Generalized orthogonal polynomials, discrete KP and Riemann-Hilbert problems*

M. Adler† P. van Moerbeke‡

August 24, 1998

Dedicated to Jürgen Moser, at the occasion of his 70th birthday

Abstract

Classically, a single weight on an interval of the real line leads to moments, orthogonal polynomials and tridiagonal matrices. Appropriately deforming this weight with times $t = (t_1, t_2, ...)$, leads to the standard Toda lattice and τ-functions, expressed as Hermitian matrix integrals.

This paper is concerned with a sequence of t-perturbed weights, rather than one single weight. This sequence leads to moments, polynomials and a (fuller) matrix evolving according to the discrete KP-hierarchy. The associated τ-functions have integral, as well as vertex operator representations. Among the examples considered, we mention: nested Calogero-Moser systems, concatenated solitons and m-*periodic* sequences of weights. The latter lead to $2m+1$-band matrices and generalized orthogonal polynomials, also arising in the context of

*Appeared in: Comm. Math. Phys., 207, 589–620 (1999)

†Department of Mathematics, Brandeis University, Waltham, Mass 02454, USA. E-mail: adler@math.brandeis.edu. The support of a National Science Foundation grant # DMS-98-4-50790 is gratefully acknowledged.

‡Department of Mathematics, Université de Louvain, 1348 Louvain-la-Neuve, Belgium and Brandeis University, Waltham, Mass 02454, USA. E-mail: vanmoerbeke@geom.ucl.ac.be and @math.brandeis.edu. The support of a National Science Foundation grant # DMS-98-4-50790, a Nato, a FNRS and a Francqui Foundation grant is gratefully acknowledged. Some of the present work was done at the Centre Emile Borel, Paris (fall 96).

a Riemann-Hilbert problem. We show the Riemann-Hilbert factorization is tantamount to the factorization of the moment matrix into the product of a lower- times upper-triangular matrix.

Contents

0 Introduction

The starting point in the standard theory of orthogonal polynomials is a single weight $\rho(z)dz$ on an interval of the real line. The latter leads to moments $\mu_{ij} = \langle z^i, z^j\rho\rangle$, depending on $i+j$ only; in turn, moments lead to polynomials $p_n(z)$, defined by the determinant (0.2) below and the spectral relation $zp_n = (Lp)_n$ defines tridiagonal semi-infinite matrices L. An important recent development in this ancient theory is that the perturbed weight $e^{\sum_1^\infty t_i z^i}\rho(z)dz$ leads to t-dependent tridiagonal matrices $L(t)$ satisfying the standard Toda lattice equations; the determinants of the principal minors of the moment matrix are τ-functions for the Toda lattice and are representable as integrals over Hermitean matrices, as developped extensively in [1].

This paper is designed to show the reader how the introduction of an infinite family of weights $\rho_j(z)dz$, rather than a family $z^j\rho(z)dz$ generated by one weight $\rho(z)dz$, leads to a theory having many features in common with the classic situation above. The weights lead to "moments" μ_{ij}, to a semi-infinite moment matrix m_∞, to polynomials $p_n(z)$, as in (0.2), and to semi-infinite matrices L of type (0.4) below, defined by $zp_n(z) = (Lp(z))_n$. We mainly deal with:
(i) t-deformations ($t = (t_1, t_2, ...)$)

$$\rho_j(t; z) = e^{\sum_1^\infty t_i z^i}\rho_j(z)dz, \qquad t = (t_1, t_2, ...) \in \mathbf{C}^\infty,\ z \in \mathbf{R},\ j = 0, 1, 2, ...$$

of the weights; they imply for the matrix L the so-called "discrete KP-hierarchy" in t; this hierarchy is fully described in [2], and a large class of solutions is explained in section 1.

Occasionally, shall we deal with
(ii) (t, s)-deformations [1]

$$\rho_j(t, s; z) = e^{\sum_1^\infty t_i z^i}\sum_{\ell=0}^{\infty} F_\ell(-s)\rho_{j+\ell}(z), \quad t, s \in \mathbf{C}^\infty,\ z \in \mathbf{R},\ j = 0, 1, 2, ...$$

of the weights ρ_j; they imply for L the 2d-Toda hierarchy, as described in [16, 3] and summarized in section 2.

To be specific, given a family of weights $\rho_0(z)dz, \rho_1(z)dz, ...$ on $\mathbf{R}$, and their t-deformations

$$\rho_j^t(z)dz := \rho_j(t; z)dz = e^{\sum_1^\infty t_k z^k}\rho_j(z)dz,$$

define the "moments", with regard to the usual integration in $\mathbf{R}$:

$$\mu_{ij} := \langle z^i, \rho_j(z)\rangle \quad \text{and} \quad \mu_{ij}(t) := \langle z^i, \rho_j^t(z)\rangle, \tag{0.1}$$

and the moment matrix

$$m_n(t) := (\mu_{ij}(t))_{0\le i,j\le n-1}.$$

Then the semi-infinite moment matrix m_∞ satisfies the linear differential equations

$$\frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty,$$

[1] where the $F_i(t)$ are the elementary Schur polynomials $e^{\sum_1^\infty t_i z^i} = \sum_{i=0}^\infty F_i(t) z^i$

where Λ denotes the standard shift matrix. They form a infinite set of commuting vector fields. Generically the semi-infinite moment matrix m_∞ admits a (unique) factorization into upper- and triangular matrices S_1 and S_2 respectively, with S_1 having 1's on the diagonal:

$$m_\infty = S_1^{-1} S_2.$$

Consider the vector $p(t,z) := (p_n(t,z))_{n\geq 0}$ of monic polynomials in z, depending[2] on $t = (t_1, t_2, ...) \in \mathbf{C}^\infty$,

$$p_n(t,z) := S_1\chi(z) := \frac{1}{\det m_n(t)} \det \left(\begin{array}{ccc|c} \mu_{00}(t) & \ldots & \mu_{0,n-1}(t) & 1 \\ \vdots & & \vdots & \vdots \\ \mu_{n-1,0}(t) & \ldots & \mu_{n-1,n-1}(t) & z^{n-1} \\ \hline \mu_{n0}(t) & \ldots & \mu_{n,n-1}(t) & z^n \end{array} \right). \tag{0.2}$$

The eigenvalue problem

$$zp(t,z) = L(t)p(t,z) \tag{0.3}$$

or, alternatively, the S_1-matrix in the factorization above, gives rise to the semi-infinite matrix

$$\begin{aligned} L &= S_1 \Lambda S_1^{-1} \\ &= \Lambda + a\Lambda^0 + \Lambda^\top b + \Lambda^{\top 2} c + \ldots = \begin{pmatrix} a_0 & 1 & 0 & 0 \\ b_0 & a_1 & 1 & 0 \\ c_0 & b_1 & a_2 & 1 \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \end{aligned} \tag{0.4}$$

The polynomials $p_n(t,z)$ also give rise to a *Grassmannian flag* of nested infinite-dimensional planes $\ldots \supset \mathcal{W}_n^t \supset \mathcal{W}_{n+1}^t \supset \ldots$, given by[3]

$$\mathcal{H}_+ \supset \mathcal{W}_n^t := \mathcal{W}_n e^{-\sum_1^\infty t_i z^i} := \operatorname{span}\{p_n(t,z), p_{n+1}(t,z)\ldots\}. \tag{0.5}$$

We shall also need the associated "Vandermonde" determinants[4],

$$\Delta_n^{(\rho)}(z) = \det\left(\rho_{\ell-1}(z_k)\right)_{1\leq \ell,k\leq n}, \quad \Delta_n(z) = \det(z_k^{\ell-1})_{1\leq \ell,k\leq n}, \tag{0.6}$$

[2] $\chi(z) := (1, z, z^2, \ldots)$ and $\chi^*(z) := \chi(z^{-1})$.

[3] $\mathcal{H}_+ :=$ span $\{1, z, z^2, \ldots\}$.

[4] $\Delta_n(z) = \prod_{1\leq j<i\leq n}(z_i - z_j)$

and the simple vertex operator,

$$X(t,z) := e^{\sum_1^\infty t_i z^i} e^{-\sum_1^\infty \frac{z^{-i}}{i}\frac{\partial}{\partial t_i}}; \tag{0.7}$$

this is, in disguise, a Darboux transform acting on KP τ-functions. We now state:

Theorem 0.1 *Given the moments (0.1) and the construction above, the semi-infinite matrix L in (0.4) satisfies the discrete KP hierarchy*

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n = 1, 2, ..., \tag{0.8}$$

and has the following τ-function representation[5]

$$L = \sum_{\ell=0}^{\infty} diag \left(\frac{F_\ell(\tilde\partial)\tau_{n+2-\ell} \circ \tau_n}{\tau_{n+2-\ell}\tau_n} \right)_{n\geq 0} \Lambda^{1-\ell} \tag{0.9}$$

in terms of a sequence of τ-functions $(\tau_0 = 1, \tau_1, \tau_2, ...)$, which enjoys many different representations:

$$\begin{aligned}
\tau_n(t) &= \det\left(\mu_{\ell,k}(t)\right)_{0\leq \ell,k\leq n-1} && \textit{(moment representation)} \quad (0.10)\\
&= \frac{1}{n!}\int \dots \int_{\mathbf{R}^n} \Delta_n(z)\Delta_n^{(\rho)}(z) \prod_{k=1}^n \left(e^{\sum t_i z_k^i} dz_k\right) && \textit{(integral representation)} \quad (0.11)\\
&= \det\left(Proj:\ e^{-\sum_1^\infty t_i z^i} z^{-n}\mathcal{W}_n \to \mathcal{H}_+\right) && \textit{(flag representation)} \quad (0.12)\\
&= \det\left(Proj:\ e^{\sum_1^\infty t_i z^i} z^{n}\mathcal{W}_n^* \to \mathcal{H}_+\right) && \textit{(dual flag representation)} \quad (0.13)\\
&= \left(\int_{\mathbf{R}} dz\ z^{n-1}\rho_{n-1}(z) X(t,z)\right) \tau_{n-1}(t) && \textit{(vertex representation)} \quad (0.14)
\end{aligned}$$

[5] where $F_\ell(\tilde\partial) = F_\ell(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, ...)$, for the elementary Schur polynomials F_ℓ. The symbol $F_\ell(\tilde\partial) f \circ g$ is the customary Hirota symbol.

where

$$\begin{aligned}\mathcal{W}_n e^{-\sum_1^\infty t_i z^i} &= \operatorname{span}\{p_n(t,z), p_{n+1}(t,z)...\} \subset \mathcal{H}_+\\ \mathcal{W}_n^*\, e^{\sum_1^\infty t_i z^i} &= \operatorname{span}\left(\left\{\int_{\mathbf{R}} \frac{\rho_j^t(u)du}{z-u}, j=0,...,n-1\right\} \oplus \mathcal{H}_+\right) \supset \mathcal{H}_+.\end{aligned} \tag{0.15}$$

The polynomials (0.2) have the following representations

$$p_n(t,z) = \frac{\det\left(z\mu_{ij}(t) - \mu_{i+1,j}(t)\right)_{0\le i,j\le n-1}}{\det(\mu_{ij}(t))_{0\le i,j\le n-1}} \tag{0.16}$$

$$= \frac{1}{n!\tau_n(t)} \int \cdots \int_{\mathbf{R}^n} \Delta_n(z)\Delta_n^{(\rho)}(z) \prod_{k=1}^n \left(e^{\sum t_i z_k^i}(z-z_k)\,dz_k\right), \tag{0.17}$$

and satisfy the eigenvalue problem $Lp = zp$.

Notice that formulae (0.16) and (0.17) go in parallel with (0.10) and (0.11). Formula (0.17) is a generalization of a formula for classical orthogonal polynomials already appearing last century in the work of Heine [11].

