Toda-Darboux maps and vertex operators*

M. Adler† P. van Moerbeke‡

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The process of Borel decomposing a matrix as a product of lower- and upper-triangular matrices and multiplying them in opposite order, (or conversely) is called the *Bäcklund-Darboux map.* The matrices considered here are bi-infinite *tridiagonal* matrices or, more generally, $2p+1$-*band* matrices $L-\lambda I$, where λ is a free parameter; their Borel decomposition and thus also the corresponding Darboux map will depend on $2p-1$ parameters, in addition to the spectral parameter λ.

Letting these matrices evolve according to the standard (commuting) Toda vector fields introduces a dependency on a time-parameter $t \in \mathbf{C}^{\infty}$. Then we show that, upon adjusting appropriately the free parameters, the Darboux transformed matrix evolves according to the *Toda lattice*, whereas, in the tridiagonal case, each of the factors evolves according to the *KM lattice*. As is well known, the entries and the eigenvectors of the t-dependent matrix can entirely be expressed in terms of a single vector of τ-*functions* $(..., \tau_{-1}(t), \tau_0(t), \tau_1(t), ...)$. Given such a Darboux map, how are the new τ-functions and eigenvectors expressed in terms of the old ones? The formulae so obtained involve certain *vertex operators*, which depend on the spectral parameter λ and which turn out to be very useful even after setting $t=0$;

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†Department of Mathematics, Brandeis University, Waltham, Mass 02254, USA. E-mail: adler@math.brandeis.edu. The support of a National Science Foundation grant # DMS-9503246 is gratefully acknowledged.

‡Department of Mathematics, Université de Louvain, 1348 Louvain-la-Neuve, Belgium and Brandeis University, Waltham, Mass 02254, USA. E-mail: vanmoerbeke@geom.ucl.ac.be and @math.brandeis.edu. The support of a National Science Foundation grant # DMS-9503246, a Nato, a FNRS and a Francqui Foundation grant is gratefully acknowledged.

indeed the τ-functions are often well-known quantities, like matrix integrals, determinants of moments, Fredholm determinants, etc...

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Consider the Toda lattice

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L] = [-(L^n)_-, L], \qquad n = 1, 2, \dots \, . \tag{0.1}$$

on bi-infinite tridiagonal matrices

$$L = \Lambda^{-1}a + b\Lambda^0 + \Lambda = \begin{pmatrix} \ddots & \ddots & & O \\ \ddots & b_{-1} & 1 & \ddots \\ & a_{-1} & b_0 & \ddots \\ O & & \ddots & \ddots \end{pmatrix}, \tag{0.2}$$

with Λ being the customary shift $\Lambda := (\delta_{i,j-1})$ and a and b being diagonal matrices. As is well known, in analogy with Sato's KP-theory, the entries a and b have the following τ-function representation

$$b_n = \frac{\partial}{\partial t_1} \log \frac{\tau_{n+1}}{\tau_n} \quad \text{and} \quad a_{n-1} = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}, \tag{0.3}$$

in terms of a τ-vector $\tau = (\tau_n)_{n\in\mathbf{Z}}$. In section 2, it is shown that, if τ is a Toda lattice τ-vector, then

$$(1 + c\,\mathbf{X}(t,z))\tau$$

is a Toda τ-vector as well, where the *vertex operator* $\mathbf{X}(t,z)$, introduced by us in [3], is given by[1]

$$\mathbf{X}(t,z) = \Lambda^{-1}\chi(z^2)e^{\sum t_i z^i}e^{-2\sum \frac{z^{-i}}{i}\frac{\partial}{\partial t_i}}. \tag{0.4}$$

For a fixed $\lambda \in \mathbf{C}^*$, we consider *lower-upper* (LU) *Borel factorizations*:

$$L(t) - \lambda = L_-(t)L_+(t),$$

with

$$L_-(t) = \Lambda^{-1}\alpha(t) + \Lambda^0 = \begin{pmatrix} \ddots & \ddots & & O \\ \ddots & \begin{array}{c|c} 1 & 0 \\ \hline \alpha_{-1} & 1 \end{array} & & \ddots \\ O & \ddots & & \ddots \end{pmatrix},$$

$$L_+(t) = \beta(t)\Lambda^0 + \Lambda = \begin{pmatrix} \ddots & \ddots & & O \\ \ddots & \begin{array}{c|c} \beta_{-1} & 1 \\ \hline 0 & \beta_0 \end{array} & & \ddots \\ O & \ddots & & \ddots \end{pmatrix}.$$

Also consider *upper-lower* (UL) *Borel factorizations*:

$$L(t) - \lambda = L'_+(t)L'_-(t) := L_-(t)\Lambda\Lambda^{-1}L_+(t),$$

with

$$\tilde{L}'_+(t) = \alpha_{\Lambda^{-1}} + \Lambda = \begin{pmatrix} \ddots & \ddots & & O \\ \ddots & \begin{array}{c|c} \alpha_{-2} & 1 \\ \hline 0 & \alpha_{-1} \end{array} & & \ddots \\ O & \ddots & & \ddots \end{pmatrix},$$

$$\tilde{L}'_-(t) = \Lambda^{-1}\beta + I = \begin{pmatrix} \ddots & \ddots & & O \\ \ddots & \begin{array}{c|c} 1 & 0 \\ \hline \beta_{-1} & 1 \end{array} & & \ddots \\ O & \ddots & & \ddots \end{pmatrix}.$$

[1] $\chi(z) := \text{diag}\,(\ldots, z^{-1}, 1, z, \ldots)$

These factorizations lead to LU and UL *Darboux transforms*[2]:

$$\begin{array}{llll} \text{LU-map} : & L(t)-\lambda = L_-L_+ & \mapsto & \tilde{L}(t)-\lambda = L_+L_-, \\ \text{UL-map} : & L(t)-\lambda = L'_+L'_- & \mapsto & \tilde{L}'(t)-\lambda = L'_-L'_+ \end{array}$$

which in coordinates read:

$$\begin{array}{rcl} b_n-\lambda=\alpha_{n-1}+\beta_n & \longmapsto & \tilde{b}_n-\lambda=\alpha_n+\beta_n=(b_n-\lambda)+(\alpha_n-\alpha_{n-1}) \\ a_{n-1}=\alpha_{n-1}\beta_{n-1} & \longmapsto & \tilde{a}_{n-1}=\alpha_{n-1}\beta_n=a_{n-1}\dfrac{\beta_n}{\beta_{n-1}}. \end{array} \tag{0.5}$$

$$\begin{array}{rcl} b_n-\lambda=\alpha_{n-1}+\beta_n & \longmapsto & \tilde{b}'_n-\lambda=\alpha_{n-1}+\beta_{n-1} \\ a_{n-1}=\alpha_{n-1}\beta_{n-1} & \longmapsto & \tilde{a}'_{n-1}=\alpha_{n-2}\beta_{n-1}. \end{array} \tag{0.6}$$

This map will be called *Toda-Darboux*, when both matrices $L-\lambda=L_-L_+$, and $\tilde{L}-\lambda=L_+L_-$ flow according to all Toda vector fields.

The following *vertex operators* will play an important role in this paper:

$$\mathbf{X}_1(t,\lambda):=\chi(\lambda)X(t,\lambda) \quad \text{and} \quad \mathbf{X}_2(t,\lambda):=\chi(\lambda^{-1})X(-t,\lambda)\Lambda, \tag{0.7}$$

where

$$X(t,\lambda):=e^{\sum_1^\infty t_i\lambda^i}e^{-\sum_1^\infty \frac{\lambda^{-i}}{i}\frac{\partial}{\partial t_i}}. \tag{0.8}$$

