Ergod. Th. & Dynam. Sys. (2002), **22**, 1–41
DOI: 10.1017/S0143385702001025

Rational solutions to the Pfaff lattice and Jack polynomials

M. ADLER†, V. B. KUZNETSOV‡ and P. VAN MOERBEKE§

† *Department of Mathematics, Brandeis University, Waltham, MA 02454, USA*
(e-mail: adler@brandeis.edu)
‡ *Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK*
(e-mail: V.B.Kuznetsov@leeds.ac.uk)
§ *Department of Mathematics, Université de Louvain, 1348 Louvain-la-Neuve, Belgium and Brandeis University, Waltham, MA 02454, USA*
(e-mail: vanmoerbeke@geom.ucl.ac.be and @brandeis.edu)

Abstract. The finite Pfaff lattice is given by commuting Lax pairs involving a finite matrix L (zero above the first subdiagonal) and a projection onto $Sp(N)$. The lattice admits solutions such that the entries of the matrix L are rational in the time parameters $t_1, t_2, \ldots$, after conjugation by a diagonal matrix. The sequence of polynomial τ-functions, solving the problem, belongs to an intriguing chain of subspaces of Schur polynomials, associated to Young diagrams, dual with respect to a finite chain of rectangles. Also, this sequence of τ-functions is given inductively by the action of a fixed vertex operator.

As an example, one such sequence is given by Jack polynomials for rectangular Young diagrams, while another chain starts with any two-column Jack polynomial.

1. *Introduction*

1.1. *Self-dual partitions.* For positive integers n and $n|k$, define the following sets of partitions,

$$\mathbb{Y} = \{\lambda = (\lambda_1, \lambda_2, \ldots),\ \lambda_1 \geq \lambda_2 \geq \cdots \geq 0\}$$

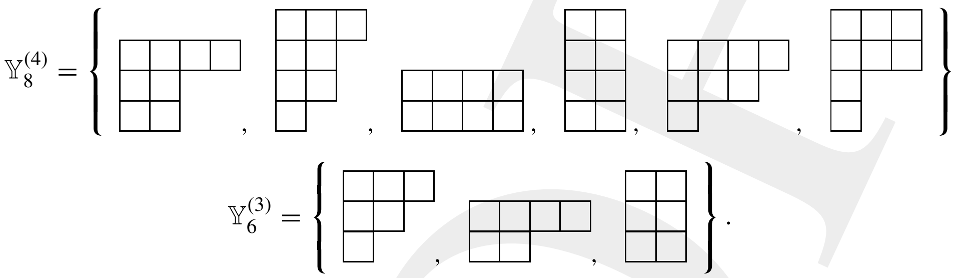
$$\mathbb{Y}_k = \left\{\lambda \in \mathbb{Y}, |\lambda| = \sum \lambda_i = k\right\}$$

$$\mathbb{Y}_k^{(n)} = \left\{\begin{array}{l}\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Y}_k,\ \hat{\lambda}_1 \leq n, \\ \lambda_i + \lambda_{n+1-i} = \dfrac{2k}{n},\ 1 \leq i \leq \left[\dfrac{n+1}{2}\right]\end{array}\right\}$$

with

$$\#\mathbb{Y}_k^{(n)} = \begin{pmatrix}\left[\dfrac{n}{2} + \dfrac{k}{n}\right] \\ \left[\dfrac{n+1}{2}\right]\end{pmatrix}.$$

These are a few examples:



Let $\mathbf{s}_\lambda(t) := \det(s_{\lambda_i - i + j}(t))_{1\le i,j}$ be the Schur polynomials corresponding to λ, with $\mathbf{s}_i(t)$ being the elementary Schur polynomials, defined by

$$\exp\left(\sum_1^\infty t_i z^i\right) = \sum_{i\ge 0} \mathbf{s}_i(t) z^i \quad \text{with } \mathbf{s}_i(t) = 0 \text{ for } i < 0.$$

The linear space

$$\mathbb{L}_k^{(n)} := \left\{ \sum_{\lambda \in \mathbb{Y}_k^{(n)}} a_\lambda \mathbf{s}_\lambda \mid a_\lambda \in \mathbb{C} \right\}$$

will play an ubiquitous role in this work.