We shall apply this theorem to a variety of examples, corresponding to sections 4 to 8 ($\delta(x)$ is the customary delta-function, i.e., $\int_{\mathbf{R}} \delta(x)f(x)dx = f(0)$) :

$\rho_j(z) := z^j\rho(z)$	*tridiagonal matrix L*
$\rho_{j+km}(z) := z^{km}\rho_j(z)$	*$2m+1$-band matrix L^m*
$\rho_k(z) = \delta(z-p_{k+1}) - \lambda_{k+1}^2\delta(z-q_{k+1})$	*concatenated solitons*
$\rho_k(z) = \delta'(z-p_{k+1}) + \lambda_{k+1}\delta(z-p_{k+1})$	*nested Calogero-Moser systems*
$\rho_k(z) = (-1)^k\delta^{(k)}(z-p) - \delta^{(k)}(z+p)$	*upper-triangular L^2.*

The first example leads to the standard Toda lattice and the the classic theory of *orthogonal polynomials*. Since the work of Fokas, Its and Kitaev [9], the *Riemann-Hilbert method* is a device to obtain asymptotics for orthogonal polynomials; for semi-classical asymptotics, see Bleher and Its [7]. We show

$$\text{Riemann-Hilbert factorization} \iff \text{factorization } m_\infty = S_1^{-1}S_2. \tag{0.18}$$

To be precise, we show the Riemann-Hilbert matrices Y_n take on the following form ($\chi(z)$ and $\chi^*(z)$ are as in footnote 2 and $h_{n-1} := \tau_n/\tau_{n-1}$):

$$\begin{aligned} Y_n(z) &= \begin{pmatrix} (S_1\chi(z))_n & \frac{1}{z}\left(S_2\chi^*(z)\right)_n \\ h_{n-1}^{-1}\left(S_1\chi(z)\right)_{n-1} & h_{n-1}^{-1}\frac{1}{z}\left(S_2\chi^*(z)\right)_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\tau_n(t-[z^{-1}])}{\tau_n(t)} z^n & \frac{\tau_{n+1}(t+[z^{-1}])}{\tau_n(t)} z^{-n-1} \\ \frac{\tau_{n-1}(t-[z^{-1}])}{\tau_n(t)} z^{n-1} & \frac{\tau_n(t+[z^{-1}])}{\tau_n(t)} z^{-n} \end{pmatrix}. \end{aligned} \tag{0.19}$$

The second example, which is novel and which is developped in section 5, involves a finite set of weights

$$\rho_0(z)dz, ..., \rho_{m-1}(z)dz$$

for $m \geq 2$, which we extend into an infinite "*m-periodic*" sequence

$$\rho_0(z)dz, ..., \rho_{m-1}(z)dz, z^m\rho_0(z)dz, ..., z^m\rho_{m-1}(z)dz,$$
$$z^{2m}\rho_0(z)dz, ..., z^{2m}\rho_{m-1}(z)dz, ...$$

This sequence leads naturally to *generalized orthogonal polynomials* $p_n(z)$ by the recipe (0.2), which enjoys the following properties:

(i) the polynomials $p_n(z)$ satisfy the *orthogonality relations* $\langle p_i(z), \rho_j(z)\rangle = 0$ for $i \geq j+1$;
(ii) Applying z^m to the vector $p(z) := (p_0(z), p_1(z), ...)$leads to a $2m+1$-band matrix L^m.
(iii) The t-evolution $e^{\sum_1^\infty t_i z^i}\rho_k(z)$ implies L evolves according to the discrete KP hierarchy.

The discrete KP-hierarchy on $2m+1$-band matrices has been studied in [17]; see also [10]. We also formulate here a Riemann-Hilbert problem, which should characterize the generalized orthogonal polynomials.

Another interesting set of examples is provided by picking as weights various combinations of (standard) δ-functions, which lead to concatenated soliton solutions, Calogero-Moser systems, etc...

We wish to thank Leonid Dickey for insightful comments and criticism, which lead to recognize the importance of the integral in (0.15) beyond its formal aspects. L. Dickey has shown in a very interesting recent paper [8] that the discrete KP hierarchy is the most natural generalization of the modified

KP. We also thank Alexander Its and Pavel Bleher for having explained to us the Riemann-Hilbert problem, and for having posed the problem of finding the connection with the matrix factorization of the moment matrix. We also thank Taka Shiota for a number of interesting conversations.

1 Vertex operator solutions to the discrete KP hierarchy

In [2], we discussed the discrete KP hierarchy and found a general method for generating its solutions, in both, the bi- and semi-infinite situations; this paper mainly deals with the semi-infinite case. In [2] and [4], we gave an application of the bi-infinite discrete KP to the q-KP equation. In general, the main features are summarized in the following statement, whose proof can be found in [2]:

Theorem 1.1 *From an arbitrary KP τ-function and a sequence of real functions $(...,\nu_{-1}(\lambda),\nu_0(\lambda),\nu_1(\lambda),...)$, defined on $\mathbf{R}$, one constructs the infinite sequence of τ-functions: $\tau_0=\tau$ and, for $n>0$,*

$$\begin{aligned}
\tau_0(t) &= \tau(t)\\
\tau_n(t) &= \left(\int X(t,\lambda)\nu_{n-1}(\lambda)d\lambda \ ... \int X(t,\lambda)\nu_0(\lambda)d\lambda\right)\tau(t), \quad n>0\\
\tau_{-n}(t) &= \left(\int X(-t,\lambda)\nu_{-n}(\lambda)d\lambda \ ... \int X(-t,\lambda)\nu_{-1}(\lambda)d\lambda\right)\tau(t), \qquad n>0.
\end{aligned} \tag{1.1}$$

Then the bi-infinite vector

$$\Psi(t,z)=\left(\frac{\tau_n(t-[z^{-1}])}{\tau_n(t)}e^{\sum_1^\infty t_iz^i}z^n\right)_{n\in\mathbf{Z}} \tag{1.2}$$

and bi-infinite matrix

$$L=\sum_{\ell=0}^{\infty} diag\left(\frac{F_\ell(\tilde\partial)\tau_{n+2-\ell}\circ\tau_n}{\tau_{n+2-\ell}\tau_n}\right)_{n\in\mathbf{Z}}\Lambda^{1-\ell} \tag{1.3}$$

satisfy the discrete KP-hierarchy equations for $n=1,2,...$:

$$\frac{\partial\Psi}{\partial t_k}=(L^k)_+\Psi \quad \textit{and} \quad \frac{\partial L}{\partial t_k}=[(L^k)_+,L], \quad \textit{with} \ \ L\Psi(t,z)=z\Psi(t,z). \tag{1.4}$$

Then $\tau_n(t)$ *is given by the following projection*

$$\tau_n(t) = \det\left(Proj : e^{-\sum_1^\infty t_i z^i} z^{-n}\mathcal{W}_n \longrightarrow \mathcal{H}_+\right), \tag{1.5}$$

where the Grassmannian flag $... \supset \mathcal{W}_n \supset \mathcal{W}_{n+1} \supset ...$ *is given by*

$$\mathcal{W}_n := span_{\mathbf{C}}\{\Psi_n(t,z), \Psi_{n+1}(t,z), ...\}. \tag{1.6}$$

Conversely, a Grassmannian flag $... \supset \mathcal{W}_n \supset \mathcal{W}_{n+1} \supset ...$, *given by (1.6), with functions* $\Psi_n(t,z)$ *satisfying the asymptotics* $\Psi_n(t,z) = e^{\sum t_i z^i} z^n (1+O(1/z))$ *leads to the discrete KP-hierarchy.*

Remark : A semi-infinite discrete KP-hierarchy with $\tau_0(t) = 1$ is equivalent to a bi-infinite discrete KP-hierarchy with $\tau_{-n}(t) = \tau_n(-t)$ and $\tau_0(t) = 1$; in the above theorem, this amounts to setting $\tau_0(t) = 1$ and $\nu_{-n}(\lambda) := \nu_{n-1}(\lambda)$, $n = 1, 2, ...$ We extend the semi-infinite flag $\mathcal{W}_0 = \mathcal{H}_+ \supset ... \supset \mathcal{W}_n \supset \mathcal{W}_{n+1} \supset ...$, by setting $\mathcal{W}_{-n} = \mathcal{W}_n^*$, for $n \geq 0$.

2 Moment matrix factorization and solutions to discrete KP and 2d-Toda

In (0.1), we considered t-deformations of the sequence of weights, with $t \in \mathbf{C}^\infty$,

$$(\rho_0^t(z), \rho_1^t(z), ...)\ ,\quad t \in \mathbf{C}^\infty \text{ with } \rho_j^t(z) = e^{\sum_0^\infty t_k z^k}\rho_j(z).$$

As announced in the introduction, we consider further deformations of the sequence of weights, in the (t,s)-direction,

$$\rho_j(t,s;z) = e^{\sum_1^\infty t_i z^i}\sum_{\ell=0}^\infty F_\ell(-s)\rho_{j+\ell}(z),\quad t,s \in \mathbf{C}^\infty,\ z \in \mathbf{R},\ j = 0,1,2,.... \tag{2.1}$$

and the corresponding moment matrix

$$m_n(t,s) = (\mu_{ij}(t,s))_{0\leq i,j<n-1},\ \text{with}\ \ \mu_{ij}(t,s) = \langle z^i, \rho_j(t,s;z)\rangle. \tag{2.2}$$

We now state the following proposition (e.g., see [3, 5]):

Proposition 2.1 *The matrix $m_\infty(t,s)$ satisfies the differential equations*

$$\frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty \ , \quad \frac{\partial m_\infty}{\partial s_k} = -m_\infty \Lambda^{k\top}. \tag{2.3}$$

Factorizing the matrix $m_\infty(t,s)$ into the product of lower- and upper-triangular matrices S_1 and S_2, with S_1 having 1's along the diagonal:

$$m_\infty(t,s) = S_1^{-1}(t,s)S_2(t,s), \tag{2.4}$$

the sequence of wave functions[6]*, derived from S_1 and S_2,*

$$\Psi_i(t,s;z) = e^{\xi_i(z)} S_i \chi(z) \quad \Psi_i^*(t,s;z) = e^{-\xi_i(z)}(S_i^\top)^{-1}\chi^*(z), \tag{2.5}$$

can be expressed in terms of τ-functions $\tau_n(t,s) = \det m_n$, as follows:

$$\begin{aligned}
\Psi_1(t,s;z) &= \left(\frac{\tau_n(t-[z^{-1}],s)}{\tau_n(t,s)} e^{\sum_1^\infty t_i z^i} z^n \right)_{n\in\mathbf{Z}} \\
\Psi_2(t,s;z) &= \left(\frac{\tau_{n+1}(t,s-[z])}{\tau_n(t,s)} e^{\sum_1^\infty s_i z^{-i}} z^n \right)_{n\in\mathbf{Z}} \\
\Psi_1^*(t,s;z) &= \left(\frac{\tau_{n+1}(t+[z^{-1}],s)}{\tau_{n+1}(t,s)} e^{-\sum_1^\infty t_i z^i} z^{-n} \right)_{n\in\mathbf{Z}} \\
\Psi_2^*(t,s;z) &= \left(\frac{\tau_n(t,s+[z])}{\tau_{n+1}(t,s)} e^{-\sum_1^\infty s_i z^{-i}} z^{-n} \right)_{n\in\mathbf{Z}},
\end{aligned} \tag{2.6}$$

with $\Psi_i(t,s)$ satisfying the following differential equations[7]

$$\frac{\partial \Psi_i}{\partial t_n} = (L_1^n)_+ \Psi_i \ , \quad \frac{\partial \Psi_i}{\partial s_n} = (L_2^n)_- \Psi_i \ \text{with } L_1 = S_1 \Lambda S_1^{-1}, \ L_2 = S_2 \Lambda^{-1} S_2^{-1}.$$

The τ-functions satisfy bilinear identities, for all $n,m \geq 0$,

$$\begin{aligned}
&\oint_{z=\infty} \tau_n(t-[z^{-1}],s)\tau_{m+1}(t'+[z^{-1}],s') e^{\sum_1^\infty (t_i-t_i')z^i} z^{n-m-1} dz \\
&\quad = \oint_{z=0} \tau_{n+1}(t,s-[z])\tau_m(t',s'+[z]) e^{\sum_1^\infty (s_i-s_i')z^{-i}} z^{n-m-1} dz,
\end{aligned} \tag{2.7}$$

and therefore the KP hierarchy in each of the variables t and s.

---

[6] $\xi_1(z) := \sum t_i z^i$ and $\xi_2(z) := \sum s_i z^{-i}$; also $\chi(z) := (1, z, z^2, ...)$ and $\chi^*(z) := \chi(z^{-1})$.

[7] A_+ and A_- denote the upper-triangular and strictly lower-triangular part of the matrix A, respectively.

The following corollary can be found in [5]:

Corollary 2.2 *2d-Toda τ-functions satisfy the following (Fay-like) identities for arbitrary $z, u, v \in \mathbf{C}$.*

$$\tau_n(t-[z^{-1}], s+[v]-[u])\tau_n(t,s) - \tau_n(t, s+[v]-[u])\tau_n(t-[z^{-1}], s)$$
$$= \frac{v-u}{z}\tau_{n+1}(t, s-[u])\tau_{n-1}(t-[z^{-1}], s+[v]). \tag{2.8}$$

Introduce now the residue pairing about $z=\infty$, between $f=\sum_{i\geq 0} a_i z^i \in \mathcal{H}^+$ and $g=\sum_{j\in\mathbf{Z}} b_j z^{-j-1} \in \mathcal{H}$:

$$\langle f, g\rangle_\infty = \oint_{z=\infty} f(z)g(z)\frac{dz}{2\pi i} = \sum_{i\geq 0} a_i b_i, \tag{2.9}$$

where the integral is taken over a small circle about $z=\infty$.