We also define a *discrete Wronskian* $\{\ ,\ \}$ on column vectors f and g

$$\{f,g\}:=(f_{n+1}g_n-f_ng_{n+1})_{n\in\mathbf{Z}}. \tag{0.9}$$

Theorem 0.1 *Each element of the 2-dimensional null-space*

$$\ker(L(t)-\lambda)=\left\{\Phi(t,\lambda)=\frac{\tilde{\tau}(t)}{\tau(t)}=\frac{\left(a\,\mathbf{X}_1(t,\lambda)+b\,e^{\sum t_i\lambda^i}\mathbf{X}_2(t,\lambda)\right)\tau(t)}{\tau(t)}\right\}, \tag{0.10}$$

where $\Phi(t,\lambda)$ *satisfies*

$$\frac{\partial\Phi}{\partial t_n}=(L^n)_+\Phi,$$

[2]Note $\tilde{L}'(t)-\lambda=L'_-L'_+=\Lambda^{-1}L_+L_-\Lambda=L_+L_-$ (with $i\mapsto i-1$)

specifies a factorization for all $t \in \mathbf{C}^\infty$*, depending on a free parameter* b/a*:*

$$L(t) - \lambda = L_- L_+ = (\Lambda^{-1}\alpha + \Lambda^0)(\beta\Lambda^0 + \Lambda),$$

with

$$\alpha_n = \frac{\partial}{\partial t_1} \log \Phi_{n+1}(t,\lambda) - \lambda \text{ and } \beta_n = -\frac{\Phi_{n+1}(t,\lambda)}{\Phi_n(t,\lambda)}, \tag{0.11}$$

and so

$$\ker L_+ = \mathbf{C}\Phi(t,\lambda).$$

The LU-Toda-Darboux transform (0.5) maps $L(t) - \lambda$ *into a new tridiagonal matrix* $\tilde{L}(t) - \lambda$*, with entries* $\tilde{b}_n$ *and* $\tilde{a}_n$ *given by (0.3) in term of* $\tilde{\tau}$*. The LU-Toda-Darboux transform on* L *induces a map on* τ*:*

$$\tau \longmapsto \tilde{\tau} = \tau\Phi = \left(a\mathbf{X}_1(t,\lambda) + b\, e^{\sum t_i\lambda^i}\mathbf{X}_2(t,\lambda)\right)\tau(t), \tag{0.12}$$

whereas the UL-Darboux transform (0.5) does the same with entries $\tilde{b}'_n = \tilde{b}_{n-1}$ *and* $\tilde{a}'_n = \tilde{a}_{n-1}$*. Thus the UL-Toda-Darboux transform on* L *induces a map on* τ*:*

$$\tau \longmapsto \tilde{\tau} = \tau\Phi = \Lambda^{-1}\left(a\mathbf{X}_1(t,\lambda) + b\, e^{\sum t_i\lambda^i}\mathbf{X}_2(t,\lambda)\right)\tau(t). \tag{0.13}$$

Remark: In the bi-infinite case, the LU or UL-factorizations and Darboux transforms are given by the same formulae, except for a backwards shift, as seen above. However, when the matrix is semi-infinite, Theorem 0.1 applies as such with $b = 0$ for LU-Darboux and with a and b arbitrary for UL-Darboux.

The kernel $\ker(L(t) - \lambda)$, as in (0.10), contains two distinguished (wave) functions, whose asymptotics is given later in (2.5),

$$\Phi^{(1)}(t,\lambda) := \frac{\mathbf{X}_1(t,\lambda)\tau(t)}{\tau(t)} \text{ and } \Phi^{(2)}(t,\lambda) := \frac{e^{\sum t_i\lambda^i}\mathbf{X}_2(t,\lambda)\tau(t)}{\tau(t)} \tag{0.14}$$

which Darboux transform, as follows:

Theorem 0.2 *The wave functions $\Phi^{(1)}$ and $\Phi^{(2)}$ for the L-operator are Darboux transformed into wave functions $\tilde{\Phi}^{(1)}$ and $\tilde{\Phi}^{(2)}$ for the Darboux transformed operator $\tilde{L}$; they are given by Wronskian formulas, also expressible in terms of the new τ-function $\tilde{\tau}$, to wit*[3]

$$\begin{aligned}\tilde{\Phi}^{(1)}(t,z) &= \frac{\mathbf{X}_1(t,z)\tilde{\tau}}{\tilde{\tau}} = \frac{1}{z}\frac{\{\Phi^{(1)}(t,z),\Phi(t,\lambda)\}}{\Phi(t,\lambda)} = \frac{1}{z}L_+\Phi^{(1)}(t,z)\\ \tilde{\Phi}^{(2)}(t,z) &= \frac{e^{\xi(t,z)}\mathbf{X}_2(t,z)\tilde{\tau}}{\tilde{\tau}} = \frac{z}{\lambda-z}\frac{\{\Phi^{(2)}(t,z),\Phi(t,\lambda)\}}{\Phi(t,\lambda)} = \frac{z}{\lambda-z}L_+\Phi^{(2)}(t,z),\end{aligned}$$

thus satisfying

$$\tilde{L}\tilde{\Phi}^{(i)} = z\tilde{\Phi}^{(i)} \text{ and } \frac{\partial\tilde{\Phi}^{(i)}}{\partial t_n} = (\tilde{L}^n)_+\tilde{\Phi}^{(i)}, \quad i=1,2.$$

Theorem 0.3 *The following holds:*

$$\text{Toda for } L \text{ and } \tilde{L} \Longleftrightarrow \text{KM-lattice for } L_- \text{ and } L_+.$$

In coordinates, the first flow of the KM-lattice takes the form

$$\dot{\alpha}_n = (\beta_{n+1}-\beta_n)\alpha_n, \quad \dot{\beta}_n = (\alpha_n - \alpha_{n-1})\beta_n; \tag{0.15}$$

moreover α_n and β_n satisfy the Ricatti equations, with coefficients given by the entries of L,

$$\begin{aligned}\dot{\alpha}_n &= -\alpha_n^2 + (b_{n+1}-\lambda)\alpha_n - a_n\\ \dot{\beta}_n &= \beta_n^2 - (b_n-\lambda)\beta_n + a_n.\end{aligned}$$

in well known analogy with the Sturm-Liouville situation.

[3]set $\xi(t,z) = \sum_1^\infty t_i z^i$.

Consider $2p+1$-*band matrices* of the form

$$\begin{aligned} L &= \sum_{-p\leq i\leq p} a_i \Lambda^i \\ &= \left(\begin{array}{cccc|cccc} \ddots & & \ddots & & \ddots & & \ddots & O \\ a_{-p+1}(-1) & \ldots & & a_0(-1) & a_1(-1) & \ldots & 1 & \\ \hline a_{-p}(0) & & \ldots & a_{-1}(0) & a_0(0) & \ldots & a_{p-1}(0) & 1 \\ 0 & & \ddots & & & \ddots & \ddots & \ddots \end{array}\right) \end{aligned} \tag{0.16}$$

with a_i being diagonal matrices and $a_p = I$; define p-reduced Toda lattice vector fields, as follows:

$$\frac{\partial L}{\partial x_i} = [(\overline{L^{i/p}})_+, L], \quad \frac{\partial L}{\partial y_i} = [(\underline{L^{i/p}})_-, L], \text{ for } i = 1, 2, \ldots, p \not| i$$
$$\frac{\partial L}{\partial t_{ip}} = [(L^i)_+, L], \quad i = 1, 2, \ldots . \tag{0.17}$$

Note $\overline{L^{i/p}}$ and $\underline{L^{i/p}}$ involve *right* p^{th} roots and *left* p^{th} roots:

$$\begin{aligned} \overline{L^{i/p}} &= (\overline{L^{1/p}})^i = \left(\Lambda + \sum_{k\leq 0} b_k \Lambda^k\right)^i \\ \underline{L^{i/p}} &= (\underline{L^{1/p}})^i = \left(c_{-1}\Lambda^{-1} + \sum_{k\geq 0} c_k \Lambda^k\right)^i; \end{aligned} \tag{0.18}$$

with this notation, the vector fields (0.16) preserve the band structure of L.

Then L can be expressed in terms of a string of τ-functions,

$$\tau_n := \tau_n(\hat{x}, \hat{y}, \hat{t}), \tag{0.19}$$

with $\hat{x}$, $\hat{y}$, and $\hat{t}$ having certain components omitted:

$$\hat{x} = (x_1, \ldots, \hat{x}_p, \ldots, \hat{x}_{2p}, \ldots), \quad \hat{y} = (y_1, \ldots, \hat{y}_p, \ldots, \hat{y}_{2p}, \ldots), \quad \hat{t} = (t_p, t_{2p}, t_{3p}, \ldots). \tag{0.20}$$

Define the following vertex operators:

$$\begin{aligned} \mathbf{X}_1(\lambda) &= \chi(\lambda) e^{\sum_1^\infty t_{pi}\lambda^{pi}} e^{-\sum_1^\infty \frac{\lambda^{-pi}}{pi}\frac{\partial}{\partial t_{pi}}} e^{\sum_{p\not| i} x_i \lambda^i} e^{-\sum_{p\not| i} \frac{\lambda^{-i}}{i}\frac{\partial}{\partial x_i}} \\ \mathbf{X}_2(\lambda) &= \chi(\lambda^{-1}) e^{-\sum_1^\infty t_{pi}\lambda^{pi}} e^{\sum_1^\infty \frac{\lambda^{-pi}}{pi}\frac{\partial}{\partial t_{pi}}} e^{\sum_{p\not| i} y_i \lambda^i} e^{-\sum_{p\not| i} \frac{\lambda^{-i}}{i}\frac{\partial}{\partial y_i}} \Lambda. \end{aligned}$$

