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Pfaff τ-functions

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Consider the two-dimensional Toda lattice, with certain skew-symmetric initial condition, which is preserved along the locus $s = -t$ of the space of time variables. Restricting the solution to $s = -t$, we obtain another hierarchy called Pfaff lattice, which has its own tau function, being equal to the square root of the restriction of 2D-Toda tau function. We study its bilinear and Fay identities, W and Virasoro symmetries, relation to symmetric and symplectic matrix integrals and quasiperiodic solutions.

0. Introduction

Consider the set of equations

$$\frac{\partial m_\infty}{\partial t_n} = \Lambda^n m_\infty\,, \qquad \frac{\partial m_\infty}{\partial s_n} = -m_\infty(\Lambda^\top)^n\,, \qquad n = 1, 2, \ldots, \tag{0.1}$$

on infinite matrices

$$m_\infty = m_\infty(t, s) = (\mu_{i,j}(t, s))_{0\le i,j<\infty}\,,$$

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where $t = (t_1, t_2, \dots)$ and $s = (s_1, s_2, \dots)$ are two sequences of scalar independent variables, $\Lambda = (\delta_{i,j-1})_{0\le i,j<\infty}$ is the shift matrix, and $\Lambda^\top$ its transpose. In [2,4], it was shown that Borel decomposing[1] m_∞ into lower- and upper-triangular matrices $S_1 = S_1(t,s)$ and $S_2 = S_2(t,s)$:

$$m_\infty(t,s) = \left(\mu_{ij}\right)_{0\le i,j<\infty} = S_1^{-1} S_2 \,, \quad \text{for "generic" } t, s \in \mathbb{C}^\infty \,, \tag{0.2}$$

leads to a two-Toda (two-dimensional Toda) system for $L_1 := S_1 \Lambda S_1^{-1}$ and $L_2 = S_2 \Lambda^\top S_2^{-1}$,

$$\frac{\partial L_i}{\partial t_n} = [(L_1^n)_+, L_i] \,, \quad \frac{\partial L_i}{\partial s_n} = [(L_2^n)_-, L_i] \,, \quad i = 1, 2,$$

with $A = A_- + A_+$ being the decomposition into lower- and strictly upper-triangular matrices. The solution L_1 and L_2 can be expressed entirely in terms of a sequence of τ-functions $\tau = (\tau_0, \tau_1, \dots)$ given by

$$\tau_n(t,s) = \det m_n(t,s) \,, \quad m_n(t,s) = (\mu_{i,j}(t,s))_{0\le i,j<n} \,, \tag{0.3}$$

for $n \in \mathbb{Z}_{\ge 0} := \{0, 1, \dots\}$.

As readily seen from formula (0.1), the 2-Toda flow then maintains the relation $m_\infty(t,s) = -m_\infty(-s,-t)^\top$, and hence, by formula (0.3)

$$\tau_n(t,s) = (-1)^n \tau_n(-s,-t) \,. \tag{0.4}$$

The main point of this paper is to study equation (0.1) with skew-symmetric initial condition $m_\infty(0,0)$ and the restriction of the system to $s = -t$. When $s \to -t$, formula (0.4) shows that in the limit the odd τ-functions vanish, whereas the even τ-functions are determinants of skew-symmetric matrices. In particular, the factorization (0.2) fails; in fact in the limit the system leaves the main stratum to penetrate a *deeper stratum* in the Borel decomposition. In this paper we show this specialization $s = -t$ leads to its own system, *the Pfaff lattice* on a successor L to the 2-Toda Lax pair (L_1, L_2), whereas in [7], we have shown this system is *integrable* by producing a Lax pair

$$\frac{\partial L}{\partial t_i} = \left[-\pi_{\mathfrak{k}} L^i, L\right] = \left[\pi_{\mathfrak{n}} L^i, L\right],$$

[1] Here "$t, s \in \mathbb{C}^\infty$" is an informal way of saying that t and s are two sequences of independent scalar variables. Under suitable assumptions, $m_\infty(t,s)$ exists for all $t, s \in \mathbb{C}^\infty$ and the decomposition holds for "generic" $t, s \in \mathbb{C}^\infty$, but in general a function of those variables may be defined only in an open subset of $\mathbb{C}^\infty \times \mathbb{C}^\infty$, or may even be a formal power series in t and s.

on semi-infinite matrices of the form

$$L = \begin{pmatrix} 0 & 1 & & & & & & \\ & -d_1 & a_1 & & & & O & \\ & & d_1 & 1 & & & & \\ & & & -d_2 & a_2 & & & \\ & & & & d_2 & 1 & & \\ & * & & & & \ddots & \ddots & \end{pmatrix}.$$

The projections $\pi_{\mathfrak{k}}$ and $\pi_{\mathfrak{n}}$ correspond to the Lie algebra splitting (in the formula below, lower-triangular with special diagonal means: the diagonal consists of 2×2 blocks, each of them proportional to the 2×2 identity)

$$gl(\infty) = \mathfrak{k} \oplus \mathfrak{n} \begin{cases} \mathfrak{k} = \{\text{lower-triangular matrices, with special diagonal }\} \\ \mathfrak{n} = sp(\infty) = \{a \text{ such that } J a^\top J = a\}, \end{cases}$$

where

$$J := \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & & O \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & \\ O & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \\ & & & \ddots \end{pmatrix} \quad \text{with } J^2 = -I.$$

The precise projections take on the following form[2]

$$\begin{aligned} a &= (a)_{\mathfrak{k}} + (a)_{\mathfrak{n}} \\ &= \left((a_- - J(a_+)^\top J) + \frac{1}{2}(a_0 - J(a_0)^\top J) \right) \\ &\quad + \left((a_+ + J(a_+)^\top J) + \frac{1}{2}(a_0 + J(a_0)^\top J) \right). \end{aligned}$$

The solution to the Pfaff lattice can be expressed in terms of "Pfaff τ-functions" $\tilde{\tau}(t)$ as follows:

$$L(t) = Q(t) \Lambda Q(t)^{-1},$$

[2] $a_\pm$ refers to projection onto strictly upper (strictly lower) triangular matrices, with all 2×2 diagonal blocks equal zero. a_0 refers to projection onto the "diagonal", consisting of 2×2 blocks.

where $Q(t)$ is the lower-triangular matrix whose entries are given by the coefficients of the polynomials in λ:

$$\begin{aligned} q_{2n}(t,\lambda) &:= \sum_{j=0}^{2n} Q_{2n,j}(t)\lambda^j \\ &= \frac{\lambda^{2n}}{\sqrt{\tilde\tau_{2n}\tilde\tau_{2n+2}}}\tilde\tau_{2n}(t-[\lambda^{-1}]) \\ q_{2n+1}(t,\lambda) &:= \sum_{j=0}^{2n+1} Q_{2n+1,j}(t)\lambda^j \\ &= \frac{\lambda^{2n}}{\sqrt{\tilde\tau_{2n}\tilde\tau_{2n+2}}}(\lambda+\frac{\partial}{\partial t_1})\tilde\tau_{2n}(t-[\lambda^{-1}]) \end{aligned}$$

where $\tilde\tau_{2n}(t)$ are *Pfaffians*:

$$\tilde\tau_{2n}(t) := \operatorname{Pf} m_{2n}(t,-t) = (\det m_{2n}(t,-t))^{1/2} = \tau_{2n}(t,-t)^{1/2}\,, \qquad (0.5)$$

for every *even* $n\in\mathbb{Z}_{\geq 0}$. The q_i are skew-orthonormal polynomials with respect to a skew inner-product $\langle,\rangle$, namely $\langle q_i,q_j\rangle = J_{ij}$ in terms of the J-matrix defined earlier; see [7]. The *"Pfaffian $\tilde\tau$-function"* is itself *not* a 2-Toda τ-function, but it ties up remarkably with the 2-Toda τ-function τ (see (0.3)) as follows[3]:

$$\begin{aligned} \tau_{2n}(t,-t-[\alpha]+[\beta]) &= \tilde\tau_{2n}(t)\tilde\tau_{2n}(t+[\alpha]-[\beta]) \\ \tau_{2n+1}(t,-t-[\alpha]+[\beta]) &= (\beta-\alpha)\tilde\tau_{2n}(t-[\beta])\tilde\tau_{2n+2}(t+[\alpha])\,. \end{aligned} \qquad (0.6)$$

When $\beta\to\alpha$, we approach the *deeper* stratum in the Borel decomposition of m_∞ in a very specific way. It shows that the odd τ-functions $\tau_{2n+1}(t,-t-[\alpha]+[\beta])$ approach zero linearly as $\beta\to\alpha$, at the rate depending on α:

$$\lim_{\beta\to\alpha}\tau_{2n+1}(t,-t-[\alpha]+[\beta])/(\beta-\alpha) = \tilde\tau_{2n}(t-[\alpha])\tilde\tau_{2n+2}(t+[\alpha])\,.$$

Equations (0.6) are crucial in establishing bilinear relations for Pfaffian $\tilde\tau$-functions: for all $t,t'\in\mathbb{C}^\infty$ and m,n positive integers

$$\begin{aligned} &\oint_{z=\infty}\tilde\tau_{2n}(t-[z^{-1}])\tilde\tau_{2m+2}(t'+[z^{-1}])e^{\sum_{i=0}^\infty(t_i-t_i')z^i}z^{2n-2m-2}dz \\ &\qquad+\oint_{z=0}\tilde\tau_{2n+2}(t+[z])\tilde\tau_{2m}(t'-[z])e^{\sum_{i=0}^\infty(t_i'-t_i)z^{-i}}z^{2n-2m}dz = 0\,, \end{aligned} \qquad (0.7)$$

[3] $[\alpha] := (\alpha,\alpha^2/2,\alpha^3/3,\dots)$.

This bilinear identity leads to different types of relations, involving nearest neighbors, like the "differential Fay identity",

$$\begin{aligned}
&\{\tilde\tau_{2n}(t-[u]),\tilde\tau_{2n}(t-[v])\}\\
&\quad+(u^{-1}-v^{-1})\Big(\tilde\tau_{2n}(t-[u])\tilde\tau_{2n}(t-[v])-\tilde\tau_{2n}(t)\tilde\tau_{2n}(t-[u]-[v])\Big)\\
&=uv(u-v)\tilde\tau_{2n-2}(t-[u]-[v])\tilde\tau_{2n+2}(t)\,,
\end{aligned}\tag{0.8}$$

and the Hirota bilinear equations[4],

$$\left(p_{k+4}(\tilde D)-\frac{1}{2}D_1D_{k+3}\right)\tilde\tau_{2n}\cdot\tilde\tau_{2n}=p_k(\tilde D)\tilde\tau_{2n+2}\cdot\tilde\tau_{2n-2}\,.\tag{0.9}$$

For $k=0$, this equation can be viewed as an inductive expression of $\tilde\tau_{2n+2}$ in terms of $\tilde\tau_{2n-2}$ and derivatives of $\tilde\tau_{2n}$.

In analogy with the 2-Toda or KP theory, we establish Fay identities for the Pfaff $\tilde\tau$-functions. In this instance, they involve Pfaffians rather than determinants:

$$\begin{aligned}
&\mathrm{Pf}\left(\frac{(z_j-z_i)\tilde\tau_{2n-2}(t-[z_i]-[z_j])}{\tilde\tau_{2n}(t)}\right)_{1\le i,j\le 2k}\\
&\qquad\qquad=\Delta(z)\frac{\tilde\tau_{2n-2k}\left(t-\sum_{i=1}^{2k}[z_i]\right)}{\tilde\tau_{2n}(t)}\,,
\end{aligned}\tag{0.10}$$

In particular for $k=2$,

$$\begin{aligned}
&\sum_{1\to2\to3\to1}(z_1-z_0)(z_2-z_3)\tilde\tau_{2n}(t-[z_0]-[z_1])\tilde\tau_{2n}(t-[z_2]-[z_3])\\
&=-\left(\prod_{0\le i<j\le3}(z_i-z_j)\right)\tilde\tau_{2n+2}(t)\tilde\tau_{2n-2}(t-[z_0]-[z_1]-[z_2]-[z_3])
\end{aligned}$$

which has a useful interpretation in terms of the Pfaffians of Christoffel-Darboux kernels of the form

$$\begin{aligned}
&K_n(\mu,\lambda)\\
&\quad=e^{\sum_{i=1}^{\infty}t_i(\mu^i+\lambda^i)}\sum_{k=0}^{n-1}\Big(q_{2k}(t,\lambda)q_{2k+1}(t,\mu)-q_{2k}(t,\mu)q_{2k+1}(t,\lambda)\Big)\,,
\end{aligned}\tag{0.11}$$

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[4] $\tilde\partial=(\partial/\partial t_1,(1/2)\partial/\partial t_2,(1/3)\partial/\partial t_3,\dots)$, $\tilde D=(D_1,(1/2)D_2,(1/3)D_3,\dots)$ is the corresponding Hirota symbol: $P(\tilde D)f\cdot g:=P(\partial/\partial y_1,(1/2)\partial/\partial y_2,\dots)f(t+y)g(t-y)|_{y=0}$, and p_k are the elementary Schur functions: $\sum_{k=0}^{\infty}p_k(t)z^k:=\exp(\sum_{i=1}^{\infty}t_iz^i)$.

where the $q_m(t, \lambda)$ form the system of skew-orthogonal polynomials, mentioned above (see [7]). This is the analogue of the Christoffel-Darboux kernel for orthogonal polynomials. So, formula (0.10) can be rewritten as

$$\mathrm{Pf}(K_n(z_i, z_j))_{1\le i,j\le 2k} = \left(\frac{1}{\tilde{\tau}} \prod_{\substack{i=1 \\ \text{ordered}}}^{2k} \mathbb{X}(t; z_i)\tilde{\tau} \right)_{2n}, \tag{0.12}$$

where

$$\mathbb{X}(t; z) := \Lambda^{-1} e^{\sum_{i=1}^{\infty} t_i z^i} e^{-\sum_{i=1}^{\infty} \frac{z^{-i}}{i} \frac{\partial}{\partial t_i}} \underline{\chi}(z),$$

with $\underline{\chi}(z) = (z^i \delta_{ij})_{i,j\in A}$ a diagonal matrix, is a vertex operator for the corresponding Pfaff lattice (see [7,6]). This vertex operator also has the remarkable property that for a Pfaffian $\tilde{\tau}$-function,

$$\begin{aligned} &\tilde{\tau}_{2n}(t) + a\mathbb{X}(t; \lambda)\mathbb{X}(t; \mu)\tilde{\tau}_{2n}(t) \equiv \\ &\quad \tilde{\tau}_{2n}(t) + a\left(1 - \frac{\mu}{\lambda}\right) \lambda^{2n-2} \mu^{2n-1} e^{\sum t_1(\lambda^i + \mu^i)} \tilde{\tau}_{2n-2}(t - [\lambda^{-1}] - [\mu^{-1}]) \end{aligned}$$

is again a Pfaffian $\tilde{\tau}$-function.

As was shown in [2,3], the 2-Toda lattice has four distinct vertex operators. Upon setting $s = -t$, the 2-Toda vertex operators reduce to vertex operators for the Pfaff lattice. This enables us to give the action of Virasoro generators on Pfaff $\tilde{\tau}$-functions, in terms of the restriction (to $s = -t$) of actions on 2-Toda τ-functions:

$$\left(J_i^{(k)}(t) + (-1)^k J_i^{(k)}(s) \right) \tau_{2n}(t, s)|_{s=-t} = 2\tilde{\tau}_{2n}(t) J_i^{(k)}(t) \tilde{\tau}_{2n}(t).$$

Finally, in Sect. 6 and 8 we discuss two examples. In the first example, inherently semi-infinite, the Pfaff $\tilde{\tau}$-functions are integrals

$$\int_{\mathcal{S}_k} e^{tr(-V(X)+\sum t_i X^i)} dX \text{ and } \int_{\mathcal{T}_k} e^{tr(-V(X)+\sum t_i X^i)} dX,$$

where dX denotes Haar measure over the spaces

$$\begin{aligned} \mathcal{S}_k &= \{k \times k \text{ symmetric matrices}\} \\ \mathcal{T}_k &= \{k \times k \text{ self-dual Hermitian matrices, with quaternionic entries}\}, \end{aligned}$$

appearing naturally in the theory of random matrices; this is extensively discussed in [6] and [25]. By studying two strings of bi-orthogonal polynomials in the 2-Toda lattice case, we find "*string equations*"; upon setting $t = -s$ and using the above formulae, we derive "*Virasoro constraints*" for the symmetric matrix integrals.

The second example, inherently bi-infinite, will be given in the context of curves with fixed point free involution ι, equipped with a line bundle $\mathcal{L}$ having a suitable antisymmetry condition with respect to ι. This example is genuinely *bi-infinite*, i.e., S_i, L_i, Λ etc., are $\mathbb{Z} \times \mathbb{Z}$ matrices, and Ψ_i, τ etc., are $\mathbb{Z}$-vectors. The bi-infinite Pfaff lattice has a quasi-periodic solution in terms of a Prym Θ-function, which is essentially the square root ofthe Riemann Θ-function. The case we discussed before, i.e., when those matrices and vectors are indexed by $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{\geq 0}$, respectively, will be called the *semi-infinite* case.

The Pfaff lattice already appear in the work of Jimbo and Miwa as one half of the D'_∞-hierarchy (compare (0.7) (or (3.2)) with the case $l = l'$ of formula (7.7) in [15]), in the work of Hirota et al., in the context of the coupled KP hierarchy (compare, e.g., (0.5) and (0.9) with formulas (3.5) and (3.25a) in [13], respectively), in the work of Kac and van de Leur [16] in the context of the DKP hierarchy (on the exact connection, see forthcoming work by J. van de Leur [24]), and in the recent work of S. Kakei [17,18], who realized Hirota et al.'s coupled KP hierarchy as a restriction of the 2-component KP hierarchy instead of the 2-Toda lattice, and studied its relation to matrix integrals among other aspects.

1. Borel decomposition and the 2-Toda lattice

In this section we recall the theory of 2-Toda lattice. While the matrix m_∞ and its Borel decomposition may look more natural in the semi-infinite case, the general theory of 2-Toda or Pfaff lattice works better in the bi-infinite case. However, since the latter is actually independent of the former, this does not affect us in developing the theory in its full generality. In what follows, unless otherwise noted, we shall treat both cases in parallel, by denoting the index set for matrices and vectors by

$$A := \begin{cases} \mathbb{Z}_{\geq 0} & \text{semi-infinite case,} \\ \mathbb{Z} & \text{bi-infinite case,} \end{cases}$$

and make brief remarks without going into details when the two cases need be treated differently.

In [4,2], we considered the following evolution equations for the (semi- or bi-infinite) moment matrix $m_\infty \in \mathrm{Mat}_{A\times A}$

$$\frac{\partial m_\infty}{\partial t_n} = \Lambda^n m_\infty\,, \quad \frac{\partial m_\infty}{\partial s_n} = -m_\infty (\Lambda^\top)^n\,, \quad n = 1, 2, \ldots, \tag{1.1}$$

where $\Lambda = (\delta_{i,j-1})_{i,j\in A}$ is the shift matrix; then (1.1) has the following solution

$$m_\infty(t, s) = e^{\sum_{n=1}^\infty t_n \Lambda^n} m_\infty(0, 0) e^{-\sum_{n=1}^\infty s_n (\Lambda^\top)^n} \tag{1.2}$$

in terms of the initial data $m_\infty(0, 0)$.

Assume $m_\infty = m_\infty(t,s)$ allows, for "generic" (t,s), the Borel decomposition $m_\infty = S_1^{-1} S_2$, for

$$S_1 \in G_- := \left\{\begin{matrix}\text{lower-triangular matrices}\\ \text{with 1's on the diagonal}\end{matrix}\right\},$$
$$S_2 \in G_+ := \left\{\begin{matrix}\text{upper-triangular matrices}\\ \text{with non-zero diagonal entries}\end{matrix}\right\},$$

with corresponding Lie algebras g_-, g_+. For any $X \in \mathrm{Mat}_{A\times A}$, denote by X_- and X_+ its strictly lower-triangular part and the upper-triangular part, respectively: $X = X_- + X_+$, $X_\pm \in g_\pm$. Setting

$$L_1 := S_1 \Lambda S_1^{-1}, \tag{1.3}$$

we have[5]

$$S_1 \frac{\partial m_\infty}{\partial t_n} S_2^{-1} \begin{cases} = S_1(\partial/\partial t_1)(S_1^{-1}S_2)S_2^{-1} = -\dot{S}_1 S_1^{-1} + \dot{S}_2 S_2^{-1}\,, \\ = S_1 \Lambda^n m_\infty S_2^{-1} = S_1 \Lambda^n S_1^{-1} = L_1^n\,. \end{cases}$$

Since $-\dot{S}_1 S_1^{-1} \in g_-$ and $\dot{S}_2 S_2^{-1} \in g_+$, the uniqueness of the decomposition $g_- + g_+$ leads to

$$-\frac{\partial S_1}{\partial t_n} S_1^{-1} = (L_1^n)_-\,, \quad \frac{\partial S_2}{\partial t_n} S_2^{-1} = (L_1^n)_+\,.$$

Similarly, setting

$$L_2 = S_2 \Lambda^\top S_2^{-1}, \tag{1.4}$$

we find

$$-\frac{\partial S_1}{\partial s_n} S_1^{-1} = -(L_2^n)_-\,, \quad \frac{\partial S_2}{\partial s_n} S_2^{-1} = -(L_2^n)_+\,.$$

This leads to the 2-Toda equations [23] for S_1, S_2 and L_1, L_2:

$$\frac{\partial}{\partial t_n} S_{\left\{\substack{1\\2}\right\}} = \mp (L_1^n)_\mp S_{\left\{\substack{1\\2}\right\}}\,, \quad \frac{\partial}{\partial s_n} S_{\left\{\substack{1\\2}\right\}} = \pm (L_2^n)_\mp S_{\left\{\substack{1\\2}\right\}}\,, \tag{1.5}$$

$$\frac{\partial L_i}{\partial t_n} = [(L_1^n)_+, L_i]\,, \quad \frac{\partial L_i}{\partial s_n} = [(L_2^n)_-, L_i]\,, \quad i = 1, 2, \tag{1.6}$$

and conversely, reading this argument backwards, we observe that the 2-Toda equations (1.5) imply the time evolutions (1.1) for m_∞.

[5] In the semi-infinite case, the left G_-- and right G_+-multiplications on $\mathrm{Mat}_{A\times A}$ are well-defined and associative: $X(YZ) = (XY)Z$ if $X, Y \in G_-$ or $Y, Z \in G_+$. In the bi-infinite case, we must require those properties, e.g., by putting conditions on the behavior of μ_{ij} as $i, j \to -\infty$, in order to make sense of the following calculation.