But setting $s=s'$ and $m \leq n-1$, the right hand integrand of (2.7) is holomorphic and so the right hand side of (2.7) vanishes. Of course, freezing $s=s'$ yields the discrete KP-hierarchy; see [2]. Therefore

$$\oint_{z=\infty} \tau_n(t-[z^{-1}], s)\tau_{m+1}(t'+[z^{-1}], s)e^{\sum_1^\infty (t_i-t'_i)z^i} z^{n-m-1}dz = 0 \text{ for } n \geq m+1, \tag{2.10}$$

and so for $n \geq m+1$,

$$\oint_{z=\infty} e^{\sum_1^\infty t_i z^i} z^n \frac{\tau_n(t-[z^{-1}], s)}{\tau_n(t,s)} e^{-\sum_1^\infty t'_i z^i} z^{-m-1} \frac{\tau_{m+1}(t'+[z^{-1}], s)}{\tau_m(t,s)} dz = 0. \tag{2.11}$$

Defining the linear space $\mathcal{W}_n^*$ as the space of functions perpendicular to $\mathcal{W}_n$ for the residue pairing (2.9), we thus have for fixed s, by virtue of (1.6), (2.6) and then (2.11),

$$\begin{aligned}
\mathcal{W}_n^t &= \operatorname{span}\{z^j \frac{\tau_j(t-[z^{-1}], s)}{\tau_j(t,s)},\ \ j \geq n\} = e^{-\sum t_i z^i}\mathcal{W}_n \\
\mathcal{W}_n^{*t} &= \operatorname{span}\{z^{-j} \frac{\tau_j(t+[z^{-1}], s)}{\tau_{j-1}(t,s)},\ \ j \leq n\} = e^{\sum t_i z^i}\mathcal{W}_n^*.
\end{aligned} \tag{2.12}$$

It also shows that $\tau_n(t,s)$ can be obtained from those spaces in two different ways (for fixed s):

$$\begin{aligned}
\tau_n(t,s) &= \det\left(\text{Proj} : e^{-\sum t_i z^i} z^{-n}\mathcal{W}_n \longrightarrow \mathcal{H}_+\right) \\
&= \det\left(\text{Proj} : e^{\sum t_i z^i} z^{n}\mathcal{W}_n^* \longrightarrow \mathcal{H}_+\right),
\end{aligned} \tag{2.13}$$

where the multiplication by z^{-n} and z^n makes the corresponding linear spaces have "genus zero", in accordance with the terminology of Segal-Wilson [14].

As a *special case (Hänkel matrices)* , consider the sequence of weights

$$\rho_j(z)dz = z^j \rho(z)dz. \tag{2.14}$$

Then the (t,s)-deformations take on the following form:

$$\rho_j(t,s;z) = e^{\sum t_i z^i} \sum_{\ell \geq 0} F_\ell(-s) z^{\ell+j} \rho(z) = e^{\sum (t_i - s_i) z^i} z^j \rho(z),$$

thus depending on $t-s$ only. Therefore $\mu_{ij}(t,s)$ depends only on $t-s$ and $i-j$ (m_∞ is a Hänkel matrix) and so $\tau_n(t,s)$ depends only on $t-s$. Therefore, in this case we may replace $t-s$ by t.

In this case, the matrix m_∞ is symmetric, which simplifies the factorization (2.4) above. Indeed:

$$m_\infty(t) = S_1^{-1} S_2 = S_1^{-1} h S_1^{-1\top} = S^{-1}(t) S^{\top -1}(t), \tag{2.15}$$

upon setting

$$S = h^{-1/2} S_1 = h^{1/2} S_2^{-1\top}. \tag{2.16}$$

3 Weights, flags and dual flags

The purpose of this section is to prove Theorem 0.1. The point is to derive the τ-functions from the Grassmannian flag (1.6). Unfortunately, the matrix associated with the projection (1.5) is *infinite*; therefore taking its determinant would be non-trivial, although possible. However, it turns out to be infinitely easier to consider the *dual flag*, which leads to a *finite projection matrix*, whose determinant is the same τ-function.

To carry out this program, we equip the space $\mathcal{H} := \operatorname{span}\{z^i,\ i \in \mathbf{Z}\}$ with two inner products: the usual one

$$\langle f, g \rangle = \int_{\mathbf{R}} f(z) g(z)\, dz, \tag{3.1}$$

and remember the residue pairing about $z = \infty$, between $f = \sum_{i \geq 0} a_i z^i \in \mathcal{H}^+$ and $g = \sum_{j \in \mathbf{Z}} b_j z^{-j-1} \in \mathcal{H}$:

$$\langle f, g \rangle_\infty = \oint_{z=\infty} f(z) g(z) \frac{dz}{2\pi i} = \sum_{i \geq 0} a_i b_i, \tag{3.2}$$

where the integral is taken over a small circle about $z = \infty$. The two pairings, which will be instrumental in linking the flag to the dual flag, are related as follows:

Lemma 3.1

$$\langle f, g\rangle = \left\langle f, \int_{\mathbf{R}} \frac{g(u)}{z-u} du \right\rangle_{\infty}. \tag{3.3}$$

<u>*Proof*</u>: Expanding the integral above into an asymptotic series, which we take as its definition,

$$\begin{aligned} \int_{\mathbf{R}} \frac{g(u)}{z-u} du &= \frac{1}{z} \int_{\mathbf{R}} g(u) \sum_{j\geq 0} \left(\frac{u}{z}\right)^j du \\ &= \frac{1}{z} \sum_{j\geq 0} z^{-j} \int_{\mathbf{R}} g(u) u^j du, \end{aligned} \tag{3.4}$$

we check that for holomorphic functions f in $\mathbf{C}$,

$$\begin{aligned} \left\langle f, \int_{\mathbf{R}} \frac{g(u)}{z-u} du \right\rangle_{\infty} &= \left\langle \sum_{i\geq 0} a_i z^i, \frac{1}{z} \sum_{j\geq 0} z^{-j} \int_{\mathbf{R}} g(u) u^j du, \right\rangle_{\infty} \\ &= \sum_{i\geq 0} a_i \int_{\mathbf{R}} g(u) u^i du \\ &= \int_{\mathbf{R}} g(u) \sum_{i\geq 0} a_i u^i du \\ &= \langle f, g\rangle. \end{aligned} \tag{3.5}$$

■

<u>*Remark*</u>: The series (3.4) only converges outside the support of $g(u)$. So, in general, the series (3.4) diverges, even for large z. In specific examples, this integral will have a precise meaning; see sections 4 and 5.

To the family of functions $\rho_0(z), \rho_1(z), \ldots$ on $\mathbf{R}$, and $\rho_j^t(z) := e^{\sum t_k z^k} \rho_j(z)$, we associate the flag of spaces $\mathcal{W}_0 = \mathcal{H}_+ \supset \ldots \supset \mathcal{W}_n \supset \mathcal{W}_{n+1} \supset \ldots$, defined by

$$\begin{aligned} \mathcal{W}_n &:= \left(\operatorname{span}\{\rho_1, \rho_1, \ldots, \rho_{n-1}\}\right)^{\perp} \\ &= \{f \in \mathcal{H}_+ \text{ such that } \langle f, \rho_i\rangle = 0,\ 0 \leq i \leq n-1\} \end{aligned} \tag{3.6}$$

with respect to the inner product (3.1). So, throughout we shall be playing with the following two representations of the moments:

$$\mu_{ij} = \langle z^i \ , \ \rho_j^t(z) \rangle = \left\langle z^i \ , \ \int_{\mathbf{R}} \frac{\rho_j^t(u)du}{z-u} \right\rangle_\infty \tag{3.7}$$

With the moments $\mu_{ij}(t) := \langle z^i, \rho_j^t \rangle$, we associate the monic polynomials $p_k(t,z)$ in z of degree k, introduced in (0.2). As usual, set

$$\mathcal{W}_n^t = e^{-\sum t_i z^i} \mathcal{W}_n \ \text{ and its dual } \ \mathcal{W}_n^{*t} = e^{\sum t_i z^i} \mathcal{W}_n^*.$$

As we showed in (2.12), for the residue pairing we have:

$$\left\langle \mathcal{W}_n^t, \mathcal{W}_n^{*t} \right\rangle_\infty = \langle \mathcal{W}_n, \mathcal{W}_n^* \rangle_\infty = 0.$$

The integral representation (3.9) below of the dual flag already appears in the work of Mulase [13], for the case $\rho_j(z) = z^j \rho(z)$.

Proposition 3.2 *The flag $\mathcal{H}_+ \supset \mathcal{W}_1 \supset \mathcal{W}_2 \supset \ldots$, defined by (3.6) at $t = 0$, evolves into*

$$\mathcal{W}_n^t = (\text{span}\{\rho_0^t, \rho_1^t, \ldots, \rho_{n-1}^t\})^\perp = \text{span}\{p_n(t,z), p_{n+1}(t,z), \ldots\} \subset \mathcal{H}_+, \tag{3.8}$$

and the dual flag $\mathcal{H}_+ \subset \mathcal{W}_1^* \subset \mathcal{W}_2^* \subset \ldots$, evolves into

$$\mathcal{W}_n^{*t} = \text{span}\left(\left\{ \int \frac{\rho_j^t(u)du}{z-u}, j = 0, \ldots, n-1 \right\} \oplus \mathcal{H}_+ \right). \tag{3.9}$$

Proof: Indeed to show (3.8), it suffices to check the following, for $k \geq j+1$ and the polynomials $p_k(t,z) = \frac{1}{a_{kk}(t)} \sum_{i=0}^k a_{ki}(t) z^i$, defined in (0.2):

$$\begin{aligned} \langle p_k(t,z), \rho_j^t \rangle &= \frac{1}{a_{kk}(t)} \sum_{i=0}^k a_{ki}(t) \langle z^i, \rho_j^t \rangle \\ &= \frac{1}{a_{kk}(t)} \sum_{i=0}^k a_{ki} \mu_{ij}(t) \\ &= \frac{1}{a_{kk}(t)} \det \left(\begin{array}{ccc|c} \mu_{00}(t) & \ldots & \mu_{0,k-1}(t) & \mu_{0j}(t) \\ \vdots & & \vdots & \vdots \\ \hline \mu_{k0}(t) & \ldots & \mu_{k,k-1}(t) & \mu_{kj}(t) \end{array} \right) = 0. \end{aligned} \tag{3.10}$$

To prove the dual statement (3.9), one checks for $k \geq j+1$

$$\left\langle p_k(t,z), \int_{\mathbf{R}} \frac{\rho_j^t(u)du}{z-u} \right\rangle_\infty = \langle p_k(t,z), \rho_j^t(z)\rangle = 0,$$

using Lemma 3.1, and, of course,

$$\left\langle p_k(t,z), z^\ell \right\rangle_\infty = 0, \quad \text{for all } k, \ell \geq 0.$$

■

Remember from (2.13), the τ-functions $\tau_n(t)$ can be computed in two different ways:

$$\begin{aligned} \tau_n(t) &= \det\left(\text{Proj} : e^{-\sum t_i z^i} z^{-n}\mathcal{W}_n \longrightarrow \mathcal{H}_+\right) \\ &= \det\left(\text{Proj} : e^{\sum t_i z^i} z^{n}\mathcal{W}_n^* \longrightarrow \mathcal{H}_+\right). \end{aligned} \tag{3.11}$$

We shall need the following lemma concerning Vandermonde-like determinants, extending a lemma mentioned in [13]:

Lemma 3.3

$$\sum_{\sigma\in\prod} \det\left(u_{\ell,\sigma(k)} v_{k,\sigma(k)}\right)_{1\leq \ell,k\leq n} = \det\left(u_{\ell,k}\right)_{1\leq \ell,k\leq n} \det\left(v_{\ell,k}\right)_{1\leq \ell,k\leq n}. \tag{3.12}$$

Proof of theorem 0.1: Since $z^n\mathcal{W}_n^* \supset z^n\mathcal{H}_+$, the matrix of the projection (3.9) onto $\mathcal{H}_+$, involving $\mathcal{W}_n^*$, reduces to a *finite* matrix, whereas the projection involving $\mathcal{W}_n$ would involve an *infinite* matrix! This is the point of using $\mathcal{W}_n^*$ rather then $\mathcal{W}_n$. Therefore the matrix of the projection

$$\text{Proj} : \ e^{\sum t_k z^k} z^n \mathcal{W}_n^{*t} \longrightarrow \mathcal{H}_+$$

is obtained by putting all coefficients of

$$e^{\sum t_k z^k} z^n \int \frac{\rho_j(u)du}{z-u} \ \text{ for } \ (0\leq j\leq n-1) \ \text{ and } \ e^{\sum t_k z^k} z^{n+j} \ \text{ for } \ (0\leq j<\infty)$$

in the jth and $n+j$th columns respectively, starting on top with $z^0, z^1, \ldots$. Since for any power series

$$z^j\text{-coef of } f = \oint_{z=\infty} z^{-j-1} f(z)\frac{dz}{2\pi i} = \langle z^{-j-1}, f(z)\rangle_\infty,$$

we have

$$\tau_n(t) = \det \left(\begin{array}{c|c} A & 0 \\ \hline B & C \end{array} \right) = \det A \det C = \det A, \tag{3.13}$$

where

$$\begin{aligned} C &= \left(\operatorname{coef}_{z^{n+i}} z^{n+j} e^{\sum t_k z^k} \right)_{0 \le i,j < \infty} \\ &= \begin{pmatrix} 1 & 0 & 0 & ... \\ F_1 & 1 & 0 & ... \\ F_2 & F_1 & 1 & ... \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} A &= \left(\operatorname{coef}_{z^i} \left(z^n e^{\sum t_k z^k} \int_{\mathbf{R}} \frac{\rho_j(u)du}{z-u} \right) \right)_{0 \le i,j \le n-1} \\ &= \left(\left\langle z^{n-i-1} e^{\sum t_k z^k}, \int_{\mathbf{R}} \frac{\rho_j(u)du}{z-u} \right\rangle_\infty \right)_{0 \le i,j \le n-1} \\ &= \left(\left\langle u^{n-i-1}, e^{\sum t_k u^k} \rho_j(u) \right\rangle \right)_{0 \le i,j \le n-1} \\ &= \left(\mu_{n-i-1,j}(t) \right)_{0 \le i,j \le n-1}, \end{aligned} \tag{3.14}$$

