Vertex operator solutions to the discrete KP-hierarchy*

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Contents

Vertex operators, which are disguised Darboux maps, transform solutions of the KP equation into new ones. In this paper, we show that the bi-infinite

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sequence obtained by Darboux transforming an arbitrary KP solution recursively forward and backwards, yields a solution to the *discrete KP-hierarchy*. The latter is a KP hierarchy where the continuous space x-variable gets replaced by a discrete n-variable. The fact that these sequences satisfy the discrete KP hierarchy is tantamount to certain bilinear relations connecting the consecutive KP solutions in the sequence. At the Grassmannian level, these relations are equivalent to a very simple fact, which is the nesting of the associated infinite-dimensional planes (flag). The discrete KP hierarchy can thus be viewed as a container for an entire ensemble of vertex or Darboux generated KP solutions.

It turns out that many new and old systems lead to such discrete (semi-infinite) solutions, like sequences of soliton solutions, with more and more solitons, sequences of Calogero-Moser systems, having more and more particles, just to mention a few examples; this is developped in [3]. In this paper, as an other example, we show that the *q-KP hierarchy* maps, via a kind of *Fourier transform*, into the discrete KP hierarchy, enabling us to write down a very large class of solutions to the q-KP hierarchy. This was also reported in a brief note with E. Horozov[4].

Given the shift operator $\Lambda = (\delta_{i,j-1})_{i,j\in\mathbf{Z}}$, consider the Lie algebra

$$\mathcal{D} = \left\{ \sum_{-\infty<i\ll\infty} a_i\Lambda^i, a_i \text{ diagonal operators} \right\} = \mathcal{D}_- + \mathcal{D}_+ \tag{0.1}$$

with the usual splitting $\mathcal{D} = \mathcal{D}_- + \mathcal{D}_+$, into subalgebras

$$\mathcal{D}_+ = \left\{ \sum_{0\leq i\ll\infty} a_i\Lambda^i \in \mathcal{D} \right\}, \mathcal{D}_- = \left\{ \sum_{-\infty<i<0} a_i\Lambda^i \in \mathcal{D} \right\}. \tag{0.2}$$

The discrete KP-hierarchy equations

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n = 1, 2, ... \tag{0.3}$$

are deformations of an infinite matrix

$$L = \sum_{-\infty<i\leq 0} a_i(t)\Lambda^i + \Lambda \in \mathcal{D}, \quad \text{with } t = (t_1, t_2, ...) \in \mathbf{C}^\infty. \tag{0.4}$$

If we represent L as a dressing up of Λ by a wave operator $S \in I + \mathcal{D}_-$

$$L = S\Lambda S^{-1} = W\Lambda W^{-1}, \quad W = Se^{\sum_1^\infty t_i\Lambda^i}, \tag{0.5}$$

then the L-deformations are induced by S-deformations and W-deformations:

$$\frac{\partial S}{\partial t_n} = -(L^n)_- S, \quad \frac{\partial W}{\partial t_n} = (L^n)_+ W, \quad n = 1, 2, ...; \tag{0.6}$$

In terms of vectors

$$\chi(z) = (z^n)_{n\in\mathbf{Z}}, \qquad \chi^*(z) = \chi(z^{-1}), \tag{0.7}$$

such that $z\chi(z) = \Lambda\chi(z), \quad z\chi^*(z) = \Lambda^\top\chi^*(z)$, let us define wave and adjoint wave vectors $\Psi(t,z)$ and $\Psi^*(t,z)$

$$\Psi(t,z) = W\chi(z) \text{ and } \Psi^*(t,z) = (W^{-1})^\top\chi^*(z). \tag{0.8}$$

We find, using (0.5), (0.8), (0.6), that

$$\begin{aligned} L\Psi(t,z) &= z\Psi(t,z) \qquad & L^\top\Psi^*(t,z) &= z\Psi^*(t,z), \\ \frac{\partial\Psi}{\partial t_n} &= (L^n)_+\Psi \qquad & \frac{\partial\Psi^*}{\partial t_n} &= -((L^n)_+)^\top\Psi^*. \end{aligned} \tag{0.9}$$

Theorem 0.1 *If L satisfies the Toda lattice, then the wave vectors $\Psi(t,z)$ and $\Psi^*(t,z)$ can be expressed in terms of one sequence of τ-functions $\tau(n,t) := \tau_n(t_1, t_2, \ldots), \quad n \in \mathbf{Z}$, to wit:*

$$\begin{aligned} \Psi(t,z) &= \left(e^{\sum_1^\infty t_iz^i}\psi(t,z)\right)_{n\in\mathbf{Z}} = \left(\frac{\tau_n(t-[z^{-1}])}{\tau_n(t)}e^{\sum_1^\infty t_iz^i}z^n\right)_{n\in\mathbf{Z}}, \\ \Psi^*(t,z) &= \left(e^{-\sum_1^\infty t_iz^i}\psi^*(t,z)\right)_{n\in\mathbf{Z}} = \left(\frac{\tau_{n+1}(t+[z^{-1}])}{\tau_{n+1}(t)}e^{-\sum_1^\infty t_iz^i}z^{-n}\right)_{n\in\mathbf{Z}}, \end{aligned} \tag{0.10}$$

satisfying the bilinear identity

$$\oint_{z=\infty}\Psi_n(t,z)\Psi_m^*(t',z)\frac{dz}{2\pi iz} = 0 \tag{0.11}$$

for all $n > m$. It follows that

$$\Psi = W\chi(z) = e^{\sum_1^\infty t_iz^i}S\chi(z),$$

$$\Psi^* = \left(W^\top\right)^{-1}\chi^*(z) = e^{-\sum_1^\infty t_i z^i}(S^{-1})^\top \chi^*(z),$$

with[1]

$$S = \sum_0^\infty \frac{p_n(-\tilde\partial)\tau(t)}{\tau(t)}\Lambda^{-n} \quad \textit{and} \quad S^{-1} = \sum_0^\infty \Lambda^{-n}\,\tilde\Lambda\left(\frac{p_n(\tilde\partial)\tau(t)}{\tau(t)}\right). \tag{0.12}$$

Then L^k has the following expression in terms of τ-functions[2],

$$L^k = \sum_{\ell=0}^\infty \textit{diag}\,\left(\frac{p_\ell(\tilde\partial)\tau_{n+k-\ell+1}\circ\tau_n}{\tau_{n+k-\ell+1}\tau_n}\right)_{n\in\mathbf{Z}}\Lambda^{k-\ell} \tag{0.13}$$

with the τ_n's satisfying

$$\left(\frac{\partial}{\partial t_k} - \sum_{r=0}^{\ell-1}(\ell-r)p_r(-\tilde\partial)p_{k-r}(\tilde\partial)\right)\tau_n\circ\tau_{n-\ell} = 0, \quad \textit{for } \ell,k = 1,2,3,... \tag{0.14}$$

and

$$\left(\frac{1}{2}\frac{\partial^2}{\partial t_1\partial t_k} - p_{k+1}(\tilde\partial)\right)\tau_n\circ\tau_n = 0, \quad \textit{for } k = 1,2,3,...$$

Remark: Equation (0.14) reads

$$\begin{aligned} L^k \;&=\; \Lambda^k + \left(\frac{\partial}{\partial t_1}\log\frac{\tau_{n+k}}{\tau_n}\right)_{n\in\mathbf{Z}}\Lambda^{k-1} + ... \\ &\quad + \left(\frac{\partial}{\partial t_k}\log\frac{\tau_{n+1}}{\tau_n}\right)_{n\in\mathbf{Z}}\Lambda^0 + \left(\frac{\partial^2}{\partial t_1\partial t_k}\log\tau_n\right)_{n\in\mathbf{Z}}\Lambda^{-1} + ...\,, \end{aligned} \tag{0.15}$$

With each component of the wave vector Ψ, or, what is the same, with each component of the τ-vector, we associate a sequence of infinite-dimensional planes in the Grassmannian $Gr^{(n)}$

$$\begin{aligned} \mathcal{W}_n \;&=\; \text{span}_{\mathbf{C}}\left\{\left(\frac{\partial}{\partial t_1}\right)^k\Psi_n(t,z), \quad k=0,1,2,...\right\} \\ &=\; e^{\sum_1^\infty t_i z^i}\,\text{span}_{\mathbf{C}}\left\{\left(\frac{\partial}{\partial t_1}+z\right)^k\psi_n(t,z), \quad k=0,1,2,...\right\} \\ &=:\; e^{\sum_1^\infty t_i z^i}\mathcal{W}_n^t. \end{aligned} \tag{0.16}$$

[1] In an expression, like $S = \sum a^{(n)}\Lambda^n$, $a^{(n)} = \text{diag}(a_k^{(n)})_{k\in\mathbf{Z}}$ and $(\tilde\Lambda a)_k = a_{k+1}\Lambda^0$.

[2] where the p_ℓ are elementary Schur polynomials and where $p_\ell(\tilde\partial)f\circ g$ refers to the usual Hirota operation, to be defined in section 1.

Note that the plane $z^{-n}\mathcal{W}_n \in Gr^{(0)}$ has so-called virtual genus zero, in the terminology of [12]; in particular, this plane contains an element of order $1+O(z^{-1})$. Setting $\{f,g\} = f'g - fg'$ for $' = \partial/\partial t_1$, we have the following statement:

Theorem 0.2 *The following six statements are equivalent*
(i) The discrete KP-equations (0.3)
(ii) Ψ and Ψ^*, with the proper asymptotic behaviour, given by (0.8), satisfy the bilinear identities for all $t,t' \in \mathbf{C}^\infty$

$$\oint_{z=\infty} \Psi_n(t,z)\Psi_m^*(t',z)\frac{dz}{2\pi i z} = 0, \qquad \textit{for all} \quad n > m; \tag{0.17}$$

(iii) the τ-vector satisfies the following bilinear identities for all $n > m$ and $t,t' \in \mathbf{C}^\infty$:

$$\oint_{z=\infty} \tau_n(t-[z^{-1}])\tau_{m+1}(t'+[z^{-1}])e^{\sum_1^\infty (t_i - t_i')z^i} z^{n-m-1} dz = 0; \tag{0.18}$$

(iv) The components τ_n of a τ-vector correspond to a flag of planes in Gr,

$$\ldots \supset \mathcal{W}_{n-1} \supset \mathcal{W}_n \supset \mathcal{W}_{n+1} \supset \ldots . \tag{0.19}$$

(v) A sequence of KP-τ-functions τ_n satisfying the equations

$$\{\tau_n(t-[z^{-1}]),\tau_{n+1}(t)\} + z(\tau_n(t-[z^{-1}])\tau_{n+1}(t) - \tau_{n+1}(t-[z^{-1}])\tau_n(t)) = 0 \tag{0.20}$$

(vi) A sequence of KP-τ-functions τ_n satisfying equations (0.14) for $\ell = 1$, i.e.,

$$\left(\frac{\partial}{\partial t_k} - p_k(\tilde{\partial})\right)\tau_{n+1} \circ \tau_n = 0 \quad \textit{for } k = 2,3,\ldots \textit{ and } n \in \mathbf{Z}. \tag{0.21}$$

Remark: The 2-Toda lattice, studied in [14], amounts to two coupled 1-Toda lattices or discrete KP-hierarchies, thus introducing two sets of times t_n's and

s_n's. Actually, every 1-Toda lattice can naturally be extended to a 2-Toda lattice; this is the content of Theorem 3.4.