1.2. *The finite Pfaff lattice.* The $(N \times N)$ skew-symmetric matrices

$$J = \begin{cases} \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & 0 \\ & \ddots & \\ 0 & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} \end{pmatrix}, & \text{for } N \text{ even}, \\ \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & 0 & \\ & \ddots & & \\ 0 & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \\ & & & 0 \end{pmatrix}, & \text{for } N \text{ odd}, \end{cases} \tag{1.1}$$

satisfy

$$J^2 = \begin{cases} -I_N, & \text{for } N \text{ even}, \\ \begin{pmatrix} -I_{N-1} & O \\ O & 0 \end{pmatrix}, & \text{for } N \text{ odd}. \end{cases} \tag{1.2}$$

Also consider the Lie algebra $\mathfrak{k}$ of lower-triangular matrices of the form

$$\mathfrak{k} = \begin{cases} \begin{pmatrix} \boxed{\begin{matrix} a_1 & 0 \\ 0 & a_1 \end{matrix}} & & O \\ & \ddots & \\ * & & \boxed{\begin{matrix} a_{[N/2]} & 0 \\ 0 & a_{[N/2]} \end{matrix}} \end{pmatrix}, & \text{for } N \text{ even}, \\ \begin{pmatrix} \boxed{\begin{matrix} a_1 & 0 \\ 0 & a_1 \end{matrix}} & & O & \\ & \ddots & & \\ & & \boxed{\begin{matrix} a_{[N/2]} & 0 \\ 0 & a_{[N/2]} \end{matrix}} & \\ * & & & \boxed{a_{(N+1)/2}} \end{pmatrix}, & \text{for } N \text{ odd}. \end{cases} \tag{1.3}$$

For each $a \in gl(N)$, consider the decomposition†

$$\begin{aligned} a &= (a)_{\mathfrak{k}} + (a)_{\mathfrak{n}} \\ &= \pi_{\mathfrak{k}} a + \pi_{\mathfrak{n}} a \\ &= ((a_- - J(a_+)^\top J) + \tfrac{1}{2}(a_0 - J(a_0)^\top J)) \\ &\quad + ((a_+ + J(a_+)^\top J) + \tfrac{1}{2}(a_0 + J(a_0)^\top J)). \end{aligned} \tag{1.4}$$

For N even, this corresponds to a Lie algebra splitting, given by

$$gl(N) = \mathfrak{k} + \mathfrak{n} \begin{cases} \mathfrak{k} = \{\text{lower-triangular matrices of the form (1.3)}\} \\ \mathfrak{n} = sp(N) = \{a \text{ such that } Ja^\top J = a\}. \end{cases} \tag{1.5}$$

For N odd, this is merely a vector space splitting

$$gl(N) = \mathfrak{k} + \mathfrak{n} \begin{cases} \mathfrak{k} = \{\text{lower-triangular matrices of the form (1.3)}\} \\ \mathfrak{n} = \text{span}\{\pi_{\mathfrak{n}}(a) \text{ with } a \in gl(N)\}. \end{cases} \tag{1.6}$$

† $a_\pm$ refers to projection onto strictly upper (strictly lower) triangular matrices, with all (2×2) diagonal blocks equal to zero. a_0 refers to projection onto the 'diagonal', consisting of (2×2) blocks.

The Pfaff lattice is defined on $(N \times N)$ matrices L of the form

$$L = \begin{cases} \begin{pmatrix} 0 & 1 & & & & & & & O \\ & -d_1 & a_1 & & & & & & \\ & & d_1 & 1 & & & & & \\ & & & -d_2 & a_2 & & & & \\ & & & & d_2 & & & & \\ & * & & & & \ddots & a_{(N-2)/2} & & \\ & & & & & & -d_{(N-2)/2} & 1 & \\ & & & & & & & 0 & \end{pmatrix}, & \text{for } N \text{ even}, \\ \begin{pmatrix} 0 & 1 & & & & & & & O \\ & -d_1 & a_1 & & & & & & \\ & & d_1 & 1 & & & & & \\ & & & -d_2 & a_2 & & & & \\ & & & & d_2 & & & & \\ & * & & & & \ddots & 1 & & \\ & & & & & & -d_{(N-1)/2} & a_{(N-1)/2} & \\ & & & & & & & d_{(N-1)/2} & \end{pmatrix}, & \text{for } N \text{ odd}, \end{cases} \tag{1.7}$$

namely,

$$\frac{\partial L}{\partial t_i} = [-(L^i)_{\mathfrak{k}}, L] \quad \text{(the Pfaff lattice).} \tag{1.8}$$

