right) D^{m+1} \left(\tilde{X}_q(x,t',z)\tau(c(x)+t')\right) dz = 0 \\
&\qquad \longrightarrow \quad \oint_{z=\infty} X(t,z)\tau(\bar{x}+t)\ X(t',z)\tau(\bar{x}+t')\ dz = 0
\end{aligned} \tag{0.13}$$

Moreover, the q-differential operator Q_q^N has the form below and tends to the differential operator $\mathcal{L}^N$ of the KdV hierarchy, when q goes to 1:

$$\begin{aligned}
Q_q^N \quad = \quad & D_q^N + \frac{\partial}{\partial t_1} \log \frac{\tau(D^N c+t)}{\tau(c+t)} D_q^{N-1} \\
& + \left(\sum_{i=0}^{N-1} \frac{\partial^2}{\partial t_1^2} \log \tau(D^i c+t) \right. \\
& - \sum_{i=0}^{N-2} \lambda_{i+1} \left(\frac{\partial}{\partial t_1} \log \frac{\tau(D^N c+t)}{\tau(D^{N-1}c+t)} - \frac{\partial}{\partial t_1} \log \frac{\tau(D^{i+1}c+t)}{\tau(D^i c+t)} \right) \\
& \left. + \sum_{0 \le i \le j \le N-2} \frac{\partial}{\partial t_1} \log \frac{\tau(D^{i+1}c+t)}{\tau(D^i c+t)} \frac{\partial}{\partial t_1} \log \frac{\tau(D^{j+1}c+t)}{\tau(D^j c+t)} \right) D_q^{N-2} + \ldots \\
\longrightarrow \ & \left(\frac{\partial}{\partial x}\right)^N + N \frac{\partial^2}{\partial t_1^2} \log \tau(\bar{x}+t) \left(\frac{\partial}{\partial x}\right)^{N-2} + \ldots\ .
\end{aligned} \tag{0.14}$$

M.A. and PvM thank Edward Frenkel for kindly discussing this problem during spring 1996. For a systematic study of discrete systems, see Kupershmidt [13] and Gieseker [8]. It is an old observation (see [11]) that the Toda lattice discretizes the KdV equation, and this in many different ways. Therefore it is not surprising that q-KdV is yet another Toda discretization of KdV. In an elegant recent preprint, Iliev [10] has obtained q-bilinear identities and q-tau functions, as well, purely within the KP theory.

1 The KP τ-functions and Grassmannians

KP τ-functions satisfy the differential Fay identity for all $y, z \in \mathbf{C}$, in terms of the Wronskian $\{f, g\} := f'g - fg'$, as shown in [1, 16]:

$$\{\tau(t-[y^{-1}]), \tau(t-[z^{-1}])\}$$
$$+(y-z)(\tau(t-[y^{-1}])\tau(t-[z^{-1}])-\tau(t)\tau(t-[y^{-1}]-[z^{-1}]) = 0. \tag{1.1}$$

In fact this identity characterizes the τ-function, as shown in [15]. We shall need the following, shown in [1]:

Proposition 1.1. *Consider τ-functions τ_1 and τ_2, the corresponding wave functions*

$$\Psi_i = e^{\sum_{i\geq 1} t_i z^i} \frac{\tau_i(t-[z^{-1}])}{\tau_i(t)} = e^{\sum_{i\geq 1} t_i z^i}\Big(1 + O(z^{-1})\Big) \tag{1.2}$$

and the associated infinite-dimensional planes, as points in the Grassmannian Gr,

$$\tilde{W}_i = span\left\{\left(\frac{\partial}{\partial t_1}\right)^k \Psi_i(t, z), \text{ for } k = 0, 1, 2, ...\right\};$$

then the following statements are equivalent
(i) $z\tilde{W}_2 \subset \tilde{W}_1$;
(ii) $z\Psi_2(t, z) = \frac{\partial}{\partial t_1}\Psi_1(t, z) - \alpha\Psi_1(t, z)$, for some function $\alpha = \alpha(t)$;
(iii)

$$\{\tau_1(t-[z^{-1}]), \tau_2(t)\} + z(\tau_1(t-[z^{-1}])\tau_2(t) - \tau_2(t-[z^{-1}])\tau_1(t)) = 0 \tag{1.3}$$