Theorem 0.4 *Each element of the 2p-dimensional null-space*[4]

$$\ker(L-\lambda^p)=\left\{\Phi(\lambda)=\frac{\tilde\tau}{\tau}=\frac{\sum_{k=0}^{p-1}\left(a_k\mathbf{X}_1(\omega^k\lambda)+b_k e^{\sum_1^\infty t_{ip}\lambda^{ip}}\mathbf{X}_2(\omega^k\lambda)\right)\tau}{\tau}\right\},$$

where $\Phi(\lambda)$ *satisfies*

$$L\Phi=\lambda^p\Phi$$

$$\frac{\partial\Phi}{\partial x_i}=(L^{i/p})_+\Phi,\quad \frac{\partial\Phi}{\partial y_i}=(L^{i/p})_-\Phi,\quad \text{for } i=1,2,\ldots \text{ with } p\not| i$$

$$\frac{\partial\Phi}{\partial t_{ip}}=(L^i)_+\Phi,\quad \text{for } i=1,2,\ldots,\tag{0.21}$$

determines a Toda-Darboux transform (depending on $2p-1$ *parameters* a_i *and* b_i*)*

$$L-\lambda^pI\longmapsto(\beta\Lambda^0+\Lambda)(L-\lambda^pI)(\beta\Lambda^0+\Lambda)^{-1}$$

with

$$\beta_n=-\frac{\Phi_{n+1}(\lambda)}{\Phi_n(\lambda)};$$

it acts on τ *as*

$$\tau\longmapsto\tilde\tau=\tau\Phi=\sum_{k=0}^{p-1}\left(a_k\mathbf{X}_1(\omega^k\lambda)+b_k e^{\sum_1^\infty t_{ip}\lambda^{ip}}\mathbf{X}_2(\omega^k\lambda)\right)\tau.$$

Remark: The map

$$L-\lambda^pI\longmapsto(\Lambda^{-1}\beta+I)(L-\lambda^pI)(\Lambda^{-1}\beta+I)^{-1}$$

acts on τ as

$$\tau\longmapsto\tilde\tau=\Lambda^{-1}(\tau\Phi).$$

For a broad account of Darboux transforms in a variety of situations, see the book of Matveev and Salle [11], which also contains a very extensive bibliography. Darboux transforms for differential operators, and their connections with τ-functions have been investigated in [9, 2, 14] for 2nd order

[4] ω is a primitive pth root of unity.

differential operators; in [2], we used the Darboux transform in order to regularize differential operators near their blow-up locus. Bakalov, Horozov and Yakimov [7] studied Darboux transforms for general differential operators. In fact, using our vertex operator methods (see [2]), the results of [7] can be made quite a bit more precise. The connection with the KM-lattice was first made in [1].

1 The 2-Toda lattice

Letting P_+ and P_- denote the upper (including diagonal) and strictly lower triangular parts of the matrix P, the two-dimensional Toda lattice equations read

$$\frac{\partial L_i}{\partial x_n} = [(L_1^n)_+, L_i] \quad \text{and} \quad \frac{\partial L_i}{\partial y_n} = [(L_2^n)_-, L_i] \quad n = 1, 2, \ldots; \tag{1.1}$$

they are deformations of a pair of infinite matrices

$$L = (L_1, L_2) = \left(\sum_{-\infty < i \leq 1} a_i^{(1)} \Lambda^i, \sum_{-1 \leq i < \infty} a_i^{(2)} \Lambda^i \right) \tag{1.2}$$

where $a_i^{(1)}$ and $a_i^{(2)}$ are diagonal matrices depending on $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, such that

$$a_1^{(1)} = I \quad \text{and} \quad (a_{-1}^{(2)})_{nn} \neq 0 \quad \text{for all } n. \tag{1.3}$$

In their 2-Toda theory, Ueno-Takasaki [13] also introduce a pair of wave vectors $\Psi = (\Psi_1, \Psi_2)$, satisfying $(L_1, L_2)\Psi = (z, z^{-1})\Psi$ and[5]

$$\begin{cases} \frac{\partial}{\partial x_n} \Psi = ((L_1^n)_+, (L_1^n)_+)\Psi \\ \frac{\partial}{\partial y_n} \Psi = ((L_2^n)_-, (L_2^n)_-)\Psi \end{cases} \tag{1.4}$$

They show that the wave vectors Ψ can be expressed in terms of one sequence of τ-functions $\tau(n, x, y) = \tau_n(x_1, x_2, \ldots; y_1, y_2, \ldots)$, $\quad n \in \mathbf{Z}$, to wit:

$$\Psi_1(x, y; z) = \left(\frac{\tau_n(x - [z^{-1}], y)}{\tau_n(x, y)} e^{\sum_1^\infty x_i z^i} z^n \right)_{n \in \mathbf{Z}}, \tag{1.5}$$

[5]Here the action is viewed componentwise, e.g., $(A, B)\Psi = (A\Psi_1, B\Psi_2)$ or $(z, z^{-1})\Psi = (z\Psi_1, z^{-1}\Psi_2)$.

$$\Psi_2(x,y;z) = \left(\frac{\tau_{n+1}(x,y-[z])}{\tau_n(x,y)} e^{\sum_1^\infty y_i z^{-i}} z^n\right)_{n\in\mathbf{Z}}, \tag{1.6}$$

with τ satisfying the following bilinear identities:

$$\oint_{z=\infty} \tau_n(x-[z^{-1}],y)\tau_{m+1}(x'+[z^{-1}],y') e^{\sum_1^\infty (x_i-x_i')z^i} z^{n-m-1} dz$$

$$= \oint_{z=0} \tau_{n+1}(x,y-[z])\tau_m(x',y'+[z]) e^{\sum_1^\infty (y_i-y_i')z^{-i}} z^{n-m-1} dz. \tag{1.7}$$

for all $m,n \in \mathbf{Z}$. Conversely, any τ-vector satisfying the bilinear identity (1.7) leads to a solution of the 2-Toda lattice; see [12].

Upon introducing appropriate shifts of x and y, and evaluating the contour integration over a contour about ∞ and the singularities, created by the shifts, one finds the following Fay identities, due to [4]; they will be useful later:

$$\tau_n(x-[z^{-1}],y+[v]-[u])\tau_n(x,y) - \tau_n(x,y+[v]-[u])\tau_n(x-[z^{-1}],y)$$

$$= \frac{v-u}{z}\tau_{n+1}(x,y-[u])\tau_{n-1}(x-[z^{-1}],y+[v]) \tag{1.8}$$

and

$$\tau_n(x,y+[v_1])\tau_{n+1}(x+[z_1^{-1}]-[z_2^{-1}],y-[v_2])\frac{z_1^{-1}}{z_1^{-1}-z_2^{-1}}$$

$$+\tau_n(x+[z_1^{-1}]-[z_2^{-1}],y+[v_1])\tau_{n+1}(x,y-[v_2])\frac{z_2^{-1}}{z_2^{-1}-z_1^{-1}}$$

$$= \tau_{n+1}(x+[z_1^{-1}],y)\tau_n(x-[z_2^{-1}],y+[v_1]-[v_2])\frac{v_1}{v_1-v_2}$$

$$+\tau_{n+1}(x+[z_1^{-1}],y+[v_1]-[v_2])\tau_n(x-[z_2^{-1}],y)\frac{v_2}{v_2-v_1}. \tag{1.9}$$

Consider the following 2-Toda vertex operators[6]

$$\mathbf{Y}_1(x,\lambda) = \chi(\lambda)X(x,\lambda) \quad \text{and} \quad \mathbf{Y}_2(y,\mu) = \chi(\mu)X(y,\mu^{-1})\Lambda,$$

where $X(x,\lambda)$ is given by (0.8). In [4], we have shown that for fixed $\lambda,\mu \in \mathbf{C}$, the vertex operator

$$a\mathbf{Y}_1(\lambda) + b\mathbf{Y}_2(\mu)$$

maps 2-Toda τ-vectors into themselves. Spelled out,

[6] $\chi(\lambda) = (\lambda^n)_{n\in\mathbf{Z}}$

$$\left(\left(a\mathbf{Y}_1(\lambda)+b\mathbf{Y}_2(\mu)\right)\tau\right)_n$$

$$= ae^{\sum_1^\infty x_i\lambda^i}\lambda^n\tau_n(x-[\lambda^{-1}],y)+be^{\sum_1^\infty y_i\mu^{-i}}\mu^n\tau_{n+1}(x,y-[\mu]) \tag{1.10}$$

is a new τ-vector for the 2-Toda lattice.