How to construct discrete KP-solutions. A wide class of examples of discrete KP-solutions is given in section 4 by the following construction, involving the simple vertex operators,

$$X(t,z) := e^{\sum_1^\infty t_i z^i} e^{-\sum_1^\infty \frac{z^{-i}}{i}\frac{\partial}{\partial t_i}}, \tag{0.22}$$

which are disguised Darboux transformations acting on KP τ-functions. We now state:

Theorem 0.3 *Consider an arbitrary τ-function for the KP equation and a family of weights $...,\nu_{-1}(z)dz, \nu_0(z)dz, \nu_1(z)dz, ...$ on* $\mathbf{R}$. *The infinite sequence of τ-functions: $\tau_0 = \tau$ and, for $n > 0$,*

$$\tau_n := \left(\int X(t,\lambda)\nu_{n-1}(\lambda)d\lambda ... \int X(t,\lambda)\nu_0(\lambda)d\lambda\right)\tau(t),$$

$$\tau_{-n} := \left(\int X(-t,\lambda)\nu_{-n}(\lambda)d\lambda ... \int X(-t,\lambda)\nu_{-1}(\lambda)d\lambda\right)\tau(t),$$

form a discrete KP-τ-vector, i.e., the bi-infinite matrix

$$L = \sum_{\ell=0}^{\infty} diag\left(\frac{p_\ell(\tilde\partial)\tau_{n+2-\ell}\circ\tau_n}{\tau_{n+2-\ell}\tau_n}\right)_{n\in\mathbf{Z}} \Lambda^{1-\ell} \tag{0.23}$$

satisfies the discrete KP hierarchy (0.3).

As an interesting special case of this situation, we study in section 6 the *q-KP equation.*

A wide variety of examples are captured by this construction, like q-approximations to KP, discussed in section 5, but also soliton formulas, matrix integrals, certain integrals leading to band matrices, the Calogero-Moser system and others, discussed in [3].
Remark: A semi-infinite discrete KP-hierarchy with $\tau_0(t) = 1$ is equivalent to a bi-infinite discrete KP-hierarchy with $\tau_{-n}(t) = \tau_n(-t)$ and $\tau_0(t) = 1$; this also amounts to $\mathcal{W}_{-n} = \mathcal{W}_n^*$, with $\mathcal{W}_0 = \mathcal{H}_+$. In such cases, one only keeps the lower right hand corner of L, while the lower left hand corner completely vanishes.

1 The KP τ-functions, Grassmannians and a residue formula

As is well known [5], the bilinear identity

$$\oint_{z=\infty} \Psi(t,z)\Psi^*(t,z)dz = 0, \tag{1.1}$$

together with the asymptotics

$$\Psi(t,z) = e^{\sum_1^\infty t_i z^i}\left(1+O\left(\frac{1}{z}\right)\right), \Psi^*(t,z) = e^{-\sum_1^\infty t_i z^i}\left(1+O\left(\frac{1}{z}\right)\right) \tag{1.2}$$

force Ψ, Ψ^* to be expressible in terms of τ-functions

$$\Psi(t,z) = e^{\sum_1^\infty t_i z^i}\frac{\tau(t-[z^{-1}])}{\tau(t)}, \Psi^*(t,z) = e^{-\sum_1^\infty t_i z^i}\frac{\tau(t+[z^{-1}])}{\tau(t)};$$

moreover the KP τ-functions satisfy the differential Fay identity[3], for all $y, z \in \mathbf{C}$, as shown in [1, 15]:

$$\begin{aligned} &\{\tau(t-[y^{-1}]), \tau(t-[z^{-1}])\} \\ &\quad + (y-z)(\tau(t-[y^{-1}])\tau(t-[z^{-1}]) - \tau(t)\tau(t-[y^{-1}]-[z^{-1}]) = 0. \end{aligned} \tag{1.3}$$

In fact this identity characterizes the τ-function, as shown in [13].

From (1.1), it follows that

$$\begin{aligned} 0 &= \oint \tau(t-a-[z^{-1}])\tau(t+a+[z^{-1}])e^{-2\sum_1^\infty a_i z^i}\frac{dz}{2\pi i} \\ &= \sum_{k=1}^\infty a_k\left(\frac{\partial^2}{\partial t_1 \partial t_k} - 2p_{k+1}(\tilde{\partial})\right)\tau \circ \tau + O(a^2). \end{aligned} \tag{1.4}$$

The Hirota notation used here is the following: Given a polynomial $p\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \ldots\right)$ in $\frac{\partial}{\partial t_i}$, define the symbol

$$p\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \ldots\right) f \circ g(t) := p\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \ldots\right) f(t+u)g(t-u)\Big|_{u=0}, \tag{1.5}$$

[3] $\{f,g\} := \frac{\partial f}{\partial t_1}g - f\frac{\partial g}{\partial t_1}$.

and
$$\tilde{\partial}_t := \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, ...\right).$$

For future use, we state the following proposition shown in [1]:

Proposition 1.1 *Consider τ-functions τ_1 and τ_2, the corresponding wave functions*
$$\Psi_j = e^{\sum_{i\geq 1} t_i z^i} \frac{\tau_j(t-[z^{-1}])}{\tau_j(t)} = e^{\sum_{i\geq 1} t_i z^i}\left(1+O(z^{-1})\right) \tag{1.6}$$
and the associated infinite-dimensional planes, as points in the Grassmannian Gr,
$$\tilde{\mathcal{W}}_i = \text{span}\left\{\left(\frac{\partial}{\partial t_1}\right)^k \Psi_i(t,z),\ \textit{for } k=0,1,2,...\right\} \quad \textit{with}\ \ \tilde{\mathcal{W}}_i^t = \tilde{\mathcal{W}}_i e^{-\sum_1^\infty t_k z^k};$$
then the following statements are equivalent
(i) $z\tilde{\mathcal{W}}_2 \subset \tilde{\mathcal{W}}_1$;
(ii) $z\Psi_2(t,z) = \frac{\partial}{\partial t_1}\Psi_1(t,z) - \alpha\Psi_1(t,z)$, for some function $\alpha = \alpha(t)$;
(iii)
$$\{\tau_1(t-[z^{-1}]), \tau_2(t)\} + z(\tau_1(t-[z^{-1}])\tau_2(t) - \tau_2(t-[z^{-1}])\tau_1(t)) = 0. \tag{1.7}$$

When (i), (ii) or (iii) holds, $\alpha(t)$ is given by
$$\alpha(t) = \frac{\partial}{\partial t_1}\log\frac{\tau_2}{\tau_1}. \tag{1.8}$$

<u>*Proof*</u>: To prove that (i) $\Rightarrow$ (ii), the inclusion $z\tilde{\mathcal{W}}_2 \subset \tilde{\mathcal{W}}_1$, hence $z\tilde{\mathcal{W}}_2^t \subset \tilde{\mathcal{W}}_1^t$, implies by (0.16) that
$$z\psi_2(t,z) = z(1+O(z^{-1})) \in \tilde{\mathcal{W}}_1^t$$
must be a linear combination[4]
$$z\psi_2 = \frac{\partial\psi_1}{\partial x} + z\psi_1 - \alpha(t)\psi_1, \text{ and thus } z\Psi_2 = \frac{\partial}{\partial t_1}\Psi_1 - \alpha(t)\Psi_1. \tag{1.9}$$

[4] ψ_i is the same as Ψ_i, but without the exponential.

The expression (1.8) for $\alpha(t)$ follows from equating the z^0-coefficient in (ii), upon using the τ-function representation (1.6). To show that (ii) $\Rightarrow$ (i), note that

$$z\Psi_2 = \frac{\partial}{\partial t_1}\Psi_1 - \alpha\Psi_1 \in \tilde{\mathcal{W}}_1$$

and taking t_1-derivatives, we have

$$z\left(\frac{\partial}{\partial t_1}\right)^j \Psi_2 = \left(\frac{\partial}{\partial t_1}\right)^{j+1}\Psi_1 + \beta_1\left(\frac{\partial}{\partial t_1}\right)^j \Psi_1 + \cdots + \beta_{j+1}\Psi_1,$$

for some $\beta_1, \cdots, \beta_{j+1}$ depending on t only; this implies the inclusion (i). The equivalence (ii) $\Longleftrightarrow$ (iii) follows from a straightforward computation using the τ-function representation (1.6) of (ii) and the expression for $\alpha(t)$. ■

Lemma 1.2 *The following integral along a clockwise circle in the complex plane encompassing $z = \infty$ and $z = \alpha^{-1}$, can be evaluated as follows*

$$\begin{aligned}
&\oint_{z=\infty} f(t + [\alpha] - [z^{-1}])g(t - [\alpha] + [z^{-1}])\frac{z^{m+1}}{(z-\alpha^{-1})^2}\frac{dz}{2\pi i z} \\
= \quad &\alpha^{1-m}\sum_{k=1}^{\infty}\alpha^k\left(-\frac{\partial}{\partial t_k} + \sum_{r=0}^{m-1}(m-r)p_r(-\tilde{\partial})p_{k-r}(+\tilde{\partial})\right) f \circ g.
\end{aligned}$$

<u>*Proof*</u>: By the residue theorem, the integral above is the sum of residue at $z = \infty$ and at $z = \alpha^{-1}$:

$$\begin{aligned}
&\oint_{z=\infty} f(t + [\alpha] - [z^{-1}])g(t - [\alpha] + [z^{-1}])\frac{z^{m+1}}{(z-\alpha^{-1})^2}\frac{dz}{2\pi i z} \\
= \quad &\frac{1}{(m-1)!}\left(\frac{d}{du}\right)^{m-1} f(t + [\alpha] - [u])g(t - [\alpha] + [u])\frac{1}{(1-u\alpha^{-1})^2}\Bigg|_{u=0} \qquad (1.10)
\end{aligned}$$

$$-\frac{d}{dz}z^m f(t + [\alpha] - [z^{-1}])g(t - [\alpha] + [z^{-1}])\Bigg|_{z=\alpha^{-1}}. \qquad (1.11)$$

Evaluating each of the pieces requires a few steps.