which provides the A-matrix in (3.13), thus establishing (0.10). Hence,

$$\begin{aligned} \tau_n(t) &= \det \left(\mu_{\ell k}(t) \right)_{0 \le \ell,k \le n-1} \\ &= \det \left(\int_{\mathbf{R}} z^\ell \rho_k(t,z) dz \right)_{0 \le \ell,k \le n-1} \\ &= \det \left(\int_{\mathbf{R}} z_{\sigma(k)}^{\ell-1} \rho_{k-1}(t, z_{\sigma(k)}) dz_{\sigma(k)} \right)_{1 \le \ell,k \le n} \quad \text{for a fixed permutation } \sigma \\ &= \int ... \int_{\mathbf{R}^n} \det \left(z_{\sigma(k)}^{\ell-1} \rho_{k-1}(t, z_{\sigma(k)}) \right)_{1 \le \ell,k \le n} dz_1 ... dz_n \\ &= \frac{1}{n!} \sum_\sigma \int ... \int_{\mathbf{R}^n} \det \left(z_{\sigma(k)}^{\ell-1} \rho_{k-1}(t, z_{\sigma(k)}) \right)_{1 \le \ell,k \le n} dz_1 ... dz_n \\ &= \frac{1}{n!} \int ... \int_{\mathbf{R}^n} \det(z_k^{\ell-1})_{1 \le k,\ell \le n} \det \left(\rho_{\ell-1}(z_k) \right)_{1 \le k,\ell \le n} \prod_{k=1}^n \left(e^{\sum t_i z_k^i} dz_k \right), \end{aligned}$$

using Lemma 3.3; this establishes (0.11). Furthermore, we have, continuing the identities above, that[8]

$$\begin{aligned}
\tau_n(t) &= \frac{1}{n!}\int\cdots\int_{\mathbf{R}^n}\Delta_n(z)\sum_\sigma(-1)^\sigma\prod_{\ell=1}^n\rho_{\ell-1}(z_{\sigma(\ell)})\prod_{k=1}^n\left(e^{\sum t_i z_k^i}dz_k\right)\\
&= \frac{1}{n!}\int\cdots\int_{\mathbf{R}^n}\sum_\sigma\Delta_n(\sigma^{-1}z)(-1)^\sigma\prod_{\ell=1}^n\rho_{\ell-1}(z_\ell)\prod_{k=1}^n\left(e^{\sum t_i z_k^i}dz_k\right)\\
&= \frac{1}{n!}\int\cdots\int_{\mathbf{R}^n}n!\Delta_n(z)\prod_{\ell=1}^n\left(\rho_{\ell-1}(z_\ell)e^{\sum t_i z_\ell^i}dz_\ell\right)\\
&= \int_{\mathbf{R}}z^{n-1}\prod_1^{n-1}\left(1-\frac{z_i}{z}\right)e^{\sum t_i z^i}\rho_{n-1}(z)\,dz\\
&\qquad\int_{\mathbf{R}^{n-1}}\Delta_{n-1}(z_1,\ldots,z_{n-1})\prod_{\ell=1}^{n-1}\left(\rho_{\ell-1}(z_\ell)e^{\sum t_i z_\ell^i}dz_\ell\right)\\
&= \int_{\mathbf{R}}dz\;z^{n-1}\rho_{n-1}(z)e^{\sum t_i z^i}e^{-\sum\frac{1}{iz^i}\frac{\partial}{\partial t_i}}\\
&\qquad\int_{\mathbf{R}^{n-1}}\Delta_{n-1}(z_1,\ldots,z_{n-1})\prod_{\ell=1}^{n-1}\left(\rho_{\ell-1}(z_\ell)e^{\sum t_i z_\ell^i}dz_\ell\right)\\
&= \left(\int_{\mathbf{R}}dz\;z^{n-1}\rho_{n-1}(z)X(t,z)\right)\tau_{n-1}(t),
\end{aligned}$$

proving (0.14). Therefore, the sequence $\tau_n(t)$ satisfies (1.1) in Theorem 1.1 with $\nu_n(z)=z^n\rho_n(z)$ and $\tau_0(t)=1$. The τ-functions lead to the expression (1.3) for L and to the expression (1.2) for Ψ, which both satisfy the discrete KP hierarchy, according to Theorem 1.1. Notice that, from (0.15), (1.5) and (1.2), we have

$$p_n(t,z)=e^{-\sum_1^\infty t_i z^i}\Psi_n(t,z); \tag{3.15}$$

therefore L defined by τ-functions (1.3) agrees with the semi-infinite L, defined by the semi-infinite polynomial relations $zp(t,z)=Lp(t,z)$, yielding (0.8) and (0.9) for this L.

Finally, using (0.10) and (0.11), the wave function (1.2) equals,

$$\begin{aligned}
\Psi_n(t,z) &= z^n e^{\sum_1^\infty t_i z^i}\frac{\tau_n(t-[z^{-1}])}{\tau_n(t)}\\
&= \frac{\det\left(z\mu_{ij}(t)-\mu_{i+1,j}(t)\right)_{0\le i,j\le n-1}}{\det(\mu_{ij}(t))_{0\le i,j\le n-1}}, \quad \text{using (0.10), (0.1), footnote 8,}
\end{aligned}$$

[8]using in the fifth identity $e^{-\sum_1^\infty a^i/i}=1-a$.

$$= \frac{z^n e^{\sum_1^\infty t_i z^i}}{n!\det(\mu_{\ell k}(t))} \int \cdots \int_{\mathbf{R}^n} \Delta_n(z)\Delta_n^{(\rho)}(z) \prod_{k=1}^n \left(e^{\sum t_i z_k^i} \left(1 - \frac{z_k}{z}\right) dz_k \right),$$

establishing (0.16) and (0.17). ■

In the subsequent sections, it is shown that many integrable solutions, when linked together, are nothing but special instances of the situation described in section 2; we mention matrix integrals, $2m+1$-band matrices, soliton formulas, the Calogero-Moser system and others in subsequent sections.

4 Toda lattice, matrix integrals and Riemann-Hilbert for orthogonal polynomials

Setting

$$\rho_j(u)du := u^j \rho(u)du, \quad \rho^t(u) := \rho(u)e^{\sum_1^\infty t_k u^k}, \tag{4.1}$$

define the moment matrix

$$m_\infty(t) := (\mu_{ij}(t))_{0\le i,j<\infty} \text{ with } \mu_{ij}(t) = \int z^{i+j} e^{\sum_1^\infty t_k z^k} \rho(z)dz, \tag{4.2}$$

and the corresponding t-dependent monic orthogonal polynomials $p_n(t,z)$ in z. Note that m_∞ is a Hänkel matrix and is therefore symmetric. From the form of the moments, the matrix $m_\infty(t)$ satisfies the the following differential equations

$$\frac{\partial m_\infty}{\partial t} = \Lambda^k m_\infty. \tag{4.3}$$

Refering to the special case of Hänkel matrices, discussed at the end of section 2, we consider the factorization of the symmetric matrix $m_\infty(t)$ into the product of a lower- and upper-triangular matrix S_1 and S_2, with 1's along the diagonal of S_1 and h's along the diagonal of S_2:

$$m_\infty(t) = S_1^{-1}S_2 = S_1^{-1}hS_1^{-1\top} = S^{-1}(t)S^{\top -1}(t), \text{ with } S = h^{-1/2}S_1 = h^{1/2}S_2^{-1\top}. \tag{4.4}$$

Theorem 4.1 *Then $S(t)$ and the tridiagonal matrix $L(t) = S(t)\Lambda S^{-1}(t)$ satisfy the standard Toda Lattice equations*[9]*:*

$$\frac{\partial S}{\partial t_k} = -\frac{1}{2}(L^k)_b S \quad \textit{and} \quad \frac{\partial L}{\partial t_k} = -\frac{1}{2}[(L^k)_b, L]. \tag{4.5}$$

[9] with regard to the splitting of $A \in gl_\infty$ into a lower-triangular A_b and skew-symmetric matrices A_{sk}.

The flag and dual flag of (0.15) take on the following form

$$
\begin{aligned}
\mathcal{W}_n^t &= \operatorname{span}\{p_n(t,z), p_{n+1}(t,z), ...\} \\
&= \operatorname{span}\{(S(t)\chi(z))_n\,, (S(t)\chi(z))_{n+1}\,, ...\} \\
&= \operatorname{span}\left\{ z^n \frac{\tau_n(t-[z^{-1}])}{\tau_n(t)}, 0 \le n < \infty \right\} \\
\mathcal{W}_n^{*t} &= \operatorname{span}\left\{ \int \frac{p_j(t,u)\rho^t(u)du}{z-u}, j=0,...,n-1 \right\} \oplus \mathcal{H}_+ \\
&= \operatorname{span}\left\{ z^{-1}((Sm_\infty(t))\,\chi^*(z))_j, j=0,...,n-1 \right\} \oplus \mathcal{H}_+ \\
&= \operatorname{span}\left\{ z^{-j-1} \frac{\tau_{j+1}(t+[z^{-1}])}{\tau_j(t)}, j=0,...,n-1 \right\} \oplus \mathcal{H}_+, \qquad (4.6)
\end{aligned}
$$

with the τ-functions having the following representation, derived from (0.10) up to (0.13),

$$
\begin{aligned}
\tau_n(t) &= \det\left(\int z^{i+j}\rho^t(z)dz \right)_{0\le i,j\le n-1} \\
&= \frac{1}{n!}\int ... \int_{\mathbf{R}^n} \Delta_n^2(z) \prod_{\ell=1}^n \left(e^{\sum t_i z_\ell^i}\rho(z_\ell)dz_\ell \right) \\
&= \int_{\mathcal{H}_n} e^{Tr(V(M)+\sum_1^\infty t_i M^i)} dM, \quad \textit{setting } \rho(z)=e^{V(z)} \\
&= \det\left(Proj : e^{-\sum_1^\infty t_i z^i} z^{-n}\mathcal{W}_n \to \mathcal{H}_+ \right) \\
&= \left(\int_{\mathbf{R}} dz \rho(z)\; z^{2(n-1)} X(t,z) \right) \tau_{n-1}(t), \qquad (4.7)
\end{aligned}
$$

and the orthogonal polynomials, having the form

$$
\begin{aligned}
p_n(t,z) &= z^n \frac{\tau_n(t-[z^{-1}])}{\tau_n(t)} \\
&= \frac{\int_{\mathcal{H}_n} \det(zI-M) e^{Tr(V(M)+\sum_1^\infty t_i M^i)} dM}{\int_{\mathcal{H}_n} e^{Tr(V(M)+\sum_1^\infty t_i M^i)} dM},
\end{aligned}
$$

where $dM = \Delta_n^2(z)\; dz_1 ... dz_n\; dU$ is Haar measure on the set of Hermitean matrices $\mathcal{H}_n$.

Before stating the corollary, some explanation is needed. The integral in the matrix below is taken over the **R** with a small upper semi-circle about

z, when $\Im z > 0$ and over $\mathbf{R}$, with a small lower semi-circle about z, when $\Im z < 0$. Moreover $Y_{n\pm}(z) = \lim_{\substack{z' \to z \\ \pm \Im z' > 0}} Y_n(z')$.

Corollary 4.2 *In view of the factorization $m_\infty(t) = S_1^{-1} S_2$ of the moment matrix $m_\infty(t)$ and setting $h_n = \tau_{n+1}(t)/\tau_n(t)$, we have the following identity of matrices:*

$$
\begin{aligned}
Y_n(z) &= \begin{pmatrix} p_n(t,z) & \int_{\mathbf{R}} \frac{p_n(t,u)}{z-u} \rho^t(u)du \\ \\ h_{n-1}^{-1} p_{n-1}(t,z) & h_{n-1}^{-1} \int_{\mathbf{R}} \frac{p_{n-1}(t,u)}{z-u} \rho^t(u)du \end{pmatrix} \\
&= \begin{pmatrix} (S_1\chi(z))_n & \frac{1}{z}\left(S_2\chi^*(z)\right)_n \\ \\ h_{n-1}^{-1}\left(S_1\chi(z)\right)_{n-1} & h_{n-1}^{-1}\frac{1}{z}\left(S_2\chi^*(z)\right)_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\tau_n(t-[z^{-1}])}{\tau_n(t)} z^n & \frac{\tau_{n+1}(t+[z^{-1}])}{\tau_n(t)} z^{-n-1} \\ \frac{\tau_{n-1}(t-[z^{-1}])}{\tau_n(t)} z^{n-1} & \frac{\tau_n(t+[z^{-1}])}{\tau_n(t)} z^{-n} \end{pmatrix}.
\end{aligned} \tag{4.8}
$$

The matrix Y_n satisfies the Riemann-Hilbert problem of Fokas, Its and Kitaev [9]:

1. $Y(z)$ holomorphic[10] *on $\mathbf{C}_+$ and $\mathbf{C}_-$.*

2. $Y_-(z) = Y_+(z)\begin{pmatrix} 1 & 2\pi i \rho^t(z) \\ 0 & 1 \end{pmatrix}$.