2 Reduction from 2- to standard Toda, and vertex operators

In the notation of section 1, consider the locus of (L_1, L_2)'s such that $L_1 = L_2$. From the equations (1.1), it follows that along that locus

$$\frac{\partial(L_1-L_2)}{\partial x_n}=0 \qquad \frac{\partial(L_1-L_2)}{\partial y_n}=0, \tag{2.1}$$

implying that the vector fields $\frac{\partial}{\partial x_n}$ and $\frac{\partial}{\partial y_n}$ are tangent to $\mathcal{D}_u$. Also when $L_1 = L_2$, the matrix $L := L_1 = L_2$ is tridiagonal. Moreover

$$\left(\frac{\partial}{\partial x_n}+\frac{\partial}{\partial y_n}\right)L_1=[(L_1^n)_+ +(L_2^n)_-,L_1]=[(L_1^n)_+ +(L_1^n)_-,L_1]=0; \tag{2.2}$$

setting

$$x_n=\frac{t_n+s_n}{2}, \qquad y_n=\frac{-t_n+s_n}{2} \qquad \text{and} \qquad t_n=x_n-y_n, \qquad s_n=x_n+y_n, \tag{2.3}$$

with

$$\frac{\partial}{\partial t_n}=\frac{1}{2}\left(\frac{\partial}{\partial x_n}-\frac{\partial}{\partial y_n}\right), \qquad \frac{\partial}{\partial s_n}=\frac{1}{2}\left(\frac{\partial}{\partial x_n}+\frac{\partial}{\partial y_n}\right), \tag{2.4}$$

equation (2.1) implies that $L = L_1 = L_2$ is independent of s; i.e., $L(x, y) = L(x - y)$ and so $\tau(x, y) = \tau(x - y)$. The converse is true as well, namely $(L_1, L_2)(x, y) = (L_1, L_2)(x - y)$ implies both $\tau(x, y) = \tau(x - y)$ and $L_1 = L_2$, as is seen by studying the vector field on $M = S_1^{-1}S_2$, where S_1 and S_2 are the wave operators; see [5].

The 2-Toda lattice wave functions Ψ_1 and Ψ_2, properly reduced, yield two distinguished eigenfunctions for the 1-Toda lattice; they have the following expressions in terms of the 1-Toda τ-functions:

$$\begin{aligned}\Phi^{(1)}(t,z) &:= \Psi_1(x,y;z)e^{-\sum_1^\infty y_i z^i} \\ &= e^{\sum_1^\infty t_i z^i}\left(z^n \frac{\tau_n(t-[z^{-1}])}{\tau_n(t)}\right)_{n\in\mathbf{Z}} \\ &= \frac{\mathbf{X}_1(z)\tau}{\tau} \\ &= e^{\sum_1^\infty t_i z^i}\left(z^n\left(1+O(z^{-1})\right)\right)_{n\in\mathbf{Z}} \end{aligned} \tag{2.5}$$

$$\begin{aligned}\Phi^{(2)}(t,z) &:= \Psi_2(x,y;z^{-1})e^{-\sum_1^\infty y_i z^i} \\ &= \left(z^{-n}\frac{\tau_{n+1}(t+[z^{-1}])}{\tau_n(t)}\right)_{n\in\mathbf{Z}} \\ &= \left(\frac{e^{\sum_1^\infty t_i z^i}\mathbf{X}_2(z)\tau}{\tau}\right)_{n\in\mathbf{Z}} \\ &= \left(z^{-n}\frac{\tau_{n+1}(t)}{\tau_n(t)}+O(z^{-1})\right)_{n\in\mathbf{Z}}, \end{aligned} \tag{2.6}$$

in terms of the vertex operators, already defined in (0.7), namely

$$\mathbf{X}_1(t,z)=\chi(z)X(t,z) \quad \text{and} \quad \mathbf{X}_2(t,z)=\chi(z^{-1})X(-t,z)\Lambda; \tag{2.7}$$

also, for later use, recall from (0.4),

$$\mathbf{X}(t,z):=\Lambda^{-1}\chi(z^2)e^{\sum t_i z^i}e^{-2\sum \frac{z^{-i}}{i}\frac{\partial}{\partial t_i}}; \tag{2.8}$$

Using (1.4), (2.2) and the fact that $L_1=L_2$, one checks

$$L\Phi^{(1)}=L_1\Phi^{(1)}=z\Phi^{(1)} \qquad L\Phi^{(2)}=L_2\Phi^{(2)}=z\Phi^{(2)}$$

and

$$\frac{\partial\Phi^{(1)}}{\partial t_n} = \frac{1}{2}\left(\frac{\partial}{\partial x_n}-\frac{\partial}{\partial y_n}\right)\Psi_1(x,y;z)e^{-\sum_1^\infty y_i z^i}=(L^n)_+\Phi^{(1)} \tag{2.9}$$

$$\frac{\partial \Phi^{(2)}}{\partial t_n} = \frac{1}{2}\left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial y_n}\right)\Psi_2(x,y;z^{-1})e^{-\sum y_i z^i} = (L^n)_+\Phi^{(2)}. \quad (2.10)$$

Therefore

$$\Phi(t,z) = a\Phi^{(1)} + b\Phi^{(2)} \quad (2.11)$$

is the most general solution to the following problem

$$L\Phi = z\Phi \quad \text{and} \quad \frac{\partial \Phi}{\partial t_n} = (L^n)_+\Phi, \quad (2.12)$$

with

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L]. \quad (2.13)$$

Proposition 2.1 *If* $(\tau_n)_{n\in\mathbf{Z}}$ *is a* τ*-vector for 1-Toda, then for arbitrary* $z \in \mathbf{C}^*$*,* $a, b \in \mathbf{C}$*, the vectors*[7]

$$\Big((a\mathbf{X}_1(t,z) + be^{\xi(t,z)}\mathbf{X}_2(t,z))\tau\Big)_{n\in\mathbf{Z}} \quad (2.14)$$

and

$$((1 + c\mathbf{X}(t,z))\tau)_{n\in\mathbf{Z}} \quad (2.15)$$

are 1-Toda vectors.

Remark: The vertex operators $\mathbf{X}_1(t,y)$ and $e^{\xi(t,z)}\mathbf{X}_2(t,z)$ satisfy the commutation relation

$$\frac{y}{1-y/z}\mathbf{X}_1(t,y)e^{\xi(t,z)}\mathbf{X}_2(t,z) = e^{\xi(t,z)}\mathbf{X}_2(t,z)\mathbf{X}_1(t,y). \quad (2.16)$$

Proof: According to the reduction above from 2- to 1-Toda, the tau-functions τ_n are independent of the sum of the arguments. Thus, we may write $\tau_n(x-y) = \tau_n(t)$ for $\tau_n(x,y)$. Set $\lambda = \mu^{-1} = z \in \mathbf{C}^*$ in (1.10); moreover, it is legitimate to multiply the 2-Toda τ-vector with an exponential in y_i with constant coefficients. By virtue of (1.7), it remains a 2-Toda τ-vector, which depends on t only; it is thus a 1-Toda τ-vector as is the following expression:

[7] $\xi(t,z) := \sum_1^\infty t_i z^i$.

$$
\begin{aligned}
& e^{-\sum_1^\infty y_i z^i}\left(\left(a\mathbf{Y}_1(z)+b\mathbf{Y}_2(z^{-1})\right)\tau\right)_n \\
&= e^{-\sum y_i z^i}\left(ae^{\sum_1^\infty x_i z^i}z^n\tau_n(x-y-[z^{-1}])+be^{\sum y_i z^i}z^{-n}\tau_{n+1}(x-y+[z^{-1}])\right) \\
&= ae^{\sum t_i z^i}z^n\tau_n(t-[z^{-1}])+bz^{-n}\tau_{n+1}(t+[z^{-1}]) \\
&= \left((a\mathbf{X}_1(z)+be^{\xi(z)}\mathbf{X}_2(z))\tau\right)_n .
\end{aligned}
$$