Step 1.

$$\frac{1}{k!}\left(\frac{d}{du}\right)^k f(t+[\alpha]-[u])g(t-[\alpha]+[u])\Big|_{u=0}=\sum_{\ell=0}^{\infty}\alpha^{\ell}p_k(-\tilde\partial)p_\ell(\tilde\partial)f\circ g.$$

At first note

$$\left(\frac{d}{du}\right)^k F([u])\Big|_{u=0}=k!p_k(\tilde\partial_s)F(s) \tag{1.12}$$

and, by (1.5) and (1.12),

$$\begin{aligned}\frac{1}{k!}\left(\frac{d}{du}\right)^k f(t+[u])g(t-[u])\Big|_{u=0} &= p_k(\tilde\partial)f\circ g\\ &= p_k(-\tilde\partial)g\circ f\\ &= \sum_{i+j=k}p_i(-\tilde\partial)g.p_j(\tilde\partial)f. \end{aligned}\tag{1.13}$$

Indeed

$$\begin{aligned}&\frac{1}{k!}\left(\frac{d}{du}\right)^k f(t+[\alpha]-[u])g(t-[\alpha]+[u])\Big|_{u=0}\\
&= p_k(\tilde\partial_s)g(t-[\alpha]+s)f(t+[\alpha]-s)\Big|_{s=0}, \quad \text{using (1.12)}\\
&= p_k(\tilde\partial_s)\sum_{\ell=0}^{\infty}\alpha^{\ell}p_\ell(\tilde\partial_t)f(t-s)\circ g(t+s)\Big|_{s=0}, \quad \text{using (1.13)}\\
&= \sum_{\ell=0}^{\infty}\alpha^{\ell}p_k(\tilde\partial_s)p_\ell(\tilde\partial_w)f(t+w-s)g(t-w+s)\Big|_{s=w=0}, \quad \text{expressing Hirota,}\\
&= \sum_{\ell=0}^{\infty}\alpha^{\ell}p_k(\tilde\partial_s)p_\ell(-\tilde\partial_w)f(t-w-s)g(t+w+s)\Big|_{s=w=0}, \quad \text{flipping signs,}\\
&= \sum_{\ell=0}^{\infty}\alpha^{\ell}p_k(\tilde\partial_v)p_\ell(-\tilde\partial_v)f(t-v)g(t+v)\Big|_{v=0}\\
&= \sum_{\ell=0}^{\infty}\alpha^{\ell}p_k(-\tilde\partial)p_\ell(\tilde\partial)f\circ g, \qquad \text{using(1.13).}\end{aligned}$$

Step 2. Residue at ∞.

Note

$$\left(\frac{d}{du}\right)^{\ell}\left(\frac{1}{1-u\alpha^{-1}}\right)^{2}\Bigg|_{u=0}=\left(\frac{d}{du}\right)^{\ell}\sum_{i=1}^{\infty}i(u\alpha^{-1})^{i-1}\Bigg|_{u=0}=(\ell+1)!\alpha^{-\ell}; \quad (1.14)$$

then we find

$$\begin{aligned}
&\frac{1}{(m-1)!}\left(\frac{d}{du}\right)^{m-1}f(t+[\alpha]-[u])g(t-[\alpha]+[u])\frac{1}{(1-u\alpha^{-1})^2}\Bigg|_{u=0}\\
&= \frac{1}{(m-1)!}\sum_{r=0}^{m-1}\begin{pmatrix} m-1\\ r\end{pmatrix}\left(\frac{d}{du}\right)^{r}f(t+[\alpha]-[u])g(t-[\alpha]+[u])\\
&\qquad\qquad\left(\frac{d}{du}\right)^{m-1-r}\frac{1}{(1-u\alpha^{-1})^2}\Bigg|_{u=0}\\
&= \sum_{r=0}^{m-1}(m-r)\sum_{\ell=0}^{\infty}\alpha^{\ell-m+r+1}p_r(-\tilde{\partial})p_\ell(\tilde{\partial})f\circ g, \quad \text{using step 1 and (1.14)}\\
&= m\alpha^{1-m}f(t)g(t)+\alpha^{1-m}\sum_{k=1}^{\infty}\alpha^{k}\sum_{r=0}^{m}(m-r)p_r(-\tilde{\partial})p_{k-r}(\tilde{\partial})f\circ g, \quad \text{using } p_0=1.
\end{aligned} \quad (1.15)$$

Step 3. Residue at $z=\alpha^{-1}$.

$$\begin{aligned}
&\frac{d}{dz}z^m f(t+[\alpha]-[z^{-1}])g(t-[\alpha]+[z^{-1}])\Bigg|_{z=\alpha^{-1}}\\
&= -u^2\frac{d}{du}u^{-m}f(t+[\alpha]-[u])g(t-[\alpha]+[u])\Bigg|_{u=\alpha}\\
&= m\alpha^{-m+1}f(t)g(t)-\alpha^{2-m}\frac{d}{du}f(t+[\alpha]-[u])g(t-[\alpha]+[u])\Bigg|_{u=\alpha}\\
&= m\alpha^{1-m}f(t)g(t)+\sum_{k=1}^{\infty}\alpha^{1-m+k}\frac{\partial}{\partial t_k}f\circ g, \quad \text{by explicit differentiation.}
\end{aligned} \quad (1.16)$$

Finally, putting step 2 and step 3 in (1.11) yields Lemma 1.2. ■

Lemma 1.3 *The Hirota symbol acts as follows on functions $f(t_1, t_2, ...)$ and $g(t_1, t_2, ...)$:*

$$\frac{1}{fg}\frac{\partial^n}{\partial t_1 ... \partial t_n} f \circ g \quad = \quad \text{a polynomial } P_n \text{ in } \begin{cases} \frac{\partial^k}{\partial t_{i_1} ... \partial t_{i_k}} \log \frac{f}{g} & \text{for } k \text{ odd} \\ \\ \frac{\partial^k}{\partial t_{i_1} ... \partial t_{i_k}} \log fg & \text{for } k \text{ even} \end{cases} \tag{1.17}$$

over all subsets $\{i_1, ..., i_k\} \subset \{1, ..., n\}$. Upon granting degree 1 *to each partial in t_i, the polynomial P_n is homogeneous of degree n.*

Proof: By induction, we assume the statement to be valid for an Hirota symbol, involving ℓ partials, and we prove the statement for a symbol involving $\ell + 1$ partials:

$$\frac{1}{fg}\frac{\partial}{\partial t_{\ell+1}}\frac{\partial^\ell}{\partial t_1 ... \partial t_\ell} f(t) \circ g(t)$$

$$\begin{aligned} &= \frac{1}{fg}\frac{\partial}{\partial u_{\ell+1}} f(t+u)g(t-u) \frac{\frac{\partial^\ell}{\partial t_1 ... \partial t_\ell} f(t+u) \circ g(t-u)}{f(t+u)g(t-u)}\Big|_{u=0} \\ &= \left(\frac{\partial}{\partial t_{\ell+1}} \log \frac{f}{g}\right) \frac{1}{fg}\frac{\partial^\ell}{\partial t_1 ... \partial t_\ell} f(t+u) \circ g(t-u) \\ &\quad + \frac{\partial}{\partial u_{\ell+1}} P\left(..., \frac{\partial^m}{\partial t_{i_1} ... t_{i_m}} \log \frac{f(t+u)}{g(t-u)}, ..., \frac{\partial^n}{\partial t_{j_1} ... \partial t_{j_n}} \log f(t+u)g(t-u), ...\right)\Big|_{u=0}, \end{aligned} \tag{1.18}$$

where m is odd and n even. The result follows from the simple computation:

$$\begin{aligned} \frac{\partial}{\partial u_{\ell+1}}\frac{\partial^m}{\partial t_{i_1} ... \partial t_{i_m}} \log \frac{f(t+u)}{g(t-u)}\Big|_{u=0} &= \frac{\partial^{m+1}}{\partial t_{i_1} ... \partial t_{i_m} . \partial t_{\ell+1}} \log f(t)g(t) \\ \frac{\partial}{\partial u_{\ell+1}}\frac{\partial^n}{\partial t_{i_1} ... \partial t_{i_n}} \log f(t+u)g(t-u)\Big|_{u=0} &= \frac{\partial^{n+1}}{\partial t_{i_1} ... \partial t_{i_n} . \partial t_{\ell+1}} \log \frac{f(t)}{g(t)} \end{aligned} \tag{1.19}$$

■

Remark: The induction formula (1.18) can be made into an explicit formula for P_n, involving partitions of the set $\{1, 2, ..., n\}$.

2 The existence of a τ-vector and the discrete KP bilinear identity

Before proving Theorem 0.1, we shall need two lemmas, which are analogues of basic lemmas in the theory of differential operators. So the main purpose of this section is threefold, namely, to prove the bilinear identities for the wave and adjoint wave vectors, to prove the existence of a τ-vector and finally to give a closed form for L^k.