When (i), (ii) or (iii) holds, $\alpha(t)$ is given by

$$\alpha(t) = \frac{\partial}{\partial t_1}\log\frac{\tau_2}{\tau_1}. \tag{1.4}$$

Proof: To prove that (i) $\Rightarrow$ (ii), the inclusion $z\tilde{W}_2 \subset \tilde{W}_1$ implies $z\tilde{W}_2^t \subset \tilde{W}_1^t$, where $\tilde{W}^t = \tilde{W}e^{-\sum_1^\infty t_i z^i}$; it follows that

$$z\psi_2(t, z) = z(1 + O(z^{-1})) \in W_1^t$$

must be a linear combination, involving the wave functions $\Psi_i = e^{\sum_1^\infty t_i z^i}\psi_i$:

$$z\psi_2 = \left(\frac{\partial}{\partial x} + z\right)\psi_1 - \alpha(t)\psi_1, \text{ and thus } z\Psi_2 = \frac{\partial}{\partial t_1}\Psi_1 - \alpha(t)\Psi_1. \tag{1.5}$$

The expression (1.4) for $\alpha(t)$ follows from equating the z^0-coefficient in (1.5), upon using the τ-function representation (1.2). To show that (ii) $\Rightarrow$ (i), note that

$$z\Psi_2 = \frac{\partial}{\partial t_1}\Psi_1 - \alpha\Psi_1 \in W_1^0$$

and taking z-derivatives, we have

$$z\left(\frac{\partial}{\partial t_1}\right)^j \Psi_2 = \left(\frac{\partial}{\partial t_1}\right)^{j+1}\Psi_1 + \beta_1\left(\frac{\partial}{\partial t_1}\right)^j \Psi_1 + \cdots + \beta_{j+1}\Psi_1,$$

for some $\beta_1, \cdots, \beta_{j+1}$ depending on t only; this implies the inclusion (i). The equivalence (ii) $\Longleftrightarrow$ (iii) follows from a straight forward computation using the τ-function representation (1.2) of (ii) and the expression for $\alpha(t)$. ■

2 The one-Toda lattice

For details on this sketchy exposition, see [3]. The one-Toda lattice equations

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \tag{2.1}$$

are deformations of an infinite matrix

$$L = \sum_{-\infty < i \leq 0} a_i \tilde{\Lambda}^i + \tilde{\Lambda}, \text{with } \tilde{\Lambda} := \lambda\Lambda = \varepsilon\Lambda\varepsilon^{-1}, \tag{2.2}$$

for diagonal matrices λ and ε, with non-zero entries, and diagonal matrices a_i, depending on $t = (t_1, t_2, \ldots)$. Note the conjugation by the constant diagonal matrix ε is harmless, but it is necessary to capture the KLR-system. One introduces wave and adjoint wave vectors $\Psi(t,z)$ and $\Psi^*(t,z)$, satisfying

$$L\Psi = z\Psi \quad \text{and} \quad L^\top\Psi^* = z\Psi^*$$

and

$$\frac{\partial\Psi}{\partial t_n} = (L^n)_+\Psi \qquad \frac{\partial\Psi^*}{\partial t_n} = -((L^n)_+)^\top\Psi^*. \tag{2.3}$$

The wave vectors Ψ and Ψ^* can be expressed in terms of one sequence of τ-functions $\tau(n,t) := \tau_n(t_1,t_2,\ldots)$, $\quad n \in \mathbf{Z}$, to wit:

$$\Psi(t,z) = \left(e^{\sum_1^\infty t_i z^i}\psi(t,z)\right)_{n\in\mathbf{Z}} = \left(\frac{\tau_n(t-[z^{-1}])}{\tau_n(t)}e^{\sum_1^\infty t_i z^i}\varepsilon_n z^n\right)_{n\in\mathbf{Z}},$$

$$\Psi^*(t,z) = \left(e^{-\sum_1^\infty t_i z^i}\psi^*(t,z)\right)_{n\in\mathbf{Z}} = \left(\frac{\tau_{n+1}(t+[z^{-1}])}{\tau_{n+1}(t)}e^{-\sum_1^\infty t_i z^i}\varepsilon_n^{-1}z^{-n}\right)_{n\in\mathbf{Z}} \tag{2.4}$$