Applying the inverse of $e^{\xi(y)}\mathbf{X}_2(y)$ to the above 1-Toda τ-vector remains within that class; so taking a limit, when $y \to z$, leads to the 1-Toda τ-vector of (2.15), after noting that

$$
\lim_{y\to z}\left(e^{\xi(y)}\mathbf{X}_2(y)\right)^{-1}\left(e^{\xi(z)}\mathbf{X}_2(z)\right)=\lim_{y\to z}\Lambda^{-1}\chi(yz^{-1})e^{\sum\frac{z^{-i}-y^{-i}}{i}\frac{\partial}{\partial t_i}}\Lambda=I
$$

and

$$
\begin{aligned}
&\lim_{y\to z}\frac{1}{1-z/y}\left(e^{\xi(y)}\mathbf{X}_2(y)\right)^{-1}\mathbf{X}_1(z) \\
&\qquad = \lim_{y\to z}\frac{1}{1-z/y}\Lambda^{-1}\chi(y)\chi(z)e^{-\sum\frac{y^{-i}}{i}\frac{\partial}{\partial t_i}}e^{\sum t_i z^i}e^{-\sum\frac{z^{-i}}{i}\frac{\partial}{\partial t_i}} \\
&\qquad = \Lambda^{-1}\chi(z^2)e^{\sum t_i z^i}e^{-2\sum\frac{z^{-i}}{i}\frac{\partial}{\partial t_i}} \\
&\qquad = \mathbf{X}(t,z);
\end{aligned}
$$

here we used the commutation relation

$$
[e^{-\sum\frac{y^{-i}}{i}\frac{\partial}{\partial t_i}},e^{\sum t_i z^i}]=-\frac{z}{y}e^{\sum t_i z^i}e^{-\sum\frac{y^{-i}}{i}\frac{\partial}{\partial t_i}}.
$$

■

3 Toda-Darboux transformations

The purpose of this section is to establish theorem 0.1, which we restate in an explicit form. We shall use throughout the paper the following notation: $\tilde\Lambda^k$ shifts the immediate expression following the symbol only; i.e.

$$
(\tilde\Lambda^k x)=(x_{n+k})_{n\in\mathbf{Z}}\neq(x_{n+k})_{n\in\mathbf{Z}}\Lambda^k\ !
$$

Proposition 3.1 *For L evolving according to the Toda lattice equations (2.13), the Borel decomposition $L(t)-\lambda = L_-(t)L_+(t)$ is given for all $t \in \mathbf{C}^\infty$, by*

$$\alpha_n = \frac{\partial}{\partial t_1} \log \left(e^{-\sum t_i \lambda^i} \Phi_{n+1}(t,\lambda) \right) = \frac{\partial}{\partial t_1} \log \Phi_{n+1}(t,\lambda) - \lambda \tag{3.1}$$

$$\beta_n = -\frac{\partial}{\partial t_1} \log \left(\frac{\tau_n}{\tau_{n+1}} \Phi_n(t,\lambda) \right) = -\frac{\Phi_{n+1}(t,\lambda)}{\Phi_n(t,\lambda)} \tag{3.2}$$

in terms of the vector Φ, defined in (2.11), i.e.,

$$L-\lambda = L_-L_+ = \left(\Lambda^{-1}(\tilde{\Lambda} \frac{\partial}{\partial t_1} \log \Phi(t,\lambda) - \lambda) + \Lambda^0 \right) \left(-\left(\frac{\tilde{\Lambda}\Phi(t,\lambda)}{\Phi(t,\lambda)} \right) \Lambda^0 + \Lambda \right);$$

so, Φ belongs to the kernel of L_+, i.e.,

$$L_+\Phi = 0.$$

Finally the Darboux map

$$L(t) - \lambda = L_-(t)L_+(t) \mapsto \tilde{L}(t) - \lambda = L_+(t)L_-(t)$$

is Toda-Darboux, i.e., $\tilde{L}(t)$ satisfies the Toda lattice as well.

<u>*Proof*</u>: Let us begin by showing the second identity in (3.2), using (0.3) and (2.12):

$$\Phi_n \frac{\partial}{\partial t_1} \log \left(\frac{\tau_n}{\tau_{n+1}} \Phi_n(t,\lambda) \right) = -b_n \Phi_n + \frac{\partial \Phi_n}{\partial t_1} = -b_n \Phi_n + (L_+\Phi)_n = \Phi_{n+1}.$$

To establish the theorem, we must check, at first, that $L - \lambda = L_-L_+$, with the entries α_n and β_n, given by (3.1) and (3.2); secondly, we must check that $\tilde{L} - \lambda = L_+L_-$ evolves according to the Toda lattice.

To begin with, we verify the transformation (0.6): $b_n - \lambda = \alpha_{n-1} + \beta_n$ and $a_{n-1} = \alpha_{n-1}\beta_{n-1}$, with α_n, β_n, and b_n equal to (3.1),(3.2) and (0.3) respectively:

$$\alpha_{n-1} + \lambda + \beta_n - b_n = \frac{\partial}{\partial t_1} \left(\log \Phi_n(t,\lambda) - \log \frac{\tau_n}{\tau_{n+1}} \Phi_n(t,\lambda) - \log \frac{\tau_{n+1}}{\tau_n} \right) = 0$$

$$\begin{aligned}\Phi_{n-1}(a_{n-1}-\alpha_{n-1}\beta_{n-1}) &= \Phi_{n-1}\left(a_{n-1}+\left(\frac{\partial}{\partial t_1}\log\Phi_n-\lambda\right)\frac{\Phi_n}{\Phi_{n-1}}\right)\\ &= a_{n-1}\Phi_{n-1}+\frac{\partial\Phi_n}{\partial t_1}-\lambda\Phi_n\\ &= \left(\left(\frac{\partial}{\partial t_1}-L+L_-\right)\Phi\right)_n=0.\end{aligned}$$

Remember from (0.6), that the Darboux map

$$L-\lambda=L_-L_+\mapsto\tilde{L}-\lambda=L_+L_- \tag{3.3}$$

reads componentwise:

$$\begin{aligned}b_n-\lambda=\alpha_{n-1}+\beta_n &\longmapsto \tilde{b}_n-\lambda=\alpha_n+\beta_n=(b_n-\lambda)+(\alpha_n-\alpha_{n-1})\\ a_{n-1}=\alpha_{n-1}\beta_{n-1} &\longmapsto \tilde{a}_{n-1}=\alpha_{n-1}\beta_n=a_{n-1}\frac{\beta_n}{\beta_{n-1}}\end{aligned}$$

with α_n and β_n given by (3.2). So, in terms of τ_n and Φ_n, we have

$$\begin{aligned}\tilde{b}_n-\lambda &= (b_n-\lambda)+(\alpha_n-\alpha_{n-1})\\ &= \frac{\partial}{\partial t_1}\log\frac{\tau_{n+1}}{\tau_n}-\lambda+\frac{\partial}{\partial t_1}\log\frac{\Phi_{n+1}(t,\lambda)}{\Phi_n(t,\lambda)}\\ &= \frac{\partial}{\partial t_1}\log\frac{\tau_{n+1}\Phi_{n+1}(t,\lambda)}{\tau_n\Phi_n(t,\lambda)}-\lambda\\ &= \frac{\partial}{\partial t_1}\log\frac{\tilde{\tau}_{n+1}}{\tilde{\tau}_n}-\lambda\end{aligned}$$

$$\begin{aligned}\tilde{a}_{n-1} &= a_{n-1}\frac{\beta_n}{\beta_{n-1}}\\ &= \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}\cdot\frac{\Phi_{n-1}(t,\lambda)\Phi_{n+1}(t,\lambda)}{\Phi_n(t,\lambda)^2}\\ &= \frac{\tilde{\tau}_{n-1}\tilde{\tau}_{n+1}}{\tilde{\tau}_n^2},\end{aligned}$$

where

$$\tilde{\tau}=\tau(t)\Phi(t,\lambda)=\left(a\mathbf{X}_1(\lambda)+be^{\xi(\lambda)}\mathbf{X}_2(\lambda)\right)\tau$$

is a new τ-vector, by virtue of Lemma 2.1. Hence, the matrix $\tilde{L}$, parametrized by this new τ-vector $\tilde{\tau}$, satisfies the symmetric 1-Toda equations. ■

4 Expressing Wronskians in terms of vertex operators

The aim of this section is to prove Theorem 0.2. The Wronskian on vectors $f = (f_n)_{n\in\mathbf{Z}}$ was defined in (0.9), the vertex operators $\mathbf{X}_1(t,y)$ and $\mathbf{X}_2(t,z)$ in (0.7), and $\xi(t,z) = \xi(z) = \sum_1^\infty t_i z^i$. The following identities are based on the 2-Toda Fay identities (1.8) and (1.9).