Given arbitrary, but fixed, parameters

$$b_0, \ldots, b_{(N-2)/2} \in \mathbb{C}, \tag{1.9}$$

consider the skew-symmetric antidiagonal initial condition,

$$m_N(0) = \begin{cases} \begin{pmatrix} & O & & b_{(N-2)/2} \\ & & \iddots & \\ & & b_0 & \\ & -b_0 & & \\ \iddots & & & \\ -b_{(N-2)/2} & & O & \end{pmatrix}, & \text{for } N \text{ even}, \\ \begin{pmatrix} & & O & & b_{(N-3)/2} \\ & & & \iddots & \\ & & & b_0 & \\ & & 0 & & \\ & -b_0 & & & \\ \iddots & & & & \\ -b_{(N-3)/2} & & & O & \end{pmatrix}, & \text{for } N \text{ odd}, \end{cases} \tag{1.10}$$

and its time evolution (respecting the skew-symmetry),

$$m_\ell(t) = E_{\ell,N}(t)m_N(0)E^\top_{\ell,N}(t), \tag{1.11}$$

where†

$$E_{\ell,N}(t) := \left(\exp\left(\sum_1^\infty t_i\Lambda^i\right)\right)_{\substack{1,\dots,\ell\\1,\dots,N}}. \tag{1.12}$$

The Pfaffian $pf\ m_\ell(t)$ of the skew-symmetric matrix $m_\ell(t)$ will play an important role in this paper.

1.3. *Rational solutions to the Pfaff lattice.*

THEOREM 1.1. *Modulo conjugation by an $(N \times N)$ diagonal matrix $D(t)$ (see the remark below), the finite Pfaff lattice*

$$\frac{\partial L}{\partial t_i} = [-(L^i)_{\mathfrak{k}}, L] \quad \text{(the Pfaff lattice)}$$

has rational solutions in $t_1, t_2, \dots$, i.e. the matrix

$$D^{-1}(t)L(t)D(t) = \tilde{Q}(t)\Lambda\tilde{Q}(t)^{-1} \tag{1.13}$$

is rational in $t_1, t_2, \dots$, with $\tilde{Q}(t)$ a lower-triangular $(N\times N)$ matrix with rational entries, obtained by Taylor expanding $\tau_{2n}(t-[z^{-1}])$ in z^{-1}, with $\tau_0 = 1$,

$$\tilde{q}_{2n}(t;z) := \sum_{j=0}^{2n} \tilde{Q}_{2n+1,j+1}(t)z^j = z^{2n}\tau_{2n}(t-[z^{-1}]) \quad \text{with } 0 \le n \le \left[\frac{N-1}{2}\right]$$

$$\tilde{q}_{2n+1}(t;z) := \sum_{j=0}^{2n+1} \tilde{Q}_{2n+2,j+1}(t)z^j = z^{2n}\left(z + \frac{\partial}{\partial t_1}\right)\tau_{2n}(t-[z^{-1}]), \tag{1.14}$$

with (see the definition of the $\mathbb{L}$-space at the beginning of this section)

$$\begin{aligned}\tau_\ell(t) &= pf(E_{\ell,N}(t)m_N(0)E^\top_{\ell,N}(t))\\ &= \sum_{\lambda\in\mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2}} \left(\prod_1^{[\ell/2]} b_{\lambda_i - i+\ell-[(N+1)/2]}\right)\mathbf{s}_\lambda(t), \quad \text{for } \begin{cases}0\le \ell\le N-1\\ \ell \text{ even}\end{cases}\\ &\in \mathbb{L}^{(\ell)}_{\ell(N-\ell)/2}.\end{aligned} \tag{1.15}$$

The polynomials $q_k = D_k\tilde{q}_k$ (in z) of degree $0 \le k \le N-1$ are 'skew-orthonormal' with respect to the skew inner-product $\langle z^i, z^j\rangle = m_{ij}(t)$, i.e.

$$\langle q_i, q_j\rangle = J_{ij}, \tag{1.16}$$

and the N-vector $(q_0,\dots,q_{N-1})^\top$ is an eigenvector for the matrix L, with modified boundary conditions. The fact that $Q_{2n,2n-1} = 0$ defines the skew-orthogonal polynomials in a unique way, up to ± 1.