Lemma 2.1 *For z-independent U, $V \in \mathcal{D}$, the following matrix identities hold* [5]

$$UV = \oint_{z=\infty} U\chi(z) \otimes V^\top \chi^*(z) \frac{dz}{2\pi iz}, \tag{2.1}$$

<u>Proof</u>: Set

$$U = \sum_\alpha u_\alpha \Lambda^\alpha \quad \text{and} \quad V = \sum_\beta \Lambda^\beta v_\beta,$$

where u_α and v_α are diagonal matrices. To prove (2.1), it suffices to compare the (i,j)-entries on each side. On the left side of (2.1), we have

$$\begin{aligned}(UV)_{ij} &= \Big(\sum_{\alpha,\beta} u_\alpha \Lambda^{\alpha+\beta} v_\beta\Big)_{ij} \\ &= \sum_{\alpha,\beta} u_\alpha(i)(\Lambda^{\alpha+\beta})_{ij} v_\beta(j) \\ &= \sum_{\substack{\alpha,\beta \\ \alpha+\beta=j-i}} u_\alpha(i) v_\beta(j).\end{aligned}$$

On the right side of (2.1), we have

$$\begin{aligned}&\oint_{z=\infty} \Big(U\chi(z)\Big)_i \Big(V^\top \chi(z^{-1})\Big)_j \frac{dz}{2\pi iz} \\ &= \oint_{z=\infty} \Big(\sum_\alpha u_\alpha z^\alpha \chi(z)\Big)_i \Big(\sum_\beta v_\beta z^\beta \chi(z^{-1})\Big)_j \frac{dz}{2\pi iz} \\ &= \oint_{z=\infty} \sum_{\alpha,\beta} u_\alpha(i) v_\beta(j) z^{\alpha+\beta+i-j} \frac{dz}{2\pi iz} \\ &= \sum_{\substack{\alpha,\beta \\ \alpha+\beta=j-i}} u_\alpha(i) v_\beta(j),\end{aligned}$$

[5] $(A \otimes B)_{ij} = A_i B_j$ and remember $\chi^*(z) = \chi(z^{-1})$. The contour in the integration below runs clockwise about ∞; i.e., opposite to the usual orientation.

establishing (2.1). ∎

Lemma 2.2 *For $W(t)$ a wave operator of the discrete KP-hierarchy,*

$$W(t)W^{-1}(t') \in \mathcal{D}_+, \quad \forall t, t'. \tag{2.2}$$

Proof: Setting $h(t,t') = W(t)W^{-1}(t')$, compute from (0.6)

$$\frac{\partial h}{\partial t_n} = (L^n(t))_+ h, \quad \frac{\partial h}{\partial t'_n} = -h(L^n(t'))_+, \tag{2.3}$$

since $h(t,t) = I \in \mathcal{D}_+$, it follows that $h(t,t')$ evolves in $\mathcal{D}_+$. ∎

Consider the wave function, already defined in the introduction, and the adjoint wave function:

$$\begin{aligned} \Psi(t,z) &= W\chi(z) = e^{\sum_1^\infty t_i z^i} S\chi(z) = e^{\sum t_i z^i}\left(z^n + \sum_{i<n} s_i(n) z^i\right)_{n\in\mathbf{Z}} \\ \Psi^*(t,z) &= (W^{-1})^\top \chi^*(z) = e^{-\sum_1^\infty t_i z^i}(S^{-1})^\top \chi^*(z) \\ &\qquad = e^{-\sum t_i z^i}\left(z^{-n} + \sum_{i<-n} s_i^*(n) z^i\right)_{n\in\mathbf{Z}}. \end{aligned} \tag{2.4}$$

Proof of Theorem 0.1:
Step 1: Setting

$$U := W(t) \quad \text{and} \quad V^\top := (W^{-1}(t'))^\top$$

in formula (2.1) of Lemma 2.1, and using formula (0.8) of Ψ and Ψ^* in terms of W, one finds for all $t, t' \in \mathbf{C}^\infty$,

$$W(t)W(t')^{-1} = \oint_{z=\infty} \Psi(t,z) \otimes \Psi^*(t',z) \frac{dz}{2\pi i z}. \tag{2.5}$$

But, according to Lemma 2.2, $W(t)W(t')^{-1} \in \mathcal{D}_+$ and thus (2.5) is upper-triangular, yielding

$$\oint_{z=\infty} \Psi_n(t,z)\Psi_m^*(t',z)\frac{dz}{2\pi i z} = 0 \quad \text{for all } n > m. \tag{2.6}$$

Defining

$$\begin{aligned}\Phi_n(t,z) &:= z^{-n}\Psi_n(t,z) = e^{\sum t_i z^i}(1+O(z^{-1}))\\ \Phi_n^*(t,z) &:= z^{n-1}\Psi_{n-1}^*(t,z) = e^{-\sum t_i z^i}(1+O(z^{-1})),\end{aligned}$$

upon using the asymptotics (0.8), we have, by setting $m=n-1$ in (2.6)

$$\oint_{z=\infty}\Phi_n(t,z)\Phi_n^*(t',z)dz = \oint_{z=\infty}\Psi_n(t,z)\Psi_{n-1}^*(t',z)\frac{dz}{z} = 0.$$

From the KP-theory, there exists a τ-function $\tau_n(t)$ for each n, such that

$$\Phi_n(t,z) = e^{\sum t_i z^i}\frac{\tau_n(t-[z^{-1}])}{\tau_n(t)}, \quad \Phi_n^*(t,z) = e^{-\sum t_i z^i}\frac{\tau_n(t+[z^{-1}])}{\tau_n(t)},$$

yielding the τ-function representation (0.10) for Ψ_n and Ψ_n^*. ■

Step 2: The following holds for $n\in\mathbf{Z}$:

$$\left(\frac{1}{2}\frac{\partial^2}{\partial t_1\partial t_k} - p_{k+1}(\tilde\partial)\right)\tau_n\circ\tau_n = 0, \quad \text{for } k=1,2,3,... \tag{2.7}$$

$$\left(\frac{\partial}{\partial t_k} - \sum_{r=0}^{\ell-1}(\ell-r)p_r(-\tilde\partial)p_{k-r}(\tilde\partial)\right)\tau_n\circ\tau_{n-\ell} = 0, \quad \text{for } \ell,k=1,2,3,... \tag{2.8}$$

Indeed the bilinear identity (2.6), upon setting $m=n-\ell-1$, shifting $t\mapsto t+[\alpha]$, $t'\mapsto t-[\alpha]$, using the τ-function representation (0.10) of Ψ and Ψ^*, and lemma 1.2 with $m=\ell$, yield[6]

$$\begin{aligned} 0 &= -\alpha^2\oint_{z=\infty}\Psi_n(t+[\alpha],z)\Psi_{n-\ell-1}^*(t-[\alpha],z)\frac{dz}{2\pi iz}\tau_n(t+[\alpha])\tau_{n-\ell}(t-[\alpha])\\ &= -\oint_{z=\infty}\tau_n(t+[\alpha]-[z^{-1}])\tau_{n-\ell}(t-[\alpha]+[z^{-1}])e^{2\sum_1^\infty(\alpha z)^i/i}\alpha^2 z^{\ell+1}\frac{dz}{2\pi iz}\\ &= \alpha^{1-\ell}\sum_{k=1}^\infty\alpha^k\left(\frac{\partial}{\partial t_k} - \sum_{r=0}^{\ell-1}(\ell-r)p_r(-\tilde\partial)p_{k-r}(\tilde\partial)\right)\tau_n\circ\tau_{n-\ell},\end{aligned}$$

establishing the second relation of (2.8). As for the first one, set $m=n-1$, $t\mapsto t-a$ and $t'\mapsto t+a$ in the bilinear identity, and use (1.4), thus yielding (0.14).

[6] $e^{m\sum_1^\infty(\alpha z)^i/i} = (1-\alpha z)^{-m}$

Step 3: To check the formulas (0.12) for S, compute

$$
\begin{aligned}
e^{\sum_1^\infty t_i z^i} S\chi(z) &=: \Psi(t,z) \\
&= e^{\sum_1^\infty t_i z^i} \frac{\tau(t-[z^{-1}])}{\tau(t)}\chi(z) \quad \text{(by (0.10))} \\
&= e^{\sum_1^\infty t_i z^i} \sum_{n=0}^{\infty} \frac{p_n(-\tilde\partial)\tau(t)}{\tau(t)} z^{-n}\chi(z) \\
&= e^{\sum_1^\infty t_i z^i} \sum_{0}^{\infty} \frac{p_n(-\tilde\partial)\tau(t)}{\tau(t)} \Lambda^{-n}\chi(z).
\end{aligned}
$$

Similarly one checks the formula for S^{-1} using the formulas for $\Psi^*(t,z)$ in terms of S^{-1} and $\tau(t)$. Finally to check the formula (0.13) for L^k, use the formulas (0.12) for S and S^{-1} (for $\tilde\Lambda$, see footnote 1):

$$
\begin{aligned}
L^k &= S\Lambda^k S^{-1} \\
&= \sum_{i,j\geq 0}^{\infty} \frac{p_i(-\tilde\partial)\tau}{\tau}\Lambda^{-i-j+k}\left(\tilde\Lambda \frac{p_j(\tilde\partial)\tau}{\tau}\right) \\
&= \sum_{i,j\geq 0}^{\infty} \frac{p_i(-\tilde\partial)\tau}{\tau}\left(\tilde\Lambda^{-i-j+k+1} \frac{p_j(\tilde\partial)\tau}{\tau}\right)\Lambda^{-i-j+k} \\
&= \sum_{\ell\geq 0}\left(\sum_{\substack{i,j\geq 0\\ i+j=\ell}} \frac{p_i(-\tilde\partial)\tau_n p_j(\tilde\partial)\tau_{n+k-\ell+1})}{\tau_n\tau_{n+k-\ell+1}}\right)_{n\in\mathbf{Z}} \Lambda^{k-\ell} \\
&= \sum_{\ell\geq 0}\left(\frac{p_\ell(\tilde\partial)\tau_{n+k-\ell+1}\circ\tau_n}{\tau_{n+k-\ell+1}\tau_n}\right)_{n\in\mathbf{Z}} \Lambda^{k-\ell} \quad \text{(using (1.13))}
\end{aligned}
$$

yielding (0.13) and (0.15), upon noting,

$$
\begin{aligned}
\mathrm{coef}_{\Lambda^{k-1}} L^k &= \left(\frac{p_1(\tilde\partial)\tau_{n+k}\circ\tau_n}{\tau_{n+k}\tau_n}\right)_{n\in\mathbf{Z}} = \left(\frac{\partial}{\partial t_1}\log\frac{\tau_{n+k}}{\tau_n}\right)_{n\in\mathbf{Z}} \\
\mathrm{coef}_{\Lambda^{0}} L^k &= \left(\frac{p_k(\tilde\partial)\tau_{n+1}\circ\tau_n}{\tau_{n+1}\tau_n}\right)_{n\in\mathbf{Z}} = \left(\frac{\partial}{\partial t_k}\log\frac{\tau_{n+1}}{\tau_n}\right)_{n\in\mathbf{Z}} \quad \text{by (2.8)} \\
\mathrm{coef}_{\Lambda^{-1}} L^k &= \left(\frac{p_{k+1}(\tilde\partial)\tau_{n}\circ\tau_n}{\tau_{n}\tau_n}\right)_{n\in\mathbf{Z}} = \left(\frac{\partial^2}{\partial t_1\partial t_k}\log\tau_n\right)_{n\in\mathbf{Z}}, \quad \text{by (2.7)},
\end{aligned}
$$

concluding the proof of the Theorem 0.1. ∎

Corollary 2.3 *Setting $\gamma(t) := (\tilde{\Lambda}\tau(t)/\tau(t))$, the wave operator $W(t)$ for the discrete KP-hierarchy has the following property*

$$(W(t)W^{-1}(t'))_- = 0, \quad (W(t)W^{-1}(t'))_0 = \frac{\gamma(t)}{\gamma(t')}.$$