Proposition 4.1 *The following identities hold:*

$$\begin{aligned}
\{\Phi^{(1)}(t,y), \Phi^{(1)}(t,z)\} &= y\frac{\mathbf{X}_1(y)\mathbf{X}_1(z)\tau}{\tau}\\
\{\Phi^{(1)}(t,y), \Phi^{(2)}(t,z)\} &= y\frac{\mathbf{X}_1(y)e^{\xi(z)}\mathbf{X}_2(z)\tau}{\tau} = \left(1-\frac{y}{z}\right)\frac{e^{\xi(z)}\mathbf{X}_2(z)\mathbf{X}_1(y)\tau}{\tau}\\
\{\Phi^{(2)}(t,y), \Phi^{(2)}(t,z)\} &= -\left(1-\frac{z}{y}\right)\frac{e^{\xi(y)}\mathbf{X}_2(y)e^{\xi(z)}\mathbf{X}_2(z)\tau}{\tau}.
\end{aligned}$$

<u>*Proof*</u>: Using, in the third equality, Fay identity (1.9) with the shift $x \mapsto x - [z_1^{-1}]$, and the limits $z_1 \to z$, $z_2 \to y$ and $v_1 = v_2 \to 0$,

$$\begin{aligned}
&\{\Phi_n^{(1)}(t,y), \Phi_n^{(1)}(t,z)\}\\
&= \frac{e^{\xi(y)+\xi(z)}}{\tau_n\tau_{n+1}}(yz)^n\\
&\qquad \Big(y\tau_{n+1}(t-[y^{-1}])\tau_n(t-[z^{-1}]) - z\tau_n(t-[y^{-1}])\tau_{n+1}(t-[z^{-1}])\Big)\\
&= \frac{e^{\xi(y)+\xi(z)}}{\tau_n\tau_{n+1}}(yz)^n(y-z)\tau_{n+1}\tau_n(t-[y^{-1}]-[z^{-1}])\\
&= e^{\xi(y)+\xi(z)}(y-z)(yz)^n\frac{\tau_n(t-[y^{-1}]-[z^{-1}])}{\tau_n(t)}\\
&= y\left(\frac{\mathbf{X}_1(y)\mathbf{X}_1(z)\tau}{\tau}\right)_n.
\end{aligned}$$

Using in the third equality Fay identity (1.8) with $z \to y$, $u \to z^{-1}$, $v \to 0$ and $n \mapsto n+1$:

$$\begin{aligned}
&\{\Phi_n^{(1)}(t,y),\Phi_n^{(2)}(t,z)\}\\
&= \frac{e^{\xi(y)}}{\tau_n\tau_{n+1}}\left(\frac{y}{z}\right)^n \\
&\qquad \Big(y\tau_{n+1}(t-[y^{-1}])\tau_{n+1}(t+[z^{-1}]) - z^{-1}\tau_n(t-[y^{-1}])\tau_{n+2}(t+[z^{-1}])\Big)\\
&= \frac{e^{\xi(y)}}{\tau_n\tau_{n+1}}\left(\frac{y}{z}\right)^n y\tau_{n+1}(t-[y^{-1}]+[z^{-1}])\tau_{n+1}\\
&= e^{\xi(y)}y\left(\frac{y}{z}\right)^n\frac{\tau_{n+1}(t-[y^{-1}]+[z^{-1}])}{\tau_n}\\
&= y\left(\frac{\mathbf{X}_1(y)e^{\xi(z)}\mathbf{X}_2(z)\tau}{\tau}\right)_n = \left(1-\frac{y}{z}\right)\left(\frac{e^{\xi(z)}\mathbf{X}_2(z)\mathbf{X}_1(y)\tau}{\tau}\right)_n,
\end{aligned}$$

using the commutation relation (2.16), in the last equality.

Using in the third equality the same identity as for the first wronskian, but with $t\mapsto t+[y^{-1}]+[z^{-1}]$ and $n\mapsto n+1$:

$$\begin{aligned}
&\{\Phi_n^{(2)}(t,y),\Phi_n^{(2)}(t,z)\}\\
&= \frac{1}{\tau_n\tau_{n+1}}(yz)^{-n-1}\\
&\qquad \Big(z\tau_{n+2}(t+[y^{-1}])\tau_{n+1}(t+[z^{-1}]) - y\tau_{n+1}(t+[y^{-1}])\tau_{n+2}(t+[z^{-1}])\Big)\\
&= -\frac{1}{\tau_n\tau_{n+1}}(yz)^{-n-1}(y-z)\tau_{n+2}(t+[y^{-1}]+[z^{-1}])\tau_{n+1}\\
&= -(y-z)(yz)^{-n-1}\frac{\tau_{n+2}(t+[y^{-1}]+[z^{-1}])}{\tau_n}\\
&= -\left(1-\frac{z}{y}\right)\left(\frac{e^{\xi(y)}\mathbf{X}_2(y)e^{\xi(z)}\mathbf{X}_2(z)\tau}{\tau}\right)_n
\end{aligned}$$

■

Proof of Theorem 0.2: Proposition 4.1 implies the following relations:

$$\{\Phi_n^{(1)}(t,z),\Phi_n(t,\lambda)\} = z\left(\frac{\mathbf{X}_1(z)(a\mathbf{X}_1(\lambda)+be^{\xi(\lambda)}\mathbf{X}_2(\lambda))\tau}{\tau}\right)_n = z\left(\frac{\mathbf{X}_1(z)\tilde{\tau}}{\tau}\right)_n,$$

and

$$\begin{aligned}\{\Phi_n^{(2)}(t,z),\Phi_n(t,\lambda)\} &= -(1-\frac{\lambda}{z})\left(\frac{e^{\xi(z)}\mathbf{X}_2(z)(a\mathbf{X}_1(\lambda)+be^{\xi(\lambda)}\mathbf{X}_2(\lambda))\tau}{\tau}\right)_n \\ &= \frac{\lambda-z}{z}\left(\frac{e^{\xi(z)}\mathbf{X}_2(z)\tilde{\tau}}{\tau}\right)_n,\end{aligned}$$

from which the proof follows.

■

5 Borel factorization, KM-lattice and Ricatti equations

This section concerns itself with the proof of Theorem 0.3. For a fixed $\lambda \in \mathbf{C}^*$, consider the Darboux map

$$L-\lambda = L_-L_+ \longmapsto \tilde{L}-\lambda = L_+L_-, \tag{5.1}$$

which in coordinates reads

$$\begin{aligned} b_n-\lambda = \alpha_{n-1}+\beta_n \quad &\longmapsto \quad \tilde{b}_n-\lambda = \alpha_n+\beta_n = (b_n-\lambda)+(\alpha_n-\alpha_{n-1}) \\ a_{n-1} = \alpha_{n-1}\beta_{n-1} \quad &\longmapsto \quad \tilde{a}_{n-1} = \alpha_{n-1}\beta_n = a_{n-1}\frac{\beta_n}{\beta_{n-1}}. \end{aligned} \tag{5.2}$$

Remember the first Toda flow ($=\partial/\partial t_1 = {}'$)

$$b_n' = a_n - a_{n-1}, \quad a_n' = a_n(b_{n+1}-b_n) \tag{5.3}$$

and the first flow of the Kac-Moerbeke lattice (see [10]), refered to with $\dot{}$:

$$\dot{\alpha}_n = (\beta_{n+1}-\beta_n)\alpha_n, \quad \dot{\beta}_n = (\alpha_n-\alpha_{n-1})\beta_n. \tag{5.4}$$

Proposition 5.1 *Toda for L and $\tilde{L}$* $\Longleftrightarrow$ KM *for L_- and L_+.*

<u>*Proof*</u>: Assuming the KM-vector field on (α,β), one computes

$$(b_n-\lambda)^{\cdot} = \dot{\alpha}_{n-1}+\dot{\beta}_n = \alpha_n\beta_n - \alpha_{n-1}\beta_{n-1} = a_n - a_{n-1} = (b_n-\lambda)' \tag{5.5}$$

$$\dot{a}_n = \dot{\alpha}_n\beta_n + \alpha_n\dot{\beta}_n = \alpha_n\beta_n(\beta_{n+1} - \beta_n + \alpha_n - \alpha_{n-1}) = a_n(b_{n+1} - b_n) = a'_n \quad (5.6)$$

and

$$(\tilde{b}_n - \lambda)^{\cdot} = \dot{\alpha}_n + \dot{\beta}_n = \alpha_n\beta_{n+1} - \alpha_{n-1}\beta_n = \tilde{a}_n - \tilde{a}_{n-1} = (\tilde{b}_n - \lambda)' \quad (5.7)$$

$$\dot{\tilde{a}}_n = \dot{\alpha}_n\beta_{n+1} + \alpha_n\dot{\beta}_{n+1} = \alpha_n\beta_{n+1}(\beta_{n+1} - \beta_n + \alpha_{n+1} - \alpha_n) = \tilde{a}_n(\tilde{b}_{n+1} - \tilde{b}_n) = \tilde{a}'_n. \quad (5.8)$$