The pairs of wave and adjoint wave functions $\Psi = (\Psi_1, \Psi_2)$ and $\Psi^* = (\Psi_1^*, \Psi_2^*)$, defined by

$$\begin{aligned}\Psi_{\left\{\substack{1\\2}\right\}}(t,s;z) &= e^{\sum_{i=1}^{\infty}\left\{\substack{t_i\\s_i}\right\}z^{\pm i}} S_{\left\{\substack{1\\2}\right\}}\chi(z)\,,\\ \Psi^*_{\left\{\substack{1\\2}\right\}}(t,s;z) &= e^{-\sum_{i=1}^{\infty}\left\{\substack{t_i\\s_i}\right\}z^{\pm i}} \left(S^\top_{\left\{\substack{1\\2}\right\}}\right)^{-1}\chi(z^{-1})\,,\end{aligned} \tag{1.7}$$

where $\chi(z)$ is the column vector $(z^n)_{n\in A}$, satisfy[6]

$$L_1\Psi_1 = z\Psi_1\,,\quad L_2\Psi_2 \doteq z^{-1}\Psi_2\,,\quad L_1^\top\Psi_1^* \doteq z\Psi_1^*\,,\quad L_2^\top\Psi_2^* = z^{-1}\Psi_2^* \tag{1.8}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial t_n}\Psi_1 = (L_1^n)_+\Psi_1\,, && \frac{\partial}{\partial t_n}\Psi_2 = (L_1^n)_+\Psi_2\,,\\
&\frac{\partial}{\partial s_n}\Psi_1 = (L_2^n)_-\Psi_1\,, && \frac{\partial}{\partial s_n}\Psi_2 \doteq (L_2^n)_-\Psi_2\,,\\
&\frac{\partial}{\partial t_n}\Psi_1^* \doteq -((L_1^n)_+)^\top\Psi_1^*\,, && \frac{\partial}{\partial t_n}\Psi_2^* = -((L_1^n)_+)^\top\Psi_2^*\,,\\
&\frac{\partial}{\partial s_n}\Psi_1^* = -((L_2^n)_-)^\top\Psi_1^*\,, && \frac{\partial}{\partial s_n}\Psi_2^* = -((L_2^n)_-)^\top\Psi_2^*\,,
\end{aligned} \tag{1.9}$$

which are equivalent to (1.5), and are further equivalent to the following bilinear identities,[7] for all $m, n \in A$ and $t, s, t', s' \in \mathbb{C}^\infty$:

$$\begin{aligned}&\oint_{z=\infty}\Psi_{1n}(t,s;z)\Psi^*_{1m}(t',s';z)\frac{dz}{2\pi i z}\\ &\qquad\qquad = \oint_{z=0}\Psi_{2n}(t,s;z)\Psi^*_{2m}(t',s';z)\frac{dz}{2\pi i z}\,.\end{aligned} \tag{1.10}$$

By 2-Toda theory [23,4], the problem is solved in terms of a sequence of tau-functions $\tau_n(t,s)$, which in the semi-infinite case (or in the bi-infinite case if we can take a "nice" m_∞) are given by

$$\tau_n(t,s) = \det m_n(t,s)\,,\quad m_n(t,s) := (\mu_{ij}(t,s))_{i,j\in A;\, i,j<n} \tag{1.11}$$

[6] Here and in what follows, we denote by "$\doteq$" any equality which is true in the bi-infinite case, but not true in general in the semi-infinite case. In the semi-infinite case $\Lambda^\top\chi(z) = \pi_+(z^{-1}\chi(z)) \neq z^{-1}\chi(z)$, where π_+ maps z^k to itself if $k \geq 0$ and 0 otherwise; so the second and third formulas in (1.8) should be replaced by $L_2\Psi_2 = \psi\pi_+(\psi^{-1}z^{-1}\Psi_2)$ and $L_1^\top\Psi_1^* = \varphi^{-1}\pi_-(\varphi z\Psi_1^*)$, respectively, where $\psi = e^{\sum_{i=1}^\infty s_i z^{-i}}$, $\varphi := e^{\sum_{i=1}^\infty t_i z^i}$, and π_- maps z^k to itself if $k \leq 0$ and 0 otherwise. In (1.9), the second formula in the second line and the first formula in the third line need similar corrections.

[7] The contour integral around $z = \infty$ is taken *clockwise* about a small circle around $z = \infty \in \mathbb{P}^1(\mathbb{C})$, while the one around $z = 0$ is taken counter-clockwise about $z = 0$.

($\tau_0 \equiv 1$ in the semi-infinite case), as

$$\begin{aligned}
\Psi_1(t,s;z) &= \left(\frac{\tau_n(t-[z^{-1}],s)}{\tau_n(t,s)} e^{\sum_1^\infty t_i z^i} z^n\right)_{n\in A}, \\
\Psi_2(t,s;z) &= \left(\frac{\tau_{n+1}(t,s-[z])}{\tau_n(t,s)} e^{\sum_1^\infty s_i z^{-i}} z^n\right)_{n\in A}, \\
\Psi_1^*(t,s;z) &= \left(\frac{\tau_{n+1}(t+[z^{-1}],s)}{\tau_{n+1}(t,s)} e^{-\sum_1^\infty t_i z^i} z^{-n}\right)_{n\in A}, \\
\Psi_2^*(t,s;z) &= \left(\frac{\tau_n(t,s+[z])}{\tau_{n+1}(t,s)} e^{-\sum_1^\infty s_i z^{-i}} z^{-n}\right)_{n\in A}.
\end{aligned} \tag{1.12}$$

Note (1.7) and (1.12) yield

$$h(t,s) := (\text{diagonal part of } S_2) = \operatorname{diag}\left(\frac{\tau_{n+1}(t,s)}{\tau_n(t,s)}\right)_{n\in A}. \tag{1.13}$$

Formulas (1.10) and (1.12) imply the following bilinear identities

$$\begin{aligned}
&\oint_{z=\infty} \tau_n(t-[z^{-1}],s)\tau_{m+1}(t'+[z^{-1}],s') e^{\sum_{i=1}^\infty (t_i-t_i')z^i} z^{n-m-1} dz \\
&\quad = \oint_{z=0} \tau_{n+1}(t,s-[z])\tau_m(t',s'+[z]) e^{\sum_{i=1}^\infty (s_i-s_i')z^{-i}} z^{n-m-1} dz,
\end{aligned} \tag{1.14}$$

where $m, n \in A$, satisfied by and characterizing the 2-Toda τ-functions.

Using the matrices $\varepsilon := (i\delta_{i,j+1})_{i,j\in A}$ and $\varepsilon^* := (-i\delta_{i,j-1})_{i,j\in A}$, which are characterized by

$$\varepsilon\chi(z) = (\partial/\partial z)\chi(z), \quad \text{and} \quad \varepsilon^*\chi(z) = (\partial/\partial(z^{-1}))\chi(z), \tag{1.15}$$

and using the notation

$$\xi(t,z) := \sum_{i=1}^\infty t_i z^i, \quad \xi'(t,z) := (\partial\xi/\partial z)(t,z) = \sum_{i=1}^\infty i t_i z^{i-1},$$

we also define[8]

$$\begin{aligned}
M_1 &:= S_1\big(\varepsilon + \xi'(t,\Lambda)\big)S_1^{-1} = W_1\varepsilon W_1^{-1}, \\
M_2 &:= S_2\big(\varepsilon^* + \xi'(s,\Lambda^\top)\big)S_2^{-1} \doteq W_2\varepsilon^* W_2^{-1}, \\
M_1^* &:= S_1^{\top-1}\big(\varepsilon^* - \xi'(t,\Lambda^\top)\big)S_1^\top \doteq W_1^{\top-1}\varepsilon^* W_1^\top, \\
M_2^* &:= S_2^{\top-1}\big(\varepsilon - \xi'(s,\Lambda)\big)S_2^\top = W_2^{\top-1}\varepsilon W_2^\top,
\end{aligned} \tag{1.16}$$

[8] See footnote 6 for the notation "$\doteq$". In the semi-infinite case the last equality in the second and third line of (1.16), and the second and third equalities in (1.18) fail because $[\Lambda^\top, \varepsilon^*] = 1$ fails (note $[\Lambda, \varepsilon] = 1$ is true both in the semi- and bi-infinite cases). The failure in the semi-infinite case of the second and the third equalities in (1.17) is due to that of the second and the third equalities in (1.8).

where $W_1 := S_1 e^{\xi(t,\Lambda)}$ and $W_2 := S_2 e^{\xi(s,\Lambda^\top)}$. The operators M_i and M_i^* satisfy

$$\begin{aligned} M_1\Psi_1 &= \frac{\partial \Psi_1}{\partial z}\,, \quad M_2\Psi_2 \doteq \frac{\partial \Psi_2}{\partial (z^{-1})}\,, \\ M_1^*\Psi_1^* &\doteq \frac{\partial \Psi_1^*}{\partial z}\,, \quad M_2^*\Psi_2^* = \frac{\partial \Psi_2^*}{\partial (z^{-1})}\,, \end{aligned} \tag{1.17}$$

$$\begin{aligned} [L_1, M_1] &= I\,, \quad [L_2, M_2] \doteq I\,, \\ [L_1^\top, M_1^*] &\doteq I\,, \quad [L_2^\top, M_2^*] = I\,. \end{aligned} \tag{1.18}$$

The symmetry vector fields[9] $\mathbb{Y}_N$ acting on Ψ and L,

$$\begin{aligned} \mathbb{Y}_{M_i^\alpha L_i^\beta}\Psi_1 &:= (-1)^i (M_i^\alpha L_i^\beta)_- \Psi_1\,, \\ \mathbb{Y}_{M_i^\alpha L_i^\beta}\Psi_2 &:= (-1)^{i-1} (M_i^\alpha L_i^\beta)_+ \Psi_2\,, \\ \mathbb{Y}_{M_i^\alpha L_i^\beta}L_1 &:= (-1)^i \big[(M_i^\alpha L_i^\beta)_-, L_1\big]\,, \\ \mathbb{Y}_{M_i^\alpha L_i^\beta}L_2 &:= (-1)^{i-1} \big[(M_i^\alpha L_i^\beta)_+, L_2\big]\,. \end{aligned} \tag{1.19}$$

for $i = 1, 2$ and $\alpha, \beta \in \mathbb{Z}, \alpha \geq 0$, lift to an action on τ, according to the Adler-Shiota-van Moerbeke formula [9,10]:

Proposition 1.1 *For $n, k \in \mathbb{Z}$, $n \geq 0$, and $i = 1, 2$, the symmetry vector fields $\mathbb{Y}_{M_i^n L_i^{n+k}}$ acting on Ψ lead to the correspondences*

$$\begin{aligned} -\frac{((M_1^n L_1^{n+k})_- \Psi_1)_m}{\Psi_{1,m}} &= \frac{1}{n+1}(e^{-\eta} - 1)\frac{W_{m,k}^{(n+1)}(\tau_m)}{\tau_m}\,, \\ \frac{((M_1^n L_1^{n+k})_+ \Psi_2)_m}{\Psi_{2,m}} &= \frac{1}{n+1}\left(e^{-\tilde\eta}\frac{W_{m+1,k}^{(n+1)}(\tau_{m+1})}{\tau_{m+1}} - \frac{W_{m,k}^{(n+1)}(\tau_m)}{\tau_m}\right), \\ \frac{((M_2^n L_2^{n+k})_- \Psi_1)_m}{\Psi_{1,m}} &= \frac{1}{n+1}(e^{-\eta} - 1)\frac{\tilde W_{m-1,k}^{(n+1)}(\tau_m)}{\tau_m}\,, \\ -\frac{((M_2^n L_2^{n+k})_+ \Psi_2)_m}{\Psi_{2,m}} &= \frac{1}{n+1}\left(e^{-\tilde\eta}\frac{\tilde W_{m,k}^{(n+1)}(\tau_{m+1})}{\tau_{m+1}} - \frac{\tilde W_{m-1,k}^{(n+1)}(\tau_m)}{\tau_m}\right), \end{aligned} \tag{1.20}$$

where $\eta = \sum_{i=1}^\infty (z^{-i}/i)(\partial/\partial t_i)$ and $\tilde\eta = \sum_{i=1}^\infty (z^i/i)(\partial/\partial s_i)$, so that

$$e^{a\eta + b\tilde\eta} f(t,s) = f(t + a[z^{-1}], s + b[z])\,.$$

[9] Note the action of $\mathbb{Y}_{M_i^\alpha L_i^\beta}$ on L follows from that on Ψ, which in turn follows from (1.20). In the semi-infinite case, note also the appearance in (1.19) of negative powers of L_i, which do not exist: It is natural to replace L_1^β and L_2^β for $\beta < 0$ by $S_1(\Lambda^\top)^{-\beta}S_1^{-1}$ and $S_2\Lambda^{-\beta}S_2^{-1}$, respectively, but the appearance of projector π_+ in $\Lambda^\top\chi(z) = \pi_+(z^{-1}\chi(z))$ (see footnote 6) makes it nontrivial to apply the method of [10]. So in the semi-infinite case, we first *define* the action of $\mathbb{Y}_{M_i^\alpha L_i^\beta}$ on Ψ by (1.20) (with $\pm(-1)^{i-1}(M_i^n L_i^{n+k})_\pm$ replaced by $\mathbb{Y}_{M_i^n L_i^{n+k}}$), and then check the validity of/deviation from (1.19).

In Proposition 1.1, the W-generators take on the following form in terms of the customary W-generators

$$W_{n,\ell}^{(k)} = \sum_{j=0}^{k} \binom{n}{j} (k)_j W_\ell^{(k-j)} \quad \text{and} \quad \tilde{W}_{n,\ell}^{(k)} = W_{-n,\ell}^{(k)}\Big|_{t\mapsto s} \tag{1.21}$$

(see (4.5)). We shall need the $W_{n,\ell}^{(k)}$-generators for $0 \le k \le 2$:

$$\begin{aligned} W_n^{(0)} &= \delta_{n,0}\,, \quad W_n^{(1)} = J_n^{(1)}\,, \\ W_n^{(2)} &= J_n^{(2)} - (n+1) J_n^{(1)}\,, \end{aligned} \qquad n \in \mathbb{Z}\,, \tag{1.22}$$

and

$$\begin{aligned} W_{m,i}^{(1)} &= W_i^{(1)} + m W_i^{(0)} = J_i^{(1)} + m\delta_{i0}\,, \\ W_{m,i}^{(2)} &= W_i^{(2)} + 2m W_i^{(1)} + m(m-1) W_i^{(0)} \\ &= J_i^{(2)} + (2m-i-1) J_i^{(1)} + m(m-1)\delta_{i0}\,, \end{aligned} \tag{1.23}$$

expressed in terms of the Virasoro generators

$$\begin{gathered} J_n^{(0)} = \delta_{n0}\,, \quad J_n^{(1)} = \begin{cases} \partial/\partial t_n & \text{if } n>0 \\ -n t_{-n} & \text{if } n<0\,, \\ 0 & \text{if } n=0 \end{cases} \\ J_n^{(2)} = \sum_{i+j=n} \frac{\partial^2}{\partial t_i \partial t_j} + 2 \sum_{-i+j=n} i t_i \frac{\partial}{\partial t_j} + \sum_{-i-j=n} (i t_i)(j t_j)\,. \end{gathered} \tag{1.24}$$

The corresponding expression $\tilde{W}_{m,i}^{(k)}$ can be read off from the above, using (1.21), with $J_n^{(k)}$ replaced by $\tilde{J}_n^{(k)} = J_n^{(k)}\big|_{t\mapsto s}$.

2. Two-Toda τ-functions versus Pfaffian $\tilde{\tau}$-functions

In this section, we exhibit the properties of the 2-Toda lattice, associated with a skew-symmetric initial matrix $m_\infty(0,0)$, or τ-functions $\tau_n(t,s)$ satisfying

$$\tau_n(t,s) = (-1)^n \tau_n(-s,-t)\,.$$

As in the last section, we use the notation $A := \mathbb{Z}_{\ge 0}$ or $\mathbb{Z}$ to treat both the semi- and bi-infinite cases at once.

Theorem 2.1 *The following five conditions for a 2-Toda solution are equivalent, where (2.1) and (2.2) assume the solution arises from the matrix m_∞ (e.g., the* *semi-infinite case), h in (2.3) and (2.4) is the diagonal matrix defined by (1.13), and ε in (2.5) is either* 0 *or* 1 *(in the semi-infinite case $\varepsilon = 0$)*[10]*:*

$$m_\infty(0,0) = -m_\infty(0,0)^\top \,, \tag{2.1}$$

$$m_\infty(t,s) = -m_\infty(-s,-t)^\top \,, \tag{2.2}$$

$$\begin{aligned} h^{-1} S_1(t,s) &= -(S_2^\top)^{-1}(-s,-t)\,, \\ h^{-1} S_2(t,s) &= (S_1^\top)^{-1}(-s,-t)\,, \end{aligned} \tag{2.3}$$

$$\begin{aligned} h^{-1}\Psi_1(t,s;z) &= -\Psi_2^*(-s,-t;z^{-1})\,, \\ h^{-1}\Psi_2(t,s;z) &= \Psi_1^*(-s,-t;z^{-1})\,, \end{aligned} \tag{2.4}$$

$$\tau_n(-s,-t) = (-1)^{n+\varepsilon}\tau_n(t,s)\,. \tag{2.5}$$

Those equivalent conditions imply, and in the semi-infinite case are equivalent to, the following two conditions (2.6) and (2.7):

$$\begin{aligned} L_1(t,s) &= hL_2^\top h^{-1}(-s,-t)\,, \\ L_2(t,s) &= hL_1^\top h^{-1}(-s,-t)\,, \end{aligned} \tag{2.6}$$

$$h(-s,-t) = -h(t,s)\,. \tag{2.7}$$

Proof. Formula (2.1) clearly follows from (2.2). Conversely, (2.2) is an immediate consequence of (1.2) and (2.1). Next, consider the Borel decomposition of $m_\infty(t,s)$ and $-m_\infty(-s,-t)$:

$$\begin{aligned} m_\infty(t,s) &= S_1^{-1}(t,s)S_2(t,s)\,, \\ -m_\infty(-s,-t)^\top &= -S_2^\top(-s,-t)S_1^{-1\top}(-s,-t) \\ &= (S_2^\top(-s,-t)h^{-1}(-s,-t)) \cdot \\ &\quad \cdot (-h(-s,-t)S_1^{-1\top}(-s,-t))\,. \end{aligned}$$

Hence (2.2) clearly follows from

$$\begin{aligned} S_1^{-1}(t,s) &= S_2^\top(-s,-t)h^{-1}(-s,-t) \in G_-\,, \\ S_2(t,s) &= -h(-s,-t)S_1^{-1\top}(-s,-t) \in G_+\,, \end{aligned} \tag{2.8}$$

which is (2.3) up to the substitution $(t,s) \to (-s,-t)$. Conversely, (2.2) and the uniqueness of the Borel decomposition imply (2.8). The equivalence of (2.3) and (2.4) follows from (1.7). By the Definitions (1.3) and (1.4) of L_i, (2.3) implies (2.6). Comparing (2.3) with (2.8) again, we have (2.7). Using (1.12) to rewrite condition (2.4) in terms of τ, we see (2.5) clearly implies (2.4), and conversely,

[10] In what follows, shifting the index n in the bi-infinite case if necessary, we assume (2.5) holds always with $\varepsilon = 0$.

(2.4) implies that $\tau_n'(t,s) := (-1)^n \tau_n(-s,-t)$ plays the same role as $\tau(t,s)$, i.e., it is also a τ-function associated to Ψ. Since Ψ determines τ uniquely up to a constant, there exists $c \in \mathbb{C} \setminus \{0\}$ such that $\tau_n'(t,s) = c\tau_n(t,s)$, i.e.,

$$(-1)^n \tau_n(-s,-t) = c\tau_n(t,s)\,.$$

Comparing this formula, with itself with (t,s) replaced by $(-s,-t)$, we have $c^2 = 1$, and hence $c = \pm 1 = (-1)^\varepsilon$, showing (2.5).