Proof: That $h(t,t') = W(t)W^{-1}(t') \in \mathcal{D}_+$ was shown in Lemma 2.2. Concerning its diagonal h_0, we deduce from (2.3) that[7]

$$\frac{\partial}{\partial t_k}\log h_0 = (L^k(t))_0, \quad \frac{\partial}{\partial t_k}\log h_0 = -(L^k(t'))_0, \quad \text{with } h_0(t,t) = I.$$

Note that $\gamma(t)/\gamma(t')$ satisfies the same differential equations as $h_0(t)$ with the same initial condition, upon using (0.15):

$$\begin{aligned}
\left(\frac{\partial}{\partial t_k}\log\frac{\gamma(t)}{\gamma(t')}\right)_n &= \frac{\partial}{\partial t_k}\log\frac{\tau_{n+1}(t)}{\tau_n(t)} = L^k(t)_{nn} \\
\left(\frac{\partial}{\partial t'_k}\log\frac{\gamma(t)}{\gamma(t')}\right)_n &= -\frac{\partial}{\partial t'_k}\log\frac{\tau_{n+1}(t')}{\tau_n(t')} = -L^k(t')_{nn},
\end{aligned}$$

with $\gamma(t)/\gamma(t')\Big|_{t=t'} = I$.

3 Sequences of τ-functions, flags and the discrete KP equation

In this section, we prove Theorem 0.2; it will be broken up into three propositions: the first one is very similar to the analogous statement for the KP theory (see [5, 15]). One could make an argument unifying both cases, in the context of Lie theory. The second statement uses Grassmannian technology.

Proposition 3.1 *The following equivalences (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) hold.*

Proof: (i) $\Rightarrow$ (ii) was already shown in Theorem 0.1. Regarding the converse (ii) $\Rightarrow$ (i), we show vectors $\Psi(t,z)$ and $\Psi^*(t,z)$ having the asymptotics (0.8) and satisfying the bilinear identity (ii) are discrete KP-hierarchy vectors.

[7] $M_0 :=$ diagonal part of M.

The point of the proof is to show that the matrices S and $T^t \in I + \mathcal{D}_-$ defined through

$$\Psi(t,z) =: e^{\sum_1^\infty t_i z^i} S\chi(z), \quad \Psi^*(t,z) =: e^{-\sum_1^\infty t_i z^i} T\chi^*(z)$$

satisfy the vector fields (0.6) with $T^t = S^{-1}$.

Step 1. $T^t = S^{-1}$.

Assuming the bilinear identities (assumption (ii) of Theorem 0.2),

$$\begin{aligned} 0 &= \left(\oint_{z=\infty} \Psi(t,z) \otimes \Psi^*(t,z) \frac{dz}{2\pi i z}\right)_- \\ &= \left(\oint_{z=\infty} e^{\sum_1^\infty t_i z^i} S\,\chi(z) \otimes e^{-\sum_1^\infty t_i z^i} T\,\chi(z^{-1}) \frac{dz}{2\pi i z}\right)_- \\ &= (ST^\top)_-, \quad \text{by (2.1)} \end{aligned}$$

but since $S, T^t \in I + \mathcal{D}_-$, $ST^t = I$, yielding $T^t = S^{-1}$.

Step 2. $W(t)W^{-1}(t') \in \mathcal{D}_+$, upon defining $W(t) := S(t)e^{\sum t_i \Lambda^i}$.

According to the bilinear identity, the left hand side of

$$\begin{aligned} &\oint_{z=\infty} \Psi(t,z) \otimes \Psi^*(t',z) \frac{dz}{2\pi i z} \\ &= \oint_{z=\infty} e^{\sum t_i z^i} S\,\chi(z) \otimes e^{-\sum_1^\infty t_i' z^i} (S^{-1})^\top \chi(z^{-1}) \frac{dz}{2\pi i z} \\ &= \oint_{z=\infty} S(t) e^{\sum t_i \Lambda^i}\, \chi(z) \otimes (S^{-1}(t'))^\top e^{-\sum t_i' \Lambda^{\top - i}} \chi(z^{-1}) \frac{dz}{2\pi i z} \\ &= S(t) e^{\sum t_i \Lambda^i} e^{-\sum t_i' \Lambda^i} S^{-1}(t'), \quad \text{using Lemma 2.1} \\ &= W(t) W^{-1}(t'); \end{aligned}$$

belongs to $\mathcal{D}_+$, and hence so is the right hand side.

Step 3.

$$\begin{aligned} \left(\frac{\partial}{\partial t_n} - (L^n)_+\right)\Psi(t,z) &= \left(\frac{\partial}{\partial t_n} - (L^n)_+\right) S\chi(z) e^{\sum_1^\infty t_i z^i} \\ &= \left(\frac{\partial S}{\partial t_n} - (L^n)_+ S + S\, z^n\right) \chi(z) e^{\sum_1^\infty t_i z^i} \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\partial S}{\partial t_n} - (L^n)_+ S + S\,\Lambda^n (S^{-1}S)\right)\chi(z)e^{\sum_1^\infty t_i z^i}\\
&= \left(\frac{\partial S}{\partial t_n} - (L^n)_+ S + L^n S\right)\chi(z)e^{\sum_1^\infty t_i z^i}\\
&= \left(\frac{\partial S}{\partial t_n} + (L^n)_- S)\right)\chi(z)e^{\sum_1^\infty t_i z^i}.
\end{aligned}
$$

Step 4. From $W(t)W^{-1}(t') \in \mathcal{D}_+$, since $\mathcal{D}_+$ is an algebra, deduce

$$
\begin{aligned}
\mathcal{D}_+ \ \ni\ & \left(\left(\frac{\partial}{\partial t_n} - (L^n)_+\right)W(t)\right)W^{-1}(t')\Big|_{t'=t}\\
=\ & \oint_{z=\infty}\left(\frac{\partial}{\partial t_n} - (L^n)_+\right)\Psi(t,z)\otimes\Psi^*(t,z)\frac{dz}{2\pi i z}, \quad \text{by step 2}\\
=\ & \oint_{z=\infty}\left(\frac{\partial S(t)}{\partial t_n} + (L^n)_- S(t)\right)\chi(z)e^{\sum_1^\infty t_i z^i}\otimes (S^\top(t))^{-1}\chi(z^{-1})e^{-\sum_1^\infty t_i z^i}\frac{dz}{2\pi i z},\\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by step 3}\\
=\ & \left(\frac{\partial S(t)}{\partial t_n} + (L^n)_- S(t)\right)S(t)^{-1}, \quad \text{by Lemma 2.1}
\end{aligned}
$$

and thus, since $S \in I + \mathcal{D}_-$ and $\mathcal{D}_-$ is an algebra,

$$
\left(\frac{\partial S(t)}{\partial t_n} + (L^n)_- S(t)\right)S(t)^{-1} \in \mathcal{D}_+ \cap \mathcal{D}_- = 0;
$$

therefore, we have the discrete KP-hierarchy equations on S

$$
\frac{\partial S(t)}{\partial t_n} + (L^n)_- S = 0, \quad n = 1, 2, ...,
$$

and on $L = S\Lambda S^{-1}$,

$$
\frac{\partial L}{\partial t_n} = [-(L^n)_-, L],
$$

ending the proof that (ii) $\Rightarrow$ (i). ■

Finally (ii) $\Longleftrightarrow$ (iii) upon using the equivalence (i) $\Longleftrightarrow$ (ii) and the τ-function representation (0.10) of Ψ and Ψ^*, shown in Theorem 0.1; this establishes Proposition 3.1.

With each component of the wave vector Ψ, given in (0.10), or, what is the same, with each component of the τ-vector, we associate a sequence of infinite-dimensional planes in the Grassmannian $Gr^{(n)}$

$$\begin{aligned}\mathcal{W}_n &= \operatorname{span}_{\mathbf{C}}\left\{\left(\frac{\partial}{\partial t_1}\right)^k \Psi_n(t,z), \quad k=0,1,2,...\right\}\\ &= e^{\sum_1^\infty t_i z^i}\operatorname{span}_{\mathbf{C}}\left\{\left(\frac{\partial}{\partial t_1}+z\right)^k \psi_n(t,z), \quad k=0,1,2,...\right\}\\ &=: e^{\sum_1^\infty t_i z^i}\mathcal{W}_n^t. \end{aligned} \tag{3.1}$$

and planes

$$\mathcal{W}_n^* = \operatorname{span}_{\mathbf{C}}\left\{\frac{1}{z}\left(\frac{\partial}{\partial t_1}\right)^k \Psi_{n-1}^*(t,z), \quad k=0,1,2,...\right\}, \tag{3.2}$$

which are the orthogonal complements of $\mathcal{W}_n$ in $Gr^{(n)}$, by the residue pairing

$$\langle f,g\rangle_\infty := \oint_{z=\infty} f(z)g(z)\frac{dz}{2\pi i}. \tag{3.3}$$

Proposition 3.2 *(ii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) holds.*

<u>*Proof*</u>: The inclusion ... $\supset \mathcal{W}_{n-1} \supset \mathcal{W}_n \supset \mathcal{W}_{n+1} \supset$ ... in (iv) implies that $\mathcal{W}_n$, given by (3.1) and (0.10), is also given by

$$\mathcal{W}_n = \operatorname{span}_{\mathbf{C}}\{\Psi_n(t,z), \Psi_{n+1}(t,z), ...\}.$$

Moreover the inclusions ... $\supset \mathcal{W}_n \supset \mathcal{W}_{n+1} \supset$ ... imply, by orthogonality, the inclusions ... $\subset \mathcal{W}_n^* \subset \mathcal{W}_{n+1}^* \subset$ ..., and thus $\mathcal{W}_n^*$, given by (3.2) and (0.10) and thus specified by Ψ_{n-1}^* and τ_n, is also given by

$$\mathcal{W}_n^* = \{\frac{\Psi_{n-1}^*(t,z)}{z}, \frac{\Psi_{n-2}^*(t,z)}{z}, ...\}.$$

The formula (0.10) for Ψ_n and Ψ_{n-1}^* imply the bilinear identities (1.1), since each τ_n is a τ-function, yielding $\mathcal{W}_n^* = \mathcal{W}_n^\perp$, with respect to the residue pairing and so:

$$\langle \Psi_n(t,z), \frac{\Psi_{n-1}^*(t',z)}{z}\rangle_\infty = \oint_{z=\infty} \Psi_n(t,z)\Psi_{n-1}^*(t',z)\frac{dz}{2\pi i z} = 0.$$

Since

$$\mathcal{W}_n \subset \mathcal{W}_{m+1} = (\mathcal{W}^*_{m+1})^*, \quad \text{all } n > m$$

we have the orthogonality $\mathcal{W}_n \perp \mathcal{W}^*_{m+1}$ for all $n > m$, with respect to the residue pairing; since $\Psi_n(t,z) \in \mathcal{W}_n$, $\frac{\Psi^*_m(t',z)}{z} \in \mathcal{W}^*_{m+1}(t',z)$, we have

$$0 = \langle \Psi_n(t,z), \frac{\Psi^*_m(t',z)}{z} \rangle_\infty = \oint_{z=\infty} \Psi_n(t,z)\Psi^*_m(t',z)\frac{dz}{2\pi i z}, \quad \text{all } n > m, \tag{3.4}$$

which is (ii).