Conversely, assuming Toda on L and $\tilde{L}$ leads to the following four equations on α and β; the first two are (5.5) and (5.8), with a shift, the last two are (5.6) and (5.7):

$$\begin{cases} \alpha'_n + \beta'_{n+1} & = \alpha_{n+1}\beta_{n+1} - \alpha_n\beta_n \\ \\ \alpha'_n\beta_{n+1} + \alpha_n\beta'_{n+1} & = \alpha_n\beta_{n+1}(\beta_{n+1} - \beta_n + \alpha_{n+1} - \alpha_n) \end{cases}$$

and

$$\begin{cases} \alpha'_n\beta_n + \alpha_n\beta'_n & = \alpha_n\beta_n(\beta_{n+1} - \beta_n + \alpha_n - \alpha_{n-1}) \\ \\ \alpha'_n + \beta'_n & = \alpha_n\beta_{n+1} - \alpha_n\beta_n. \end{cases}$$

Solving the first system in α'_n and β'_{n+1} or the second in α'_n and β'_n leads to the KM-lattice equations (5.4); so, now we may replace the differentiation $\cdot$ by $'$. ∎

The KM-lattice equations (5.4)

$$\alpha'_n = (\beta_{n+1} - \beta_n)\alpha_n \quad \text{and} \quad \beta'_n = (\alpha_n - \alpha_{n-1})\beta_n,$$

upon using $\beta_n = \frac{a_n}{\alpha_n}$, $\beta_{n+1} = b_{n+1} - \alpha_n - \lambda$ in the first equation and $\alpha_n = \frac{a_n}{\beta_n}$ and $\alpha_{n-1} = b_n - \lambda - \beta_n$ in the second equation, yield

$$\begin{array}{rcl} \alpha'_n & = & -\alpha_n^2 + (b_{n+1} - \lambda)\alpha_n - a_n \\ \beta'_n & = & \beta_n^2 - (b_n - \lambda)\beta_n + a_n \qquad \text{(Ricatti equations);} \end{array} \quad (5.9)$$

the transformations

$$\alpha_n = \frac{\partial}{\partial t_1}\log\gamma_n \quad \text{and} \quad \beta_n = -\frac{\partial}{\partial t_1}\log\varepsilon_n \quad (5.10)$$

yield the second order linear equations

$$\gamma_n'' - (b_{n+1} - \lambda)\gamma_n' + a_n\gamma_n = 0 \tag{5.11}$$

$$\varepsilon_n'' + (b_n - \lambda)\varepsilon_n' + a_n\varepsilon_n = 0. \tag{5.12}$$

Proposition 5.2 *The expressions*

$$\alpha_n = \frac{\partial}{\partial t_1}\log\gamma_n := \frac{\partial}{\partial t_1}\log\left(e^{-\sum t_i\lambda^i}\Phi_{n+1}(t,\lambda)\right) \tag{5.13}$$

$$\beta_n = -\frac{\partial}{\partial t_1}\log\varepsilon_n := -\frac{\partial}{\partial t_1}\log\left(\frac{\tau_n}{\tau_{n+1}}\Phi_n(t,\lambda)\right), \tag{5.14}$$

given by the 2-dimensional family $\Phi(t,z) = a\Phi^{(1)} + b\Phi^{(2)}$*, provide the most general solution to the Ricatti equations (5.9).*

Proof: Since the parametrization of α_n and β_n in Proposition 3.1 provide a Toda-Darboux transformation, then, by proposition 5.1, α_n and β_n satisfy the KM-Lattice, and hence provide a 2-dimensional solution to the two Ricatti equations (5.9).

An *alternate proof* not using τ-function theory proceeds as follows: The Ricatti equations (5.9) are equivalent via the transformation (5.2) and the Toda vector fields (5.3). We check, for instance, that ε_n satisfies (5.12); at first, we compute:

$$\Phi_n' = (L_+\Phi)_n = b_n\Phi_n + \Phi_{n+1} = \lambda\Phi_n - a_{n-1}\Phi_{n-1}, \text{ using } L\Phi = \lambda\Phi. \tag{5.15}$$

Using the first expression for Φ_n' and Φ_{n-1}' and the Toda lattice equation for a_{n-1}', we then find

$$\begin{aligned}\Phi_n'' &= \lambda\Phi_n' - a_{n-1}'\Phi_{n-1} - a_{n-1}\Phi_{n-1}' \\ &= \lambda(b_n\Phi_n + \Phi_{n+1}) - a_{n-1}(b_n - b_{n-1})\Phi_{n-1} - a_{n-1}(b_{n-1}\Phi_{n-1} + \Phi_n) \\ &= (b_n + \lambda)\Phi_{n+1} + (b_n^2 - a_{n-1})\Phi_n, \text{ using } \lambda\Phi_n = (L\Phi)_n.\end{aligned} \tag{5.16}$$

Then, using

$$\frac{\tau_{n+1}}{\tau_n}\left(\frac{\tau_n}{\tau_{n+1}}\right)' = -b_n \quad \text{and} \quad \frac{\tau_{n+1}}{\tau_n}\left(\frac{\tau_n}{\tau_{n+1}}\right)'' = -b_n' + b_n^2 = a_{n-1} - a_n + b_n^2, \tag{5.17}$$

we find

$$\begin{aligned}
&\frac{\tau_{n+1}}{\tau_n}(\varepsilon_n'' + (b_n - \lambda)\varepsilon_n' + a_n\varepsilon_n) \\
&\quad = \Phi_n'' - 2b_n\Phi_n' + (-b_n' + b_n^2)\Phi_n + (b_n - \lambda)(\Phi_n' - b_n\Phi_n) + a_n\Phi_n \\
&\quad \stackrel{*}{=} \Phi_n'' - (b_n + \lambda)\Phi_{n+1} + (-b_n^2 + a_{n-1})\Phi_n \\
&\quad = 0, \quad \text{by (5.18)}
\end{aligned}$$

in $\stackrel{*}{=}$, we have used (5.3), (5.17) and $\lambda\Phi = L\Phi$. ■

6 Toda flows and Toda-Darboux transforms for band matrices

In this section, we factorize band matrices of the form $L = \sum_{-p \leq i \leq p} a_i \Lambda^i$, as in (0.16), and study their Toda-Darboux transforms. The first Lemma will be stated for general band matrices; in its statement, we use the following typical difference operators:

$$I - \Lambda^{-1}\beta = \left(\begin{array}{c|cc}
\ddots & \ddots & \ddots \\
1 & 0 & 0 \\
\hline
-\beta(-1) & 1 & 0 \\
0 & -\beta(0) & 1 \\
\ddots & \ddots & \ddots
\end{array}\right)$$

$$\Lambda - \beta I = \left(\begin{array}{c|cc}
\ddots & \ddots & \ddots \\
-\beta(-1) & 1 & 0 \\
\hline
0 & -\beta(0) & 1 \\
0 & 0 & -\beta(1) \\
\ddots & \ddots & \ddots
\end{array}\right).$$

Lemma 6.1 *Consider the difference operator*

$$L = a_{-r}\Lambda^{-r} + a_{-r+1}\Lambda^{-r+1} + ... + a_{n-r-1}\Lambda^{n-r-1} + \Lambda^{n-r} \tag{6.1}$$

with $n \geq 2$, $r \geq 0$, diagonal matrices a_j, with leading term $a_{-r}(j) \neq 0$ for j sufficiently small. Then any choice of basis $\Phi^{(1)}, ...\Phi^{(n)} \in \ker L$ leads to a *factorization of*[8] *L:*

$$L = \left(I - \Lambda^{-1}(\tilde{\Lambda}^{-r+1}\beta_n)\right)\left(I - \Lambda^{-1}(\tilde{\Lambda}^{-r+2}\beta_{n-1})\right)\ldots\left(I - \Lambda^{-1}(\beta_{n-r+1})\right)\cdot$$
$$\cdot\,(\Lambda - \beta_{n-r}I)\,(\Lambda - \beta_{n-r-1}I)\ldots(\Lambda - \beta_1 I)\,, \tag{6.2}$$

with

$$\beta_k(\ell) = \frac{\alpha_k(\ell+1)}{\alpha_k(\ell)}, \quad \alpha_k(\ell) = \frac{W_k(\ell)}{W_{k-1}(\ell)}, \quad W_k(\ell) = W[\Phi^{(1)}, \Phi^{(2)}, ..., \Phi^{(k)}](\ell).$$

Proof: Step 1: $r = 0$

First we prove the statement for $r = 0$ by induction on the degree of L. Define the operator L_i

$$L_i(f) := \frac{W[\Phi^{(1)}, ..., \Phi^{(i)}, f]}{W[\Phi^{(1)}, ..., \Phi^{(i)}]},$$

which is the unique operator of the form (6.1) with $r = 0$, $n = i$ such that $\ker L_i = \{\Phi^{(1)}, ..., \Phi^{(i)}\}$. But clearly

$$L_{i+1}(f) = \frac{W[L_i(\Phi^{(i+1)}), L_i(f)]}{L_i(\Phi^{(i+1)})} = \left(\Lambda - \frac{\tilde{\Lambda}\alpha_i}{\alpha_i} I\right) L_i(f),$$

with $\alpha_i = L_i(\Phi^{(i+1)}) = W_{i+1}/W_i$. So, by induction $L_i(f)$ factors, thus leading to (7.2) for $r = 0$.