Finally, in the semi-infinite case, relation (2.5) with $\varepsilon = 0$ follows from (1.11), (2.2), and the multilinearity of determinant; or from (2.7), using $\tau_0(t,s) = 1$:

$$\frac{\tau_n(t,s)}{\tau_n(-s,-t)} = -\frac{\tau_{n-1}(t,s)}{\tau_{n-1}(-s,-t)} = \cdots = (-1)^n \frac{\tau_0(t,s)}{\tau_0(-t,-s)} = (-1)^n\,.$$

In particular, in the semi-infinite case (2.7) implies, and hence is equivalent to, (2.5). Note also that in the semi-infinite case L_i determines S_i uniquely, so (2.3) and (2.6) are also equivalent. □

For a skew-symmetric initial matrix $m_\infty(0,0)$, relation (2.2) implies the skew-symmetry of $m_\infty(t,-t)$. Therefore the odd τ-functions vanish and the even ones have a natural square root, the *Pfaffian* $\tilde{\tau}_{2n}(t)$:

$$\tau_{2n+1}(t,-t) = 0\,, \quad \tau_{2n}(t,-t) = \tilde{\tau}_{2n}^2(t)\,, \tag{2.9}$$

where the Pfaffian, together with its sign specification, is also determined by the formula:

$$\tilde{\tau}_{2n}(t)dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{2n-1} := \frac{1}{n!}\left(\sum_{0\le i<j\le 2n-1} \mu_{ij}(t,-t)dx_i \wedge dx_j\right)^n\,. \tag{2.10}$$

Theorem 2.2 *For τ satisfying (2.5), and hence for a skew-symmetric initial condition $m_\infty(0,0)$, the 2-Toda τ-function $\tau(t,s)$ and the Pfaffians $\tilde{\tau}(t)$ are related by*

$$\begin{aligned} \tau_{2n}(t+[\alpha]-[\beta],-t) &= \tilde{\tau}_{2n}(t)\tilde{\tau}_{2n}(t+[\alpha]-[\beta])\,, \\ \tau_{2n+1}(t+[\alpha]-[\beta],-t) &= (\alpha-\beta)\tilde{\tau}_{2n}(t-[\beta])\tilde{\tau}_{2n+2}(t+[\alpha])\,, \end{aligned} \tag{2.11}$$

or alternatively

$$\begin{aligned} \tau_{2n}(t-[\beta],-t+[\alpha]) &= \tilde{\tau}_{2n}(t-[\alpha])\tilde{\tau}_{2n}(t-[\beta])\,, \\ \tau_{2n}(t+[\alpha],-t-[\beta]) &= \tilde{\tau}_{2n}(t+[\alpha])\tilde{\tau}_{2n}(t+[\beta])\,, \\ \tau_{2n+1}(t-[\beta],-t+[\alpha]) &= (\alpha-\beta)\tilde{\tau}_{2n}(t-[\alpha]-[\beta])\tilde{\tau}_{2n+2}(t)\,, \\ \tau_{2n+1}(t+[\alpha],-t-[\beta]) &= (\alpha-\beta)\tilde{\tau}_{2n}(t)\tilde{\tau}_{2n+2}(t+[\alpha]+[\beta])\,. \end{aligned} \tag{2.12}$$

Proof. In formula (1.14), set $n = m - 1$, $s = -t + [\beta]$, $t' = t + [\alpha] - [\beta]$ and $s' = s - [\alpha] - [\beta] = -t - [\alpha]$; then using

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_{z=\infty} \tau_n(t - [z^{-1}], s)\tau_{m+1}(t' + [z^{-1}], s')e^{\sum_{i=1}^{\infty}(t_i - t_i')z^i} z^{n-m-1} dz \\
&\quad = \frac{1}{2\pi i}\oint_{z=\infty} \tau_{m-1}(t - [z^{-1}], s)\tau_{m+1}(t' + [z^{-1}], s')\frac{1-\alpha z}{1-\beta z}\frac{dz}{z^2} \\
&\quad = -\operatorname{Res}_{z=\beta^{-1}} \tau_{m-1}(t - [z^{-1}], s)\tau_{m+1}(t' + [z^{-1}], s')\frac{1-\alpha z}{1-\beta z}\frac{dz}{z^2} \\
&\quad = (\beta - \alpha)\tau_{m-1}(t - [\beta], s)\tau_{m+1}(t' + [\beta], s') \\
&\quad = (\beta - \alpha)\tau_{m-1}(t - [\beta], -t + [\beta])\tau_{m+1}(t + [\alpha], -t - [\alpha])\,,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_{z=0} \tau_m(t, s - [z])\tau_m(t', s' + [z])e^{\sum_{i=1}^{\infty}(s_i - s_i')z^{-i}} z^{n-m-1} dz \\
&\quad = \frac{1}{2\pi i}\oint_{z=0} \tau_m(t, s - [z])\tau_m(t', s' + [z])\frac{1}{1-\alpha/z}\,\frac{1}{1-\beta/z}\,\frac{dz}{z^2} \\
&\quad = (\operatorname{Res}_{z=\alpha} + \operatorname{Res}_{z=\beta})\tau_m(t, s - [z])\tau_m(t', s' + [z])\frac{dz}{(z-\alpha)(z-\beta)} \\
&\quad = \frac{1}{\alpha - \beta}\big(\tau_m(t, s - [\alpha])\tau_m(t', s' + [\alpha]) \\
&\qquad\qquad - \tau_m(t, s - [\beta])\tau_m(t', s' + [\beta])\big) \\
&\quad = \frac{1}{\alpha - \beta}\big(\tau_m(t, -t + [\beta] - [\alpha])\tau_m(t + [\alpha] - [\beta], -t) \\
&\qquad\qquad - \tau_m(t, -t)\tau_m(t + [\alpha] - [\beta], -t - [\alpha] + [\beta])\big)\,,
\end{aligned}$$

and (2.5), we have

$$\begin{aligned}
&-(\beta - \alpha)^2\tau_{m-1}(t - [\beta], -t + [\beta])\tau_{m+1}(t + [\alpha], -t - [\alpha]) \\
&\qquad = (-1)^m\tau_m(t + [\alpha] - [\beta], -t)^2 \\
&\qquad\qquad - \tau_m(t, -t)\tau_m(t + [\alpha] - [\beta], -t - [\alpha] + [\beta])\,.
\end{aligned}$$

Setting first $m = 2l$ and then $m = 2l + 1$, we find respectively, since odd τ-functions vanish on $\{s = -t\}$ in view of (2.5):

$$\begin{aligned}
0 = {}&\tau_{2l}(t + [\alpha] - [\beta], -t)^2 \\
&- \tau_{2l}(t, -t)\tau_{2l}(t + [\alpha] - [\beta], -t - [\alpha] + [\beta])\,, \quad (2.13)
\end{aligned}$$

and

$$\begin{aligned}
&-(\beta - \alpha)^2\tau_{2l}(t - [\beta], -t + [\beta])\tau_{2l+2}(t + [\alpha], -t - [\alpha]) \\
&\qquad = -\tau_{2l+1}(t + [\alpha] - [\beta], -t)^2\,. \quad (2.14)
\end{aligned}$$

Taking the square root, with the consistent choice of sign[11] (2.10) yields (2.11), and then (2.12) upon setting $t \to t - [\alpha]$ or $t \to t + [\beta]$. $\square$

Corollary 2.3 *Under the assumption of Theorem 2.2, the wave and adjoint wave functions Ψ, Ψ^* along the locus $\{s = -t\}$ satisfy the relations*

$$\begin{aligned}
\Psi_{1,2n}(t,-t;z) &= -\lim_{s\to -t}\left(\frac{\tau_{2n+1}}{\sqrt{\tau_{2n}\tau_{2n+2}}}\Psi_{1,2n+1}(t,s;z)\right)\\
&= \lim_{s\to -t}\left(\frac{\tau_{2n+1}}{\tau_{2n}}\Psi^*_{2,2n}(t,s;z^{-1})\right)\\
&= \lim_{s\to -t}\left(\sqrt{\frac{\tau_{2n+2}}{\tau_{2n}}}\Psi^*_{2,2n+1}(t,s;z^{-1})\right)\\
&= \frac{\tilde\tau_{2n}(t-[z^{-1}])}{\tilde\tau_{2n}(t)}z^{2n}e^{\sum_{i=1}^\infty t_i z^i},
\end{aligned}$$

$$\begin{aligned}
\Psi^*_{1,2n-1}(t,-t;z) &= \lim_{s\to -t}\left(\frac{\tau_{2n-1}}{\sqrt{\tau_{2n-2}\tau_{2n}}}\Psi^*_{1,2n-2}(t,s;z)\right)\\
&= \lim_{s\to -t}\left(\frac{\tau_{2n-1}}{\tau_{2n}}\Psi_{2,2n-1}(t,s;z^{-1})\right)\\
&= -\lim_{s\to -t}\left(\sqrt{\frac{\tau_{2n-2}}{\tau_{2n}}}\Psi_{2,2n-2}(t,s;z^{-1})\right)\\
&= \frac{\tilde\tau_{2n}(t+[z^{-1}])}{\tilde\tau_{2n}(t)}z^{-(2n-1)}e^{-\sum_{i=1}^\infty t_i z^i}.
\end{aligned}$$

Proof. These follow from (1.12), (2.11) and (2.12) by straightforward calculations. $\square$

Corollary 2.4 *Under the assumption of Theorem 2.2, we have*
(i) for $k \geq 1$:

$$\begin{aligned}
\left.\frac{\partial\tau_{2n}}{\partial t_k}\right|_{s=-t} &= \tilde\tau_{2n}(t)\frac{\partial\tilde\tau_{2n}}{\partial t_k}(t),\\
\left.\frac{\partial\tau_{2n+1}}{\partial t_k}\right|_{s=-t} &= p_{k-1}(-\tilde D_t)\tilde\tau_{2n}\cdot\tilde\tau_{2n+2}(t)\\
&\equiv \sum_{i+j=k-1}(p_i(-\tilde\partial_t)\tilde\tau_{2n}(t))(p_j(\tilde\partial_t)\tilde\tau_{2n+2}(t)),
\end{aligned}$$

[11] It suffices to check that (2.10) yields the correct sign in the second equation of (2.11) at $\beta = 0$, $t = 0$ and modulo $O(\alpha^2)$, i.e.,

$$(\partial/\partial t_1)\tau_{2n+1}(0,0) = \tilde\tau_{2n}(0)\tilde\tau_{2n+2}(0),$$

for some $m_\infty(0,0)$ for which the right hand side does not vanish. This can be checked easily, e.g., for $m_\infty(0,0)$ made of 2×2 blocks $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ on the diagonal.

(ii) for $m \geq 2$:

$$\sum_{k+l=m} \frac{\partial^2 \tau_{2n}}{\partial t_k \partial t_l}\bigg|_{s=-t} = \tilde{\tau}_{2n}(t) \sum_{k+l=m} \frac{\partial^2 \tilde{\tau}_{2n}}{\partial t_k \partial t_l}(t)\,, \tag{2.15}$$

$$\sum_{k+l=m} \frac{\partial^2 \tau_{2n}}{\partial t_k \partial s_l}\bigg|_{s=-t} = -\sum_{k+l=m} \frac{\partial \tilde{\tau}_{2n}}{\partial t_k}(t) \frac{\partial \tilde{\tau}_{2n}}{\partial t_l}(t)\,, \tag{2.16}$$

$$\sum_{k+l=m} \frac{\partial^2 \tau_{2n+1}}{\partial t_k \partial t_l}\bigg|_{s=-t} = \sum_{k+l=m-1} (k-l) p_k(-\tilde{\partial}_t)\tilde{\tau}_{2n}(t) \cdot p_l(\tilde{\partial}_t)\tilde{\tau}_{2n+2}(t)\,,$$

(iii) for $k, l \geq 0$:

$$\begin{aligned} p_k(\tilde{\partial}_t) p_l(-\tilde{\partial}_t)\tau_{2n}(t,s)|_{s=-t} &= \tilde{\tau}_{2n}(t) p_k(\tilde{\partial}_t) p_l(-\tilde{\partial}_t)\tilde{\tau}_{2n}(t)\,, \\ p_k(\tilde{\partial}_t) p_l(-\tilde{\partial}_t)\tau_{2n+1}(t,s)|_{s=-t} &= p_l(-\tilde{\partial}_t)\tilde{\tau}_{2n}(t) \cdot p_{k-1}(\tilde{\partial}_t)\tilde{\tau}_{2n+2}(t) \\ &\quad - p_{l-1}(-\tilde{\partial}_t)\tilde{\tau}_{2n}(t) \cdot p_k(\tilde{\partial}_t)\tilde{\tau}_{2n+2}(t)\,, \end{aligned}$$

where $p_k(\cdot)$ are the elementary Schur functions, with $p_{-1}(\cdot) = 0$, and $D_t = (D_{t_1}, (1/2)D_{t_2}, \dots)$ are Hirota's symbols.

Proof. Relations (i) are obtained by differentiating formulas (2.11) in α, setting $\beta = \alpha$ and identifying the coefficients of α^{k-1}. The first two relations in (ii) are obtained by differentiating formulas (2.11) in α and β (i.e., applying $\partial^2/\partial\alpha\partial\beta$), setting $\beta = \alpha$ and identifying the coefficients of α^{m-2}. The last relation in (ii) is obtained by differentiating the first formula in (2.12) in α and β, setting $\beta = \alpha$, substituting $t + [\alpha]$ for t, and then identifying the coefficients of α^{m-2}. Finally, expanding both identities (2.11) in α and β, e.g.,

$$\tau_{2n}(t + [\alpha] - [\beta], s) = \sum_{k,l=0}^{\infty} \alpha^k \beta^l p_k(\tilde{\partial}_t) p_l(-\tilde{\partial}_t)\tau_{2n}(t,s)$$

and identifying the powers of α and β yields relations (iii). □

Variants of formulas (2.11) and the formulas in the corollary can be obtained by using (2.5) and the following consequence of it:

$$\frac{\partial^{|I|+|J|}}{\partial t^I \partial s^J}\tau_n\bigg|_{s=-t} = (-1)^{|I|+|J|+n} \frac{\partial^{|J|+|I|}}{\partial t^J \partial s^I}\tau_n\bigg|_{s=-t}\,, \tag{2.17}$$

where $I = (i_1, i_2, \dots)$ and $J = (j_1, j_2, \dots)$ are multiindices, $|I| = i_1 + i_2 + \cdots$, $\partial t^I = \partial t_1^{i_1} \partial t_2^{i_2} \cdots$, etc. E.g., $(\partial^2/\partial t_k \partial s_l + \partial^2/\partial t_l \partial s_k)\tau_{2n+1} = 0$, so we get the following (trivial) counterpart of (2.16):

$$\sum_{k+l=m} \frac{\partial^2 \tau_{2n+1}}{\partial t_k \partial s_l}\bigg|_{s=-t} = 0\,.$$

3. Equations satisfied by Pfaffian $\tilde{\tau}$-functions

In this section, we exhibit the properties of the Pfaffian $\tilde{\tau}$-function introduced above.

Theorem 3.1 *For all $t, t' \in \mathbb{C}^\infty$ and m, n positive integers, the $\tilde{\tau}$-functions satisfy the bilinear relations*

$$\oint_{z=\infty} \tilde{\tau}_{2n}(t-[z^{-1}])\tilde{\tau}_{2m+2}(t'+[z^{-1}])e^{\sum_{i=0}^{\infty}(t_i-t_i')z^i} z^{2n-2m-2}dz \\ + \oint_{z=0} \tilde{\tau}_{2n+2}(t+[z])\tilde{\tau}_{2m}(t'-[z])e^{\sum_{i=0}^{\infty}(t_i'-t_i)z^{-i}} z^{2n-2m}dz = 0\,, \quad (3.1)$$

or equivalently

$$\sum_{\substack{j,k\geq 0\\ j-k=-2n+2m+1}} p_j(2y)e^{\sum_{i=1}^{\infty} y_i D_i} p_k(-\tilde{D})\tilde{\tau}_{2n}\cdot\tilde{\tau}_{2m+2} \\ + \sum_{\substack{j,k\geq 0\\ k-j=-2n+2m-1}} p_j(-2y)e^{\sum_{i=1}^{\infty} y_i D_i} p_k(\tilde{D})\tilde{\tau}_{2n+2}\cdot\tilde{\tau}_{2m} = 0\,. \quad (3.2)$$

Proof. Formula (3.1) follows from (1.14) upon replacing[12] n by $2n$ and m by $2m$, using (2.11) and (2.12), with $\beta = 0$, to eliminate $\tau_{2n}(t-[z^{-1}], -t)$, $\tau_{2m+1}(t', -t'-[z])$, $\tau_{2n+1}(t, -t-[z])$ and $\tau_{2m}(t'-[z], -t')$ and, upon dividing both sides by $\tilde{\tau}_{2n}(t)\tilde{\tau}_{2m}(t)$.

Substituting $t+y$ and $t-y$ for t and t', respectively, into the left hand side of (3.1) and Taylor expanding it in y, we obtain

$$\oint_{z=\infty} e^{\sum_{i=1}^{\infty} 2y_i z^i}\tilde{\tau}_{2n}(t+y-[z^{-1}])\tilde{\tau}_{2m+2}(t-y+[z^{-1}])z^{2n-2m-2}dz \\ + \oint_{z=0} e^{-\sum_{i=1}^{\infty} 2y_i z^{-i}}\tilde{\tau}_{2n+2}(t+y+[z])\tilde{\tau}_{2m}(t-y-[z])z^{2n-2m}dz \\ = \oint_{z=\infty} e^{\sum_{i=1}^{\infty} 2y_i z^i} e^{\sum_{i=1}^{\infty} y_i D_i} e^{-\sum_{i=1}^{\infty} z^{-i} D_i/i}\tilde{\tau}_{2n}\cdot\tilde{\tau}_{2m+2} z^{2n-2m-2}dz$$

[12] One can check that all the other choices of parities of n and m, i.e., the cases where one or both of n, m are odd, yield the same bilinear identities.

$$
\begin{aligned}
&+\oint_{z=0} e^{-\sum_{i=1}^{\infty} 2y_i z^{-i}} e^{\sum_{i=1}^{\infty} y_i D_i} e^{\sum_{i=1}^{\infty} z^i D_i/i}\,\tilde\tau_{2n+2}\cdot\tilde\tau_{2m} z^{2n-2m} dz\\
&=\oint_{z=\infty}\sum_{j,k=0}^{\infty} p_j(2y) z^j e^{\sum_{i=1}^{\infty} y_i D_i} p_k(-\tilde D) z^{-k}\,\tilde\tau_{2n}\cdot\tilde\tau_{2m+2} z^{2n-2m-2} dz\\
&+\oint_{z=0}\sum_{j,k=0}^{\infty} p_j(-2y) z^{-j} e^{\sum_{i=1}^{\infty} y_i D_i} p_k(\tilde D) z^{k}\,\tilde\tau_{2n+2}\cdot\tilde\tau_{2m} z^{2n-2m} dz\\
&=2\pi i\Bigg(\sum_{j-k=-2n+2m+1} p_j(2y) e^{\sum_{i=1}^{\infty} y_i D_i} p_k(-\tilde D)\tilde\tau_{2n}\cdot\tilde\tau_{2m+2}\\
&\qquad+\sum_{k-j=-2n+2m-1} p_j(-2y) e^{\sum_{i=1}^{\infty} y_i D_i} p_k(\tilde D)\tilde\tau_{2n+2}\cdot\tilde\tau_{2m}\Bigg),
\end{aligned}
$$

showing the equivalence of (3.1) and (3.2). □

The identity (3.1) gives various bilinear relations satisfied by $\tilde\tau$. We show that the Pfaffian $\tilde\tau$-functions satisfy identities reminiscent of the Fay and differential Fay identities for the KP or 2-Toda τ-functions (e.g., see [1]). From this we deduce a sequence of Hirota bilinear equations for $\tilde\tau$, which can be interpreted as a recursion relation for $\tilde\tau_{2n}(t)$.

Theorem 3.2 *The functions $\tilde\tau_{2n}(t)$ satisfy the following "Fay identity":*

$$
\begin{aligned}
&\sum_{i=1}^{r}\tilde\tau_{2n}\Bigg(t-\sum_{j=1}^{l}[z_j]-[\zeta_i]\Bigg)\tilde\tau_{2m+2}\Bigg(t-\sum_{\substack{1\le j\le r\\ j\ne i}}[\zeta_j]\Bigg)\frac{\prod_{k=1}^{l}(\zeta_i-z_k)}{\prod_{\substack{1\le k\le r\\ k\ne i}}(\zeta_i-\zeta_k)}\\
&+\sum_{i=1}^{l}\tilde\tau_{2n+2}\Bigg(t-\sum_{\substack{1\le j\le l\\ j\ne i}}[z_j]\Bigg)\tilde\tau_{2m}\Bigg(t-\sum_{j=1}^{r}[\zeta_j]-[z_i]\Bigg)\frac{\prod_{k=1}^{r}(z_i-\zeta_k)}{\prod_{\substack{1\le k\le l\\ k\ne i}}(z_i-z_k)}\\
&=0\,,
\end{aligned}
\tag{3.3}
$$

the "differential Fay identity":

$$
\begin{aligned}
&\{\tilde\tau_{2n}(t-[u]),\tilde\tau_{2n}(t-[v])\}\\
&\quad+(u^{-1}-v^{-1})(\tilde\tau_{2n}(t-[u])\tilde\tau_{2n}(t-[v])-\tilde\tau_{2n}(t)\tilde\tau_{2n}(t-[u]-[v]))\\
&\qquad=uv(u-v)\tilde\tau_{2n-2}(t-[u]-[v])\tilde\tau_{2n+2}(t)\,,
\end{aligned}
\tag{3.4}
$$

and Hirota bilinear equations, involving nearest neighbors:

$$
\left(p_{k+4}(\tilde D)-\frac{1}{2}D_1 D_{k+3}\right)\tilde\tau_{2n}\cdot\tilde\tau_{2n}=p_k(\tilde D)\tilde\tau_{2n+2}\cdot\tilde\tau_{2n-2}\,. \tag{3.5}
$$

In (3.3), $2n$, $2m \in A$, l, $r \geq 0$ such that $r - l = 2n - 2m$, z_i $(1 \leq i \leq l)$ and ζ_i $(1 \leq i \leq r)$ are scalar parameters near 0; in (3.4), $2n - 2 \in A$ (hence $2n$, $2n+2 \in A$), and u and v are scalar parameters near 0; and in (3.5), $2n-2 \in A$, $k = 0, 1, 2, \ldots$, and $\{f, g\} := f'g - fg' = D_1 f \cdot g$ is the Wronskian of f and g, where $' = \partial/\partial t_1$.

Proof. The Fay identity (3.3) follows from the bilinear identity (3.1) by substitutions

$$t \mapsto t - [z_1] - \cdots - [z_l] \quad \text{and} \quad t' \mapsto t - [\zeta_1] - \cdots - [\zeta_r] \,.$$

Indeed, since $r - l = 2n - 2m$, we have

$$\begin{aligned}\exp\left(\sum_{i=1}^{\infty}(t_i - t_i')z^i\right) z^{2n-2m-2} dz &= \frac{\prod_{k=1}^{l}(1 - zz_k)}{\prod_{k=1}^{r}(1 - z\zeta_k)} z^{r-l-2} dz \\ &= -\frac{\prod_{k=1}^{l}((1/z) - z_k)}{\prod_{k=1}^{r}((1/z) - \zeta_k)} d(1/z)\end{aligned}$$

and

$$\begin{aligned}\exp\left(\sum_{i=1}^{\infty}(t_i' - t_i)z^{-i}\right) z^{2n-2m} dz &= \frac{\prod_{k=1}^{r}(1 - \zeta_k/z)}{\prod_{k=1}^{l}(1 - z_k/z)} z^{r-l} dz \\ &= \frac{\prod_{k=1}^{r}(z - \zeta_k)}{\prod_{k=1}^{l}(z - z_k)} dz \,,\end{aligned}$$

so the first and second terms on the left hand side of (3.1), divided by $2\pi i$, become, respectively,

$$\begin{aligned}&\frac{1}{2\pi i}\oint_{z=\infty} \tilde{\tau}_{2n}(t - [z^{-1}])\tilde{\tau}_{2m+2}(t' + [z^{-1}]) e^{\sum_{i=0}^{\infty}(t_i - t_i')z^i} z^{2n-2m-2} dz \\ &= -\sum_{i=1}^{r} \mathrm{Res}_{z=\zeta_i^{-1}}\, \tilde{\tau}_{2n}\left(t - \sum_{j=1}^{l}[z_j] - [z^{-1}]\right) \\ &\qquad\qquad \tilde{\tau}_{2m+2}\left(t - \sum_{j=1}^{r}[\zeta_j] + [z^{-1}]\right) \frac{\prod_{k=1}^{l}(1 - zz_k)}{\prod_{k=1}^{r}(1 - z\zeta_k)} dz\end{aligned}$$

$$= \sum_{i=1}^{r} \mathrm{Res}_{\zeta=\zeta_i}\, \tilde{\tau}_{2n}\Big(t - \sum_{j=1}^{l}[z_j] - [\zeta]\Big)\tilde{\tau}_{2m+2}\Big(t - \sum_{j=1}^{r}[\zeta_j] + [\zeta]\Big)$$
$$\cdot \frac{\prod_{k=1}^{l}(\zeta - z_k)}{\prod_{k=1}^{r}(\zeta - \zeta_k)} d\zeta \qquad (\zeta := z^{-1})$$
$$= \sum_{i=1}^{r} \tilde{\tau}_{2n}\Big(t - \sum_{j=1}^{l}[z_j] - [\zeta_i]\Big)\tilde{\tau}_{2m+2}\Big(t - \sum_{j=1}^{r}[\zeta_j] + [\zeta_i]\Big)$$
$$\cdot \frac{\prod_{k=1}^{l}(\zeta_i - z_k)}{\prod_{\substack{1\le k\le r\\ k\ne i}}(\zeta_i - \zeta_k)} ,$$

and

$$\frac{1}{2\pi i}\oint_{z=0} \tilde{\tau}_{2n+2}(t + [z])\tilde{\tau}_{2m}(t' - [z]) e^{\sum_{i=0}^{\infty}(t_i'-t_i)z^{-i}} z^{2n-2m} dz$$
$$= \sum_{i=1}^{l} \mathrm{Res}_{z=z_i}\, \tilde{\tau}_{2n+2}\Big(t - \sum_{j=1}^{l}[z_j] + [z]\Big)$$
$$\tilde{\tau}_{2m}\Big(t - \sum_{j=1}^{r}[\zeta_j] - [z]\Big)\frac{\prod_{k=1}^{r}(z - \zeta_k)}{\prod_{k=1}^{l}(z - z_k)} dz$$
$$= \sum_{i=1}^{l} \tilde{\tau}_{2n+2}\Big(t - \sum_{j=1}^{l}[z_j] + [z_i]\Big)\tilde{\tau}_{2m}\Big(t - \sum_{j=1}^{r}[\zeta_j] - [z_i]\Big)$$
$$\cdot \frac{\prod_{k=1}^{r}(z_i - \zeta_k)}{\prod_{\substack{1\le k\le l\\ k\ne i}}(z_i - z_k)} ,$$

showing (3.3).