It follows that, in terms of $\chi(z) := (z^n)_{n\in\mathbf{Z}}$ and the notation $a_\Lambda := \operatorname{diag}(a_{k+1})_{k\in\mathbf{Z}}$:

$$\Psi = e^{\sum_1^\infty t_i z^i} S\varepsilon\chi(z), \quad \text{with} \quad S = \sum_0^\infty \frac{p_n(-\tilde\partial)\tau(t)}{\tau(t)}\tilde\Lambda^{-n},$$

$$\Psi^* = e^{-\sum_1^\infty t_i z^i}(S^\top)^{-1}\varepsilon^{-1}\chi(z^{-1}), \quad \text{with} \quad S^{-1} = \sum_0^\infty \tilde\Lambda^{-n}\left(\frac{p_n(\tilde\partial)\tau(t)}{\tau(t)}\right)_\Lambda.$$

Moreover, as will follow from Proposition 2.1 below, Ψ and Ψ^* satisfy the bilinear identities:

$$\oint_{z=\infty}\Psi_n(t,z)\Psi_m^*(t',z)\frac{dz}{2\pi i z} = 0, \qquad \text{for all} \quad n > m.$$

From the representation of S and S^{-1} above, it follows that

$$\begin{aligned} L^k &= S\tilde\Lambda^k S^{-1} \\ &= \sum_{\ell=0}^\infty \operatorname{diag}\left(\frac{p_\ell(\tilde\partial)\tau_{n+k-\ell+1}\circ\tau_n}{\tau_{n+k-\ell+1}\tau_n}\right)_{n\in\mathbf{Z}}\tilde\Lambda^{k-\ell} \\ &= \tilde\Lambda^k + \operatorname{diag}\left(\frac{\partial}{\partial t_1}\log\frac{\tau_{n+k}}{\tau_n}\right)_{n\in\mathbf{Z}}\tilde\Lambda^{k-1} + \ldots \\ &\qquad + \operatorname{diag}\left(\frac{\partial}{\partial t_k}\log\frac{\tau_{n+1}}{\tau_n}\right)_{n\in\mathbf{Z}}\tilde\Lambda^0 + \operatorname{diag}\left(\frac{\partial^2}{\partial t_1\partial t_k}\log\tau_n\right)_{n\in\mathbf{Z}}\tilde\Lambda^{-1} + \ldots \end{aligned} \tag{2.5}$$

For instance, the Λ^0-term in the last expression follows from setting $m = n-2,\ t \mapsto t+[\alpha],\ t' \mapsto t-[\alpha]$ in the bilinear identity above, yielding

$$\begin{aligned} 0 &= \frac{\tau_n(t+[\alpha])\tau_{n-1}(t-[\alpha])}{\tau_n(t)\tau_{n-1}(t)} \oint_{z=\infty} \Psi_n(t+[\alpha],z)\Psi^*_{n-2}(t-[\alpha],z)\frac{dz}{2\pi i z} \\ &= \frac{1}{\tau_n\tau_{n-1}} \sum_{j\geq 0} \alpha^j \left(\frac{\partial}{\partial t_{j+2}} - p_{j+2}(\tilde\partial)\right) \tau_n \circ \tau_{n-1}, \end{aligned}$$

and thus,

$$\frac{\partial}{\partial t_k} \log \frac{\tau_{n+1}}{\tau_n} = \frac{p_k(\tilde\partial)\tau_{n+1}\circ\tau_n}{\tau_{n+1}\tau_n} = \left(L^k\right)_{nn}.$$