Now assume (ii); then, for fixed $n > m$, we have

$$0 = \oint_{z=\infty} \left(\frac{\partial}{\partial t_1}\right)^k \Psi_n(t,z) \left(\frac{\partial}{\partial t'_1}\right)^\ell \Psi^*_m(t',z)\frac{dz}{2\pi i z}, \quad n > m$$

and thus by (3.1) and (3.2),

$$\mathcal{W}_n \subseteq (\mathcal{W}^*_{m+1})^* = \mathcal{W}_{m+1}, \quad \text{for } n > m,$$

which implies the flag condition ... $\supset \mathcal{W}_{n-1} \supset \mathcal{W}_n \supset \mathcal{W}_{n+1} \supset$ ..., stated in (iv).

(iv) $\Longleftrightarrow$ (v), follows from the equivalence of (i) and (iii) in Proposition 1.1, by setting $\tau_1 := \tau_{n-1}$, $\tau_2 = \tau_n$, $\tilde{\mathcal{W}}_1 = z^{-n+1}\mathcal{W}_{n-1}$ and $\tilde{\mathcal{W}}_2 = z^{-n}\mathcal{W}_n$ and noting

$$z(z^{-n}\mathcal{W}_n) \subset (z^{-n+1}\mathcal{W}_{n-1}), \quad \text{i.e. } \mathcal{W}_n \subset \mathcal{W}_{n-1},$$

concluding the proof of the proposition. ■

Proposition 3.3 *(v)* $\Longleftrightarrow$ *(vi) holds.*

Proof:

Step 1. For a given $n \in \mathbf{Z}$, statement (v), namely

$$R_k^{(n)} := \{p_{k-1}(-\tilde{\partial})\tau_n, \tau_{n+1}\} + \tau_{n+1}p_k(-\tilde{\partial})\tau_n - \tau_n p_k(-\tilde{\partial})\tau_{n+1} = 0, \quad k \geq 2$$

implies

$$R_k^{(n)'} = \left(\frac{\partial}{\partial t_k} - p_k(\tilde{\partial})\right)\tau_{n+1} \circ \tau_n = 0, \quad k \geq 2.$$

Since $R_k^{(n)}$ are the Taylor coefficients of relation (v) in Theorem 0.2, statement $(\mathrm{v})_n$ is equivalent to $(\mathrm{iv})_n$ (i.e. $\mathcal{W}_n \supset \mathcal{W}_{n+1}$). The latter is equivalent to the bilinear identity $(\mathrm{iii})_n$ (i.e., (0.18) with $n \to n+1$ and $m \to n-1$). According to the arguments used in the proof of Theorem 0.1, $(\mathrm{iii})_n$ implies $R_k^{(n)'} = 0$.

Step 2. The converse holds, because, upon using an inductive argument,

$$R_k^{(n)} = \alpha R_k^{(n)'} + \text{partials of } (R_1^{(n)'}, ..., R_{k-1}^{(n)'});$$

thus the vanishing of the $R_1^{(n)'}, ..., R_k^{(n)'}$ implies the vanishing of $R_k^{(n)}$. ∎

Theorem 3.4 *Every 1-Toda lattice is equivalent to a 2-Toda lattice.*

Proof: The 1-Toda theory implies for $S_1 := S \in I + \mathcal{D}_-$, $L_1 := L$

$$\frac{\partial S_1}{\partial t_n} = -(L_1^n)_- S_1(t), \quad \text{where } L_1 = S_1 \Lambda S_1^{-1}.$$

Then, in view of the 2-Toda theory, define $S_2(t) \in \mathcal{D}_+$ by means of the differential equations

$$\frac{\partial S_2(t)}{\partial t_n} = (L_1^n)_+ S_2(t), \quad n = 1, 2, ...,$$

with initial condition $S_2(0) =$ (an invertible element $d_+ \in \mathcal{D}_+$). Then define[8] $S_{1,2}(t,s)$ and $L_{1,2} = S_{1,2}\Lambda^{\pm 1} S_{1,2}^{-1}$, flowing according to the commuting differential equations

$$\frac{\partial S_{1,2}(t,s)}{\partial s_n} = \pm (L_2^n(t,s))_{\mp} S_{1,2}(t,s) \quad \text{with} \quad S_{1,2}(t,0) = S_{1,2}(t). \tag{3.5}$$

$S_{1,2}(t,s)$ satisfies the t-equations of 2-Toda for $s = 0$, by construction; now we must check that this holds for $s \neq 0$; therefore, set

$$F_{1,2}^{(n)}(t,s) = \frac{\partial S_{1,2}}{\partial t_n}(t,s) \pm (L_1^n(t,s))_{\mp} S_{1,2}(t,s), \quad \text{for } n = 1, 2, ... \tag{3.6}$$

Compute, using (3.5) and $[\partial/\partial t_n, \partial/\partial s_n] = 0$, the system of two differential equations

$$\frac{\partial F_{1,2}^{(n)}}{\partial s_k}(t,s) = \pm [F_{2,1}^{(n)} S_2^{-1}, L_2^k]_{\mp} S_{1,2} \pm (L_2^k)_{\mp} F_{1,2}^{(n)}, \quad k, n = 1, 2, ...;$$

since $F_{1,2}^{(n)}(t,0) = 0$, we have $F_{1,2}^{(n)}(t,s) = 0$ for all s. Thus, by (3.5) and (3.6), $S_{1,2}(t,s)$ flow according to 2-Toda. ∎

[8]The first index in $L_{1,2}$ and $S_{1,2}$ corresponds to the upper-sign.

4 Discrete KP-solutions generated by vertex operators

An important construction leading to Toda solutions is contained in Theorem 0.3, which is based on the following Lemma:

Lemma 4.1 *Particular solutions to equation*

$$\{\tau_1(t-[z^{-1}]),\tau_2(t)\}+z(\tau_1(t-[z^{-1}])\tau_2(t)-\tau_2(t-[z^{-1}])\tau_1(t))=0 \quad (4.1)$$

are given, for arbitrary $\lambda \in \mathbf{C}^*$, by pairs (τ_1,τ_2), defined by:

$$\tau_2(t)=\left(\int X(t,\lambda)\nu(\lambda)d\lambda\right)\tau_1(t)=\int e^{\sum t_i\lambda^i}\tau_1(t-[\lambda^{-1}])\nu(\lambda)d\lambda, \qquad (4.2)$$

or

$$\tau_1(t)=\left(\int X(-t,\lambda)\nu'(\lambda)d\lambda\right)\tau_2(t)=\int e^{-\sum t_i\lambda^i}\tau_2(t+[\lambda^{-1}])\nu'(\lambda)d\lambda. \quad (4.3)$$

<u>*Proof*</u>: Using

$$e^{-\sum_1^\infty \frac{1}{i}(\frac{\lambda}{z})^i}=1-\frac{\lambda}{z},$$

it suffices to check,before even integrating, that $\tau_2(t)=X(t,\lambda)\tau_1(t)$ satisfies the above equation (4.1)

$$\begin{aligned}
&e^{-\sum t_i\lambda^i}\left(\{\tau_1(t-[z^{-1}]),\tau_2(t)\}+z(\tau_1(t-[z^{-1}])\tau_2(t)-\tau_2(t-[z^{-1}])\tau_1(t))\right)\\
&\quad=e^{-\sum t_i\lambda^i}\{\tau_1(t-[z^{-1}]),e^{\sum t_i\lambda^i}\tau_1(t-[\lambda^{-1}])\}\\
&\qquad+z(\tau_1(t-[z^{-1}])\tau_1(t-[\lambda^{-1}])-(1-\frac{\lambda}{z})\tau_1(t)\tau_1(t-[z^{-1}]-\lambda^{-1}]))\\
&\quad=\{\tau_1(t-[z^{-1}]),\tau_1(t-[\lambda^{-1}])\}\\
&\qquad+(z-\lambda)(\tau_1(t-[z^{-1}])\tau_1(t-[\lambda^{-1}])-\tau_1(t)\tau_1(t-[z^{-1}]-[\lambda^{-1}]))\\
&\quad=0,
\end{aligned}$$

using the differential Fay identity (1.3) for the τ-function τ_1; a similar proof works for the second solution, given by $\tau_1(t)=X(-t,\lambda)\tau_2(t)$. Since equation (4.1) is linear in $\tau_1(t)$, and also in $\tau_2(t)$, the equation remains valid after integrating with regard to λ. ■

Proof of Theorem 0.3: Note, from the definition of $\tau_{\pm n}$ in Theorem 3, that each τ_n is defined inductively by

$$\tau_{n+1} = \int X(t,\lambda)\nu_n(\lambda)d\lambda\ \tau_n \text{ and } \tau_{-n-1} = \int X(-t,\lambda)\nu_{-n-1}(\lambda)d\lambda\ \tau_{-n};$$

thus by Lemma 4.1, the functions τ_{n+1} and τ_n are a solution of equation (v) of Theorem 0.2. Therefore, theorem 0.2 implies that the τ_n's form a τ-vector of the discrete KP hierarchy. ■