3. $Y(z) = (I + O(z^{-1}))\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$, when $z \to \infty$.

(4.9)

Note the first column of $Y(z)$ relates to the Grassmannian $\mathcal{W}_n$ and the lower-triangular matrix S_1, whereas the second column to the dual $\mathcal{W}_n^*$ and the upper-triangular matrix S_2 in the decomposition of $m_\infty = S_1^{-1} S_2$.

Proof of Theorem 4.1 The vertex representation (4.7) of $\tau_n(t)$ shows that the τ-vector $\tau(t) = (\tau_n(t))_{n \geq 0}$ is a solution of the discrete KP equation (1.4). But more is true: $L = S\Lambda S^{-1}$ is tridiagonal; so, S and L satisfy the *standard Toda lattice* (4.5). Some of the arguments are contained in [1].

[10] $\mathbf{C}_+$ and $\mathbf{C}_-$ denote the Siegel upper- and lower half plane.

Notice that the Borel decomposition (4.4) is tantamount to finding the orthogonal polynomials $p_n(t,z)$ with respect to the inner-product $\langle z^i, z^j\rangle = \mu_{ij}$, to be precise:

$$m_\infty = S^{-1}S^{\top -1} \Longleftrightarrow Sm_\infty S^\top = I \Longleftrightarrow \langle h_i^{-1/2}p_i, h_j^{-1/2}p_j\rangle = \delta_{ij}. \tag{4.10}$$

It follows that the coefficients of the *orthonormal* polynomials $h_i^{-1/2}p_i$ are given by the ith row of the matrix $S(t)$ and so

$$S_1(t) = h^{1/2}S(t) = (p_{ij}(t))_{0\le i,j\le\infty}, \text{ where } p_n(t) = \sum_{0\le j\le n} p_{nj}(t)z^j. \tag{4.11}$$

(i) So, the monic polynomials $p_n(t,z)$ of (0.17) have the following form:

$$\begin{aligned} h_n^{1/2}\left(S(t)\chi(z)\right)_n &= \left(S_1(t)\chi(z)\right)_n \\ &= p_n(t,z) \\ &= z^n\frac{\tau_n(t-[z^{-1}])}{\tau_n(t)} \\ &= \frac{1}{n!\tau_n(t)}\int\cdots\int_{\mathbf{R}^n}\Delta^2(u)\prod_{k=1}^n\left((z-u_k)e^{\sum t_iu_k^i}\rho(u_k)du_k\right), \end{aligned} \tag{4.12}$$

leading to the formula in the statement of Theorem 4.1. The p_n's are the standard monic *orthogonal* polynomials with regard to the weight $\rho^t(u) = \rho(u)e^{\sum t_iu^i}$.

(ii) But, we now prove

$$\begin{aligned} h_n^{1/2}\left(S^{\top -1}(t)\chi^*(z)\right)_n = \left(S_2(t)\chi^*(z)\right)_n &= z\int\frac{p_n(t,u)\rho^t(u)}{z-u}du \\ &= z^{-n}\frac{\tau_{n+1}(t+[z^{-1}])}{\tau_n(t)}. \end{aligned} \tag{4.13}$$

Indeed, we compute, on the one hand,

$$\begin{aligned} h_n^{1/2}\sum_{j\ge 0}\left(Sm_\infty\right)_{nj}z^{-j} &= \sum_{j\ge 0}\left(S_1m_\infty\right)_{nj}z^{-j} \\ &= \sum_{j\ge 0}z^{-j}\sum_{\ell\ge 0}p_{n\ell}(t)\mu_{\ell j}, \quad \text{using (4.11)} \end{aligned}$$

$$\begin{aligned}
&= \sum_{j\geq 0} z^{-j} \sum_{\ell \geq 0} p_{n\ell}(t) \int_{\mathbf{R}} u^{\ell+j} \rho^t(u) du \\
&= \int_{\mathbf{R}} \sum_{\ell\geq 0} p_{n\ell}(t) u^{\ell} \sum_{j\geq 0} \left(\frac{u}{z}\right)^j \rho^t(u) du \\
&= z \int_{\mathbf{R}} \frac{p_n(t,u)\rho^t(u)}{z-u} du.
\end{aligned}$$

On the other hand, as we have seen in the special case following (2.14), the 2d-Toda τ-function $\tau(t', s')$ depends on $t = t' - s'$ only, enabling us to write (here ψ stands for Ψ without the exponential),

$$\begin{aligned}
h_n^{1/2} \sum_{j\geq 0} \left(S^{\top -1}(t)\right)_{nj} z^{-j} &= \left(h^{1/2} S^{\top -1}(t) \chi(z^{-1})\right)_n \\
&= \left(S_2(t', s')\chi(z^{-1})\right)_n \ , \text{ using } (4.4), \\
&= \psi_{2,n}(t', s'; z^{-1}) \ , \text{ using } (2.5), \\
&= \frac{\tau_{n+1}(t', s' - [z^{-1}])}{\tau_n(t', s')} z^{-n} \ , \text{ using } (2.6), \\
&= \frac{\tau_{n+1}(t + [z^{-1}])}{\tau_n(t)} z^{-n},
\end{aligned}$$

from which (4.12) follows, upon using $Sm_\infty = S^{\top -1}$; see (4.9).

Theorem 4.1 is established by remembering Proposition 3.2 and using (3.8) and (3.9); i.e.,

$$\begin{aligned}
\mathcal{W}_n^t &= \operatorname{span}\{p_n(t,z), p_{n+1}(t,z), \ldots\} \\
\mathcal{W}_n^{*t} &= \operatorname{span}\left\{\int \frac{u^j \rho^t(u) du}{z-u}, j = 0, \ldots, n-1\right\} \oplus \mathcal{H}_+ \\
&= \operatorname{span}\left\{\int \frac{p_j(t,u)\rho^t(u) du}{z-u}, j = 0, \ldots, n-1\right\} \oplus \mathcal{H}_+,
\end{aligned}$$

together with (4.12). ■

Proof of Corollary 4.2: Following the arguments of Bleher and Its [7], the first matrix in (4.8) has the desired properties taking into account the following integrals:

$$\frac{1}{2\pi i} \lim_{\substack{z' \to z \\ \Im z' < 0}} \int_{\mathbf{R}} \frac{p_n(t,u)}{z'-u} \rho^t(u) du = p_n(t,z)\rho^t(z) + \frac{1}{2\pi i} \lim_{\substack{z' \to z \\ \Im z' > 0}} \int_{\mathbf{R}} \frac{p_n(t,u)}{z'-u} \rho^t(u) du.$$

The formulas (4.11) and (4.12) lead to the desired result. ∎

Remark: From the fact that $\det Y_{n-} = \det Y_{n+}$, it follows that $\det Y(z)$ is holomorphic in $\mathbf{C}$ and since $\det Y(z) = 1 + O(z^{-1})$, it follows from Liouville's theorem that $\det Y(z) = 1$, i.e.,

$$\begin{aligned}\det Y_n &= h_{n-1}^{-1}\left(p_n(t,z)\int_{\mathbf{R}}\frac{p_{n-1}(t,u)}{z-u}\rho^t(u)du - p_{n-1}(t,z)\int_{\mathbf{R}}\frac{p_n(t,u)}{z-u}\rho^t(u)du\right)\\ &= \frac{1}{\tau_n^2(t)}\left(\tau_n(t-[z^{-1}])\tau_n(t+[z^{-1}]) - z^{-2}\tau_{n-1}(t-[z^{-1}])\tau_{n+1}(t+[z^{-1}])\right)\\ &= 1. \end{aligned} \tag{4.14}$$

This is not surprising, in view of the fact that the first expression for $\det Y_n$ is nothing but the Wronskian of the the two fundamental solutions of the second order difference equation; see Akhiezer [6]. The second expression, involving τ-functions follows also from Corollary 2.2, by setting $u = z^{-1}$ and $v \to 0$ and by using the fact that, for the standard Toda lattice, we have $\tau(t,s) = \tau(t-s)$.

5 Periodic sequences of weights, $2m+1$-band matrices and Riemann-Hilbert problems

The results of section 4 about tridiagonal matrices will be extended in this section to $2m+1$-band matrices. As usual, we set

$$\mu_{ij}(t) = \left\langle z^i, \rho_j^t(z)\right\rangle, \text{ with } \rho_j^t(z) = e^{\sum t_k z^k}\rho_j(z). \tag{5.1}$$

In proving and stating the results below, we shall also consider the s-deformations, as in (2.1). Here we consider m-periodic sequences of weights $\rho_0, \rho_1, ...$, defined by

$$\rho_{j+km}(z) = z^{km}\rho_j(z), \quad \text{for all } j = 0,1,2,.... \tag{5.2}$$

Theorem 5.1 *For the weights (5.2), the polynomials*

$$p_n(t,z) = \frac{1}{\det\left(\mu_{\ell,k}(t)\right)_{0\le \ell,k\le n-1}}\det\left(\begin{array}{ccc|c} \mu_{00}(t) & \dots & \mu_{0,n-1}(t) & 1\\ \vdots & & \vdots & \vdots\\ \mu_{n-1,0}(t) & \dots & \mu_{n-1,n-1}(t) & z^{n-1}\\ \hline \mu_{n0}(t) & \dots & \mu_{n,n-1}(t) & z^n \end{array}\right)$$

$$
\begin{aligned}
&= z^n \frac{\tau_n(t-[z^{-1}],0)}{\tau_n(t,0)}, \ \textit{where} \ \ \tau_n(t,0) = \det\, m_n(t) \\
&= \frac{\det\left(z\mu_{ij}(t) - \mu_{i+1,j}(t)\right)_{0\le i,j\le n-1}}{\det(\mu_{ij}(t))_{0\le i,j\le n-1}} \\
&= \frac{1}{n!\tau_n(t)} \int \dots \int_{\mathbf{R}^n} \Delta_n(z)\Delta_n^{(\rho)}(z) \prod_{k=1}^{n} \left(e^{\sum_i t_i z_k^i}(z-z_k)\, dz_k\right), \quad (5.3)
\end{aligned}
$$

lead to matrices L, defined by $zp = Lp$,

$$
\left\{
\begin{array}{l}
\textit{(i) which evolve according to the discrete KP hierarchy} \\
\textit{(ii) such that } L^m \textit{ is a } \underline{2m+1\textit{-band matrix}}. \\
\textit{(iii) the polynomials } p_n(z) \textit{ satisfy the } \text{generalized orthogonality relations} \\
\qquad \langle p_i(z), \rho_j(z)\rangle = 0 \textit{ for } i \ge j+1.
\end{array}
\right.
$$

Remark: It is interesting to point out that the condition (5.2) is equivalent to a seemingly weaker one:

$$
z^m \rho_j \in \operatorname{span}\{\rho_0, ..., \rho_{m+j}\}, \quad \text{for all } j = 0,1,2,..., \tag{5.4}
$$

where ρ_{m+j} must appear in the span. Indeed, the p_n's only depend on the moments μ_{ij} by means of the determinantal formulae (5.3), which allow for column operations.

Corollary 5.2 *The following 2×2 matrices are all equal*

$$
\begin{aligned}
Y_n(z) &= \begin{pmatrix} p_n(t,z) & \displaystyle\int_{\mathbf{R}} \frac{p_n(t,u)}{z^m-u^m}\left(\sum_{k=1}^m z^{m-k}\rho^t_{k-1}(u)\right) du \\ h_{n-1}^{-1} p_{n-1}(t,z) & \displaystyle h_{n-1}^{-1}\int_{\mathbf{R}} \frac{p_{n-1}(t,u)}{z^m-u^m}\left(\sum_{k=1}^m z^{m-k}\rho^t_{k-1}(u)\right) du \end{pmatrix} \\
&= \begin{pmatrix} (S_1\chi(z))_n & \frac{1}{z}\left(S_2\chi^*(z)\right)_n \\ h_{n-1}^{-1}\left(S_1\chi(z)\right)_{n-1} & h_{n-1}^{-1}\frac{1}{z}\left(S_2\chi^*(z)\right)_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{\tau_n(t-[z^{-1}],0)}{\tau_n(t,0)} z^n & \dfrac{\tau_{n+1}(t,-[z^{-1}])}{\tau_n(t,0)} z^{-n-1} \\ \dfrac{\tau_{n-1}(t-[z^{-1}],0)}{\tau_n(t,0)} z^{n-1} & \dfrac{\tau_n(t,-[z^{-1}])}{\tau_n(t,0)} z^{-n} \end{pmatrix};
\end{aligned}
\tag{5.5}
$$

they solve the following Riemann-Hilbert problem:
1. $Y_n(z)$ *holomorphic on* $\mathbf{C}_+$ *and* $\mathbf{C}_-$.

2. $Y_{n-}(z) = Y_{n+}(z)\begin{pmatrix} 1 & \frac{2\pi i}{m} e^{\sum t_k z^k} \sum_1^m z^{-j}\rho_j(z) \\ 0 & 1 \end{pmatrix}$.