Note that when $2m = 2n - 2$, $l = 1$ and $r = 3$, denoting $z_i = \zeta_{i-1}$ for $2 \le i \le 4$, and multiplying both sides of (3.3) by $\prod_{2\le j<k\le 4}(z_j - z_k)$, we obtain

$$(z_2 - z_1)(z_3 - z_4)\tilde{\tau}_{2n}(t - [z_1] - [z_2])\tilde{\tau}_{2n}(t - [z_3] - [z_4]) \tag{3.6}$$
$$-(z_3 - z_1)(z_2 - z_4)\tilde{\tau}_{2n}(t - [z_1] - [z_3])\tilde{\tau}_{2n}(t - [z_2] - [z_4])$$
$$+(z_4 - z_1)(z_2 - z_3)\tilde{\tau}_{2n}(t - [z_1] - [z_4])\tilde{\tau}_{2n}(t - [z_2] - [z_3]) \tag{3.7}$$
$$+\Big(\prod_{1\le i<j\le 4}(z_i - z_j)\Big)\tilde{\tau}_{2n+2}(t)\tilde{\tau}_{2n-2}(t - [z_1] - [z_2] - [z_3] - [z_4]) = 0 .$$

The differential Fay identity (3.4) follows from (3.6) by taking a limit (set $z_4 = 0$, divide by z_3 and let $z_3 \to 0$). Alternatively, we can prove (3.4) directly from (3.1): Set $t - t' = [u] - [v]$, $2m = 2n - 2$ in (3.1) and in the clockwise integral[13] about

[13] See footnote 7.

$z = \infty$, set $z \mapsto 1/z$ (and reverse the orientation of contour), yielding

$$\oint_{z=0} \tilde{\tau}_{2n}(t-[z])\tilde{\tau}_{2n}(t'+[z])\frac{1-v/z}{1-u/z}\frac{dz}{z^2} = -\oint_{z=0} \tilde{\tau}_{2n+2}(t+[z])\tilde{\tau}_{2n-2}(t'-[z])\frac{1-u/z}{1-v/z}z^2 dz\,.$$

The first integral has a simple pole at $z = u$ and a double pole at $z = 0$, while the second integral has a simple pole at $z = v$ only, yielding, after substitution $t' = t - [u] + [v]$,

$$\tilde{\tau}_{2n}(t-[u])\tilde{\tau}_{2n}(t+[v])(u-v)\frac{1}{u^2} + \frac{d}{dz}\left(\tilde{\tau}_{2n}(t-[z])\tilde{\tau}_{2n}(t-[u]+[v]+[z])\frac{z-v}{z-u}\right)\Bigg|_{z=0} = -\tilde{\tau}_{2n+2}(t+[v])\tilde{\tau}_{2n-2}(t-[u])(v-u)v^2,$$

or, after carrying out $d/dz|_{z=0}$ on the left hand side,

$$\tilde{\tau}_{2n}(t-[u])\tilde{\tau}_{2n}(t+[v])(u-v)\frac{1}{u^2} + \tilde{\tau}_{2n}(t)\cdot\tilde{\tau}_{2n}(t-[u]+[v])\frac{v-u}{u^2} - D_1\,\tilde{\tau}_{2n}(t)\cdot\tilde{\tau}_{2n}(t-[u]+[v])\frac{v}{u} = -\tilde{\tau}_{2n+2}(t+[v])\tilde{\tau}_{2n-2}(t-[u])(v-u)v^2\,. \quad (3.8)$$

Shifting $t \mapsto t - [v]$ and multiplying both sides by u/v yield (3.4).

Since $P(-D)f \cdot f = P(D)f \cdot f$ by the definition of Hirota operator, (3.5) is the same as (0.8)nothing but the coefficients of y_{k+3} in (3.2). It also follows from (3.4), since, for any power series $F(t, t')$ which satisfies $F(t, t) \equiv 0$,

$$\text{coefficient of } y_{k+3} \text{ in } F(t-y, t+y) = \left(\frac{\partial}{\partial t'_n} - \frac{\partial}{\partial t_n}\right)F(t,t) = 2\frac{\partial}{\partial t'_n}F(t,t') = 2 \times \text{coefficient of } u^{k+2} \text{ in } \frac{d}{dv}F(t, t-[u]+[v])\Bigg|_{v=u}\,.$$

Indeed, differentiating (3.8), which is equivalent to (3.4), in v, setting $v = u$ and using $D_1 f \cdot f = 0$,

$$\frac{\partial}{\partial v}(D_1 f(t)\cdot g(t+[v])) = -\frac{1}{2}D_1 D_v f(t)\cdot g(t+[v]) = -\frac{1}{2}\sum_{j=1}^{\infty} v^{j-1} D_1 D_j f(t)\cdot g(t+[v])\,,$$

etc., we have

$$-\tilde{\tau}_{2n}(t-[u])\tilde{\tau}_{2n}(t+[u])\frac{1}{u^2}+\tilde{\tau}_{2n}(t)^2\frac{1}{u^2}+\frac{1}{2}\sum_{j=1}^{\infty}u^{j-1}D_1D_j\tilde{\tau}_{2n}\cdot\tilde{\tau}_{2n}(t)$$
$$=-\tilde{\tau}_{2n+2}(t+[u])\tilde{\tau}_{2n-2}(t-[u])u^2\,,$$

which, noting $f(t+[u])g(t-[u])=\sum_{k=0}^{\infty}u^k p_k(\tilde{D})f\cdot g$, is a generating function for (3.5). □

As in the case of KP or 2-Toda τ-functions, Pfaffian $\tilde{\tau}$-functions satisfy higher degree identities:

Theorem 3.3

$$\mathrm{Pf}\left(\frac{(z_j-z_i)\tilde{\tau}_{2n-2}(t-[z_i]-[z_j])}{\tilde{\tau}_{2n}(t)}\right)_{1\le i,j\le 2k}$$
$$=\Delta(z)\frac{\tilde{\tau}_{2n-2k}\left(t-\sum_{i=1}^{2k}[z_i]\right)}{\tilde{\tau}_{2n}(t)}\,, \quad (3.9)$$

where $k\ge 1$, $2n-2k\in A$, $z_1,\dots,z_{2k}$ are scalar parameters near 0*, and $\Delta(z)$ is the Vandermonde determinant $\prod_{1\le i<j\le 2k}(z_j-z_i)$.*

Proof. This may be obtained, up to the sign, from the second identity in Theorem 4.2 of [3]:

$$\det\left(\frac{\tau_{N-1}(t-[z_i],s+[y_j])}{\tau_N(t,s)}\right)_{1\le i,j\le k}$$
$$=\Delta(y)\Delta(z)\frac{\tau_{N-k}\left(t-\sum_{i=1}^{k}[z_i],s+\sum_{i=1}^{k}[y_i]\right)}{\tau_N(t,s)}\,,$$

by setting $N\mapsto 2n$, $k\mapsto 2k$, $y_i=z_i$, taking the square roots of both sides and using (2.12). Rather than taking this route, here we prove (3.9) by induction on k, using the bilinear Fay identity (3.3). First, (3.9) is trivial when $k=1$. (Note also that it gives (3.6) when $k=2$.) Suppose (3.9) holds for $k-1$. Then we have, for every $p\in\{2,\dots,2k\}$,

$$\mathrm{Pf}\left(\frac{(z_j-z_i)\tilde{\tau}_{2n-2}(t-[z_i]-[z_j])}{\tilde{\tau}_{2n}(t)}\right)_{\substack{2\le i,j\le 2k\\ i,j\ne p}}$$
$$=\Delta(z_2,\dots,\widehat{z_p},\dots,z_{2k})\frac{\tilde{\tau}_{2n-2k+2}\left(t-\sum_{2\le i\le 2k,i\ne p}[z_i]\right)}{\tilde{\tau}_{2n}(t)}\,.$$

Multiplying both sides by $(-1)^p(z_p-z_1)\tilde{\tau}_{2n-2}(t-[z_1]-[z_p])/\tilde{\tau}_{2n}(t)$, summing it up for $p=2,\dots,2k$, and using

$$(-1)^p(z_p-z_1)\Delta(z_2,\dots,\widehat{z_p},\dots,z_{2k}) \\ = \frac{\Delta(z)}{\prod_{2\le i\le 2k}(z_i-z_1)}\frac{(z_p-z_1)}{\prod_{\substack{2\le i\le 2k\\ i\ne p}}(z_p-z_i)}$$

and the identity

$$\mathrm{Pf}(a_{ij})_{1\le i,j\le 2k}=\sum_{p=2}^{2k}(-1)^p a_{1p}\,\mathrm{Pf}(a_{ij})_{\substack{2\le i,j\le 2k\\ i,j\ne p}}\,,\quad \forall(a_{ij})\text{ s.t. }a_{ji}=-a_{ij}$$

which follows from Definition (2.10) of the Pfaffian, we have

$$\begin{aligned}&\mathrm{Pf}\left(\frac{(z_j-z_i)\tilde{\tau}_{2n-2}(t-[z_i]-[z_j])}{\tilde{\tau}_{2n}(t)}\right)_{1\le i,j\le 2k}\\ &\quad=\frac{\Delta(z)}{\prod_{2\le i\le 2k}(z_i-z_1)}\cdot\frac{1}{\tilde{\tau}_{2n}(t)^2}\sum_{p=2}^{2k}\frac{(z_p-z_1)}{\prod_{\substack{2\le i\le 2k\\ i\ne p}}(z_p-z_i)}\\ &\qquad\cdot\tilde{\tau}_{2n-2}(t-[z_1]-[z_p])\tilde{\tau}_{2n-2k+2}\left(t-\sum_{\substack{2\le i\le 2k\\ i\ne p}}[z_i]\right)\end{aligned}$$

using the bilinear Fay identity (3.3) with $r=2k-1$, $l=1$, $\zeta_i:=z_{i+1}$ $(1\le i\le 2k-1)$ and $(2n,2m)$ replaced by $(2n-2,2n-2k)$ this becomes

$$\begin{aligned}&=\frac{\Delta(z)}{\prod_{2\le i\le 2k}(z_i-z_1)}\cdot\frac{1}{\tilde{\tau}_{2n}(t)^2}\\ &\qquad\cdot(-1)\left(\prod_{i=2}^{2k}(z_1-z_i)\right)\tilde{\tau}_{2n}(t)\tilde{\tau}_{2n-2k}\left(t-\sum_{i=1}^{2k}[z_i]\right)\\ &=\Delta(z)\frac{\tilde{\tau}_{2n-2k}\left(t-\sum_{i=1}^{2k}[z_i]\right)}{\tilde{\tau}_{2n}(t)}\,,\end{aligned}$$

completing the proof of (3.9) by induction. □

4. Vertex operators for Pfaffian $\tilde{\tau}$-functions

In terms of the operators

$$X(t,\lambda):=e^{\sum_{k=1}^{\infty}t_k\lambda^k}e^{-\sum_{k=1}^{\infty}\frac{\lambda^{-k}}{k}\frac{\partial}{\partial t_k}}\,,$$
$$X^*(t,\lambda):=e^{-\sum_{k=1}^{\infty}t_k\lambda^k}e^{\sum_{k=1}^{\infty}\frac{\lambda^{-k}}{k}\frac{\partial}{\partial t_k}}$$

acting on functions $f(t)$ of $t = (t_1, t_2, \dots) \in \mathbb{C}^\infty$, define the following four operators[14] acting on column vectors $g = (g_n(t))_{n\in A}$,

$$\begin{aligned} \mathbb{X}_1(\mu) &:= X(t,\mu)\underline{\chi}(\mu)\,, & \mathbb{X}_1^*(\lambda) &:= -\underline{\chi}^*(\lambda)X^*(t,\lambda)\,,\\ \mathbb{X}_2(\mu) &:= -X(s,\mu)\underline{\chi}^*(\mu)\Lambda\,, & \mathbb{X}_2^*(\lambda) &:= \Lambda^\top\underline{\chi}(\lambda)X^*(s,\lambda)\,, \end{aligned}$$

and their compositions[15]

$$\mathbb{X}_{ij}(\mu,\lambda) := \mathbb{X}_j^*(\lambda)\mathbb{X}_i(\mu)\,, \quad i, j = 1, 2.$$

They form a set of four vertex operators associated with the 2-Toda lattice. Among those, $\mathbb{X}_{12}$ is important in the semi-infinite case, related to the study of orthogonal polynomials. In [3], we showed that

$$\sum_{m\le j<n} \Psi_{1,j}(\mu)\Psi_{2,j}^*(\lambda^{-1}) = \frac{(\mathbb{X}_{12}(\mu,\lambda)\tau)_n}{\tau_n} - \frac{(\mathbb{X}_{12}(\mu,\lambda)\tau)_m}{\tau_m} \tag{4.1}$$

for any $n, m \in A$, $n \ge m$. Note on the right hand side the limit exists as $s \to -t$ if n and m are even, so in particular, taking $n = m+1$, we see the poles along $s = -t$ cancel out in $\Psi_{1,2m}(\mu)\Psi_{2,2m}^*(\lambda^{-1}) + \Psi_{1,2m+1}(\mu)\Psi_{2,2m+1}^*(\lambda^{-1})$. We shall come back to this point after proving the following theorem and its corollary.

Suppose τ satisfies (2.5), and let $\tilde\tau$ be the vector of corresponding Pfaffian $\tilde\tau$-functions. Let $\mathbb{X}_1$, $\mathbb{X}_1^*$ and $\mathbb{X}_{11}$ act on $\tilde\tau$ as if they are acting on the vector $(\tilde\tau_n)_{n\in A}$ padded with zeros, i.e., $\tilde\tau_n \equiv 0$ if n is odd, so that $\underline{\chi}(\mu)$ (resp. $\underline{\chi}^*(\lambda)$) acts on $\tilde\tau_{2n}$ by multiplication of μ^{2n} (resp. λ^{-2n}). Then we have[16]

---

[14] Here $X(s,\lambda)$ has s_i in place of t_i, as well as $\partial/\partial s_i$ in place of $\partial/\partial t_i$, in the definition of $X(t,\lambda)$, etc.; $\underline{\chi}(\mu) := (\mu^i\delta_{ij})_{i,j\in A}$, and $\underline{\chi}^*(\mu) = \underline{\chi}(\mu^{-1})$.

[15] When $i = j$, $\mathbb{X}_j^*$ interacts with $\mathbb{X}_i$ nontrivially, yielding the factor $\exp(\sum(\mu/\lambda)^k/k) = 1/(1-\mu/\lambda)$ if we bring the multiplication operators to the left and the shift operators in t or s to the right. So if we denote by : : the usual normal ordering of operators in t, s (but not in the discrete index n), we have

$$\mathbb{X}_{ii}(\mu,\lambda) = 1/(1-\mu/\lambda))\,{:}\mathbb{X}_i^*(\lambda)\mathbb{X}_i(\mu){:} = -1/(1-\mu/\lambda)\phi(\mu,\lambda)X(u,\mu,\lambda)\,,$$

where $u = t$ and $\phi(\mu,\lambda) = \underline{\chi}(\mu/\lambda)$ if $i = 1$; $u = s$ and $\phi(\mu,\lambda) = \Lambda^\top\underline{\chi}(\lambda/\mu)\Lambda = \mu/\lambda\underline{\chi}(\lambda/\mu)$ if $i = 2$; and

$$X(u,\mu,\lambda) := e^{\sum_{k=1}^\infty u_k(\mu^k-\lambda^k)}e^{\sum_{k=1}^\infty \frac{\lambda^{-k}-\mu^{-k}}{k}\frac{\partial}{\partial u_k}}\,.$$

[16] The product $\mathbb{X}_1(\lambda)\mathbb{X}_1(\mu)$ in (4.4) is computed in the same way as in footnote 15.

Theorem 4.1

$$(\mathbb{X}_{11}(\mu,\lambda)\tau)_N|_{s=-t} = \begin{cases} \tilde\tau_{2n}(t)\mathbb{X}_{11}(\mu,\lambda)\tilde\tau_{2n}(t) & (N=2n) \\ -\lambda(\mathbb{X}_1(\mu)\tilde\tau_{2n}(t))\mathbb{X}_1^*(\lambda)\tilde\tau_{2n+2} & (N=2n+1) \end{cases} \tag{4.2}$$

$$(\mathbb{X}_{22}(\mu,\lambda)\tau)_N|_{s=-t} = \begin{cases} -\tilde\tau_{2n}(t)\mathbb{X}_{11}(\lambda,\mu)\tilde\tau_{2n}(t) & (N=2n) \\ -\mu(\mathbb{X}_1(\lambda)\tilde\tau_{2n})(\mathbb{X}_1^*(\mu)\tilde\tau_{2n+2}) & (N=2n+1) \end{cases} \tag{4.3}$$

$$(\mathbb{X}_{12}(\mu,\lambda)\tau)_N|_{s=-t} = \begin{cases} -\lambda\tilde\tau_{2n}(t)\mathbb{X}_1(\lambda)\mathbb{X}_1(\mu)\tilde\tau_{2n-2}(t) & (N=2n) \\ (\mathbb{X}_1(\mu)\tilde\tau_{2n})(\mathbb{X}_1^*(\lambda)\tilde\tau_{2n}) & (N=2n+1) \end{cases} \tag{4.4}$$

Corollary 4.2 *For $k=1,2$, the following holds: (for notation see (1.24))*

$$\begin{aligned} J_i^{(k)}(t)\tau_{2n}(t,s)|_{s=-t} &= \tilde\tau_{2n}(t)J_i^{(k)}(t)\tilde\tau_{2n}(t)\,, \\ J_i^{(k)}(s)\tau_{2n}(t,s)|_{s=-t} &= (-1)^k\tilde\tau_{2n}(t)J_i^{(k)}(t)\tilde\tau_{2n}(t)\,, \end{aligned}$$

and so

$$(J_i^{(k)}(t)+(-1)^kJ_i^{(k)}(s))\tau_{2n}(t,s)|_{s=-t} = 2\tilde\tau_{2n}(t)J_i^{(k)}(t)\tilde\tau_{2n}(t)\,.$$

Remark 4.3 The appendix (Sect. 9) contains an alternate proof of this corollary.