5 Example of vertex generated solutions: the q-KP equation

Consider the class of q-pseudo-difference operators, with y-dependent coefficients, acting on functions $f(y)$

$$\mathcal{D}_q = \{\sum a_i(y)D^i\}, \quad \text{with} \quad Df(y) := f(qy).$$

and the q-derivative D_q, defined by

$$D_q f(y) := \frac{f(qy)-f(y)}{(q-1)y} = -\lambda(y)(D-1)f(y), \text{ with } \lambda(y) := -\frac{1}{(q-1)y};$$

Consider the following q-pseudo-difference operators

$$Q = D + u_0(x)D^0 + u_{-1}D^{-1} + \dots \text{ and } \quad Q_q = D_q + v_0(x)D_q^0 + v_{-1}(x)D_q^{-1} + \dots$$

and the following q-deformations, which were proposed respectively by E. Frenkel [6] and Khesin, Lyubashenko and Roger [10], for $n = 1, 2, \dots$:

$$\frac{\partial Q}{\partial t_n} = [(Q^n)_+, Q] \qquad \textit{(Frenkel system)} \qquad (5.1)$$

$$\frac{\partial Q_q}{\partial t_n} = [\left(Q_q^n\right)_+, Q_q], \qquad \textit{(KLR system)} \qquad (5.2)$$

where $(\ )_+$ and $(\ )_-$ refer to the q-difference and strictly q-pseudo-differential part of $(\ )$. Define

$$c(x) = \left(\frac{(1-q)x}{1-q}, \frac{(1-q)^2x^2}{2(1-q^2)}, \frac{(1-q)^3x^3}{3(1-q^3)}, \dots\right) \in \mathbf{C}^\infty \text{ and } \lambda_n^{-1} = (1-q)xq^{n-1}, \qquad (5.3)$$

one checks for $n \geq 1$, $D^n \lambda_0(x) = \lambda_n(x)$, and

$$\begin{aligned} D^n c(x) &= c(x) - \sum_1^n [\lambda_i^{-1}(x)] \\ D^{-n} c(x) &= c(x) + \sum_1^n [\lambda_{-i+1}^{-1}(x)] \end{aligned} \tag{5.4}$$

Details about these theorems were reported in a joint work with E. Horozov[4].

Theorem 5.1 *There is an algebra isomorphism*

$$\hat{} : \mathcal{D}_q \longrightarrow \mathcal{D},$$

which maps the Frenkel and KLR system into the discrete KP-hierarchy

$$\frac{\partial L}{\partial t_n} = [(L^n)_+ \, , L], \quad n = 1, 2, ... \tag{5.5}$$

Theorem 5.2 *The matrices*

$$L = \Lambda + \sum_{-\infty < \ell \leq 0} \operatorname{diag} \left(\frac{p_{1-\ell}(\tilde{\partial}) \tau_{n+\ell+1} \circ \tau_n}{\tau_{n+\ell+1} \tau_n} \right)_{n \in \mathbf{Z}} \Lambda^\ell$$

and

$$\tilde{L} = \varepsilon L \varepsilon^{-1}$$

with ε as in (5.11), $\tau_0 = \tau(c(x) + t)$ and

$$\begin{aligned} \tau_n &= X(t, \lambda_n) ... X(t, \lambda_1) \tau(c(x) + t) \\ &= r_n(\lambda) \left(\prod_{k=1}^n e^{\sum_{i=1}^\infty t_i \lambda_k^i} \right) D^n \tau(c(x) + t) \end{aligned} \tag{5.6}$$

$$\begin{aligned} \tau_{-n} &= X(-t, \lambda_{-n+1}) ... X(-t, \lambda_0) \tau(c(x) + t) \\ &= r_{-n}(\lambda) \left(\prod_{k=1}^n e^{-\sum_{i=1}^\infty t_i \lambda_{-k+1}^i} \right) D^{-n} \tau(c(x) + t) \end{aligned}$$

transform, using the map $\hat{}$, respectively into solutions to the q-KP deformations (5.1) and (5.2) of

$$Q = D + \sum_{-\infty < i \leq 0} a_i(y) D^i \quad \text{or} \quad Q_q = D_q + \sum_{-\infty < i \leq 0} b_i(y) D_q^i,$$

where the b_i are related to the a_i by (5.12) and[9]

$$a_\ell(y) = \textit{ polynomial in} \begin{cases} \dfrac{\partial^k}{\partial t_{i_1}...\partial t_{i_k}} \log \left(\tau(c(y)+t)^{\pi(k)} D^{\ell+1}\tau(c(y)+t)\right) \textit{ for } k \geq 2 \\ \\ \displaystyle\sum_{i=1}^{\ell+1} \lambda_i^j(y) + \frac{\partial}{\partial t_j} \log \frac{D^{\ell+1}\tau(c(y)+t)}{\tau(c(y)+t)}, \textit{ for } k=1 \end{cases}$$

The proofs of these theorems, which rely heavily on the next lemma, will be given later. In an elegant recent paper, Iliev [9] has obtained q-bilinear identities and q-tau functions, as well, purely within the KP theory.

Consider an appropriate space of functions $f(y)$ representable by "Fourier" series

$$f(y) = \sum_{-\infty}^{\infty} f_n \varphi_n(y)$$

in the basis[10] $\varphi_n(y) := \delta(q^{-n}x^{-1}y)$ for fixed $q \neq 1$, and a parameter $x \in \mathbf{R}$. Also, remember

$$\lambda_i := D^i \lambda_0 = \lambda(xq^i). \tag{5.7}$$

Lemma 5.3 *Then the Fourier transform,*

$$f \longmapsto \mathcal{F}f = (..., f_n, ...)_{n\in\mathbf{Z}},$$

induces an algebra isomorphism $\hat{}$*, mapping D-operators into Λ-operators*

$$\begin{array}{rcl} \hat{} : \mathcal{D}_q & \longrightarrow & \mathcal{D} \\ \displaystyle\sum_i a_i(y) D^i & \longmapsto & \displaystyle\sum \hat{a}_i \Lambda^i := \sum_i \ \mathit{diag}\,(..., a_i(xq^n), ...)_{n\in\mathbf{Z}} \Lambda^i. \end{array} \tag{5.8}$$

Moreover

$$\sum_{i=0}^n b_i(y) D_q^i = \sum_{i=0}^n a_i(y)(-\lambda D)^i \quad \longmapsto \quad \varepsilon \left(\sum_{i=0}^n \hat{a}_i \Lambda^i\right) \varepsilon^{-1}, \tag{5.9}$$

where the Λ-operator in brackets is monic, with[11]

$$\hat{\lambda} = (..., \lambda_{-1}(x), \lambda_0(x), \lambda_1(x), ...) = (..., D^{-1}\lambda, \lambda, D\lambda, ...) \tag{5.10}$$

[9] $\pi(k) =$ parity of $k = 1$, when k is even, and $= -1$, when k is odd.

[10] The δ-function $\delta(z) := \sum_{i\in\mathbf{Z}} z^i$; enjoys the property $f(za)\delta(z) = f(a)\delta(z)$

[11] with $[j] := \frac{1-q^j}{1-q}$ and $\left[{n \atop k}\right] := \frac{[n]\ [n-1]\ ...[n-k+1]}{[k]\ [k-1]\ ...[1]}$

$$\varepsilon := diag\left(..., \lambda_{-2}\lambda_{-1}, -\lambda_{-1}, 1, -\frac{1}{\lambda_0}, \frac{1}{\lambda_0\lambda_1}, -\frac{1}{\lambda_0\lambda_1\lambda_2}, ...\right) \quad with\ \varepsilon_0 = 1, \tag{5.11}$$

$$a_i(y) := \sum_{0\leq k\leq n-i} \frac{\left[{k+i \atop k}\right]}{(-y(q-1)q^i)^k} b_{k+i}(y). \tag{5.12}$$

<u>*Proof*</u>: The operators D and multiplication by a function $a(y)$ act on basis elements, as follows:

$$D\varphi_n(y) = \varphi_{n-1}(y) \quad \text{and} \quad a(y)\varphi_n(y) = a(xq^n)\varphi_n(y).$$

Therefore D^k and $a(y)$ act on functions $f(y)$, as

$$\begin{aligned} f(y) = \sum_{n\in\mathbf{Z}} f_n\varphi_n(y) \longmapsto D^k f(y) &= \sum_{n\in\mathbf{Z}} f_n D^k \varphi_n(y) \\ &= \sum_{n\in\mathbf{Z}} f_n \varphi_{n-k}(y) \\ &= \sum_{n\in\mathbf{Z}} f_{n+k}\varphi_n(y), \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} f(y) = \sum_{n\in\mathbf{Z}} f_n\varphi_n(y) \longmapsto a(y)f(y) &= \sum_{n\in\mathbf{Z}} f_n a(y)\varphi_n(y) \\ &= \sum_{n\in\mathbf{Z}} f_n a(xq^n)\varphi_n(y), \end{aligned} \tag{5.14}$$

from which it follows that

$$(D^k)\hat{} = \Lambda^k \tag{5.15}$$

$$\hat{a}(y) = \text{diag}\ (..., a(xq^n), ...)_{n\in\mathbf{Z}}. \tag{5.16}$$

To establish the algebra isomorphism (5.8), one checks that

$$\begin{aligned} \left(a(y)D^i\right)\hat{}\ \left(b(y)D^j\right)\hat{} &= \hat{a}(y)\Lambda^i\ \hat{b}(y)\Lambda^j \\ &= \hat{a}(y)\left(\Lambda^i\hat{b}(y)\Lambda^{-i}\right)\Lambda^{i+j} \\ &= \text{diag}(..., a(xq^n)b(xq^{n+i}), ...)_{n\in\mathbf{Z}}\Lambda^{i+j} \\ &= \left(a(y)b(yq^i)D^{i+j}\right)\hat{} \\ &= \left(a(y)D^i\ \ b(y)D^j\right)\hat{}. \end{aligned} \tag{5.17}$$

Using the inductively established identity

$$D_q^n = \frac{1}{y^n(q-1)^n q^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} D^{n-k},$$

the first identity (5.9) is immediate.