3. $Y_n(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \longrightarrow I$, *as* $z \to \infty$.

4. $Y_n(z)\begin{pmatrix} 1 & 0 \\ 0 & z^{m-1} \end{pmatrix}$ *finite , as* $z \to 0$.

The polynomials p_n *are such that* $z^m p_n$ *satisfies* $\underline{2m+1\text{-step relations}}$.

Note

$$\det Y_n = \frac{\tau_n(t-[z^{-1}], -[z^{-1}])}{\tau_n(t,0)}$$

and the first column of Y_n *is related to the Grassmannian plane* $\mathcal{W}_n$ *and the second column to a plane*[11] *related to* $\Psi_2(t,s,z)$.

Remark: In the matrix (5.5), $\tau_n(t-[z^{-1}],0)$ is given in formula (5.3), whereas by (0.10) and (5.11),

$$\tau_n(t, -[z^{-1}]) = \det\left(\left\langle u^i, e^{\sum_1^\infty t_i u^i} \sum_{k=0}^{m-1} \frac{z^{m-k}\rho_{k+j}(u)}{z^m - u^m}\right\rangle\right)_{0\le i,j\le n-1}. \tag{5.6}$$

Also note the right hand column of (5.5) behaves as $1/z^{n+1}$; this follows from the τ-function representation, but also from the "generalized orthogonality", mentioned in (iii) (Theorem 5.1).

Proving Theorem 5.1 requires the following lemma:

Lemma 5.3 *Fix* $m \ge 1$; *the polynomials* $p_n(t,z)$, *defined in (0.2), satisfy* $2m+1$*-step recursion relations, i.e.,*

$$z^m p(t,z) = L^m p(t,z) \text{ with } L^m \text{ a } 2m+1\text{-band matrix,}$$

if and only if every ρ_j, $j = 0,1,\ldots$ *satisfies the following requirement:*

$$\left\{\begin{array}{l} \text{For every } \ell = 0,1,\ldots,\ j=0,1,\ldots \text{ there exist constants } c_r,\ r=0,\ldots,m+j+\ell \\ \text{depending on } j \text{ and } \ell \text{ such that} \\ \qquad \langle u^m\rho_j - \sum_0^{m+j+\ell} c_r\rho_r, u^i\rangle = 0 \text{ for } 0\le i\le m+j+\ell+1 \end{array}\right.$$

[11] Unlike the orthogonal polynomial case, the second column does not contain elements of the dual Grassmannian $\mathcal{W}^*$.

Proof: Note the following equivalences:

$$\begin{array}{lll} z^m p_n(t,z) = \displaystyle\sum_{n-m\leq r\leq n+m} A(t)_{nr} p_r(t,z) & & \text{for some matrix } A(t) \\ \\ \Longleftrightarrow z^m p_n(t,z) \in \mathcal{W}^t_{\max(n-m,0)} & & \text{for all } n \geq 0 \\ \\ \Longleftrightarrow z^m \mathcal{W}^t_n \subset \mathcal{W}^t_{\max(n-m,0)} & & \text{for all } n \geq 0\text{, because} \\ & & \text{of the inclusion} \\ & & ... \supset \mathcal{W}_n \supset \mathcal{W}_{n+1} \supset ..., \\ \\ \Longleftrightarrow z^m \mathcal{W}_n \subset \mathcal{W}_{\max(n-m,0)}. & & \end{array}$$

Since

$$\mathcal{W}_n = (\operatorname{span}\{\rho_0, \rho_1, ..., \rho_{n-1}\})^\perp = \operatorname{span}\{p_n(z), p_{n+1}(z), ...\}$$

the latter is equivalent to

$$\begin{aligned} 0 &= \langle u^m p_n(u), \rho_j(u)\rangle \quad \text{for all } 0 \leq j \leq n-m-1,\ n \geq 0 \\ &= \langle p_n(u), u^m \rho_j(u)\rangle \\ &= \frac{1}{a_{nn}(t)} \sum_{i=0}^{n} a_{ni} \langle u^i, u^m \rho_j(u)\rangle \\ &= \frac{1}{a_{nn}(t)} \det \left(\begin{array}{ccc|c} \mu_{00} & \cdots & \mu_{0,n-1} & \mu_{mj} \\ \vdots & & \vdots & \vdots \\ \hline \mu_{n0} & \cdots & \mu_{n,n-1} & \mu_{m+n,j} \end{array} \right), \end{aligned}$$

where we have used the fact that $p_n(t,z) = \frac{1}{a_{nn}(t)} \sum a_{ni}(t) z^i$ is represented by (0.2). The vanishing of the determinant above is equivalent to the statement that the last column depends on prior columns; namely there exist $c_0, ..., c_{n-1}$ depending on m, n, j such that

$$\begin{aligned} 0 &= \mu_{m+i,j} - \sum_{r=0}^{n-1} c_r \mu_{ir} \quad \text{for } 0 \leq i \leq n,\ 0 \leq j \leq n-m-1 \\ &= \langle u^{i+m}, \rho_j\rangle - \sum_{r=0}^{n-1} c_r \langle u^i, \rho_r\rangle \\ &= \langle u^i, u^m \rho_j - \sum_{r=0}^{n-1} c_r \rho_r\rangle \end{aligned}$$

$$= \quad \langle u^i, u^m \rho_j - \sum_{r=0}^{j+m+\ell} c_r \rho_r \rangle \quad \text{for } 0 \le i \le m+j+\ell+1,$$

where ℓ was defined such that $j + \ell = n - m - 1$. ■

<u>*Proof of Theorem 5.1*</u>: The fact that L^m is a $2m+1$-band matrix follows at once from Lemma 5.3. That the matrix L evolves according to the discrete KP-hierarchy follows straightforwardly from the general statement in Theorem 0.1. ■

<u>*Proof of Corollary 5.2*</u>: For the sake of this proof, we shall be using the (t, s)-deformations of the weights ρ_j and the corresponding matrix $m_\infty(t, s)$ of (t, s)-dependent moments

$$\mu_{ij}(t,s) = \left\langle z^i, \rho_j(t,s;z) \right\rangle, \text{ with } \rho_j(t,s;z) = e^{\sum_1^\infty t_i z^i} \sum_{\ell=0}^{\infty} F_\ell(-s) \rho_{j+\ell}(z). \tag{5.7}$$

In section 2, it was mentioned that m_∞ satisfies the differential equations

$$\frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty \ , \quad \frac{\partial m_\infty}{\partial s_k} = -m_\infty \Lambda^{k\top}. \tag{5.8}$$

Factorizing the matrix

$$m_\infty(t,s) = S_1^{-1}(t,s) S_2(t,s) \tag{5.9}$$

into the product of lower- and upper-triangular matrices S_1 and S_2 then leads to the 2d Toda lattice.

For later use, we also compute $\rho_k(t, -[z^{-1}]; u)$, making specific use of the periodicity of the sequence of weights $\rho_{j+m}(u) = u^m \rho_j(u)$ and the identity[12]

$$F_n([z^{-1}]) = z^{-n}. \tag{5.10}$$

We find:

$$\rho_k(t, -[z^{-1}]; u) \quad = \quad e^{\sum_1^\infty t_i u^i} \sum_{j=0}^{\infty} F_j([z^{-1}]) \rho_{j+k}(u)$$

[12]obtained, by expanding the following expression in elementary Schur polynomials, by setting $t = 0$ and by comparing the powers of y:

$$\sum_{n \ge 0} y^n F_n(t + [z^{-1}]) = e^{\sum \left(t_i + \frac{z^{-i}}{i}\right) y^i} = e^{\sum t_i y^i} \left(1 - \frac{y}{z}\right)^{-1} = \sum_{n \ge 0} y^n F_n(t) \sum_0^\infty \left(\frac{y}{z}\right)^k.$$

$$
\begin{aligned}
&= e^{\sum_1^\infty t_i u^i} \sum_{j=0}^{\infty} \frac{\rho_{j+k}(u)}{z^j}, \quad \text{using again (),} \\
&= e^{\sum_1^\infty t_i u^i} \left(\sum_{j=0}^{m-1} \frac{\rho_{j+k}(u)}{z^j} + \sum_{j=m}^{2m-1} \frac{\rho_{j+k}(u)}{z^j} + ... \right) \\
&= e^{\sum_1^\infty t_i u^i} \sum_{j=0}^{m-1} \frac{\rho_{j+k}(u)}{z^j} \left(1 + (\frac{u}{z})^m + (\frac{u}{z})^{2m} + ... \right) \\
&= e^{\sum_1^\infty t_i u^i} \sum_{j=0}^{m-1} \frac{\rho_{j+k}(u)}{z^j(1-(\frac{u}{z})^m)} \\
&= e^{\sum_1^\infty t_i u^i} \sum_{j=0}^{m-1} \frac{z^{m-j}\rho_{j+k}(u)}{z^m - u^m}. \qquad (5.11)
\end{aligned}
$$

From (5.9), we have $S_1 m_\infty = S_2$ and hitting $\chi^*(z)$ with this matrix, we compute, on the one hand,

$$
\begin{aligned}
\sum_{j\geq 0} (S_1 m_\infty)_{nj}\, z^{-j}\Big|_{s=0} &= \sum_{j\geq 0} z^{-j} \sum_{\ell \geq 0} p_{n\ell}(t,s)\mu_{\ell j}\Big|_{s=0} \\
&= \sum_{j\geq 0} z^{-j} \sum_{\ell \geq 0} p_{n\ell}(t,s) \int_{\mathbf{R}} u^\ell \rho_j(t,s;u)du\Big|_{s=0} \\
&= \sum_{j\geq 0} z^{-j} \int_{\mathbf{R}} p_n(t,s;u) e^{\sum_1^\infty t_i u^i} \sum_{\ell\geq 0} F_\ell(-s)\rho_{j+\ell}(u)du\Big|_{s=0} \\
&= \int_{\mathbf{R}} p_n(t,0;u) e^{\sum_1^\infty t_i u^i} \sum_{j\geq 0} z^{-j}\rho_j(u)du \\
&= \int_{\mathbf{R}} p_n(t,0;u) e^{\sum_1^\infty t_i u^i} \sum_{j\geq 0} F_j([z^{-1}])\rho_j(u)du \\
&= \int_{\mathbf{R}} p_n(t,0;u) \rho_0(x, -[z^{-1}];u)du \\
&= \int_{\mathbf{R}} p_n(t,0;u) e^{\sum_1^\infty t_i u^i} \sum_{j=0}^{m-1} \frac{z^{m-j}\rho_j(u)}{z^m-u^m} du, \qquad (5.12)
\end{aligned}
$$

using (5.11) in the last identity. On the other hand, we have

$$
\begin{aligned}
\sum_{j\geq 0} (S_2(t,s))_{nj}\, z^{-j}\Big|_{s=0} &= \Big(S_2(t,s)\chi(z^{-1})\Big)_n\Big|_{s=0} \\
&= \psi_{2,n}(t,0;z^{-1}) \\
&= \frac{\tau_{n+1}(t,-[z^{-1}])}{\tau_n(t,0)} z^{-n}. \qquad (5.13)
\end{aligned}
$$

The right hand sides of (5.12) and (5.13) coincide (using $S_1 m_\infty = S_2$); so, we find the following identity, together with the desired asymptotics for $z \to \infty$:

$$\int_{\mathbf{R}} \frac{p_n(t;u)}{z^m - u^m} \left(\sum_{j=0}^{m-1} z^{m-j} \rho_j^t(u) \right) du = z^{-n} \frac{\tau_{n+1}(t, -[z^{-1}])}{\tau_n(t,0)} = z^{-n}(h_n + O(1)). \tag{5.14}$$

leading to condition 3. The jump condition 2. follows from the following

$$\frac{1}{2\pi i} \lim_{\substack{z' \to z \\ \Im z' < 0}} \int \frac{p_n(t,u)}{z^m - u^m} \left(\sum_{j=0}^{m-1} z^{m-j} \rho_j^t(u) \right) du$$

$$= p_n(t,z) \frac{1}{m} \sum_{j=0}^{m-1} z^{1-j} \rho_j^t(z) + \frac{1}{2\pi i} \lim_{\substack{z' \to z \\ \Im z' > 0}} \int \frac{p_n(t,u)}{z^m - u^m} \left(\sum_{j=0}^{m-1} z^{m-j} \rho_j^t(u) \right) du. \tag{5.15}$$

The asymptotics 3. follows from the τ-function representation of the first integral.