Proof. The theorem follows from (2.11) and (2.12) by straightforward calculations:

$$\begin{aligned} &(\mathbb{X}_{11}(\mu,\lambda)\tau)_N|_{s=-t} \\ &= -\left(\frac{\mu}{\lambda}\right)^N \frac{1}{1-\mu/\lambda} e^{\sum_{i=1}^\infty t_i(\mu^i-\lambda^i)}\tau_N(t-[\mu^{-1}]-[\lambda^{-1}],-t) \end{aligned}$$

for $N=2n$:

$$\begin{aligned} &= -\left(\frac{\mu}{\lambda}\right)^N \frac{1}{1-\mu/\lambda} e^{\sum_{i=1}^\infty t_i(\mu^i-\lambda^i)}\tilde\tau_{2n}(t)\tilde\tau_{2n}(t-[\mu^{-1}]+[\lambda^{-1}]) \\ &= \tilde\tau_{2n}(t)\mathbb{X}_{11}(\mu,\lambda)\tilde\tau_{2n}(t)\,, \end{aligned}$$

for $N=2n+1$:

$$\begin{aligned} &= \left(\frac{\mu}{\lambda}\right)^N \frac{\lambda^{-1}-\mu^{-1}}{1-\mu/\lambda} e^{\sum_{i=1}^\infty t_i(\mu^i-\lambda^i)}\tilde\tau_{2n}(t-[\mu^{-1}])\tilde\tau_{2n+2}(t+[\lambda^{-1}]) \\ &= -\lambda(\mathbb{X}_1(\mu)\tilde\tau_{2n})(\mathbb{X}_1^*(\lambda)\tilde\tau_{2n+2})\,; \end{aligned}$$

$$
\begin{aligned}
&(\mathbb{X}_{22}(\mu,\lambda)\tau)_N|_{s=-t}\\
&= -\left(\frac{\lambda}{\mu}\right)^{N-1}\frac{1}{1-\mu/\lambda}e^{-\sum_{i=1}^{\infty}t_i(\mu^i-\lambda^i)}\tau_N(t,-t-[\mu^{-1}]+[\lambda^{-1}])\\
&= -\left(\frac{\lambda}{\mu}\right)^{N-1}\frac{(-1)^N}{1-\mu/\lambda}e^{-\sum_{i=1}^{\infty}t_i(\mu^i-\lambda^i)}\tau_N(t+[\mu^{-1}]-[\lambda^{-1}],-t)
\end{aligned}
$$

for $N=2n$:

$$
\begin{aligned}
&= -\left(\frac{\lambda}{\mu}\right)^{2n-1}\frac{1}{1-\mu/\lambda}e^{-\sum_{i=1}^{\infty}t_i(\mu^i-\lambda^i)}\tilde{\tau}_{2n}(t)\tilde{\tau}_{2n}(t+[\mu^{-1}]-[\lambda^{-1}])\\
&= -\tilde{\tau}_{2n}(t)\mathbb{X}_{11}(\lambda,\mu)\tilde{\tau}_{2n}(t)\,,
\end{aligned}
$$

for $N=2n+1$:

$$
\begin{aligned}
&= \left(\frac{\lambda}{\mu}\right)^{2n}\frac{\mu^{-1}-\lambda^{-1}}{1-\mu/\lambda}e^{-\sum_{i=1}^{\infty}t_i(\mu^i-\lambda^i)}\tilde{\tau}_{2n}(t-[\lambda^{-1}])\tilde{\tau}_{2n+2}(t+[\mu^{-1}])\\
&= \frac{\lambda^{2n}}{\mu^{2n+1}}e^{-\sum_{i=1}^{\infty}t_i(\mu^i-\lambda^i)}\tilde{\tau}_{2n}(t-[\lambda^{-1}])\tilde{\tau}_{2n+2}(t+[\mu^{-1}])\\
&= -\mu(\mathbb{X}_1(t,\lambda)\tilde{\tau}_{2n})(\mathbb{X}_1(t,\mu)\tilde{\tau}_{2n+2})\,;
\end{aligned}
$$

$$
\begin{aligned}
&(\mathbb{X}_{12}(\mu,\lambda)\tau)_N|_{s=-t}\\
&= (\Lambda^{\top}\underline{\chi}(\mu\lambda)X^*(s,\lambda)X(t,\mu)\tau)_N|_{s=-t}\\
&= (\mu\lambda)^{N-1}e^{\sum_{i=1}^{\infty}(t_i\mu^i-(-t_i)\lambda^i)}\tau_{N-1}(t-[\mu^{-1}],-t+[\lambda^{-1}])
\end{aligned}
$$

for $N=2n$:

$$
\begin{aligned}
&= (\mu\lambda)^{2n-1}e^{\sum_{i=1}^{\infty}t_i(\mu^i+\lambda^i)}(\lambda^{-1}-\mu^{-1})\tilde{\tau}_{2n-2}(t-[\lambda^{-1}]-[\mu^{-1}])\tilde{\tau}_{2n}(t)\\
&= -\lambda(\mathbb{X}_1(\lambda)\mathbb{X}_1(\mu)\tilde{\tau}_{2n-2}(t))\tilde{\tau}_{2n}(t)\,,
\end{aligned}
$$

for $N=2n+1$:

$$
\begin{aligned}
&= (\mu\lambda)^{2n}e^{\sum_{i=1}^{\infty}t_i(\mu^i+\lambda^i)}\tilde{\tau}_{2n}(t-[\mu^{-1}])\tilde{\tau}_{2n}(t-[\lambda^{-1}])\\
&= (\mathbb{X}_1(\mu)\tilde{\tau}_{2n})(\mathbb{X}_1(\lambda)\tilde{\tau}_{2n})\,.
\end{aligned}
$$

The corollary is shown by expanding $\mathbb{X}_{11}$ in λ and $\mu-\lambda$. Recall that

$$
\begin{aligned}
\mathbb{X}_{11}(\mu,\lambda) &= -\frac{1}{1-\mu/\lambda}\left(\left(\frac{\mu}{\lambda}\right)^n X(\mu,\lambda)\right)_{n\in A}\\
&= -\frac{\lambda}{\lambda-\mu}\left(\sum_{k=0}^{\infty}\frac{(\mu-\lambda)^k}{k!}\sum_{l=-\infty}^{\infty}\lambda^{-l-k}W_{n,l}^{(k)}(t)\right)_{n\in A}, \qquad (4.5)
\end{aligned}
$$

where $X(\mu,\lambda) = e^{\sum_{i=1}^{\infty} t_i(\mu^i-\lambda^i)} e^{\sum_{i=1}^{\infty}(1/i)(\lambda^{-i}-\mu^{-i})(\partial/\partial t_i)}$ is the vertex operator in the KP theory [11][17], and

$$W_{n,l}^{(k)}(t) = \sum_{j=0}^{k} \binom{n}{j} (k)_j W_l^{(k-j)} ,$$

with $W_l^{(k)}$ the coefficients of similar expansion of $X(\mu,\lambda)$.

Expanding $\mathbb{X}_{11}$ in (4.2) as above leads to

$$W_{2n,l}^{(k)}(t)\tau_{2n}(t,s)|_{s=-t} = \tilde{\tau}_{2n}(t) W_{2n,l}^{(k)}(t)\tilde{\tau}_{2n}(t) .$$

In particular, since $J_i^{(k)}$ $(k \le 2)$ and $W_{(n,i)}^{(k)}$ $(k \le 2)$ are linear combinations of each other [3]:

$$W_{n,i}^{(0)} = J_i^{(0)} = \delta_{i,0} , \qquad W_{n,i}^{(1)} = J_i^{(1)} + nJ_i^{(0)} ,$$
$$W_{n,i}^{(2)} = J_i^{(2)} + (2n-i-1)J_i^{(1)} + n(n-1)J_i^{(0)} ,$$

we see for $k = 1, 2$ that

$$J_i^{(k)}(t)\tau_{2n}(t,s)|_{s=-t} = \tilde{\tau}_{2n}(t) J_i^{(k)} \tilde{\tau}_{2n}(t) .$$

□

Consider the following vertex operator[18]

$$\mathbb{X}(z) := \Lambda^\top \mathbb{X}_1(z) = \Lambda^\top e^{\sum_{i=1}^{\infty} t_i z^i} e^{-\sum_{i=1}^{\infty} \frac{z^{-i}}{i}\frac{\partial}{\partial t_i}} \underline{\chi}(z) ,$$

and define the kernel

$$K_n(y,z) := \left(\frac{1}{\tilde{\tau}} \mathbb{X}(y)\mathbb{X}(z)\tilde{\tau}\right)_{2n} .$$

It is easy to see that $(\mathbb{X}(y)\mathbb{X}(z)\tilde{\tau})_{2n} = y\mathbb{X}_1(y)\mathbb{X}_1(z)\tilde{\tau}_{2n-2}$, so by (4.3)

$$K_n(y,z) = -\left.\frac{(\mathbb{X}_{12}(z,y)\tau)_{2n}}{\tau_{2n}}\right|_{s=-t} = \left.\frac{(\mathbb{X}_{12}(y,z)\tau)_{2n}}{\tau_{2n}}\right|_{s=-t} ,$$

and by (4.1)

$$\left.\left(\sum_{2m \le j < 2n} \Psi_{1,j}(\mu)\Psi_{2,j}^*(\lambda^{-1})\right)\right|_{s=-t} = K_n(\mu,\lambda) - K_m(\mu,\lambda) .$$

[17] The order of variables is reversed: our $X(\mu,\lambda)$ is $X(\lambda,\mu)$ in [11].

[18] As noted in p. 25, $\mathbb{X}$ treats $\tilde{\tau}$ as a vector $(\tilde{\tau}_n)_{n\in A}$ padded with zeros, i.e., $\tilde{\tau}_n \equiv 0$ for n odd. So $\underline{\chi}(z)$ appearing in $\mathbb{X}(z)$ acts on $\tilde{\tau}_n$ as multiplication by z^n, and $\Lambda^\top$ acts on $\tilde{\tau}$ as $(\Lambda^\top\tilde{\tau})_n = \tilde{\tau}_{n-1}$. In practice, we always apply $\mathbb{X}$'s to $\tilde{\tau}$ even number of times, so there is always an even power of $\Lambda^\top$, and $\tilde{\tau}_n$ for odd n will never appear.

Here each term $\Psi_{1,j}(\mu)\Psi^*_{2,j}(\lambda^{-1})$ on the left hand side blows up along $s = -t$, but the poles from two successive terms (for $j = 2k$ and $j = 2k+1$) cancel, as we saw earlier.

For $n \in A$, let

$$q_n(t,\lambda) := \begin{cases} \lambda^{2m} \dfrac{\tilde\tau_{2m}(t-[\lambda^{-1}])}{\sqrt{\tilde\tau_{2m}(t)\tilde\tau_{2m+2}(t)}} & \text{if } n = 2m\,, \\ \lambda^{2m} \dfrac{(\partial/\partial t_1 + \lambda)\tilde\tau_{2m}(t-[\lambda^{-1}])}{\sqrt{\tilde\tau_{2m}(t)\tilde\tau_{2m+2}(t)}} & \text{if } n = 2m+1\,. \end{cases} \tag{4.6}$$

In the semi-infinite case, the q_n's form a system of skew-orthogonal polynomials [7].

Theorem 4.4 *The following holds:*

$$\mathrm{Pf}(K_n(z_i,z_j))_{1\le i,j\le 2k} = \left(\frac{1}{\tilde\tau}\prod_{\substack{i=1\\ \text{ordered}}}^{2k} \mathbb{X}(z_i)\tilde\tau\right)_{2n}, \tag{4.7}$$

$$\begin{aligned} &K_{n+1}(\mu,\lambda) - K_n(\mu,\lambda) \\ &\quad = e^{\sum_{i=1}^\infty t_i(\mu^i+\lambda^i)}\left(q_{2n}(t,\lambda)q_{2n+1}(t,\mu) - q_{2n}(t,\mu)q_{2n+1}(t,\lambda)\right)\,, \end{aligned} \tag{4.8}$$

so in the semi-infinite case

$$\begin{aligned} &K_N(\mu,\lambda) \\ &\quad = e^{\sum_{i=1}^\infty t_i(\mu^i+\lambda^i)} \sum_{n=0}^{N-1}\left(q_{2n}(t,\lambda)q_{2n+1}(t,\mu) - q_{2n}(t,\mu)q_{2n+1}(t,\lambda)\right)\,. \end{aligned} \tag{4.9}$$

Proof. Using (3.9) and

$$\begin{aligned} K_n&(\mu,\lambda) \\ &= \left(\frac{\mathbb{X}(\mu)\mathbb{X}(\lambda)\tilde\tau}{\tilde\tau}\right)_{2n} \\ &= (\mu-\lambda)(\mu\lambda)^{2n-2}e^{\sum_{i=1}^\infty t_i(\mu^i+\lambda^i)}\frac{\tilde\tau_{2n-2}(t-[\mu^{-1}]-[\lambda^{-1}])}{\tilde\tau_{2n}(t)} \\ &= (\lambda^{-1}-\mu^{-1})(\mu\lambda)^{2n-1}e^{\sum_{i=1}^\infty t_i(\mu^i+\lambda^i)}\frac{\tilde\tau_{2n-2}(t-[\mu^{-1}]-[\lambda^{-1}])}{\tilde\tau_{2n}(t)}\,, \end{aligned} \tag{4.10}$$

the left hand side of (4.7) becomes

$$(z_1\cdots z_{2k})^{2n-1}e^{\sum_{j=1}^{2k}\sum_{i=1}^\infty t_i z_j^i}\Delta(z^{-1})\frac{\tilde\tau_{2n-2k}(t-\sum_{j=1}^{2k}[z_j^{-1}])}{\tilde\tau_{2n}(t)}\,.$$

This equals the right hand side of (4.7), because

$$
\begin{aligned}
&(\mathbb{X}(z_{2k})\mathbb{X}(z_{2k-1})\cdots\mathbb{X}(z_2)\mathbb{X}(z_1)\tilde{\tau})_{2n}\\
&\quad = z_{2k}^{2n-1}z_{2k-1}^{2n-2}\cdots z_1^{2n-2k}e^{\sum_{i=1}^{\infty}t_i(z_1^i+\cdots+z_{2k}^i)}\\
&\qquad \cdot\left[\prod_{1\le i<j\le 2k}\left(1-\frac{z_i}{z_j}\right)\right]\tilde{\tau}_{2n-2k}\left(t-\sum_{i=1}^{2k}[z_i^{-1}]\right)\\
&\quad = (z_1\cdots z_{2k})^{2n-1}\Delta(z^{-1})e^{\sum_{i=1}^{\infty}t_i(z_1^i+\cdots+z_{2k}^i)}\tilde{\tau}_{2n-2k}\left(t-\sum_{i=1}^{2k}[z_i^{-1}]\right).
\end{aligned}
$$

To prove (4.8), we have

$$
\begin{aligned}
&(\mu-\lambda)\Bigg((\mu\lambda)^{2n}\frac{\tilde{\tau}_{2n}(t-[\mu^{-1}]-[\lambda^{-1}])}{\tilde{\tau}_{2n+2}(t)}\\
&\qquad -(\mu\lambda)^{2n-2}\frac{\tilde{\tau}_{2n-2}(t-[\mu^{-1}]-[\lambda^{-1}])}{\tilde{\tau}_{2n}(t)}\Bigg)\\
&= \frac{(\mu-\lambda)(\mu\lambda)^{2n}}{\tilde{\tau}_{2n+2}(t)\tilde{\tau}_{2n}(t)}\Big(\tilde{\tau}_{2n}(t)\tilde{\tau}_{2n}(t-[\mu^{-1}]-[\lambda^{-1}])\\
&\qquad\qquad -(\mu\lambda)^{-2}\tilde{\tau}_{2n-2}(t-[\mu^{-1}]-[\lambda^{-1}])\tilde{\tau}_{2n+2}(t)\Big)\\
&= \frac{(\mu\lambda)^{2n}}{\tilde{\tau}_{2n+2}(t)\tilde{\tau}_{2n}(t)}\Big(\{\tilde{\tau}_{2n}(t-[\mu^{-1}]),\tilde{\tau}_{2n}(t-[\lambda^{-1}])\}\\
&\qquad\qquad +(\mu-\lambda)\tilde{\tau}_{2n}(t-[\mu^{-1}])\tilde{\tau}_{2n}(t-[\lambda^{-1}])\Big)
\end{aligned}
$$

using (3.4),

$$
\begin{aligned}
&= \left(\lambda^{2n}\frac{\tilde{\tau}_{2n}(t-[\lambda^{-1}])}{\sqrt{\tilde{\tau}_{2n}(t)\tilde{\tau}_{2n+2}(t)}}\,\frac{\mu^{2n}(\partial/\partial t_1+\mu)\tilde{\tau}_{2n}(t-[\mu^{-1}])}{\sqrt{\tilde{\tau}_{2n}(t)\tilde{\tau}_{2n+2}(t)}}-(\lambda\leftrightarrow\mu)\right)\\
&= (q_{2n}(t,\lambda)q_{2n+1}(t,\mu)-q_{2n}(t,\mu)q_{2n+1}(t,\lambda))
\end{aligned}
$$

in terms of the skew-orthogonal polynomials (4.6). Multiplying this with an exponential and noting (4.11), we obtain (4.8). Summing up this telescoping sum yields (4.9). ◻

5. The exponential of vertex operator maintains $\tilde{\tau}$-functions

The purpose of this section is to show the following theorem:

Theorem 5.1 *For any constant a and a Pfaffian $\tilde{\tau}$-function $\tilde{\tau}$,*

$$\tilde{\tau}+a\mathbb{X}(\lambda)\mathbb{X}(\mu)\tilde{\tau} \tag{5.1}$$

is again a Pfaffian $\tilde{\tau}$-function.

Remember that $\mathbb{X}(\lambda)\mathbb{X}(\mu)$ acts on $\tilde{\tau}_{2n}(t)$, as follows:

$$\begin{aligned}
&(\mathbb{X}(\lambda)\mathbb{X}(\mu)\tilde{\tau})_{2n}(t)\\
&= \left(1-\frac{\mu}{\lambda}\right)\lambda^{2n-2}\mu^{2n-1}e^{\sum_{i=1}^{\infty}t_i(\lambda^i+\mu^i)}\tilde{\tau}_{2n-2}(t-[\lambda^{-1}]-[\mu^{-1}])\\
&= \frac{\mu}{\lambda}(\lambda-\mu)\left(\varLambda^{-1}e^{\sum_{i=1}^{\infty}t_i(\lambda^i+\mu^i)}e^{-\sum_{i=1}^{\infty}\left(\frac{\lambda^{-i}+\mu^{-i}}{i}\right)\frac{\partial}{\partial t_i}}\underline{\chi}(\lambda^2\mu^2)\tilde{\tau}\right)_{2n}\\
&=: \frac{\mu}{\lambda}(\lambda-\mu)\widetilde{\mathbb{X}}(\lambda,\mu)\tilde{\tau}_{2n}\,. \qquad (5.2)
\end{aligned}$$

Lemma 5.2 *We have:*

$$\begin{aligned}
&(1-\lambda z)^{-1}(1-\mu z)^{-1}-\frac{1}{\lambda\mu z^2}\left(1-\frac{1}{\lambda z}\right)^{-1}\left(1-\frac{1}{\mu z}\right)^{-1}\\
&\qquad= \frac{1}{\mu-\lambda}\delta\left(z-\frac{1}{\lambda}\right)+\frac{1}{\lambda-\mu}\delta\left(z-\frac{1}{\mu}\right)\,.
\end{aligned}$$

Proof. See for instance [8, p. 248], [11, p. 62]. □

Lemma 5.3

$$\begin{aligned}
&\oint_{z=\infty}\widetilde{\mathbb{X}}\tilde{\tau}_{2n}(t-[z^{-1}])\tilde{\tau}_{2m+2}(t'+[z^{-1}])e^{\sum_{i=1}^{\infty}(t_i-t_i')z^i}z^{2n-2m-2}dz\\
&+\oint_{z=0}\widetilde{\mathbb{X}}\tilde{\tau}_{2n+2}(t+[z])\tilde{\tau}_{2m}(t'-[z])e^{\sum_{i=1}^{\infty}(t_i'-t_i)z^{-i}}z^{2n-2m}dz\\
&= \frac{1}{\mu-\lambda}\Big(\mu^{2n}\lambda^{2m}\tilde{\tau}_{2n}(t-[\mu^{-1}])\tilde{\tau}_{2m}(t'-[\lambda^{-1}])e^{\sum_{i=1}^{\infty}(t_i'\lambda^i+t_i\mu^i)}\\
&\qquad -\lambda^{2n}\mu^{2m}\tilde{\tau}_{2n}(t-[\lambda^{-1}])\tilde{\tau}_{2m}(t'-[\mu^{-1}])e^{\sum_{i=1}^{\infty}(t_i\lambda^i+t_i'\mu^i)}\Big) \qquad (5.3)
\end{aligned}$$

Proof. Upon performing the following operations

$$\begin{cases}\text{replacing } n \text{ by } n-1, \text{ and } t \text{ by } t-[\mu^{-1}]-[\lambda^{-1}],\\ \text{multiplication by } (\lambda\mu)^{2n-1}e^{\sum_1^{\infty}t_i(\mu^i+\lambda^i)},\end{cases}$$

the bilinear identity (3.1) yields

$$\begin{aligned}
0 = &\oint_{z=\infty}\tilde{\tau}_{2n-2}(t-[z^{-1}]-[\mu^{-1}]-[\lambda^{-1}])\tilde{\tau}_{2m+2}(t'+[z^{-1}])(\lambda\mu)^{2n-2}\\
&\quad\left(1-\frac{z}{\lambda}\right)\left(1-\frac{z}{\mu}\right)\frac{\lambda\mu}{z^2}\,e^{\sum_{i=1}^{\infty}((t_i-t_i')z^i+t_i(\mu^i+\lambda^i))}z^{2n-2m-2}dz\\
&+\oint_{z=0}\tilde{\tau}_{2n}(t+[z]-[\mu^{-1}]-[\lambda^{-1}])\tilde{\tau}_{2m}(t'+[z])(\lambda\mu)^{2n}\frac{1}{1-1/\lambda z}\,\cdot\\
&\quad\cdot\frac{1}{1-1/\mu z}\,\frac{1}{\lambda\mu z^2}\,e^{\sum_{i=1}^{\infty}((t_i'-t_i)z^{-i}+t_i(\mu^i+\lambda^i))}z^{2n-2m}dz\,.
\end{aligned}$$

Subtracting this expression (which is $= 0$), the left hand side of (5.3) equals

$$
\begin{aligned}
&\oint_{z=\infty} \tilde{\tau}_{2n-2}(t-[z^{-1}]-[\lambda^{-1}]-[\mu^{-1}])\tilde{\tau}_{2m+2}(t'+[z^{-1}]) \\
&\qquad e^{\sum_{i=1}^{\infty}\left((t_i-t_i')z^i+\left(t_i-\frac{z^{-i}}{i}\right)(\lambda^i+\mu^i)\right)}(\lambda\mu)^{2(n-1)}z^{2n-2m-2}dz \\
&+\oint_{z=0} \tilde{\tau}_{2n}(t+[z]-[\lambda^{-1}]-[\mu^{-1}])\tilde{\tau}_{2m}(t'-[z]) \\
&\qquad e^{\sum_{i=1}^{\infty}\left((t_i'-t_i)z^{-i}+\left(t_i+\frac{z^i}{i}\right)(\lambda^i+\mu^i)\right)}(\lambda\mu)^{2n}z^{2n-2m}dz \\
&=\oint_{z=\infty} \tilde{\tau}_{2n-2}(t-[z^{-1}]-[\lambda^{-1}]-[\mu^{-1}])\tilde{\tau}_{2m+2}(t'+[z^{-1}]) \\
&\qquad e^{\sum_{i=1}^{\infty}\left((t_i-t_i')z^i+t_i(\lambda^i+\mu^i)\right)}(\lambda\mu)^{2n-2}z^{2n-2m-2} \\
&\qquad \left(\left(1-\frac{\lambda}{z}\right)\left(1-\frac{\mu}{z}\right)-\frac{\lambda\mu}{z^2}\left(1-\frac{z}{\lambda}\right)\left(1-\frac{z}{\mu}\right)\right)dz \\
&\quad+\oint_{z=0} \tilde{\tau}_{2n}(t+[z]-[\lambda^{-1}]-[\mu^{-1}])\tilde{\tau}_{2m}(t'-[z]) \\
&\qquad e^{\sum_{i=1}^{\infty}\left((t_i'-t_i)z^{-i}+t_i(\lambda^i+\mu^i)\right)}(\lambda\mu)^{2n}z^{2n-2m} \\
&\qquad \left(\frac{1}{1-\lambda z}\,\frac{1}{1-\mu z}-\frac{1}{\lambda\mu z^2}\,\frac{1}{1-1/\lambda z}\,\frac{1}{1-1/\mu z}\right)dz \\
&=\frac{1}{\mu-\lambda}\Big(\mu^{2n}\lambda^{2m}\tilde{\tau}_{2n}(t-[\mu^{-1}])\tilde{\tau}_{2m}(t'-[\lambda^{-1}])e^{\sum_{i=1}^{\infty}(t_i'\lambda^i+t_i\mu^i)} \\
&\qquad -\lambda^{2n}\mu^{2m}\tilde{\tau}_{2n}(t-[\lambda^{-1}])\tilde{\tau}_{2m}(t'-[\mu^{-1}])e^{\sum_{i=1}^{\infty}(t_i\lambda^i+t_i'\mu^i)}\Big),
\end{aligned}
$$

ending the proof of the lemma. $\square$

Proof of Theorem 5.1. It suffices to prove

$$
\begin{aligned}
0=&\oint_{z=\infty}(a+b\widetilde{\mathbb{X}})\tilde{\tau}_{2n}(t-[z^{-1}])(a+b\widetilde{\mathbb{X}})\tilde{\tau}_{2m+2}(t'+[z^{-1}]) \\
&\qquad e^{\sum_{i=1}^{\infty}(t_i-t_i')z^i}z^{2n-2m-2}dz \\
&+\oint_{z=0}(a+b\widetilde{\mathbb{X}})\tilde{\tau}_{2n+2}(t+[z])(a+b\widetilde{\mathbb{X}})\tilde{\tau}_{2m}(t'-[z]) \\
&\qquad e^{\sum_{i=1}^{\infty}(t_i'-t_i)z^{-i}}z^{2n-2m}dz\,.
\end{aligned}
$$