With each component of the wave vector Ψ, we associate a sequence of infinite-dimensional planes in the Grassmannian Gr

$$\begin{aligned} W_n &= \operatorname{span}_{\mathbf{C}} \left\{ \left(\frac{\partial}{\partial t_1}\right)^k \Psi_n(t,z), \quad k=0,1,2,... \right\} \\ &= e^{\sum_1^\infty t_i z^i} \operatorname{span}_{\mathbf{C}} \left\{ \left(\frac{\partial}{\partial t_1}+z\right)^k \psi_n(t,z), \quad k=0,1,2,... \right\} \end{aligned} \tag{2.6}$$

and planes

$$W_n^* = \operatorname{span}_{\mathbf{C}} \left\{ \left(\frac{\partial}{\partial t_1}\right)^k \Psi^*_{n-1}(t,z), \quad k=0,1,2,... \right\},$$

which are orthogonal to W_n by the residue pairing

$$\oint_{z=\infty} f(z)g(z)\frac{dz}{2\pi i z}. \tag{2.7}$$

Note that the plane $z^{-n}W_n$ has so-called virtual genus zero, in the terminology of [14]; in particular, this plane contains an element of order $1+O(z^{-1})$. The following statement is contained in [3]:

Proposition 2.1. *The following five statements are equivalent*
(i) The 1-Toda lattice equations (2.1)

(ii) Ψ and Ψ^*, with the proper asymptotic behaviour, given by (2.4), satisfy the bilinear identities for all $t, t' \in \mathbf{C}^\infty$

$$\oint_{z=\infty} \Psi_n(t,z)\Psi^*_m(t',z)\frac{dz}{2\pi iz} = 0, \qquad \textit{for all} \quad n > m; \tag{2.8}$$

(iii) the τ-vector satisfies the following bilinear identities for all $n > m$ and $t, t' \in \mathbf{C}^\infty$:

$$\oint_{z=\infty} \tau_n(t - [z^{-1}])\tau_{m+1}(t' + [z^{-1}])e^{\sum_1^\infty (t_i - t'_i)z^i} z^{n-m-1} dz = 0; \tag{2.9}$$

(iv) The components τ_n of a τ-vector correspond to a flag of planes in Gr,

$$\supset W_{n-1} \supset W_n \supset W_{n+1} \supset .... \tag{2.10}$$

(v) A sequence of KP-τ-functions τ_n satisfying the equations

$$\{\tau_n(t - [z^{-1}]), \tau_{n+1}(t)\} + z(\tau_n(t - [z^{-1}])\tau_{n+1}(t) - \tau_{n+1}(t - [z^{-1}])\tau_n(t)) = 0 \tag{2.11}$$

Proof: The proof that (i) is equivalent to (ii) follows from the methods in [4, 16]. That (ii) is equivalent to (iii) follows from the representation (2.4) of wave functions in terms of τ-functions. Finally, we sketch the proof that (ii) is equivalent to (iv). The inclusion in (iv) implies that W_n, given by (2.6), is also given by

$$W_n = \mathrm{span}_{\mathbf{C}} \, \{\Psi_n(t,z), \Psi_{n+1}(t,z), ...\};$$

Since each τ_n is a τ-function, we have that

$$\oint_{z=\infty} \Psi_n(t,z)\Psi^*_{n-1}(t',z)\frac{dz}{2\pi iz} = 0,$$

implying that, for each $n \in \mathbf{Z}$, $\Psi^*_{n-1}(t,z) \in W^*_n$. Moreover the inclusions $... \supset W_n \supset W_{n+1} \supset ...$ imply, by orthogonality, the inclusions $... \subset W^*_n \subset W^*_{n+1} \subset ...$, and thus

$$W^*_n = \{\Psi^*_{n-1}(t,z), \Psi^*_{n-2}(t,z), ...\}.$$

Since

$$W_n \subset W_m = (W_m^*)^*, \quad \text{all } n \geq m,$$

we have the orthogonality $W_n \perp W_m^*$ by the residue pairing (2.7) for all $n \geq m$, i.e.,

$$\oint_{z=\infty} \Psi_n(t,z)\Psi_{m-1}^*(t',z)\frac{dz}{2\pi iz} = 0, \text{ all } n \geq m.$$