The formula concerning $\det Y_n$ follows from setting $u = 0$ and $v = z^{-1}$ in identity (2.8). ∎

6 Soliton formula

For future use, we define the vertex operator:

$$X(t; \lambda, \mu) = \frac{1}{\lambda - \mu} e^{\sum_1^n (\lambda^k - \mu^k) t_k} e^{\sum_1^n (\mu^{-k} - \lambda^{-k}) \frac{1}{k} \frac{\partial}{\partial t_k}}. \tag{6.1}$$

Theorem 6.1 *Given points p_k, q_k and λ_k, $k = 1, \ldots$, and the weights*

$$\rho_k = \delta(z - p_{k+1}) - \lambda_{k+1}^2 \delta(z - q_{k+1}), \quad k = 0, 1, \ldots,$$

the τ-functions

$$\begin{aligned} \tau_n(t) &= \det \left(p_i^{j-1} e^{\sum_{k=1}^\infty t_k p_i^k} - \lambda_i^2 q_i^{j-1} e^{\sum_{k=1}^\infty t_k q_i^k} \right)_{1 \le i,j \le n} \\ &= \left(p_n^{n-1} X(t, p_n) - \lambda_n^2 q_n^{n-1} X(t, q_n) \right) \tau_{n-1}(t) \end{aligned}$$

$$\begin{aligned} &= c_n \prod_{k=1}^n e^{\sum t_i p_k^i} \det\left(\delta_{ij} - \frac{a_i}{q_i - p_j} e^{\sum_{k=1}^\infty t_k (q_i^k - p_j^k)}\right)_{1\le i,j\le n} \\ &= c_n \prod_{k=1}^n e^{\sum t_i p_k^i} \prod_1^n e^{-a_i X(t; q_i, p_i)} 1, \end{aligned}$$

form a τ-vector of the discrete KP hierarchy, for appropriately chosen functions a_i and c_n of p, q, λ. The matrix L, constructed by (0.9) from the τ's above, satisfies

$$L(t)p(t,z) = zp(t,z),$$

with polynomial eigenvectors (in z):

$$p_n(z) = \frac{\det\left(\delta_{ij}(z-p_i) - \frac{a_i(z-q_i)}{q_i - p_j} e^{\sum_{k=1}^\infty t_k(q_i^k - p_j^k)}\right)_{1\le i,j\le n}}{\det\left(\delta_{ij} - \frac{a_i}{q_i-p_j} e^{\sum_{k=1}^\infty t_k (q_i^k - p_j^k)}\right)_{1\le i,j\le n}}.$$

Then

$$\begin{aligned} \mathcal{W}_n &= (\mathrm{span}\{\rho_0, ..., \rho_{n-1}\})^\perp \\ &= \{f \in \mathcal{H}^+, \text{ such that } f(p_i) = \lambda_i^2 f(q_i), i = 1, ..., n\} \quad (6.2) \end{aligned}$$

and

$$\mathcal{W}_n^* = \mathrm{span}\left\{\frac{1}{z-p_i} - \frac{\lambda_i^2}{z-q_i}, i = 1, ..., n\right\} \oplus \mathcal{H}_+. \quad (6.3)$$

Proof: Consider the space

$$\begin{aligned} \mathcal{H}^+/z^n\mathcal{H}^+ &= \mathrm{span}\{1, ..., z^{n-1}\} \\ &= \mathrm{span}\{v_1, ..., v_n\}, \end{aligned} \quad (6.4)$$

where the polynomials

$$v_k(z) = \prod_{\substack{j\ne k \\ 1\le j\le n}} (z - p_j)$$

of degree $n-1$ form an alternative basis; the determinant of the transformation between the two bases being the Vandermonde determinant $\Delta_n(p) = \prod_{1\le j<k\le n}(p_k - p_j)$. Define

$$P(z) := \prod_1^n (z - p_i) \quad (6.5)$$

and

$$g(z)=e^{\sum_1^\infty t_i z^i},\quad a_k=\lambda_k^2\frac{P(q_k)}{v_k(p_k)},\quad c_n(t) = \frac{\prod_1^n\left(P'(p_k)e^{\sum_{i=1}^\infty t_i p_k^i}\right)}{\Delta_n(p)}$$
$$= c_n\prod_{k=1}^n e^{\sum_{i=1}^\infty t_i p_k^i}.$$

With this notation

$$v_k(p_i)=\delta_{ik}v_k(p_k)=\delta_{ik}P'(p_k),\qquad v_k(q_i)=\frac{P(q_i)}{q_i-p_k}. \tag{6.6}$$

Using the second line of (3.14) and the formula for ρ_i, one computes

$$\begin{aligned}
\tau_n(t) &= \det(\mu_{ji})_{0\le i,j\le n-1}\\
&= \det\left(\oint_{z=\infty} z^{-j-1}g(z)z^n\left(\frac{1}{z-p_i}-\frac{\lambda_i^2}{z-q_i}\right)\frac{dz}{2\pi i}\right)_{\substack{1\le i\le n\\ 0\le j\le n-1}}\\
&= \frac{1}{\Delta(p)}\det\left(\oint_{z=\infty} v_j(z)g(z)\left(\frac{1}{z-p_i}-\frac{\lambda_i^2}{z-q_i}\right)\frac{dz}{2\pi i}\right)_{1\le i,j\le n},
\end{aligned}$$

using (6.4). The second identity above leads to the first formula in the statement above about τ, whereas (0.14) is responsible for the second formula. The last formula on the right hand side,just above, leads to

$$\begin{aligned}
\tau_n(t) &= \frac{1}{\Delta(p)}\det\Big(v_j(p_i)g(p_i)-v_j(q_i)g(q_i)\lambda_i^2\Big)_{1\le i,j\le n}\\
&= \frac{1}{\Delta(p)}\det\Big[\mathrm{diag}(g(p_i))_{1\le i\le n}\\
&\qquad \Big(\delta_{ij}v_i(p_i)g(p_i)-v_j(q_i)g(q_i)\lambda_i^2\Big)_{1\le i,j\le n}\,\mathrm{diag}\left(g(p_i)^{-1}\right)_{1\le i\le n}\Big]\\
&= \frac{1}{\Delta(p)}\det\left(\delta_{ij}v_i(p_i)g(p_i)-\lambda_i^2 v_j(q_i)g(q_i)\frac{g(p_i)}{g(p_j)}\right)_{1\le i,j\le n},\ \text{using (6.4)}\\
&= c_n(t)\ \det\left(\delta_{ij}-\lambda_i^2\frac{v_j(q_i)}{v_i(p_i)}\frac{g(q_i)}{g(p_j)}\right)_{1\le i,j\le n}\\
&= c_n(t)\ \det\left(\delta_{ij}-\frac{a_i}{q_i-p_j}\frac{g(q_i)}{g(p_j)}\right)_{1\le i,j\le n}\\
&= c_n(t)\ \det\left(\delta_{ij}-a_iX(t,q_i,p_j)1\right)_{1\le i,j\le n}\\
&= c_n(t)\prod_{i=1}^n e^{-a_iX(t,q_i,p_i)}1,
\end{aligned}$$

using in the last equality the vanishing of the square of the vertex operator.

The formula for $p_n(t,z)$ is derived from the third expression for $\tau(t)$, using the standard representation (2.6) for the wave vector $\Psi(t,z)$. ■

Remark: When $q_i = -p_i$, the formula for the KdV τ-function reads:

$$\begin{aligned}\tau_n(t) &= \left(\prod_1^n p_i\right)^n \prod_1^n \lambda_i e^{\sum_{k,i} t_{2k} p_i^{2k}} \\ &\qquad \det\left(p_i^{-j}\left(\lambda_i^{-1} e^{\sum_{\text{odd}} t_k p_i^k} - (-1)^{n-j}\lambda_i e^{-\sum_{\text{odd}} t_k p_i^k}\right)\right)_{1\le i,j\le n} \\ &= c_n(t)\det\left(\delta_{ij} + \frac{a_i}{p_i+p_j} e^{-\sum_{\text{odd}} t_k (p_i^k + p_j^k)}\right)_{1\le i,j\le n}.\end{aligned}$$

Note that Segal and Wilson have used, in [14], the infinite matrix representation of the projection of (6.2), rather than (6.3), in order to compute KdV solitons.

7 Calogero-Moser system

Theorem 7.1 *Given points p_k, λ_k $(k = 1, 2, ...)$, the weights*

$$\rho_k = \delta'(z - p_{k+1}) + \lambda_{k+1}\delta(z - p_{k+1}), \quad k = 0, 1, ..., n-1$$

determine a sequence of τ-functions for the discrete KP equation[13],

$$\begin{aligned}\tau_n(t) &= \frac{1}{n!}\int \cdots \int_{\mathbf{R}^n} \prod_{k=1}^n e^{\sum t_i z_k^i} \Delta_n(z)\Delta_n^{(\rho)}(z) dz_1 ... dz_n \\ &= e^{tr \sum_1^\infty t_i Y^i} \det\left(-X + \sum_1^\infty k \bar{t}_k Y^{k-1}\right), \end{aligned} \tag{7.1}$$

with appropriate matrices X and Y, functions of p_k and λ_k's, satisfying the commutation relation[14] $[X, Y] = I_e$, *and having the form*

$$X = \operatorname{diag}(x_1, ..., x_n) \quad \text{and} \quad Y = \left(\frac{1-\delta_{ij}}{x_i - x_j}\right)_{ij} + \operatorname{diag}(\xi_1, ..., \xi_n). \tag{7.2}$$

[13] $\bar{t} = (x, t_2, t_3, ...)$.

[14] where $I_e = (1 - \delta_{ij})_{1\le i,j\le n}$.

The matrix L, constructed by (0.9) from the τ's above, satisfies

$$L(t)p(t,z) = zp(t,z),$$

with eigenvectors, polynomial in z,

$$p_n(\bar{t},z) = \det\left(zI - Y - \left(xI + \sum_1^\infty kt_k Y^{k-1} - X\right)^{-1}\right). \tag{7.3}$$

The Grassmannian flag corresponding to this construction is given by

$$\begin{aligned}\mathcal{W}_n &= \{f \in \mathcal{H}^+, \text{ such that } f'(p_i) = \lambda_i f(p_i), 1 \le i \le n\},\\ \mathcal{W}_n^* &= \left\{\frac{1}{(z-p_i)^2} - \frac{\lambda_i}{z-p_i}, i = 1, ..., n\right\} \oplus \mathcal{H}_+.\end{aligned} \tag{7.4}$$

Proof: As before, we introduce the basis $v_k(z)$ of $\mathcal{H}^+/z^n\mathcal{H}^+$; note

$$\frac{\partial}{\partial z} v_k = \sum_{i=1}^n \prod_{j\neq k,i}(z-p_j) = \sum_{\substack{i=1\\ i\neq k}} \frac{v_k(z) - v_i(z)}{p_k - p_i}. \tag{7.5}$$

and the matrices

$$\begin{aligned}\tilde{X} &= -\operatorname{diag}\left(\sum_{\substack{1\le\alpha\le n\\ \alpha\neq i}} \frac{1}{p_i - p_\alpha} - \lambda_i\right)_{1\le i\le n} - \left(\frac{1-\delta_{ij}}{p_i - p_j}\right)_{1\le i,j\le n}\\ \tilde{Y} &= \operatorname{diag}(p_1, ..., p_n)\end{aligned} \tag{7.6}$$

with commutation relation

$$[\tilde{X}, \tilde{Y}] = I_e, \quad I_e = (1 - \delta_{ij})_{1\le i,j\le n}. \tag{7.7}$$

Then, by (3.7) and the choice[15] of ρ_i,

$$\begin{aligned}&\tau_n(t)\\ &= \det(\mu_{ij})_{0\le i,j\le n-1}\\ &= \det\left(\oint_{z=\infty} z^{j-1} g(z)\left(\frac{1}{(z-p_i)^2} - \frac{\lambda_i}{z-p_i}\right)\frac{dz}{2\pi i}\right)_{1\le i,j\le n}\end{aligned}$$

[15] $c_n := \frac{\prod_1^n v_i(p_i)}{\Delta_n(p)}$ in the expressions below.

$$
\begin{aligned}
&= \frac{1}{\Delta_n(p)} \det\left(\oint v_j(z) g(z)\left(\frac{1}{(z-p_i)^2}-\frac{\lambda_i}{z-p_i}\right)\frac{dz}{2\pi i}\right)_{1\le i,j\le n}\\
&= \frac{1}{\Delta_n(p)} \det\left((v_j g)'\Big|_{z=p_i} - \lambda_i v_j(p_i) g(p_i)\right)_{1\le i,j\le n}\\
&= \frac{1}{\Delta_n(p)} \det\left(g(p_i)\sum_{\substack{\alpha=1\\ \alpha\neq j}}^{n}\frac{v_j(p_i)-v_\alpha(p_i)}{p_j-p_\alpha} + v_j(p_i)g(p_i)\left(\sum_1^\infty k t_k p_i^{k-1}-\lambda_i\right)\right)_{1\le i,j\le n}\\
&= c_n \prod_{k=1}^n e^{\sum_1^\infty t_i p_k^i} \det\left(\frac{1-\delta_{ij}}{p_i-p_j} + \delta_{ij}\left(\sum_{\substack{\alpha=1\\ \alpha\neq i}}^{n}\frac{1}{p_i-p_\alpha}+\sum_1^\infty k t_k p_i^{k-1}-\lambda_i\right)\right)_{1\le i,j\le n}\\
&= c_n e^{\operatorname{tr}\sum_1^\infty t_i \tilde Y^i} \det\left(-\tilde X + \sum_1^\infty k t_k \tilde Y^{k-1}\right),
\end{aligned}
$$

yielding the formula for $\tau_n(t)$; According to theorem 0.1, the $p_n(t,z)$ are polynomials, which we now compute:

$$
\begin{aligned}
p_n(t,z) &= z^n \frac{\tau_n(\bar t - [z^{-1}])}{\tau_n(\bar t)}\\
&= z^n \prod_1^n \left(1-\frac{p_k}{z}\right) \frac{\det\left(-\tilde X + \sum_1^\infty k\bar t_k \tilde Y^{k-1} - z^{-1}\sum_1^\infty \left(\frac{\tilde Y}{z}\right)^{k-1}\right)}{\det\left(-\tilde X + \sum_1^\infty k\bar t_k Y^{k-1}\right)}\\
&= \det(zI-Y)\,\frac{\det\left(-\tilde X + \sum_1^\infty k\bar t_k \tilde Y^{k-1} - z^{-1}\left(1-z^{-1}\tilde Y\right)^{-1}\right)}{\det\left(-\tilde X + \sum_1^\infty k\bar t_k \tilde Y^{k-1}\right)}\\
&= \det(zI-\tilde Y)\ \det\left(I - \left(xI + \sum_1^\infty k t_k Y^{k-1} - \tilde X\right)^{-1}(z-\tilde Y)^{-1}\right),
\end{aligned}
\tag{7.8}
$$

yielding (7.3), but also an expression for the wave functions $\Psi_n(t,z)$, upon multiplying by an exponential. The formulae for $\mathcal{W}_n$ and $\mathcal{W}_n^*$ follow from (3.8) and (3.9) and the choice of ρ_k.