Then, using, by virtue of (5.10) and (5.11), $\hat{\lambda}\Lambda = -\varepsilon\Lambda\varepsilon^{-1}$ and $\varepsilon\hat{a}\varepsilon^{-1} = \hat{a}$ (since $\hat{a}$ is diagonal), one computes, using the established isomorphism,

$$\begin{aligned} \left(a_i(y)(-\lambda(y)D)^i\right)^{\wedge} &= \hat{a}_i \left(-\hat{\lambda}\hat{D}\right)^i \\ &= \hat{a}_i \left(-\hat{\lambda}\Lambda\right)^i \\ &= \hat{a}_i \left(\varepsilon\Lambda\varepsilon^{-1}\right)^i \\ &= \varepsilon \left(\hat{a}_i\Lambda^i\right) \varepsilon^{-1} \qquad (5.18) \end{aligned}$$

establishing (5.9). ■

<u>*Proof of Theorem 5.1*</u>: Indeed the Frenkel system (5.1) maps at once into (5.5), whereas, using (5.9), the KLR-system maps into

$$\begin{aligned} \frac{\partial \varepsilon L \varepsilon^{-1}}{\partial t_n} &= [\left(\varepsilon L^n \varepsilon^{-1}\right)_+, \varepsilon L \varepsilon^{-1}] \qquad (5.19) \\ &= \varepsilon[(L^n)_+, L]\varepsilon^{-1}, \qquad (5.20) \end{aligned}$$

which upon conjugation by ε leads to (5.5) as well. ■

<u>*Proof of Theorem 5.2*</u>: From Theorem 0.3, it follows that L with the τ_n's defined by (5.6), satisfies the Toda lattice; the second equality in (5.6) follows from (5.4). According to Lemma 1.3,

$$\frac{p_{1-\ell}(\tilde{\partial})\tau_{n+\ell+1} \circ \tau_n}{\tau_{n+\ell+1}\tau_n} = \text{a polynomial in } \left(\frac{\partial^k}{\partial t_{i_1} \ldots \partial t_{i_k}} \log(\tau_{n+\ell+1}\tau_n^{\pi(k)})\right),$$

where by (5.6), for $k \geq 2$,

$$\begin{aligned} &\left(\frac{\partial^k}{\partial t_{i_1} \ldots \partial t_{i_k}} \log(\tau_{n+\ell+1}\tau_n^{\pi(k)})\right)_{n\in\mathbf{Z}} \\ &\quad = \left(D^n \frac{\partial^k}{\partial t_{i_1} \ldots \partial t_{i_k}} \log\left(\tau(c(y)+t)^{\pi(k)} D^{\ell+1}\tau(c(y)+t)\right)\right)_{n\in\mathbf{Z}} \\ &\quad = \left(\frac{\partial^k}{\partial t_{i_1} \ldots \partial t_{i_k}} \log \tau(c(y)+t)^{\pi(k)} D^{\ell+1}\tau(c(y)+t)\right)^{\wedge}, \end{aligned}$$

and

$$\left(\frac{\partial}{\partial t_j}\log\frac{\tau_{n+\ell+1}}{\tau_n}\right)_{n\in\mathbf{Z}}$$

$$= \left(\frac{\partial}{\partial t_j}\log\frac{\left(\prod_{\alpha=1}^{n+\ell+1}e^{\sum_{i=1}^{\infty}t_i\lambda_\alpha^i}\right)D^{n+\ell+1}\tau(c(y)+t)}{\left(\prod_{\alpha=1}^{n}e^{\sum_{i=1}^{\infty}t_i\lambda_\alpha^i}\right)D^{n}\tau(c(y)+t)}\right)_{n\in\mathbf{Z}}$$

$$= \left(\sum_{\alpha=n+1}^{n+\ell+1}\lambda_\alpha^j(y)+\frac{\partial}{\partial t_j}\log\frac{D^{n+\ell+1}\tau(c(y)+t)}{D^n\tau(c(y)+t)}\right)_{n\in\mathbf{Z}}$$

$$= \left(D^n\left(\sum_{i=1}^{\ell+1}\lambda_i^j(y)+\frac{\partial}{\partial t_j}\log\frac{D^{\ell+1}\tau(c(y)+t)}{\tau(c(y)+t)}\right)\right)_{n\in\mathbf{Z}}$$

$$= \left(\sum_{i=1}^{\ell+1}\lambda_i^j(y)+\frac{\partial}{\partial t_j}\log\frac{D^{\ell+1}\tau(c(y)+t)}{\tau(c(y)+t)}\right)^{\wedge},$$

establishing Theorem 5.2. ■

Remark: Note the ε-conjugation has no counterpart in $\mathcal{D}_q$-world.

Defining the simple q-vertex operators:

$$X_q(x,t,z) := e_q^{xz}X(t,z) \quad\text{and}\quad \tilde{X}_q(x,t,z) := (e_q^{xz})^{-1}X(-t,z)$$

in terms of the vertex operator (6.1) and the q-exponential $e_q^x = e^{\sum_1^\infty \frac{(1-q)^k x^k}{k(1-q^k)}}$ we now state:

Corollary 5.4 *Any K-P τ-function leads to a q-K-P τ-function $\tau(c(x)+t)$ satisfying q-bilinear relations below for all $x\in\mathbf{R}$, $t,t'\in\mathbf{C}^\infty$ and all $n>m$, which tends to the standard K-P bilinear identity when q goes to 1:*

$$\oint_{z=\infty} D^n(X_q(x,t,z)\tau(c(x)+t))D^{m+1}(\tilde{X}_q(x,t',z)\tau(c(x)+t')dz=0$$

$$\longrightarrow \int_{z=\infty} X(t,z)\tau(\bar{x}+t)X(t',z)\tau(\bar{x}+t')dz=0.$$

Proof: The τ-functions τ_n defined in Theorem 5.2 satisfy the usual bilinear identity (0.18), and so, using the following identity

$$\begin{aligned}\frac{z^{n-m-1}}{\prod_{k=m+2}^{n}(-\lambda)^k}\prod_{k=m+2}^{n} e^{-\sum_{i=1}^{\infty}\frac{1}{i}\left(\frac{\lambda_k}{z}\right)^i} &= \prod_{k=m+2}^{n}\left(1-\frac{z}{\lambda_k}\right)\\ &= \prod_{k=m+2}^{n} e^{-\sum_{i=1}^{\infty}\frac{1}{i}\left(\frac{z}{\lambda_k}\right)^i}\\ &= D^n e_q^{xz} D^{m+1}(e_q^{xz})^{-1}\end{aligned}$$

in computing $\tau_n(t-[z^{-1}])$ in the usual bilinear identity yields, up to a multiplicative factor $\alpha(\lambda,\nu)$:

$$\begin{aligned}&\alpha(\lambda,\nu)\oint_{z=\infty}\tau_n(t-[z^{-1}])\tau_{m+1}(t'+[z^{-1}])e^{\sum_1^\infty(t_i-t'_i)z^i}z^{n-m}\frac{dz}{z}\\ =&\oint_{z=\infty}\tau(c(x)+t-[z^{-1}]-\sum_1^n[\lambda_i^{-1}])\tau(c(x)+t'+[z^{-1}]+\sum_1^{m+1}[\lambda_i^{-1}])\\ &\qquad\prod_{k=m+2}^{n}\left(1-\frac{z}{\lambda_k}\right)e^{\sum_1^\infty(t_i-t'_i)z^i}dz\\ =&\oint_{z=\infty}D^n(X_q(x,t,z)\tau(c(x)+t))D^{m+1}(\tilde{X}_q(x,t',z)\tau(c(x)+t'))dz=0,\end{aligned}$$

the latter tending as $q\to 1$ to the usual KP bilinear identity, upon using (5.3). ■

Corollary 5.5 *If we take $\tau_0(t)=\tau(c(x)+t)$ in Theorem 5.2, with $\tau(t)$ a N-KdV τ-function, i.e., $\partial\tau/\partial t_{iN}=0$, $i=1,2,...$, then*

$$(L^N)=(L^N)_+ \quad \textit{and} \quad \tilde{L}^N=(\tilde{L}^N)_+ \tag{5.21}$$

yielding the N-Frenkel and N-KLR system:

$$Q^N=(Q^N)_+ \quad \textit{and} \quad Q_q^N=(Q_q^N)_+. \tag{5.22}$$

The q-differential operator Q_q^N has the form below and tends to the differential operator of the N-KdV hierarchy as q goes to 1:

$$Q_q^N=D_q^N \quad + \quad \frac{\partial}{\partial t_1}\log\frac{\tau(D^Nc+t)}{\tau(c+t)}D_q^{N-1}$$

$$
\begin{aligned}
&+ \quad \left(\sum_{i=0}^{N-1} \frac{\partial^2}{\partial t_1^2} \log \tau(D^i c + t) \right. \\
&- \quad \sum_{i=0}^{N-2} \lambda_{i+1} \left(\frac{\partial}{\partial t_1} \log \frac{\tau(D^N c + t)}{\tau(D^{N-1} c + t)} - \frac{\partial}{\partial t_1} \log \frac{\tau(D^{i+1} c + t)}{\tau(D^i c + t)} \right) \\
&+ \quad \left. \sum_{0 \le i \le j \le N-2} \frac{\partial}{\partial t_1} \log \frac{\tau(D^{i+1} c + t)}{\tau(D^i c + t)} \frac{\partial}{\partial t_1} \log \frac{\tau(D^{j+1} c + t)}{\tau(D^j c + t)} \right) D_q^{N-2} + \ldots \\
&\longrightarrow \left(\frac{\partial}{\partial x} \right)^N + N \frac{\partial^2}{\partial t_1^2} \log \tau(\bar{x} + t) \left(\frac{\partial}{\partial x} \right)^{N-2} + \ldots \qquad (5.23)
\end{aligned}
$$

<u>*Proof*</u>: Note that for $W \in Gr^{(0)}$, $z^N W \subset W$ if and only if its tau function is of the form $e^{\sum_1^\infty c_i t_{iN}} \tau(t)$, with $\partial \tau(t)/\partial t_{iN} = 0$, $i = 1, 2, \ldots$. Thus by hypothesis, we have for each

$$
W_k = \operatorname{span}\{\psi_k(t,z), \psi_{k+1}(t,z), \ldots\}
$$

$z^N W_k \subset W_k$ and since $L\psi = z\psi$,

$$
z^N \psi_k = \sum_{j=0}^{N-1} a_j \psi_{k+j} + \psi_{k+N} = (L^N \psi)_k,
$$

and so L^N is upper-triangular, yielding (5.21), which by the isomorphism $\wedge$ of Lemma 5.3 yields (5.22). From (0.13) and the relationship between $a_i(y)$ and $b_i(y)$ given in (5.12), deduce (5.23).

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