Note (ii) implies $W_m^* \subset W_n^*, \quad n > m$, hence $W_n \subset W_m, \quad n > m$, yielding (iv). That (iv)$\Longleftrightarrow$(v) follows from proposition 1.1, by setting $\tau_1 := \tau_n$ and $\tau_2 = \tau_{n+1}$. Then (v) is equivalent to the inclusion property

$$z(z^{-n-1}W_{n+1}) \subset (z^{-n}W_n), \quad \text{i.e.} \quad W_{n+1} \subset W_n,$$

thus ending the proof of proposition 2.1. ∎

3 Proof of Theorems 0.1 and 0.2

At first, we exhibit particular solutions to equation (2.11), explained in [1].

Lemma 3.1. *Particular solutions to equation*

$$\{\tau_1(t-[z^{-1}]), \tau_2(t)\} + z(\tau_1(t-[z^{-1}])\tau_2(t) - \tau_2(t-[z^{-1}])\tau_1(t)) = 0$$

are given, for arbitrary $\lambda \in \mathbf{C}^*$, *by pairs* (τ_1, τ_2), *defined by:*

$$\tau_2(t) = X(t,\lambda)\tau_1(t) = e^{\sum t_i\lambda^i}\tau_1(t-[\lambda^{-1}]), \tag{3.1}$$

or

$$\tau_1(t) = X(-t,\lambda)\tau_2(t) = e^{-\sum t_i\lambda^i}\tau_2(t+[\lambda^{-1}]). \tag{3.2}$$

Proof: Using

$$e^{-\sum_1^\infty \frac{1}{i}(\frac{\lambda}{z})^i} = 1 - \frac{\lambda}{z},$$

it suffices to check that $\tau_2(t)$ satisfies the above equation (2.11)

$$\begin{aligned} &e^{-\sum t_i\lambda^i}\Big(\{\tau_1(t-[z^{-1}]), \tau_2(t)\} + z(\tau_1(t-[z^{-1}])\tau_2(t) - \tau_2(t-[z^{-1}])\tau_1(t))\Big) \\ &\quad = e^{-\sum t_i\lambda^i}\{\tau_1(t-[z^{-1}]), e^{\sum t_i\lambda^i}\tau_1(t-[\lambda^{-1}])\} \end{aligned}$$

$$
\begin{aligned}
&\quad + z(\tau_1(t-[z^{-1}])\tau_1(t-[\lambda^{-1}]) - (1-\frac{\lambda}{z})\tau_1(t)\tau_1(t-[z^{-1}]-\lambda^{-1}]))\\
&= \{\tau_1(t-[z^{-1}]), \tau_1(t-[\lambda^{-1}])\}\\
&\quad + (z-\lambda)(\tau_1(t-[z^{-1}])\tau_1(t-[\lambda^{-1}]) - \tau_1(t)\tau_1(t-[z^{-1}]-[\lambda^{-1}]))\\
&= 0,
\end{aligned}
$$

using the differential Fay identity (1.1) for the τ-function τ_1; a similar proof works for the second solution, given by (3.2). ■

Proof of Theorems 0.1 and 0.2: From an arbitrary N-KdV τ-function, construct, for $\lambda,\ c,\ \nu \in \mathbf{C}^\infty$, the following sequence of τ-functions, for $n \geq 0$, as announced in Theorem 0.1:

$$
\tau_0(t) = \tau(c+t)
$$

$$
\begin{aligned}
\tau_n &= X(t,\lambda_n)...X(t,\lambda_1)\tau(c+t)\\
&= \frac{\Delta(\lambda_1,...,\lambda_n)}{\prod_1^n \lambda_i^{i-1}} \prod_{k=1}^n e^{\sum_{i=1}^\infty t_i\lambda_k^i}\tau(c+t-\sum_1^n[\lambda_i^{-1}]),\\
\tau_{-n} &= X(-t,\lambda_{-n+1})...X(-t,\lambda_0)\tau(c+t)\\
&= \frac{\Delta(\lambda_0,...,\lambda_{-n+1})}{\prod_1^n \lambda_{-i+1}^{i-1}} \prod_{k=1}^n e^{-\sum_{i=1}^\infty t_i\lambda_{-k+1}^i}\tau(c+t+\sum_1^n[\lambda_{-i+1}^{-1}])
\end{aligned}
\tag{3.3}
$$