Step 2: $r \neq 0$

The case $r \neq 0$ is taken care of by multiplying (6.2) to the left with Λ^r:

$$\Lambda^r L = (\Lambda - \beta_n I)...(\Lambda - \beta_1 I),$$

on which you apply step 1. ■

In the next Lemma, we use definition (0.19) for $\hat{x}, \hat{y}, \hat{t}$; also define

$$\hat{x} - [\lambda^{-1}] := \left(x_i - \frac{\lambda^{-i}}{i}\right)_{i\neq p}, \hat{t} - [\lambda^{-1}] = \left(t_{ip} - \frac{\lambda^{-ip}}{ip}\right)_{i=1,2,...}.$$

[8] Wronskian $= W_k(\ell) = W[\Phi^{(1)}, \Phi^{(2)}, ..., \Phi^{(k)}](\ell) = \det\left(\Phi^{(j)}_{i+\ell-1}\right)_{1\leq i,j\leq k}$. Remember the notation $\tilde{\Lambda}^k$ from the beginning of section 3

Lemma 6.2 *On the locus $L := L_1^p = L_2^p$, the 2-Toda vector fields (1.1) on (L_1, L_2) take the form (0.17); the τ-functions are of the form (0.19) and the 2p-dimensional null-space* $\ker(L-\lambda^p)$ *is given by*

$$\Phi(\hat{x},\hat{y},\hat{t};\lambda) = \sum_{0\leq i\leq p-1} a_i \Phi^{(1)}(\omega^i\lambda) + \sum_{0\leq i\leq p-1} b_i \Phi^{(2)}(\omega^i\lambda), \tag{6.3}$$

flowing according to the equations (0.21); in (6.3), we use the expressions

$$\begin{aligned}\Phi^{(1)}(\hat{x},\hat{y},\hat{t};\lambda) &:= \Psi_1(x,y;\lambda)e^{-\sum_1^\infty y_{ip}\lambda^{ip}}\\ &= e^{\sum_{i\not| p} x_i\lambda^i} e^{\sum_{i=1}^\infty t_{ip}\lambda^{ip}} \left(\lambda^n \frac{\tau_n(\hat{x}-[\lambda^{-1}],\hat{y},\hat{t}-[\lambda^{-1}])}{\tau_n(\hat{x},\hat{y},\hat{t})}\right)_{n\in\mathbf{Z}}\\ &= \frac{\mathbf{X}_1(\lambda)\tau}{\tau}\end{aligned}$$

$$\begin{aligned}\Phi^{(2)}(\hat{x},\hat{y},\hat{t};\lambda) &:= \Psi_2(x,y;\lambda^{-1})e^{-\sum_1^\infty y_{ip}\lambda^{ip}}\\ &= e^{\sum_{i\not| p} y_i\lambda^i} \left(\lambda^{-n} \frac{\tau_{n+1}(\hat{x},\hat{y}-[\lambda^{-1}],\hat{t}+[\lambda^{-1}])}{\tau_n(\hat{x},\hat{y},\hat{t})}\right)_{n\in\mathbf{Z}}\\ &= \frac{e^{\sum_1^\infty t_{ip}\lambda^{ip}}\mathbf{X}_2(\lambda)\tau}{\tau}.\end{aligned}$$

Moreover

$$\tilde{\tau} = \tau\Phi = \sum_{k=0}^{p-1}\left(a_k\mathbf{X}_1(\omega^k\lambda) + b_k e^{\sum_1^\infty t_{ip}\lambda^{ip}}\mathbf{X}_2(\omega^k\lambda)\right)\tau \tag{6.4}$$

is a τ-vector, having the property that $L_1^p = L_2^p$. Proposition 4.1 holds for the above choice of $\Phi^{(1)}$, $\Phi^{(2)}$, $\mathbf{X}_1$ and $\mathbf{X}_2$ with $\xi(z) := \sum_1^\infty t_{ip}z^{ip}$.

Proof: The proof proceeds as in section 2; for instance, (2.2) gets replaced by

$$\frac{\partial(L_1^p - L_2^p)}{\partial x_n} = 0, \quad \frac{\partial(L_1^p - L_2^p)}{\partial y_n} = 0$$

and (2.3) by

$$\left(\frac{\partial}{\partial x_{np}} + \frac{\partial}{\partial y_{np}}\right) L_i = [(L_1^{np})_+ + (L_2^{np})_-, L_i] = [(L_i^{np})_+ + (L^{np})_-, L_i] = 0.$$

Thus L_i depends on $\hat{x}, \hat{y}, \hat{t}$ only and so does τ. Using the Toda flow on $M = S_1^{-1}S_2$ (see [5, 6]), one shows this characterizes 2-Toda on the locus $L_1^p = L_2^p$. That Φ satisfies the differential equations (0.21) proceeds along the same lines as (2.9) and (2.10). That $\tilde{\tau} = \tau\Phi$ is a 2-Toda τ-vector follows from the 2-Toda bilinear relations $B(\tau,\tau)$, as in (1.7). So, in order to check the vanishing of

$$0 = B(\tilde{\tau},\tilde{\tau}) = B(\sum_1^{2p} \tau_i, \sum_1^{2p} \tau_j) = \sum_{i,j} B(\tilde{\tau}_i, \tilde{\tau}_j),$$

it suffices to check

$$B(\tilde{\tau}_i,\tilde{\tau}_i) = 0 \quad \text{and} \quad B(\tilde{\tau}_i,\tilde{\tau}_j) + B(\tilde{\tau}_j,\tilde{\tau}_i) = 0$$

for all i, j. The first relation is obvious, since each τ_i is a τ-function. The second relation follows from the relation

$$B\left(a\mathbf{Y}_i(\lambda) + b\mathbf{Y}_j(\mu), a\mathbf{Y}_i(\lambda) + b\mathbf{Y}_j(\mu)\right) = 0 \quad \text{for } 1 \le i, j \le 2,$$

where the vertex operators are defined just prior to (1.10). This fact was shown in [4], using the four basic vertex operators $\mathbf{Y}_i^{-1}(\lambda)\mathbf{Y}_j(\mu)$ for $1 \le i, j \le 2$ for 2-Toda. Thus $\tilde{\tau}$ is a 2-Toda τ-vector depending on $\hat{x}, \hat{y}, \hat{t}$ and so is p-reduced. The proof of the very last statement invokes the 2-Toda Fay identities, just as in the proof of Proposition 4.1. ■

Proof of Theorem 0.4: Lemma 6.2 implies Theorem 0.4, except for the statement about the Darboux transform. For that, one needs to factor L according to the recipe of Lemma 6.1, where we set $\Phi = \Phi^{(1)}$, in the precise notation of Lemma 6.1. Thus the Darboux transformation corresponds to bringing the factor most to the right all the way to the left. Then we use the last part of Lemma 6.2 (analogue of Proposition 4.1) to prove the analogue of Theorem 0.2, with $\mathbf{X}_i(t,z)$ in the Theorem replaced by the expressions $\mathbf{X}_i(t,\omega^k z)$ for $i = 1,2$ of this section. This shows the Darboux transformed L has wave functions $\Phi^{(i)}(\hat{x},\hat{y},\hat{t};\omega^k z)$, specified by the p-reduced 2-Toda τ-vector $\tilde{\tau}(\hat{x},\hat{y},\hat{t})$. In turn, this specifies L, which thus satisfies the p-reduced 2-Toda equations. ■

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