In order to connect with the form of the matrices announced in (7.2), consider the hyperplane V perpendicular to $e = (1, ..., 1) \in \mathbf{C}^n$

$$\mathbf{C}^n \supset V = \{\langle z, e\rangle = 0\}$$

and the isotropy subgroup $G_e \in U(N)$ of I_e, i.e., the U's such that $U^\top e = e$, thus preserving V. That $I_e = -I\Big|_V$ follows at once, from

$$I_e z = \left(\sum_{k\neq i} z_k\right)_{1\leq i\leq n} = -z.$$

Since G_e stabilizes I_e, there exists a unitary matrix $U \in G_e$, diagonalizing X, having the property

$$[U\tilde{X}U^{-1}, U\tilde{Y}U^{-1}] = [X, Y] = UI_eU^{-1} = I_e,$$

with

$$X = \operatorname{diag}(x_1, ..., x_n);$$

i.e.,

$$(x_i - x_j)y_{ij} = 1 - \delta_{ij} \qquad i \neq j,$$

implying Y must have the form announced in (7.2). Introducing these new matrices into the expressions for τ_n and $p_n(t, z)$ yields

$$\begin{aligned}
\tau_n(t) &= e^{\operatorname{tr}\sum_1^\infty t_iY^i} \det\left(-X + \sum_1^\infty kt_kY^{k-1}\right) \\
&= \det e^{\sum_1^\infty t_iY^i} \det(-X + t_1I + 2t_2Y + ...) \\
&= \det e^{\sum_1^\infty t_iY^i} \prod_{i=1}^n (t_1 + x_i(t_2, t_3, ...)),
\end{aligned}$$

and $p_n(t, z)$, as announced in (7.1) and (7.3). Note that the n roots $x_i(t_2, t_3, ...)$ of the characteristic equation in t_1 are solutions in $(t_2, t_3, ...)$ of the n-particle Calogero-Moser system with initial configuration coordinates $(x_1, ..., x_n, \xi_1, ..., \xi_n)$; see T. Shiota's paper [15]. Thus, a solution of the discrete KP system corresponds to a flag of Calogero-Moser system generated by one pair of semi-infinite matrices X and Y, given by (7.2), for arbitrary large n. ∎

Remark: Observe that for $t = 0$, the parameters x and z in

$$p_n(0, z) = \det(zI - Y)\det\left(I - (xI - X)^{-1}(zI - Y)^{-1}\right).$$

are interchangeable (except for the trivial factor $\det(zI - Y)$). This must be compared to the results in [15] and [14].

8 Discrete KdV-solutions, with upper - triangular L^2

Letting all points p_i in the soliton example converge to p, all points q_i converge to $-p$, and all λ_i converge to 1, the weights $\rho_k(z)$ take on the form (8.2) below. For future use, define the functions:

$$\begin{aligned} f_\ell &= p^\ell \sinh \sum_{\text{odd}} t_i p^i \qquad \ell \text{ even} \\ &= p^\ell \cosh \sum_{\text{odd}} t_i p^i \qquad \ell \text{ odd} \end{aligned}$$

and

$$\begin{aligned} g_\ell &= p^\ell \left(z \sinh \sum_{\text{odd}} t_i p^i - p \cosh \sum_{\text{odd}} t_i p^i \right) \qquad \ell \text{ even} \\ &= p^\ell \left(z \cosh \sum_{\text{odd}} t_i p^i - p \sinh \sum_{\text{odd}} t_i p^i \right) \qquad \ell \text{ odd}. \end{aligned} \tag{8.1}$$

Theorem 8.1 *The family of weights,*

$$\rho_k(z) = (-1)^k \delta^{(k)}(z - p) - \delta^{(k)}(z + p), \quad \text{for } 0 \le k \le n - 1, \tag{8.2}$$

leads to discrete KdV solutions, with KdV τ-functions[16]

$$\tau_n(t) = 2^n e^{n \sum_{\text{even}} t_i p^i} W[f_0, f_1, ..., f_{n-1}]. \tag{8.3}$$

[16] $W[...]$ denotes a Wronskian with respect to the parameter p.

The matrix L has the property that L^2 is <u>*upper-triangular*</u>*, with polynomial eigenvectors $L(t)p(t,z) = zp(t,z)$, given by*

$$p_n(t,z) = \frac{W[f_0, f_1, ..., f_{n-1}]}{W[g_0, g_1, ..., g_{n-1}]} \tag{8.4}$$

in terms of (8.1); i.e., the polynomials $p_n(t,z)$ satisfy <u>*3-step relations*</u> *of the following nature:*

$$z^2 p_n(t,z) = \alpha_n p_n + \beta_n p_{n+1} + p_{n+2}.$$

Then

$$\begin{aligned}\mathcal{W}_n &= \left\{ f = \sum_0^\infty a_i z^i \text{ such that } f^{(k)}(p) - (-1)^k f^{(k)}(-p) = 0, \quad 0 \le k \le n-1 \right\} \\ &= \{1, z^2, z^4, ...\} \oplus z(z^2 - p^2)^n \{1, z^2, z^4, ...\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{W}_n^* &= \left\{ \int \frac{\rho_k(u)du}{z-u} = \left(\frac{\partial}{\partial p}\right)^k \left(\frac{2p}{z^2 - p^2}\right), k = 0, ..., n-1 \right\} \oplus \mathcal{H}_+ \\ &= \left\{ \left(\frac{\partial}{\partial p}\right)^k \left(\frac{1}{z^2 - p^2}\right), k = 0, ..., n-1 \right\} \oplus \mathcal{H}_+ .\end{aligned}$$

<u>*Proof*</u>: Indeed, the form of the flags $\mathcal{W}_n$ and $\mathcal{W}_n^*$ follow from the general formulae (3.8) and (3.9). Therefore

$$\begin{aligned}\tau_n(t) &= \det(\mu_{ij})_{0 \le i,j \le n-1} \\ &= \det \left(\left(\frac{\partial}{\partial p}\right)^k \oint_{z=\infty} z^{n-j-1} g(z) \frac{2p}{z^2 - p^2} dz \right)_{0 \le k,j \le n-1} \text{, using (3.8)} \\ &= \det \left(\left(\frac{\partial}{\partial p}\right)^k \oint z^{n-j-1} g(z) \left(\frac{1}{z-p} - \frac{1}{z+p}\right) dz \right)_{0 \le k,j \le n-1} \\ &= \det \left(\left(\frac{\partial}{\partial p}\right)^k \left(p^{n-j-1} e^{\sum t_i p^i} - (-p)^{n-j-1} e^{\sum t_i (-p)^i} \right) \right)_{0 \le k,j \le n-1} \\ &= \det \left(\left(\frac{\partial}{\partial p}\right)^k p^{n-j-1} e^{\sum_{\text{even}} t_i p^i} \left(e^{\sum_{\text{odd}} t_i p^i} + (-1)^{n-j} e^{-\sum_{\text{odd}} t_i p^i} \right) \right)_{0 \le k,j \le n-1} \\ &= 2^n e^{n \sum_{\text{even}} t_i p^i} W[f_0, f_1, ..., f_{n-1}],\end{aligned}$$

which is formula (8.3).

In order to express the wave vector, one needs to compute

$$
\begin{aligned}
& z^n \tau_n(t-[z^{-1}]) \\
&= z^n \det\left(\left(\frac{\partial}{\partial p}\right)^k \left(p^{n-j-1} e^{\sum t_i p^i}(1-\frac{p}{z}) - (-p)^{n-j-1} e^{\sum t_i (-p)^i}(1+\frac{p}{z})\right)\right)_{0\le k,j\le n-1} \\
&= \det\left(\left(\frac{\partial}{\partial p}\right)^k p^{n-j-1} e^{\sum_{\text{even}} t_i p^i}\left(e^{\sum_{\text{odd}} t_i p^i}(z-p) + (-1)^{n-j} e^{-\sum_{\text{odd}} t_i p^i}(z+p)\right)\right)_{0\le k,j\le n-1} \\
&= 2^n e^{n\sum_{\text{even}} t_i p^i} W[g_0, g_1, \ldots, g_{n-1}],
\end{aligned}
$$

from which formula (8.4) follows.

Also notice that from the form of $\mathcal{W}_n$, we have

$$
z^2\mathcal{W}_n \subset \mathcal{W}_n \quad \text{and thus} \quad z^2\mathcal{W}_n^t \subset \mathcal{W}_n^t .
$$

It shows that the $\tau_n(t)$'s are KdV τ-functions. This fact, combined with

$$
\mathcal{W}_n^t = (\operatorname{span}\{\rho_0^t, \rho_1^t, \ldots, \rho_{n-1}^t\})^\perp = \operatorname{span}\{p_n(t,z), p_{n+1}(t,z), \ldots\} \subset \mathcal{H}_+,
$$

leads to the 3-step relation:

$$
z^2 p_n(t,z) = \alpha_n p_n + \beta_n p_{n+1} + p_{n+2},
$$

establishing the upper-triangular nature of L^2. ■

Remark: Letting $p \to 0$ in $p^{-n(n+1)/2}\tau_n(t)$ leads to the rational KdV solutions, i.e., the Schur polynomials with Young diagrams of type $\nu = (n, n-1, \ldots, 1)$.

References

[1] M. Adler and P. van Moerbeke: *Matrix integrals, Toda symmetries, Virasoro constraints and orthogonal polynomials* Duke Math.J., **80** (3), 863–911 (1995)

[2] M. Adler and P. van Moerbeke: *Vertex operator solutions to the discrete KP-hierarchy*, Comm. Math. Phys. – (1999).

[3] M. Adler and P. van Moerbeke: *String orthogonal Polynomials, String Equations and two-Toda Symmetries*, Comm. Pure and Appl. Math., **L** 241–290 (1997).

[4] M. Adler, E. Horozov and P. van Moerbeke: *The solution to the q-KdV equation*, Phys. Letters, **L** 241–290 (1998).

[5] M. Adler and P. van Moerbeke: *The spectrum of coupled random matrices*, Ann. of Math. (1998).

[6] N.I. Akhiezer: The classical moment problem , University Mathematical monographs, Hafner, New York 1965 .

[7] P. Bleher, A. Its: *Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem and universality in the matrix model*, , Ann. of Math. **?** 1–71 (1999).

[8] L. Dickey: *Modified KP and Discrete KP* , preprint (1999).

[9] A.S. Fokas, A.R. Its, A.V. Kitaev: *The isomonodromy approach to matrix models in 2d quantum gravity*, Comm. Math. Phys., **147**, 395–430 (1992).

[10] D. Gieseker: *The Toda hierarchy and the KdV hierarchy* , preprint, alg-geom/9509006.

[11] E. Heine: Handbuch der Kugelfunktionen, Vols I and II, 2nd edition, Berlin 1878, 1881.

[12] D. Kazhdan, B. Kostant and S. Sternberg: *Hamiltonian group actions and dynamical systems of Calogero-Moser type* , Comm. Pure Appl. Math., **31** 481–508 (1978).

[13] M. Mulase: *Algebraic theory of the KP equations* , Perspectives in Math. Phys., Ed. R. Penner and S.T. Yau, International press company, 157–223 (1994).

[14] G. Segal, G. Wilson: *Loop groups and equations of KdV type*, Publ. Math. IHES **61**, 5–65 (1985).

[15] T. Shiota: *Calogero-Moser hierarchy and KP hierarchy*, J.. of Math. Phys. **35**, 5844–5849 (1994).

[16] K. Ueno, K. Takasaki: *Toda Lattice Hierarchy*, Adv. Studies in Pure Math. **4**,1-95 (1984) .

[17] P. van Moerbeke, D. Mumford: *The spectrum of difference operators*, Acta Math., **143**, 93–154 (1979).

[18] G. Wilson: *Collisions of Calogero-Moser particles and an adelic Grassmannian*, Inv. Math. (1998).