and so, each τ_n is defined inductively by

$$
\tau_{n+1} = X(t,\lambda_{n+1})\tau_n;
$$

thus by Lemma 3.1, the functions τ_{n+1} and τ_n are a solution of equation (v) of proposition 2.1. Therefore, by the same proposition 2.1, the τ_n's form a τ-vector of the 1-Toda lattice. By removing the harmless exponential factor $\prod_{k=1}^n \exp(\sum_1^\infty t_{iN}\lambda_k^{iN})$, each τ_n has the property that $\partial\tau_n/\partial t_{iN} = 0$ for $i = 1,2,...$; therefore

$$
z^N W_n \subset W_n.
$$

In particular, the representation

$$
W_n = \text{span}\{\Psi_n(t,z), \Psi_{n+1}(t,z), ...\},
$$

which follows from the inclusion ... $\supset W_n \supset W_{n+1} \supset$ ..., implies that, since $L\Psi = z\Psi$,

$$z^N \Psi_k = \sum_{j\geq k} a_j \Psi_j = (L^N \Psi)_k,$$

and thus L^N is upper-triangular.

Therefore, we conclude that the matrix L, defined by (2.5), from the sequence of τ-functions (3.3),

$$\begin{aligned} L &= \tilde{\Lambda} + \left(\frac{\partial}{\partial t_1} \log \frac{\tau_{n+1}}{\tau_n}\right)_{n\in\mathbf{Z}} + \left(\left(\frac{\partial}{\partial t_1}\right)^2 \log \tau_n\right)_{n\in\mathbf{Z}} \tilde{\Lambda}^{-1} + ... \\ &= \tilde{\Lambda} + (\lambda_{n+1} + b_{n+1})_{n\in\mathbf{Z}} \tilde{\Lambda}^0 + (a_n)_{n\in\mathbf{Z}} \tilde{\Lambda}^{-1} + ..., \end{aligned}$$

satisfies the 1-Toda lattice equations, where

$$\begin{aligned} b_{n+1} &= \frac{\partial}{\partial t_1} \log \frac{\tau(c+t-\sum_1^{n+1}[\lambda_i^{-1}])}{\tau(c+t-\sum_1^n[\lambda_i^{-1}])} \quad \text{for} \quad n \geq 1 \\ &= \frac{\partial}{\partial t_1} \log \frac{\tau(c+t-[\lambda_1^{-1}])}{\tau(c+t)}, \quad \text{for} \quad n=0, \\ &= \frac{\partial}{\partial t_1} \log \frac{\tau(c+t+\sum_0^{n+2}[\lambda_i^{-1}](1-\delta_{-1,n}))}{\tau(c+t+\sum_0^{n+1}[\lambda_i^{-1}])}, \quad \text{for} \quad n \leq -1, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} a_n &= \frac{\partial^2}{\partial t_1^2} \log \tau(c+t-\sum_1^n [\lambda_i^{-1}]) \quad \text{for} \quad n \geq 1 \\ &= \frac{\partial^2}{\partial t_1^2} \log \tau(c+t) \quad \text{for} \quad n=0 \\ &= \frac{\partial^2}{\partial t_1^2} \log \tau(c+t+\sum_0^{n+1}[\lambda_i^{-1}]) \quad \text{for} \quad n \leq -1, \end{aligned} \tag{3.5}$$

confirming (0.7). Using the fact that, in view of (2.5), the diagonal terms of L^N are given by

$$\frac{\partial}{\partial t_N} \log \frac{\tau_{n+1}}{\tau_n} = \lambda_{n+1}^N,$$