The coefficient of a^2 and b^2 vanishes, on view of the fact that $\tilde{\tau}_{2n}$ and $\widetilde{\mathbb{X}}\tilde{\tau}_{2n}$ are Pfaffian τ-functions. So it suffices to show the vanishing of the ab-term.

$$\begin{aligned}
&\text{coefficient of } ab \\
&= \oint_{z=\infty} \Big(\widetilde{\mathbb{X}}\tilde{\tau}_{2n}(t-[z^{-1}])\tilde{\tau}_{2m+2}(t'+[z^{-1}]) \\
&\qquad\qquad + \tilde{\tau}_{2n}(t-[z^{-1}])\widetilde{\mathbb{X}}\tilde{\tau}_{2m+2}(t'+[z^{-1}])\Big)e^{\sum_{i=1}^{\infty}(t_i-t_i')z^i} z^{2n-2m-2}dz \\
&\quad + \oint_{z=0} \Big(\widetilde{\mathbb{X}}\tilde{\tau}_{2n+2}(t+[z])\tilde{\tau}_{2m}(t'-[z]) \\
&\qquad\qquad + \tilde{\tau}_{2n+2}(t+[z])\widetilde{\mathbb{X}}\tilde{\tau}_{2m}(t'-[z])\Big)e^{\sum_{i=1}^{\infty}(t_i'-t_i)z^{-i}} z^{2n-2m}dz\,.
\end{aligned}$$

The first terms in each of the integrals can be evaluated by means of lemma. The sum of the two terms equals

$$\begin{aligned}
&\frac{1}{\mu-\lambda}\Big(\mu^{2n}\lambda^{2m}\tilde{\tau}_{2n}(t-[\mu^{-1}])\tilde{\tau}_{2m}(t'-[\lambda^{-1}])e^{\sum_{i=1}^{\infty}(t_i'\lambda^i+t_i\mu^i)} \\
&\quad -\lambda^{2n}\mu^{2m}\tilde{\tau}_{2n}(t-[\lambda^{-1}])\tilde{\tau}_{2m}(t'-[\mu^{-1}])e^{\sum_{i=1}^{\infty}(t_i\lambda^i+t_i'\mu^i)}\Big)\,.
\end{aligned}\tag{5.4}$$

Performing the exchange

$$n \longleftrightarrow m\,, \quad t \longleftrightarrow t'\,, \quad z \longleftrightarrow z^{-1}$$

gives an expression for the sum of the second terms in the integrals; the sum of expression (5.4) and the same expression with the exchange above is obviously zero. □

6. Example 1: symmetric and symplectic matrix integrals

Consider the matrix $m_n(t,s)$ of (t,s)-dependent moments,

$$\mu_{k\ell}(t,s) := \iint_{\mathbb{R}^2} x^k y^\ell e^{\sum_1^\infty(t_i x^i - s_i y^i)} F(x,y)dx\,dy\,, \quad t,s\in\mathbb{C}^\infty\,, \tag{6.1}$$

with regard to a weight function $F(x,y)$. Then m_n satisfies (0.1), so we get a 2-Toda τ-function

$$\begin{aligned}
&\tau_n(t,s) := \det m_n(t,s) \\
&\qquad = \int\cdots\int_{\mathbb{R}^{2n}} \prod_{k=1}^{n}\Big(e^{\sum_{i=1}^{\infty}(t_i x_k^i - s_i y_k^i)}F(x_k,y_k)\Big)\Delta_n(x)\Delta_n(y)\mathbf{dx}\,\mathbf{dy}\,,
\end{aligned}$$

where the last equality is due to an identity involving Vandermonde determinants that can be found in [2, Sect. 3]. If F is skew-symmetric, $F(x,y) = -F(y,x)$, then $m_\infty(0,0)$ is also skew-symmetric; so by Theorem 2.1 we have $\mu_{ij}(t,s) =$

$-\mu_{ji}(-s,-t)$, and get a solution of the Pfaff lattice, with the corresponding Pfaff $\tilde{\tau}$-function

$$\tilde{\tau}_{2n}(t) := \operatorname{Pf} m_{2n}(t,-t)\,.$$

Specializing the inner-product above to the case where $F(x,y) = 2(D^k\delta) \cdot (y-x)\rho(x)\rho(y)$ leads to the three typical cases of symmetric and skew-symmetric weight ($F(x,y) = \pm F(y,x)$), which are known to be related to the *Hermitian, symmetric* and *"symplectic"* matrix integrals.

Namely, the inner-product

$$\langle f,g\rangle_t = \iint_{\mathbb{R}^2} f(x)g(y)(2D^\alpha\delta)(y-x)e^{\sum_1^\infty t_i(x^i+y^i)}\rho(x)\rho(y)dx\,dy.$$

for $\alpha = 0, 1, -1$ leads to

$$TT_k(t) = \begin{cases} \det(\langle x^i, y^j\rangle_{t/2})_{0\le i,j\le k-1} = \int_{\mathcal{H}_k} e^{tr(-2V(X)+\sum t_iX^i)}dX \\ \qquad\qquad\qquad\qquad\qquad \text{for } \alpha = 0,\ k \ge 0 \\ \operatorname{Pf}(\langle x^i, y^j\rangle_t)_{0\le i,j\le k-1} = \int_{\mathcal{S}_k} e^{tr(-V(X)+\sum t_iX^i)}dX \\ \qquad\qquad\qquad\qquad\qquad \text{for } \alpha = -1,\ k \ge 0 \text{ even} \\ \operatorname{Pf}(\langle x^i, y^j\rangle_t)_{0\le i,j\le k-1} = \int_{\mathcal{T}_k} e^{tr(-2V(X)+\sum 2t_iX^i)}dX \\ \qquad\qquad\qquad\qquad\qquad \text{for } \alpha = +1,\ k \ge 0 \text{ even} \end{cases} \tag{6.2}$$

where dX denotes Haar measure on

$\mathcal{H}_k = \{k \times k \text{ Hermitian matrices}\}$
$\mathcal{S}_k = \{k \times k \text{ symmetric matrices}\}$
$\mathcal{T}_k = \{k \times k \text{ self-dual Hermitian matrices, with quaternionic entries}\}$

Ther second and third cases ($\alpha = \pm 1$) are solutions to the Pfaff lattice, whereas, for the first case ($\alpha = 0$), the τ-functions are solutions to the Toda lattice. For more details, see [5] and [25].

– *For* $\alpha = 0$, we have (omitting $e^{\sum t_i(x^i+y^i)}$)

$$\iint f(x)g(y)(2D^0\delta)(y-x)\rho(x)\rho(y)dx\,dy = 2\int f(x)g(x)\rho(x)^2dx\,,$$

leading to the first integral in (6.2).

– *For* $\alpha = -1$, let $\varepsilon(x) = \mathrm{sign}(x) = x/|x|$. Denoting $\varepsilon(x) = 2(\partial/\partial x)^{-1}\delta(x)$, we have

$$\begin{aligned}
&\iint_{\mathbb{R}^2} f(x)g(y)(2D^{-1}\delta)(y-x)\rho(x)\rho(y)dx\,dy \\
&= \iint_{\mathbb{R}^2} f(x)g(y)\varepsilon(x-y)\rho(x)\rho(y)dx\,dy \\
&= \iint_{x>y} f(x)g(y)\rho(x)\rho(y)dx\,dy - \iint_{x<y} f(x)g(y)\rho(x)\rho(y)dx\,dy \\
&= \iint_{x>y} (f(x)g(y) - f(y)g(x))\rho(x)\rho(y)dx\,dy \\
&= \int_{\mathbb{R}} dx \left(f(x)\rho(x)\int_{-\infty}^{x} g(y)\rho(y)dy - g(x)\rho(x)\int_{-\infty}^{x} f(y)\rho(y)dy \right),
\end{aligned}$$

leading to the second integral in (6.2).

– *For* $\alpha = +1$, since

$$\int f(x)\delta'(y-x)dx = f'(y)\,,$$

we compute

$$\begin{aligned}
&\iint f(x)g(y)(2D\delta)(y-x)\rho(x)\rho(y)dx\,dy \\
&= \int \left(\int f(x)\rho(x)\delta'(y-x)dx \right) g(y)\rho(y)dy \\
&\quad + \int f(x)\rho(x) \left(\int g(y)\rho(y)\delta'(y-x)dy \right) dx \\
&= \int (f(y)\rho(y))' g(y)\rho(y)dy - \int (g(x)\rho(x))' f(x)\rho(x)dx \\
&= \int \Big(f'(y)g(y) - f(y)g'(y) \Big) \rho(y)^2 dy\,,
\end{aligned}$$

leading to the third integral in (6.2).

In [5], we worked out the Virasoro constraints satisfied by integrals of the type (6.2), but integrated over subspaces of matrices $\subseteq \mathcal{H},\ \mathcal{S}$ or $\mathcal{T}$ having their spectrum $\leq x$, which then leads to Painlevé-like differential equations for those integrals. In the next section, we give an alternative derivation of the Virasoro constraints for symmetric matrix integrals, via the string equation.

7. String equations and Virasoro constraints for symmetric matrix integrals

In this section we consider the moments (6.1), with regard to the skew-symmetric weight

$$F(x, y) := e^{V(x)+V(y)}\varepsilon(x-y)\,, \tag{7.1}$$

assuming the following form for the potential V:

$$V'(z) = \frac{g}{f} = \frac{\sum_{i=0}^{\infty} b_i z^i}{\sum_{i=0}^{\infty} a_i z^i}\,, \tag{7.2}$$

with $e^{V(z)}$ decaying to 0 fast enough at the boundary of its domain.

According to [2,4], in the semi-infinite case the Borel decomposition of the moment matrix, $m_\infty(t,s) = S_1^{-1}S_2$, leads to the (monic) string-orthogonal polynomials

$$p^{(1)}(z) := S_1\chi(z) \ \text{ and } \ p^{(2)}(z) := hS_2^{\top-1}\chi(z)\,, \tag{7.3}$$

satisfying the orthogonality relations

$$\langle p_n^{(1)}, p_m^{(2)}\rangle = \delta_{n,m}h_n$$

for the skew-symmetric inner product

$$\langle f, g\rangle := \iint_{\mathbb{R}^2} dydz\varepsilon(y-z)e^{V(y)+V(z)+\sum_{i=1}^{\infty}(t_i y^i - s_i z^i)} f(y)g(z)\,. \tag{7.4}$$

Besides L_1 and L_2, we also define strictly lower-triangular (i.e., with zero diagonal) matrices Q_1, Q_2 by

$$Q_1 := S_1\varepsilon S_1^{-1}\,,\quad Q_2 := hS_2^{\top-1}\varepsilon S_2^{\top}h^{-1}\,, \tag{7.5}$$

where $\varepsilon = (i\delta_{i,j+1})_{i,j\geq 0}$ as in Sect. 1, satisfying (1.15).

Note that (1.7) and (7.3) imply

$$\Psi_1 = e^{\sum_{k=1}^{\infty} t_k z^k} p^{(1)}(z)\,,\quad \Psi_2^* = e^{-\sum_{k=1}^{\infty} s_k z^{-k}} h^{-1}p^{(2)}(z^{-1})\,, \tag{7.6}$$

so from (1.8)

$$zp^{(1)}(z) = L_1 p^{(1)}(z)\,,\quad zp^{(2)}(z) = hL_2^{\top}h^{-1}p^{(2)}(z)\,. \tag{7.7}$$

Also, from (1.15), (7.3) and (7.5)

$$\frac{\partial}{\partial z}p^{(1)}(z) = Q_1 p^{(1)}(z)\,,\quad \frac{\partial}{\partial z}p^{(2)}(z) = Q_2 p^{(2)}(z)\,. \tag{7.8}$$

Comparing (1.16) with (7.5), we have

$$\begin{aligned} M_1 &= Q_1 + S_1 \xi'(t,\Lambda) S_1^{-1} = Q_1 + \xi'(t, L_1)\,, \\ M_2^* &= h^{-1} Q_2 h - S_2^{\top -1} \xi'(s,\Lambda) S_2^{\top} = h^{-1} Q_2 h - \xi'(s, L_2^{\top})\,, \end{aligned} \tag{7.9}$$

where we set

$$\xi(t,z) := \sum_{i=1}^{\infty} t_i z^i\,, \quad \xi'(t,z) := \frac{\partial \xi}{\partial z}(t,z)$$

as in Sect. 1. Note that (1.17), (7.6) and (7.9) yield (7.8) again. Note also that, since $\varepsilon^* = -\varepsilon^{\top} + \Lambda$, we have

$$M_2 \equiv S_2\big(\varepsilon^* + \xi'(s,\Lambda^{\top})\big)S_2^{-1} = -h Q_2^{\top} h^{-1} + L_2^{-1} + \xi'(s, L_2)\,, \tag{7.10}$$

where L_2^{-1} is *defined* to be $S_2 \Lambda S_2^{-1}$. Since $\Lambda\Lambda^{\top} = I$, this is a left inverse of L_2, i.e., $L_2^{-1} L_2 = I$.

We now state the two main theorems of this section, namely string and Virasoro equations for the symmetric case. Similar equations can be obtained for the "symplectic" case (third integral (6.2)).

Theorem 7.1 (String equations) *The semi-infinite matrices L_i and M_i satisfy the following matrix identities in terms of f and g in $V' = g/f$, for all $k \geq -1$:*

$$\begin{aligned} &M_1 L_1^{k+1} f(L_1) - M_2 L_2^{k+1} f(L_2) \\ &\quad + L_1^{k+1} g(L_1) + L_2^{k+1} g(L_2) + (L_1^{k+1} f(L_1))' + L_2^{k} f(L_2) = 0\,, \end{aligned} \tag{7.11}$$

where $'$ means $\partial/\partial L_1$.

This fact, together with the ASV-correspondence (Proposition 1.1) and corollary 4.2 (Proposition 9.1), leads at once to the constraints for the 2-Toda τ-functions and the Pfaffian $\tilde{\tau}$-functions:

Theorem 7.2 (Virasoro constraints) *The multiple integrals*

$$\begin{aligned} \tau_n(t,s) &= \det\big(\mu_{ij}(t,s)\big)_{0 \le i,j \le n-1} \\ &= \int \cdots \int_{\mathbb{R}^{2n}} \prod_{k=1}^{n} \Big(e^{V(x_k)+V(y_k)+\sum_{i=1}^{\infty}(t_i x_k^i - s_i y_k^i)} \varepsilon(x_k - y_k)\Big) \cdot \\ &\quad \cdot \Delta_n(x)\Delta_n(y)\,\mathbf{dx}\,\mathbf{dy}\,, \qquad \tau_0 = 1\,, \end{aligned}$$

form a τ-vector for the 2-Toda lattice and satisfy the following Virasoro constraints for all $k \geq -1$ and $n \geq 0$:

$$\sum_{i \ge 0} \Big\{ \frac{a_i}{2} \Big(\mathbb{J}^{(2)}_{i+k,n} + \tilde{\mathbb{J}}^{(2)}_{i+k,n} \Big) + b_i \Big(\mathbb{J}^{(1)}_{i+k+1,n} - \tilde{\mathbb{J}}^{(1)}_{i+k+1,n} \Big) \Big\} \tau_n = 0\,, \tag{7.12}$$

where (with $J_k^{(i)}$ and $\tilde{J}_k^{(i)}$ defined in (1.24))

$$
\begin{aligned}
\mathbb{J}_{k,n}^{(2)} &:= J_k^{(2)} + (2n+k+1)J_k^{(1)} + n(n+1)J_k^{(0)}\,,\\
\mathbb{J}_{k,n}^{(1)} &:= J_k^{(1)} + nJ_k^{(0)}\,,\\
\tilde{\mathbb{J}}_{k,n}^{(2)} &:= \tilde{J}_k^{(2)} - (2n+k+1)\tilde{J}_k^{(1)} + n(n+1)J_k^{(0)}\,,\\
\tilde{\mathbb{J}}_{k,n}^{(1)} &:= \tilde{J}_k^{(1)} - nJ_k^{(0)}\,.
\end{aligned}
\tag{7.13}
$$

The Pfaffian

$$
\tilde{\tau}_N(t) = \tau_N(t,-t)^{1/2} = \int_{\mathbb{R}^N} \prod_{k=1}^{N} \left(e^{V(x_k)+\sum_{i=1}^{\infty} t_i x_k^i} \right) |\Delta_N(x)| \mathbf{dx}\,, \qquad N \text{ even}\,,
$$

satisfy the Pfaff lattice, together with the following Virasoro constraints, for all $k \geq -1$ and even $N \geq 0$:

$$
\sum_{\ell=0}^{\infty} \left(\frac{a_\ell}{2} \mathbb{J}_{k+\ell,N}^{(2)} + b_\ell \mathbb{J}_{k+\ell+1,N}^{(1)} \right) \tilde{\tau}_N(t) = 0\,,
\tag{7.14}
$$

where $\mathbb{J}_{k,n}^{(i)}$ are defined by the same formulas as in (7.13).

Proof of Theorem 7.1. Using

$$
\frac{\partial}{\partial y} \varepsilon(y-z) = 2\delta(y-z)\,,
$$

setting $V_t(z) = V(z) + \xi(t,z)$, and using the hypothesis that e^V vanishes fast enough at the boundary of its domain[19], we first compute

$$
\begin{aligned}
0 &= \int_{\mathbb{R}} dy \frac{\partial}{\partial y} \left\{ y^k f(y) \left(\int_{\mathbb{R}} dz\, \varepsilon(y-z) e^{V_{-s}(z)} p_m^{(2)}(z) \right) e^{V_t(y)} p_n^{(1)}(y) \right\}\\
&= \int_{\mathbb{R}} dy \left(\int_{\mathbb{R}} dz\, \varepsilon(y-z) e^{V_{-s}(z)} p_m^{(2)}(z) \right) e^{V_t(y)}\\
&\qquad \left\{ \left(V_t'(y) f(y) y^k + (y^k f(y))' \right) p_n^{(1)}(y) + p_n^{(1)\prime}(y) y^k f(y) \right\}\\
&\quad + 2 \iint_{\mathbb{R}^2} e^{V_t(y)+V_{-s}(z)} y^k f(y) p_n^{(1)}(y) p_m^{(2)}(z) \delta(y-z)\, dy\, dz
\end{aligned}
$$

[19] We imagine doing the calculation for all t_i and s_j vanishing beyond t_{2k} and s_{2k} and letting the latter be strictly negative and positive respectively.

$$
\begin{aligned}
&= \iint_{\mathbb{R}^2} dydz\, \varepsilon(y-z) e^{V_t(y)+V_{-s}(z)} \Big[\big\{ \big(g(L_1) + \xi'(t, L_1) f(L_1) \big) L_1^k \\
&\qquad\qquad + (L_1^k f(L_1))' + Q_1 L_1^k f(L_1) \big\} p^{(1)}(y) \Big]_n p_m^{(2)}(z) \\
&\quad + 2 \int_{\mathbb{R}} e^{V_t(y)+V_{-s}(y)} p_n^{(1)}(y) p_m^{(2)}(y) y^k f(y) dy \\
&= \big\{ (Q_1 + \xi'(t, L_1)) L_1^k f(L_1) + g(L_1) L_1^k + (L_1^k f(L_1))' \big\}_{nm} h_m \\
&\quad + 2 \int_{\mathbb{R}} e^{V_t(y)+V_{-s}(y)} p_n^{(1)}(y) p_m^{(2)}(y) y^k f(y) dy \,.
\end{aligned}
$$

Next, setting $\bar{L}_2 := h L_2^{\top} h^{-1}$ so that $zp^{(2)} = (\bar{L}_2 p^{(2)})_n$, we find similarly

$$
\begin{aligned}
0 &= \int_{\mathbb{R}} dz \frac{\partial}{\partial z} \left\{ z^k f(z) \left(\int_{\mathbb{R}} dy\, \varepsilon(y-z) e^{V_t(y)} p_n^{(1)}(y) \right) e^{V_{-s}(z)} p_m^{(2)}(z) \right\} \\
&= \int_{\mathbb{R}} dz \left(\int_{\mathbb{R}} p_n^{(1)}(y) e^{V_t(y)} \varepsilon(y-z) dy \right) e^{V_{-s}(z)} \\
&\qquad \big\{ \big(V_{-s}'(z) f(z) z^k + (z^k f(z))' \big) p_m^{(2)}(z) + p_m^{(2)\prime}(z) z^k f(z) \big\} \\
&\quad - 2 \iint_{\mathbb{R}^2} e^{V_t(y)+V_{-s}(z)} f(z) z^k p_n^{(1)}(y) p_m^{(2)}(z) \delta(y-z) dydz \\
&= \iint_{\mathbb{R}^2} dydz\, \varepsilon(y-z) e^{V_t(y)+V_{-s}(z)} \Big[\big\{ \big(g(\bar{L}_2) - \xi'(s, \bar{L}_2) f(\bar{L}_2) \big) \bar{L}_2^k \\
&\qquad\qquad + (f(\bar{L}_2) \bar{L}_2^k)' + Q_2 f(\bar{L}_2) \bar{L}_2^k \big\} p^{(2)}(z) \Big]_m p_n^{(1)}(y) \\
&\quad - 2 \iint_{\mathbb{R}^2} e^{V_t(y)+V_{-s}(y)} f(y) y^k p_n^{(1)}(y) p_m^{(2)}(y) dy \\
&= \big\{ (Q_2 - \xi'(s, \bar{L}_2)) \bar{L}_2^k f(\bar{L}_2) + g(\bar{L}_2) \bar{L}_2^k + (\bar{L}_2^k f(\bar{L}_2))' \big\}_{mn} h_n \\
&\quad - 2 \iint_{\mathbb{R}^2} e^{V_t(y)+V_{-s}(y)} f(y) y^k p_n^{(1)}(y) p_m^{(2)}(y) dy \,.
\end{aligned}
$$