and the fact that, in the notation of footnote 2,

$$\tilde{\tilde{\Lambda}}^n = (\tilde{\Lambda} + \lambda)^n = \tilde{\Lambda}^n + \left(\sum_1^n \lambda_i\right) \tilde{\Lambda}^{n-1} + \left(\sum_{1\leq i\leq j\leq n-1} \lambda_i \lambda_j\right) \tilde{\Lambda}^{n-2} + ...,$$

one finds that the upper-triangular matrix L^N has the following expression:

$$\begin{aligned}
L^N &= \tilde{\Lambda}^N + \sum_1^N (\lambda_j + b_j)\tilde{\Lambda}^{p-1} + \left(\sum_0^{N-1} a_j + \sum_{1\le i\le j\le N-1} (\lambda_i + b_i)(\lambda_j + b_j)\right)\tilde{\Lambda}^{N-2} \\
&\qquad\qquad + ... + \lambda_1^N \tilde{\Lambda}^0 \\
&= \tilde{\tilde{\Lambda}}^N + \left(\sum_1^N b_j\right)\tilde{\tilde{\Lambda}}^{N-1} + \left(\sum_0^{N-1} a_j - \sum_1^{N-1}(b_N - b_i)\lambda_i + \sum_{1\le i\le j\le N-1} b_i b_j\right)\tilde{\tilde{\Lambda}}^{N-2} + ...
\end{aligned} \tag{3.6}$$

in terms of b_k and a_k defined in (0.7), thus proving Theorem 0.1. ∎

To prove Theorem 0.2, note at first:

$$\begin{aligned}
\frac{z^{n-m-1}}{\prod_{k=m+2}^n (-\lambda_k)} \prod_{k=m+2}^n e^{-\sum_{i=1}^\infty \frac{1}{i}\left(\frac{\lambda_k}{z}\right)^i} &= \frac{z^{n-m-1}}{\prod_{k=m+2}^n (-\lambda_k)} \prod_{k=m+2}^n \left(1 - \frac{\lambda_k}{z}\right) \\
&= \prod_{k=m+2}^n \left(1 - \frac{z}{\lambda_k}\right) \\
&= \prod_{k=m+2}^n e^{-\sum_{i=1}^\infty \frac{1}{i}\left(\frac{z}{\lambda_k}\right)^i} \\
&= \frac{e_q^{xzq^n}}{e_q^{xzq^{m+1}}} = D^n e_q^{xz}\ D^{m+1}\left(e_q^{xz}\right)^{-1}.
\end{aligned}$$

The function τ_n, defined in Theorem 0.1, satisfies the bilinear identity of Theorem 0.1; therefore, using (3.3) and the above in the computation of $\tau_n(t-[z^{-1}])$, the following relations hold, up to a multiplicative factor depending on λ and ν:

$$\begin{aligned}
&\alpha(\lambda,\nu) \oint_{z=\infty} \tau_n(t - [z^{-1}])\tau_{m+1}(t' + [z^{-1}]) e^{\sum_1^\infty (t_i - t_i')z^i} z^{n-m} \frac{dz}{z} \\
&= \oint_{z=\infty} \tau(c(x) + t - [z^{-1}] - \sum_1^n [\lambda_i^{-1}])\tau(c(x) + t' + [z^{-1}] + \sum_1^{m+1} [\lambda_i^{-1}]) \\
&\qquad\qquad\qquad\qquad \prod_{k=m+2}^n \left(1 - \frac{z}{\lambda_k}\right) e^{\sum_1^\infty (t_i - t_i')z^i} dz \\
&= \oint_{z=\infty} D^n \left(X_q(x,t,z)\tau(c(x)+t)\right)\ D^{m+1}\left(\tilde{X}_q(x,t',z)\tau(c(x)+t')\right) dz = 0.
\end{aligned}$$

When $q \longrightarrow 1$, the second expression above tends to the standard KP-bilinear equation, upon using (0.10). Moreover, one checks by induction, using the expression (2.5) for L and (3.3), that $(L^N)_+$ for $N = 1, 2, 3, ...$ has the q-form (0.3). Also, note that a_k and b_k can be expressed in terms of the D-operator, using (0.7); to wit:

$$b_k = \frac{\partial}{\partial t_1} \log \frac{\tau(D^k c + t)}{\tau(D^{k-1} c + t)}, \qquad a_k = \left(\frac{\partial}{\partial t_1}\right)^2 \log \tau \left(D^k c + t\right).$$

So, the expression for Q_q^N in Theorem 0.2 follows at once from (3.6). The fact that

$$-\lambda_1 \frac{\partial}{\partial t_1} \log \frac{\tau(D^{j+1} c + t)}{\tau(D^j c + t)} \longrightarrow \frac{\partial^2}{\partial x^2} \log \tau(\bar{x} + t)$$

implies that all terms in (0.14) vanish in the limit $q \longrightarrow 1$, except for the term $\sum_{i=0}^{N-1} \frac{\partial^2}{\partial t_1^2} \log \tau(D^i c + t)$; so we have that

$$\lim_{q \to 1} Q_p^N = \left(\frac{\partial}{\partial x}\right)^N + N \frac{\partial^2}{\partial x^2} \log \tau(\bar{x} + t) \left(\frac{\partial}{\partial x}\right)^{N-2} + ...,$$

thus ending the proof of theorem 0.2. ■

4 Examples and vertex operators

The isomorphism (0.3) enables one to translate every 1-Toda statement, having the form (0.3) into a D or D_q statement. Also every τ-function of the KdV hierarchy leads automatically to a solution of q-KdV. For instance, by replacing $t \mapsto c(x) + t$ in the Schur polynomials, one finds q-Schur polynomials. The latter were obtained by Haine and Iliev [9] by using the q-Darboux transforms; the latter had been studied by Horozov and coworkers in [5, 6].

The n-soliton solution to the KdV (for $N = 2$) (for this formulation, see [4]),

$$\tau(t) = \det \left(\delta_{i,j} - \frac{a_j}{y_i + y_j} e^{-\sum_{k:\mathrm{odd}} t_k (y_i^k + y_j^k)} \right)_{1 \le i,j \le n},$$

leads to a q-soliton by the shift $t \mapsto c(x) + t$, with $c(x)$ as in (0.8), namely

$$\tau(x,t) = \det \left(\delta_{ij} - a_j \frac{\left(e_q^{xy_i} e_q^{xy_i}\right)^{-1}}{y_i + y_j} e^{-\sum_{k=1}^{\infty} t_k (y_i^k + y_j^k)} \right)_{1 \le i,j \le n}.$$

Moreover the vertex operator for the 1-Toda lattice is a reduction of the 2-Toda lattice vertex operator (see [2]), given by

$$\begin{aligned}\mathbf{X}(t,y,z) &= -\chi^*(z)X(-t,z)X(t,y)\chi(y)\\ &= \frac{z}{y-z}e^{\sum_1^\infty t_i(y^i-z^i)}e^{-\sum_1^\infty (y^{-i}-z^{-i})\frac{1}{i}\frac{\partial}{\partial t_i}}\left(\frac{y^n}{z^n}\right)_{n\in\mathbf{Z}};\end{aligned}$$

in particular, if τ is a 1-Toda vector, then $a\tau+b\mathbf{X}(t,y,z)\tau$ is a 1-Toda vector as well. Using the dictionary, this leads to q-vertex operators

$$\mathbf{X}_q(x,t;y,z) = e_q^{xy}(e_q^{xz})^{-1}e^{\sum t_i(y^i-z^i)}e^{-\sum(y^{-i}-z^{-i})\frac{1}{i}\frac{\partial}{\partial t_i}} \quad \text{for } q\text{-KP},$$

and, for any Nth root ω of 1,

$$\mathbf{X}_q(x,t;z) = e_q^{x\omega z}(e_q^{xz})^{-1}e^{\sum t_i z^i(\omega^i-1)}e^{-\sum z^{-i}(\omega^{-i}-1)\frac{1}{i}\frac{\partial}{\partial t_i}} \quad \text{for } q\text{-KdV},$$

having the typical vertex operator properties.

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