Adding the two expressions yields the matrix identity

$$
\begin{aligned}
&\{(Q_1 + \xi'(t, L_1)) L_1^k f(L_1) + g(L_1) L_1^k + (L_1^k f(L_1))'\} h \\
&\quad + h \{(Q_2 - \xi'(s, \bar{L}_2)) \bar{L}_2^k f(\bar{L}_2) + g(\bar{L}_2) \bar{L}_2^k + (\bar{L}_2^k f(\bar{L}_2))'\}^{\top} = 0 \,. \qquad (7.15)
\end{aligned}
$$

Replacing k by $k+1$, and using

$$
\begin{aligned}
h^{-1} \bar{L}_2 h = L_2^{\top} \,, \quad & Q_1 + \xi'(t, L_1) = M_1 \,, \\
(h^{-1} Q_2 h)^{\top} - \xi'(s, L_2) &= M_2^{*\top} = L_2^{-1} - M_2 \,,
\end{aligned}
$$

we observe that identity (7.15) leads to

$$M_1 L_1^{k+1} f(L_1) + g(L_1) L_1^{k+1} + \big(L_1^{k+1} f(L_1)\big)' \\ + L_2^{k+1} f(L_2) M_2^{*\top} + L_2^{k+1} g(L_2) + \big(L_2^{k+1} f(L_2)\big)' = 0\,;$$

Finally, since $[L_2, M_2^{*\top}] = -I$ by the last identity in (1.18), and since L_2^{-1} is a left inverse of L_2 (see the comment after formula (7.10)), we have

$$\begin{aligned} L_2^{k+1} f(L_2) M_2^{*\top} &= M_2^{*\top} L_2^{k+1} f(L_2) - (L_2^{k+1} f(L_2))' \\ &= (L_2^{-1} - M_2) L_2^{k+1} f(L_2) - (L_2^{k+1} f(L_2))' \\ &= L_2^k f(L_2) - M_2 L_2^{k+1} f(L_2) - (L_2^{k+1} f(L_2))'\,, \end{aligned}$$

leading to the identity, announced in Theorem 7.1. □

Proof of Theorem 7.2. Using the a_i and b_i as in representation (7.2) of $V'(z)$, we obtain from (7.11) that

$$\sum_{i\geq 0} a_i (M_1 L_1^{k+i+1} - M_2 L_2^{k+i+1} + (i+k+1) L_1^{i+k} + L_2^{i+k}) \\ + \sum_{i \geq 0} b_i (L_1^{i+k+1} + L_2^{i+k+1}) = 0\,.$$

We now apply Proposition 1.1. The vanishing of the matrix expression above implies obviously that the $(\ )_-$ and $(\ )_+$ parts vanish as well, so that acting respectively on the wave vectors Ψ_1 and Ψ_2 lead to the vanishing of the four right hand sides of (1.20) in Proposition 1.1, for the corresponding combination of W's. Therefore we have

$$\begin{aligned} &\mathcal{L}_{k,m} \tau_m \\ &:= \sum_{i\geq 0} \Big\{ a_i (W^{(2)}_{m,k+i} + \tilde{W}^{(2)}_{m-1,k+i} + 2(i+k+1) W^{(1)}_{m,i+k} - 2\tilde{W}^{(1)}_{m-1,i+k}) \\ &\qquad + 2 b_i (W^{(1)}_{m,k+i+1} - W^{(1)}_{m-1,k+i+1}) \Big\} \tau_m \\ &= c_k \tau_m\,; \end{aligned}$$

the point is that c_k is independent of t, using the first and third relations of Proposition 1.1, and independent of s and n using the second and fourth relations.

Finally, in view of the relations (1.23), we have

$$
\begin{aligned}
&\mathcal{L}_{k,m}\tau_m \\
&= \Big\{\sum_{i\ge 0} a_i\Big(J^{(2)}_{i+k} + \tilde{J}^{(2)}_{i+k} + (2m-i-k-1)J^{(1)}_{i+k} \\
&\qquad\qquad + (2(1-m)-i-k-1)\tilde{J}^{(1)}_{i+k} + 2m(m-1)\delta_{i+k,0}\Big) \\
&\qquad + 2\sum_{i\ge 0} a_i\Big((i+k+1)(J^{(1)}_{i+k} + m\delta_{i+k,0}) \\
&\qquad\qquad - (\tilde{J}^{(1)}_{i+k} + (1-m)\delta_{i+k,0})\Big) \\
&\qquad + 2\sum_{i\ge 0} b_i\Big((J^{(1)}_{i+k+1} + m\delta_{i+k+1,0}) \\
&\qquad\qquad - (\tilde{J}^{(1)}_{i+k+1} + (1-m)\delta_{i+k+1,0})\Big)\Big\}\tau_m \\
&= \Big\{\sum_{i\ge 0} a_i\Big(J^{(2)}_{i+k} + \tilde{J}^{(2)}_{i+k} + (2m+i+k+1)(J^{(1)}_{i+k} - \tilde{J}^{(1)}_{i+k}) \\
&\qquad\qquad + (2m(m+1)-2)\delta_{i+k,0}\Big) \\
&\qquad + 2\sum_{i\ge 0} b_i\Big((J^{(1)}_{i+k+1} - \tilde{J}^{(1)}_{i+k+1}) + (2m-1)\delta_{i+k+1,0}\Big)\Big\}\tau_m\,.
\end{aligned}
$$

Since c_k is independent of m and $\tau_0 = 1$, and since most of $\mathcal{L}_{k,m}$ vanish when acting on a constant, we have

$$
\frac{\mathcal{L}_{k,m}\tau_m}{\tau_m} = \frac{\mathcal{L}_{k,0}\tau_0}{\tau_0} = -2\sum_{i\ge 0}(a_i\delta_{i+k,0} + b_i\delta_{i+k+1,0})\,,
$$

and so

$$
\Big(\mathcal{L}_{k,m} + 2\sum_{i\ge 0}(a_i\delta_{i+k,0} + b_i\delta_{i+k+1,0})\Big)\tau_m = 0\,,
$$

yielding the identity (7.12). The proof of the Virasoro constraints (7.14) for $\tilde{\tau}(t)$ follows at once from (7.12) and corollary 4.2 (or Proposition 9.1). □

8. Example 2: Quasiperiodic solutions

In this section, we shall combine the construction of quasi-periodic solutions of 2-Toda lattice [20,23] and the theory of Prym varieties [21] to obtain quasiperiodic solutions of the Pfaff lattice. While we put stress on the semi-infinite case in the present paper, this gives a non-trivial example in the bi-infinite case.

A 2-Toda quasiperiodic solution is given by some deformation of a line bundle $\mathcal{L}$ on a complex curve (Riemann surface) C, with the time variables playing the role of deformation parameters, so the orbit under the 2-Toda flows is parametrized by the Jacobian of C. If C is equipped with an involution $\iota\colon C \to C$, and if $\mathcal{L}$ satisfies a suitable antisymmetry condition with respect to ι, then the 2-Toda flows can be restricted to preserve the antisymmetry of $\mathcal{L}$, giving a solution of Pfaff lattice. The Prym variety P of (C, ι) naturally appears as the restricted parameter space. The vanishing of every other $\tau_n(t, -t)$ (see (0.4) or (2.5)) indicates that the space of $\mathcal{L}$'s which satisfy the antisymmetry condition must consist of two connected components, P and P^-. This means the involution ι has no fixed points. So, in general a quasiperiodic solution of the Pfaff lattice does not satisfy the BKP equation and vice versa, since the orbit of a quasiperiodic solution of the BKP equation is isomorphic to the Prym variety of a curve with involution having at least two fixed points.

Preliminary on the geometry of curves

A line bundle on a complex curve C is defined by a divisor $D = \sum_i m_i p_i, m_i \in \mathbb{Z}$, $p_i \in C$, i.e., a set of points p_i on C with (positive or negative) multiplicities m_i, as $\mathcal{L} = \mathcal{O}(D)$. Its local sections (on an open set $U \subset C$, say) are meromorphic functions on U which have poles of order at most m_i (zeros of order at least $-m_i$) at p_i. The number $d := \sum_i m_i$ is called the degree of $\mathcal{L}$. For $\mathcal{L} = \mathcal{O}(D)$ and $m, n \in \mathbb{Z}$, $p, q \in C$, we denote $\mathcal{L}(mp + nq) = \mathcal{O}(D + mp + nq)$ etc. A deformation of $\mathcal{L}$ can be described as a deformation of D, like $D_{t,s} = \sum_i m_i p_i(t, s)$, but in the 2-Toda theory it is more convenient to describe it by requiring its local sections to have some exponential behaviors at prescribed points, as we shall see later.

Two line bundles $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$ are isomorphic if the divisors D_1 and D_2 are "linearly equivalent," i.e., if they differ by the divisor of a global meromorphic function on C. Jacobian[20] J of C is the space (Lie group) of isomorphism classes of degree 0 line bundles on C. It becomes a principally polarized abelian variety of dimension $g :=$ genus of C, i.e., J is a complex torus $\mathbb{C}^g/\varGamma$, $\mathbb{C}^g \supset \varGamma \simeq \mathbb{Z}^{2g}$, for which there is a divisor (codimension 1 subvariety) $\varTheta \subset J$, such that some positive integer multiple of $\varTheta$ defines an embedding of J into a complex projective space, and $\varTheta$ is "rigid" in the sense that it has no deformation in J except parallel translations. A complex torus $\mathbb{C}^g/\varGamma$ is a principally polarized abelian variety if and only if, after some change of coordinates by $GL(g, \mathbb{C})$, the lattice $\varGamma$ becomes $\mathbb{Z}^g + \varOmega\mathbb{Z}^g$ for some complex symmetric $g \times g$ matrix $\varOmega$ with positive definite imaginary part. On a principally polarized abelian variety $\mathbb{C}^g/(\mathbb{Z}^g + \varOmega\mathbb{Z}^g)$, there is a special quasiperiodic function (i.e., holomorphic

[20] In this section J means Jacobian, not a Virasoro generator.

function on $\mathbb{C}^g$ that satisfies some quasiperiodicity condition with respect to $\mathbb{Z}^g + \Omega\mathbb{Z}^g$) called Riemann's theta function ϑ, defined by

$$\vartheta(z) = \sum_{m \in \mathbb{Z}^g} \exp(2\pi i m^t z + \pi i m^t \Omega m) .$$

The theta divisor Θ becomes the zero divisor of ϑ.

If C has a (holomorphic) involution $\iota \colon C \to C$ (i.e., $\iota^2 = \mathrm{id}$), J gets an involution ι^* induced by ι. The Jacobian J' of the quotient curve $C' = C/\iota$, and the Prym variety P of the pair (C, ι) (or (C, C')) appear in J roughly as the ± 1 eigenspaces of ι: $\tilde{J}' := J'/(\text{some subgroup of order } 2) \subset J$ and $P \subset J$ are subabelian varieties of J, such that $\iota|_{\tilde{J}'} = +1$, $\iota|_P = -1$, and $J \simeq (J' \times P)/(\text{finite subgroup})$. When ι has at most two fixed points, the restriction of Θ on P gives twice some principal polarization on P (the restriction $\vartheta|_P$ becomes the square of the Riemann theta function on P defined by this polarization).

Quasiperiodic solutions of 2-Toda lattice

Let C be a nonsingular complete curve on $\mathbb{C}$ (compact Riemann surface) of genus g, let $\mathcal{L}$ be a line bundle of degree $g-1$ on C, let $p, q \in C$ be distinct points. Let us choose local coordinates z^{-1} at p and z at q, and trivializations of $\mathcal{L}(p)$ at p and q,

$$\sigma_p \colon \mathcal{L}_p(p) \simeq \mathcal{O}_p \quad \text{and} \quad \sigma_q \colon \mathcal{L}_q \simeq \mathcal{O}_q .$$

For $t, s \in \mathbb{C}^\infty$, let $\mathcal{L}_{t,s}$ be the line bundle whose (local holomorphic) sections are (local holomorphic) sections of $\mathcal{L}$ away from p and q, and at p (resp. q) have singularities of the form $e^{\sum_{i=1}^\infty t_i z^i} \cdot$ (holomorphic) (resp. $e^{\sum_{i=1}^\infty s_i z^{-i}} \cdot$ (holomorphic)). For "generic"[21] $(n, t, s) \in \mathbb{Z} \times \mathbb{C}^\infty \times \mathbb{C}^\infty$, the wave functions $\Psi_{1,n}$, $\Psi_{2,n}$ are obtained from a (unique) section $\varphi_n(t, s)$ of

$$\mathcal{L}_{t,s}((n+1)p - nq) ,$$

which has the form $z^n e^{\sum_{i=1}^\infty t_i z^i}(1 + O(z^{-1}))$ at p via σ_p, i.e.,

$$\Psi_{1,n}(t, s; z) := \sigma_p(\varphi_n(t, s)) = z^n e^{\sum_{i=1}^\infty t_i z^i}(1 + O(z^{-1})) ,$$
$$\Psi_{2,n}(t, s; z) := \sigma_q(\varphi_n(t, s)) = z^n e^{\sum_{i=1}^\infty s_i z^{-i}}(h_n(t, s) + O(z)) .$$

The adjoint wave functions

$$\Psi^*_{1,n} = z^{-n} e^{-\sum_{i=1}^\infty t_i z^i}(1 + O(z^{-1})) ,$$
$$\Psi^*_{2,n} = z^{-n} e^{-\sum_{i=1}^\infty s_i z^{-i}}(h_n(t, s)^{-1} + O(z))$$

[21] Here generic means that $\Gamma(\mathcal{L}_{t,s}(np - nq)) = \{0\}$ holds. For a degree $g-1$ line bundle $\mathcal{L}$, this condition holds for almost all $(n, t, s) \in \mathbb{Z} \times \mathbb{C}^\infty \times \mathbb{C}^\infty$, and implies that $\dim \Gamma(\mathcal{L}_{t,s}((n+1)p - nq)) = 1$.

are defined similarly, by using

$$(\mathcal{L}_{t,s})^*(-np+(n+1)q) = (\mathcal{L}^*)_{-t,-s}(-np+(n+1)q)\,,$$

in place of $\mathcal{L}_{t,s}((n+1)p-nq)$, where we denote

$$\mathcal{L}^* := \mathcal{H}om(\mathcal{L},\omega) = \mathcal{L}^{-1}\otimes\omega\,,$$

with ω being the dualizing sheaf (the canonical bundle, i.e., the line bundle of holomorphic 1-forms), and, in place of σ_p and σ_q, trivializations

$$\sigma_p^*\colon \mathcal{L}_p^* \simeq \mathcal{O}_p \quad\text{and}\quad \sigma_q^*\colon \mathcal{L}_q^*(q) \simeq \mathcal{O}_q\,,$$

for which the maps

$$\begin{aligned} \mathcal{L}_p(p)\otimes\mathcal{L}_p^* \ni (\phi,\psi) &\mapsto \sigma_p(\phi)\sigma_p^*(\psi)dz/z \in \omega(p)_p\,,\\ \mathcal{L}_q\otimes\mathcal{L}_q^*(q) \ni (\phi,\psi) &\mapsto \sigma_q(\phi)\sigma_q^*(\psi)dz/z \in \omega(q)_q \end{aligned} \tag{8.1}$$

extend to the canonical map

$$\mathcal{L}(p)\otimes\mathcal{L}^*(q) \xrightarrow{\simeq} \omega(p+q)\,.$$

Hence for general (n,t,s), $(m,t',s')\in\mathbb{Z}\times\mathbb{C}^\infty\times\mathbb{C}^\infty$,

$$\Psi_{i,n}(t,s;z)\Psi_{i,m}^*(t',s';z)dz/z\,,\quad i=1,\ 2$$

become expansions at p and q, respectively, of a holomorphic 1-form on $C\setminus\{p,q\}$, so by the residue calculus the pair Ψ, Ψ^* satisfies the bilinear identities (1.10).

Quasiperiodic solutions of Pfaff lattice

In the above construction, suppose C has an involution $\iota\colon C\to C$ with no fixed point. In this case g is odd, $g=2g'-1$, with g' being the genus of the quotient curve $C'=C/\iota$. Suppose $q=\iota(p)$, and $\mathcal{L}$ satisfies

$$\iota^*(\mathcal{L})\simeq\mathcal{L}^*\,,\quad\text{so that}\quad \mathcal{L}\otimes\iota^*\mathcal{L}\simeq\omega\,. \tag{8.2}$$

Choose the local coordinates $z^{\mp1}$ and the trivializations σ_p, σ_q, σ_p^*, σ_q^* at p and $q=\iota(p)$, such that $z\cdot\iota^*z\equiv1$ and $\sigma_q=\iota^*\circ\sigma_p^*\circ\iota^*$ hold. (We then have $\sigma_q^*=-\iota^*\circ\sigma_p\circ\iota^*$, with the minus sign due to the fact that dz/z, which appear in (8.1), satisfy $\iota^*(dz/z)=-dz/z$.) Then the wave and adjoint wave functions constructed above satisfy (2.4), so they lead to a quasiperiodic solution of the Pfaff lattice when $s=-t$ (and skipping every other n).

The orbit of the 2-Toda flows is parametrized by the Jacobian J of C, and the τ-functions are written in terms of Riemann's theta function of J. The orbit of the Pfaff flows will become the Prym variety P of (C,ι), with $\tilde\tau$ given by the Prym theta function. To be more precise, let J_{g-1} be the moduli space of the isomorphism classes of line bundles of degree $g-1$ on C. This is a principal homogeneous space[22] over J, on which the theta divisor

$$\Theta := \left\{ \mathcal{L} \in J_{g-1} \;\middle|\; \Gamma(\mathcal{L}) \neq (0) \right\}$$

is canonically defined. The set of $\mathcal{L} \in J_{g-1}$ satisfying (8.2) becomes the disjoint union $P_{g-1} \cup P_{g-1}^{-}$, where

$$\begin{aligned} P_{g-1} &:= \left\{ \mathcal{L} \in J_{g-1} \;\middle|\; \mathcal{L} \text{ satisfies (8.2) and } \dim \Gamma(\mathcal{L}) \text{ is even} \right\}, \\ P_{g-1}^{-} &:= \left\{ \mathcal{L} \in J_{g-1} \;\middle|\; \mathcal{L} \text{ satisfies (8.2) and } \dim \Gamma(\mathcal{L}) \text{ is odd} \right\} \end{aligned}$$

are principal homogeneous spaces over the Prym P. We have

$$P_{g-1}^{-} \subset \Theta \quad \text{and} \quad P_{g-1} \cdot \Theta = 2\Xi ,$$

for some divisor $\Xi \subset P_{g-1}$ which gives a principal polarization on P_{g-1}. Since Θ is the zero locus of Riemann's theta function ϑ of the Jacobian J, this means ϑ vanishes identically on P_{g-1}^{-}, and the restriction $\vartheta|_{P_{g-1}}$ becomes the square of Riemann's theta function ϑ_P of (P, Ξ), which is called the Prym theta function.

For a 2-Toda quasiperiodic solution, the discrete time flow (shift of n by 1) is given by the shift $\mathcal{L} \mapsto \mathcal{L}(p-q)$. In the present case, since $q = \iota(p)$, this flow preserves condition (8.2). Moreover, we have

$$\forall p \in C,\ \forall \mathcal{L} \in J_{g-1} : \begin{cases} \mathcal{L} \in P_{g-1} \Rightarrow \mathcal{L}(p - \iota(p)) \in P_{g-1}^{-} , \\ \mathcal{L} \in P_{g-1}^{-} \Rightarrow \mathcal{L}(p - \iota(p)) \in P_{g-1} , \end{cases}$$

so that $\mathcal{L}(np - n\iota(p))$'s alternate between P_{g-1} and P_{g-1}^{-}, and every other τ function vanishes identically when $s = -t$. Shifting the discrete index n by 1 if necessary, we may assume that $\tau_n(t, s)$ satisfies (0.4) or (2.5).

Explicit formulas

Explicit formulas for Ψ, Ψ^* and τ can be given in terms of Riemann's theta function for J, and hence explicit formulas for $\tilde{\tau}$ can be given in terms of the Prym theta function for P.

Taking a basis A_i, B_i $(i = 1, \ldots, g)$ of $H_1(C, \mathbb{Z})$ such that $A_i \cdot B_j = \delta_{i,j}$ and $A_i \cdot A_j = B_i \cdot B_j = 0$, let ω_i $(i = 1, \ldots, g)$ be a basis of the space of holomorphic 1-forms such that

$$\int_{A_i} \omega_j = \delta_{i,j} .$$

[22] Hence J_{g-1} is (non-canonically) isomorphic to J. We choose this isomorphism in such a way that $\Theta \subset J_{g-1}$ is identified with the zero locus of Riemann's theta function for J.

Then

$$\int_{B_i} \omega_j = \Omega_{i,j}$$

gives a complex symmetric matrix Ω with positive definite imaginary part, and $J = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$ becomes the Jacobian of C. Choosing a point $p \in C$, the map

$$\alpha\colon C \ni x \mapsto \left(\int_p^x \omega_1, \dots, \int_p^x \omega_g\right) \in J$$

is well-defined and gives an embedding of C into J. Composing α with a translate of Riemann's theta function:

$$\underline{\vartheta}(x) := \vartheta(\alpha(x) + a)\,, \quad a \in \mathbb{C}^g, \tag{8.3}$$

we obtain a multi-valued function on C which is single-valued around the A-cycles.

Next, let $\zeta_n^{(p)}$, $n = 1, 2, \dots$, be the differentials of the second kind (meromorphic 1-forms with no residues) with poles only at p of the form $d(z^n + O(1))$ and no A-periods ($\int_{A_i} \zeta_n^{(p)} = 0$), and let $\zeta_n^{(q)}$, $n = 1, 2, \dots$, be defined similarly, with p replaced by q and z by z^{-1} (recall that z^{-1} (resp. z) is the local coordinate at p (resp. q)). Let ζ_0 be the differential of the third kind (meromorphic 1-form with simple poles) with no A-periods and poles only at p and q of the form $dz/z + O(1)$. Then, given $(n, t, s) \in \mathbb{Z} \times \mathbb{C}^\infty \times \mathbb{C}^\infty$, the multi-valued holomorphic function

$$C \ni x \mapsto \varepsilon(x) := \exp\left(\int^x \left(n\zeta_0 + \sum_{i=1}^\infty t_i \zeta_i^{(p)} + \sum_{i=1}^\infty s_i \zeta_i^{(q)}\right)\right) \tag{8.4}$$

has singularities at p and q of the form $z^n e^{\sum_{i=1}^\infty t_i z^i}$ and $z^n e^{\sum_{i=1}^\infty s_i z^{-i}}$, respectively, and is single-valued around A-cycles. The product of the form

$$\phi_n(t, s; x) := \varepsilon(x)\underline{\vartheta}(x)/\underline{\vartheta}(p)\,,$$

where $\underline{\vartheta}(x)$ and $\varepsilon(x)$ are as in (8.3) and (8.4), with

$$a = a(n, t, s) = n\alpha(q) + \sum_{i=1}^\infty t_i U_i + \sum_{i=1}^\infty s_i V_i + a_0\,, \quad \forall a_0 \in \mathbb{C}^g\,, \tag{8.5}$$

and $U_i = -(d/d(z^{-1}))^i \alpha(p)/(i-1)!$, $V_i = -(d/dz)^i \alpha(q)/(i-1)!$, has the desired properties:

- the function $\varphi_n(t, s; x)$ is single-valued around the A_i, and when x goes around the B_i, it is multiplied by a factor independent of (n, t, s);
- we have $\varphi_n(t, s; x) \simeq z^n e^{\sum_{i=1}^\infty t_i z^i}(1 + O(z^{-1}))$ at $x \simeq p$, and $\varphi_n(t, s; x) \simeq z^n e^{\sum_{i=1}^\infty s_i z^{-i}}(h_n(t, s) + O(z))$ at $x \simeq q$.

and hence gives the wave functions Ψ via its expansions around p and q. The adjoint wave functions Ψ^* are obtained similarly, from $\varepsilon(x)^{-1}\vartheta(\alpha(x)-a)/\vartheta(-a)$ with the same a as above.

The 2-Toda τ-function can be computed from those formulas as

$$\tau_n(t,s)=\exp(Q(n,t,s))\vartheta(a(n,t,s)) \tag{8.6}$$

for some quadratic form $Q(n,t,s)$, i.e.,

$$Q(n,t,s)=\sum_{i,j=1}^{\infty}Q_{i,j}t_it_j+\sum_{i,j=1}^{\infty}Q'_{i,j}s_is_j+\sum_{i=1}^{\infty}n(q_it_i+q'_is_i)\,,$$

with $Q_{i,j}=Q_{j,i}$ appearing in the Laurent expansion of the integral of $\zeta_i^{(p)}$ or $\zeta_j^{(p)}$ as

$$\int^x\zeta_i^{(p)}=z^i-2\sum_{j=1}^{\infty}Q_{i,j}z^{-j}/j\quad\text{for}\quad x\simeq p\,,$$

$Q'_{i,j}=Q'_{j,i}$ appearing similarly in the Laurent expansion of the integral of $\zeta_i^{(q)}$ or $\zeta_j^{(q)}$ as

$$\int^x\zeta_i^{(q)}=z^{-i}-2\sum_{j=1}^{\infty}Q'_{i,j}z^{j}/j\quad\text{for}\quad x\simeq q\,,$$

and q_i and q'_i appearing similarly in the expansions

$$\int^x\zeta_0=\log z-\sum_{j=1}^{\infty}q_jz^{-j}/j\quad\text{for}\quad x\simeq p$$

and

$$\int^x\zeta_0=\log z-\sum_{j=1}^{\infty}q'_jz^{j}/j\quad\text{for}\quad x\simeq q\,.$$

Suppose C has an involution ι with no fixed points, so that $g=2g'-1$ with g' being the genus of the quotient curve $C'=C/\iota$. Suppose $q=\iota(p)$. Take the cycles A_i,B_i $(i=1,\dots,g)$ in such a way that $\iota(A_i)\simeq A_{g+1-i}$, $\iota(B_i)\simeq B_{g+1-i}$. Then $\iota^*(\omega_i)=\omega_{g+1-i}$, and Ω satisfies $\Omega_{i,j}=\Omega_{g+1-i,g+1-j}$. The map $\tilde\iota\colon\mathbb{C}^g\ni(z_1,\dots,z_g)\mapsto(z_g,\dots,z_1)\in\mathbb{C}^g$ maps the lattice $\Gamma:=\mathbb{Z}^g+\Omega\mathbb{Z}^g$ onto itself, and the embeddings

$$\tilde J'=J'/(\mathbb{Z}/2\mathbb{Z})\subset J\quad\text{and}\quad P\subset J$$

are given by the images under $\pi_L\colon\mathbb{C}^g\to\mathbb{C}^g/\Gamma$ of the ±1-eigenspaces of $\tilde\iota$: Setting

$$R':=(\delta_{i,j}+\delta_{i,g+1-j})_{1\le i\le g,1\le j\le g'}\,,$$
$$R'':=(\delta_{i,j}-\delta_{i,g+1-j})_{1\le i\le g,1\le j\le g'-1}\,,$$

so that $\mathbb{C}^g_+ := R'\mathbb{C}^{g'}$ and $\mathbb{C}^g_- := R''\mathbb{C}^{g'-1}$ are the ± 1-eigenspaces of $\tilde{\iota}$, and for any $z \in \mathbb{C}^g$, $z' := (1/2)(R')^t z$ and $z'' := (1/2)(R'')^t z$ give the decomposition $z = R'z' + R''z'' \in \mathbb{C}^g_+ \oplus \mathbb{C}^g_-$, we have

$$\begin{aligned}\tilde{J}' = \mathbb{C}^{g'}/(\varepsilon\mathbb{Z}^{g'} + \Omega'\mathbb{Z}^{g'}) \simeq \pi_L(R'\mathbb{C}^{g'}) &\subset \mathbb{C}^g/(\mathbb{Z}^g + \Omega'\mathbb{Z}^g)\\ z' \mapsto R'z'&\end{aligned}$$

and

$$\begin{aligned}P = \mathbb{C}^{g'-1}/(\mathbb{Z}^{g'-1} + \Omega''\mathbb{Z}^{g'-1}) \simeq \pi_L(R''\mathbb{C}^{g'-1}) &\subset \mathbb{C}^g/(\mathbb{Z}^g + \Omega'\mathbb{Z}^g)\\ z'' \mapsto R''z''\,,&\end{aligned} \tag{8.7}$$

where $\varepsilon = \mathrm{diag}(1, 1, \ldots, 1, 1/2)$,

$$\begin{aligned}\Omega' &= \left(\frac{\Omega_{i,j} + \Omega_{i,g+1-j}}{(1+\delta_{i,g'})(1+\delta_{j,g'})}\right)_{1\le i,j\le g'} \quad \text{and}\\ \Omega'' &= (\Omega_{i,j} - \Omega_{i,g+1-j})_{1\le i,j\le g'-1}\,.\end{aligned}$$

In (8.5), suppose $a_0 = R''a''_0 \in R''\mathbb{C}^{g'-1}$. Since, by definition, $\alpha(q) = \alpha(q) - \alpha(p) \in \pi_L(\mathbb{C}^g_-)$ and $\tilde{\iota}(U_i) = V_i$, we then have $a(n, t, -t) = R''a''(n, t)$, where

$$a''(n,t) = \frac{1}{2}(R'')^t a(n,t,-t) = (R'')^t\Big(\frac{1}{2}n\alpha(q) + \sum_{i=1}^{\infty} t_i U_i\Big) + a''_0\,.$$

Hence by using (8.6) and (8.7), and noting that $Q'_{i,j} = Q_{i,j}$ and $q'_i = -q_i$, we have

$$\tilde{\tau}(t) = \exp(\tilde{Q}(n,t))\vartheta_P(a''(n,t))\,,$$

where

$$\tilde{Q}(n,t) = \sum_{i,j=1}^{\infty} Q_{i,j}t_i t_j + \sum_{i=1}^{\infty} q_i n t_i\,,$$

and

$$\vartheta_P(z) = \sum_{m\in\mathbb{Z}^{g'-1}} \exp(2\pi i m^t z + \pi i m^t \Omega'' m)\,, \quad \text{for} \quad z \in \mathbb{C}^{g'-1}\,.$$

9. Appendix: 2-Toda and Pfaff Virasoro constraints (another proof of Corollary 4.2)

In this appendix we give an alternative proof of corollary 4.2 in the *semi-infinite* case. The three formulas in the corollary are equivalent to each other due to (2.17), so it suffices to prove only the last one:

Proposition 9.1 *For $k = 1, 2$,*

$$\left(J_\ell^{(k)} + (-1)^i \tilde{J}_\ell^{(k)}\right) \tau_{2n}(t,s)|_{s=-t} = 2\tilde{\tau}_{2n}(t) J_\ell^{(k)} \tilde{\tau}_{2n}(t)\,. \tag{9.1}$$

The proof is based on identities, involving skew-symmetric matrices and Pfaffians. To a skew-symmetric matrix A_{2n-1} of size $2n-1$ augmented with an arbitrary row and column

$$M = \left(\begin{array}{ccc|c} & & & x_0 \\ & A_{2n-1} & & \vdots \\ & & & x_{2n-2} \\ \hline -y_0 & \ldots & -y_{2n-2} & z \end{array}\right),$$

we associate, in a natural way, skew-symmetric matrices

$$A = \left(\begin{array}{ccc|c} & & & x_0 \\ & A_{2n-1} & & \vdots \\ & & & x_{2n-2} \\ \hline -x_0 & \ldots & -x_{2n-2} & 0 \end{array}\right), \quad B = \left(\begin{array}{ccc|c} & & & y_0 \\ & A_{2n-1} & & \vdots \\ & & & y_{2n-2} \\ \hline -y_0 & \ldots & -y_{2n-2} & 0 \end{array}\right).$$

Similarly, to a skew-symmetric matrix A_{2n-2} of size $2n-2$ augmented with two arbitrary rows and columns

$$N = \left(\begin{array}{ccc|cc} & & & x_0 & y_0 \\ & A_{2n-2} & & \vdots & \vdots \\ & & & x_{2n-3} & y_{2n-3} \\ \hline -u_0 & \ldots & -u_{2n-3} & -u_{2n-2} & y_{2n-2} \\ -v_0 & \ldots & -v_{2n-3} & x_{2n-1} & -v_{2n-1} \end{array}\right),$$

we associate the four skew-symmetric matrices

$$C = \left(\begin{array}{ccc|cc} & & & x_0 & v_0 \\ & A_{2n-2} & & \vdots & \vdots \\ & & & x_{2n-3} & v_{2n-3} \\ \hline -x_0 & \ldots & -x_{2n-3} & 0 & -x_{2n-1} \\ -v_0 & \ldots & -v_{2n-3} & x_{2n-1} & 0 \end{array}\right),$$

$$D = \left(\begin{array}{ccc|cc} & & & u_0 & y_0 \\ & A_{2n-2} & & \vdots & \vdots \\ & & & u_{2n-3} & y_{2n-3} \\ \hline -u_0 & \ldots & -u_{2n-3} & 0 & y_{2n-2} \\ -y_0 & \ldots & -y_{2n-3} & -y_{2n-2} & 0 \end{array}\right),$$

$$E = \left(\begin{array}{ccc|cc} & & & x_0 & u_0 \\ & A_{2n-2} & & \vdots & \vdots \\ & & & x_{2n-3} & u_{2n-3} \\ \hline -x_0 & \dots & -x_{2n-3} & 0 & u_{2n-2} \\ -u_0 & \dots & -u_{2n-3} & -u_{2n-2} & 0 \end{array} \right),$$

$$F = \left(\begin{array}{ccc|cc} & & & v_0 & y_0 \\ & A_{2n-2} & & \vdots & \vdots \\ & & & v_{2n-3} & y_{2n-3} \\ \hline -v_0 & \dots & -v_{2n-3} & 0 & -v_{2n-1} \\ -y_0 & \dots & -y_{2n-3} & v_{2n-1} & 0 \end{array} \right).$$

Lemma 9.2 *Given the matrices M and N above, we have*

$$\begin{aligned} \det M &= \operatorname{Pf}(A)\operatorname{Pf}(B)\,, \\ \det N &= \operatorname{Pf}(C)\operatorname{Pf}(D) - \operatorname{Pf}(E)\operatorname{Pf}(F)\,. \end{aligned}$$

Proof of Proposition 9.1. Note that $J_\ell^{(k)}$ and $\tilde{J}_\ell^{(k)}$ are differential operators of order k (see (1.24)). For each $i = 0, \dots, k$, we call the ith order part of (9.1) the equality obtained by replacing $J_\ell^{(k)}$ and $\tilde{J}_\ell^{(k)}$ on both sides of (9.1) by their ith order terms.

Since $\tau_{2n}(t, -t) = \tilde{\tau}_{2n}(t)^2$, the 0th order part of (9.1) is obvious. Since

$$\left(\frac{\partial}{\partial t_i} - \frac{\partial}{\partial s_i} \right) \tau_{2n}(t, s) \Big|_{s=-t} = \frac{d}{dt_i} \tau_{2n}(t, -t) = 2\tilde{\tau}_{2n}(t) \left(\frac{\partial}{\partial t_i} \right) \tilde{\tau}_{2n}(t)\,,$$

the same is true for the first order part. For instance, for $k = 2$ the left hand side of the first order part of (9.1):

$$\begin{aligned} &\sum_{i=1}^{\infty} \left(i t_i \frac{\partial}{\partial t_{i+\ell}} + i s_i \frac{\partial}{\partial s_{i+\ell}} \right) \tau_{2n}(t, s) \Big|_{s=-t} \\ &= \sum_{i=1}^{\infty} i t_i \left(\frac{\partial}{\partial t_{i+\ell}} - \frac{\partial}{\partial s_{i+\ell}} \right) \tau_{2n}(t, s) \Big|_{s=-t} \\ &= \sum_{i=1}^{\infty} i t_i \frac{d}{dt_{i+\ell}} \tau_{2n}(t, -t) \\ &= 2\tilde{\tau}_{2n}(t) \left(\sum_{i=1}^{\infty} i t_i \frac{\partial}{\partial t_{i+\ell}} \right) \tilde{\tau}_{2n}(t)\,, \end{aligned}$$

thus becomes the right hand side of the first order part of the same.

Let us proceed to the second, the highest, order part of (9.1), which appears for $k = 2$ and $\ell \geq 2$. This time we shall use

$$\left(\frac{\partial}{\partial t_i} - \frac{\partial}{\partial s_i}\right)\left(\frac{\partial}{\partial t_j} - \frac{\partial}{\partial s_j}\right)\tau_{2n}(t, s)\Big|_{s=-t} = \frac{d^2}{dt_i dt_j}\tau_{2n}(t, -t) = \frac{\partial^2}{\partial t_i \partial t_j}\tilde{\tau}_{2n}^2(t)\,.$$

The second order part of (9.1) is equivalent to the vanishing of the first line (9.2) of the following:

$$\begin{aligned}
&\sum_{i+j=k}\left\{\left(\frac{\partial^2}{\partial t_i \partial t_j} + \frac{\partial^2}{\partial s_i \partial s_j}\right)\tau_{2n}(t, s)\Big|_{s=-t} - 2\tilde{\tau}_{2n}(t)\frac{\partial^2}{\partial t_i \partial t_j}\tilde{\tau}_{2n}(t)\right\} \qquad (9.2)\\
&= \sum_{i+j=k}\left\{\left(\frac{\partial^2}{\partial t_i \partial t_j} + \frac{\partial^2}{\partial s_i \partial s_j}\right)\tau_{2n}(t, s)\Big|_{s=-t}\right.\\
&\qquad\left. - \frac{\partial^2}{\partial t_i \partial t_j}\tilde{\tau}_{2n}^2(t) + 2\frac{\partial \tilde{\tau}_{2n}}{\partial t_i}\frac{\partial \tilde{\tau}_{2n}}{\partial t_j}\right\}\\
&= \sum_{i+j=k}\left\{\left(\frac{\partial^2}{\partial t_i \partial t_j} + \frac{\partial^2}{\partial s_i \partial s_j}\right)\tau_{2n}(t, s)\Big|_{s=-t}\right.\\
&\qquad\left. - \left(\frac{\partial}{\partial t_i} - \frac{\partial}{\partial s_i}\right)\left(\frac{\partial}{\partial t_j} - \frac{\partial}{\partial s_j}\right)\tau_{2n}(t, s)\Big|_{s=-t} + 2\frac{\partial \tilde{\tau}_{2n}}{\partial t_i}\frac{\partial \tilde{\tau}_{2n}}{\partial t_j}\right\}\\
&= \sum_{i+j=k}\left\{\left(\frac{\partial^2}{\partial t_i \partial s_j} + \frac{\partial^2}{\partial s_i \partial t_j}\right)\tau_{2n}(t, s)\Big|_{s=-t} + 2\frac{\partial \tilde{\tau}_{2n}}{\partial t_i}\frac{\partial \tilde{\tau}_{2n}}{\partial t_j}\right\}\\
&= 2\sum_{i+j=k}\left\{\frac{\partial^2 \tau_{2n}}{\partial t_i \partial s_j}(t, s)\Big|_{s=-t} + \frac{\partial \tilde{\tau}_{2n}}{\partial t_i}\frac{\partial \tilde{\tau}_{2n}}{\partial t_j}\right\}. \qquad (9.3)
\end{aligned}$$

The vanishing of (9.2) follows from (2.15) and (2.17), and the vanishing of the last line (9.3) follows from (2.16). Here we shall prove, in the semi-infinite case, the vanishing of (9.3) using only the identities on Pfaffians in Lemma 9.2.

The action of $\partial/\partial t_i$ (respectively, $\partial/\partial s_j$) on the determinant of matrix m_{2n} amounts to a sum (over $0 \leq k \leq 2n-1$) of determinants of the same matrices, but with the kth row (respectively, the kth column) replaced by $(\mu_{k+i,0}, \ldots, \mu_{k+i,2n-1})$ (respectively, by $-(\mu_{0,k+j}, \ldots, \mu_{2n-1,k+j})^\top$). Thus, the matrices in the sum are matrices of size $2n$, which are skew-symmetric except for one row and column. So, using the first relation of Lemma 9.2, we find[23]

$$\left.\frac{\partial \tau_{2n}}{\partial t_i}\right|_{s=-t} = \tilde{\tau}_{2n}(t) \sum_{\ell=0}^{2n-1} \mathrm{Pf}_{2n}(\ell \mapsto \ell + i)\,,$$

and hence

$$\begin{aligned}\frac{\partial \tilde{\tau}_{2n}}{\partial t_i}\frac{\partial \tilde{\tau}_{2n}}{\partial t_j} &= \left.\frac{1}{\tilde{\tau}_{2n}}\frac{\partial \tau_{2n}}{\partial t_i}\cdot\frac{1}{\tilde{\tau}_{2n}}\frac{\partial \tau_{2n}}{\partial t_j}\right|_{s=-t} \\ &= \sum_{\ell,m=0}^{2n-1} \mathrm{Pf}_{2n}(m \mapsto m+j)\,\mathrm{Pf}_{2n}(\ell \mapsto \ell + i)\,. \end{aligned} \tag{9.4}$$

Similarly, the second derivative $\partial^2/\partial s_i \partial t_j$ amounts to a sum of determinants (over $0 \le m, \ell \le 2n-1$) of skew-symmetric matrices, except that the ℓth row and mth column got replaced by the $\ell + i$th row and the negative of the $m + j$th column, respectively. So, all in all, we get a sum of determinants of the second type (and the first type when $\ell = m$) in Lemma 9.2, thus leading to

$$\begin{aligned} &-\left.\frac{\partial^2 \tau_{2n}}{\partial t_i \partial s_j}\right|_{s=-t} \\ &= \sum_{\ell,m} \det \begin{pmatrix} \ell\text{th row} \mapsto (\ell+i)\text{th row} \\ m\text{th column} \mapsto (m+j)\text{th column}\end{pmatrix} \\ &= \sum_{\ell \neq m} \big\{ \mathrm{Pf}_{2n}(m \mapsto m, \ell \mapsto \ell)\,\mathrm{Pf}_{2n}(m \mapsto m+j, \ell \mapsto \ell+i) \\ &\qquad + \mathrm{Pf}_{2n}(m \mapsto m, \ell \mapsto m+j)\,\mathrm{Pf}_{2n}(m \mapsto \ell+i, \ell \mapsto \ell)\big\} \\ &\quad + \sum_{\ell} \mathrm{Pf}_{2n}(\ell \mapsto \ell+i)\,\mathrm{Pf}_{2n}(\ell \mapsto \ell+j) \\ &= \sum_{\ell \neq m} \tilde{\tau}_{2n}(t)\,\mathrm{Pf}_{2n}(m \mapsto m+j, \ell \mapsto \ell+i) \\ &\quad + \sum_{\ell,m} \mathrm{Pf}_{2n}(\ell \mapsto m+j)\,\mathrm{Pf}_{2n}(m \mapsto \ell+i)\,. \end{aligned} \tag{9.5}$$

23 We denote by $\mathrm{Pf}_{2n}(\ell \mapsto k)$ the Pfaffian of the skew-symmetric matrix $m_{2n}(t,-t)$, with the ℓth row and column replaced by the kth row and column, respectively, of $m_\infty(t,-t)$. We define $\mathrm{Pf}_{2n}(\ell \mapsto p, m \mapsto q)$ etc., similarly.

Therefore, summing both contributions (9.4) and (9.5), we find:

$$
\begin{aligned}
&-\sum_{i+j=k}\left\{\frac{\partial\tilde{\tau}_{2n}}{\partial t_i}\frac{\partial\tilde{\tau}_{2n}}{\partial t_j}+\frac{\partial^2\tau_{2n}}{\partial t_i\partial s_j}\right\}_{s=-t}\\
&=\sum_{\ell\neq m,i+j=k}\tilde{\tau}_{2n}\,\mathrm{Pf}_{2n}(m\mapsto m+j,\ell\mapsto\ell+i)\\
&\quad+\sum_{\ell,m,i+j=k}\Big\{\mathrm{Pf}_{2n}(\ell\mapsto m+j)\,\mathrm{Pf}_{2n}(m\mapsto\ell+i)\\
&\qquad\qquad-\mathrm{Pf}_{2n}(m\mapsto m+j)\,\mathrm{Pf}_{2n}(\ell\mapsto\ell+i)\Big\} \qquad (9.6)
\end{aligned}
$$

The expression above consists of two sums; we now show each of them vanishes separately. The first sum vanishes, because it is a sum of zero pairs[24]

$$
\mathrm{Pf}_{2n}(m\mapsto m+j,\ell\mapsto\ell+i)+\mathrm{Pf}_{2n}(m\mapsto m+j',\ell\mapsto\ell+i')=0\,,
$$

upon picking $m+j'=\ell+i,\ \ell+i'=m+j$, thus respecting the requirement[25] $i+j=i'+j'=k$. The argument is similar for the second sum in (9.6). $\square$

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[24] A Pfaffian flips the sign, upon permuting two indices.

[25] If $k<k'<2n$ and $m<m'<2n$, then $\mathrm{Pf}_{2n}(k\mapsto k',m\mapsto m')=0$. In particular, here we may assume $m+j\geq 2n>\ell$ and $\ell+i\geq 2n>m$, so that $i',j'>0$.

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