† Λ is the finite shift matrix $\Lambda := (\delta_{i,j-1})_{1\le i,j\le N}$ and $(A)_{\substack{1,\dots,\ell\\1,\dots,N}}$ denotes the matrix formed by the first ℓ rows and first N columns of A.

Example. For $\ell = 2$, we have

$$\tau_2(t) = \begin{cases} \sum_{i=0}^{(N-2)/2} b_i \mathbf{s}_{[(N-2)/2]+i,[(N-2)/2]-i}(t), & \text{for } N \text{ even}, \\ \sum_{i=0}^{(N-3)/2} b_i \mathbf{s}_{[(N-1)/2]+i,[(N-3)/2]-i}(t), & \text{for } N \text{ odd}. \end{cases} \tag{1.17}$$

Remark.

$$D(t) = \begin{cases} \operatorname{diag}\left(\frac{1}{\sqrt{\tau_0\tau_2}}, \frac{1}{\sqrt{\tau_0\tau_2}}, \frac{1}{\sqrt{\tau_2\tau_4}}, \frac{1}{\sqrt{\tau_2\tau_4}}, \dots, \frac{1}{\sqrt{\tau_{N-2}\tau_N}}, \frac{1}{\sqrt{\tau_{N-2}\tau_N}}\right), \\ \qquad \text{for } N \text{ even}, \\ \operatorname{diag}\left(\frac{1}{\sqrt{\tau_0\tau_2}}, \frac{1}{\sqrt{\tau_0\tau_2}}, \dots, \frac{1}{\sqrt{\tau_{N-3}\tau_{N-1}}}, \frac{1}{\sqrt{\tau_{N-3}\tau_{N-1}}}, \frac{1}{\sqrt{\tau_{N-1}}}\right), \\ \qquad \text{for } N \text{ odd}. \end{cases}$$

1.4. *Duality.* For the case of odd N, we can even define $\tau_\ell(t)$ for odd ℓ, by slightly deforming the initial moment matrix $m_N(0)$. In §6, we prove a duality between these τ_k's for k even and odd, as follows:

$$\tilde{\tau}_\ell(t) = (-1)^{\ell(N-\ell)/2}\left(\prod_0^{(N-3)/2} b_i\right)(\tau_{N-\ell}(-t)|_{b_i \to b_i^{-1}}), \quad \text{for } \ell \text{ odd}.$$

1.5. *Fay identities.*

THEOREM 1.2. *The sequence of functions*

$$\tau_\ell(t) = \sum_{\lambda \in \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2}} \left(\prod_1^{[\ell/2]} b_{\lambda_i - i + \ell - [(N+1)/2]}\right) \mathbf{s}_\lambda(t), \qquad 0 \le \ell \le N-1,\ \ell \text{ even}, \tag{1.18}$$

together with the 'boundary condition'

$$\tau_0 = 1 \quad \text{and} \quad \begin{cases} \tau_N = \prod_0^{(N-2)/2} b_i, & \text{for } N \text{ even}, \\ \tau_{N+1} = 0, & \text{for } N \text{ odd}, \end{cases} \tag{1.19}$$

satisfies the the 'differential Fay identity'†:

$$\{\tau_{2n}(t-[u]), \tau_{2n}(t-[v])\} + (u^{-1}-v^{-1})(\tau_{2n}(t-[u])\tau_{2n}(t-[v]) - \tau_{2n}(t)\tau_{2n}(t-[u]-[v])) = uv(u-v)\tau_{2n-2}(t-[u]-[v])\tau_{2n+2}(t). \tag{1.20}$$

† Define the Wronskian $\{f, g\} = (\partial f/\partial t_1)g - (\partial g/\partial t_1)f$.

1.6. *Vertex operator constructions of the rational solutions.* Consider the vertex operator acting on functions $f(t)$ of $t = (t_1, t_2, \dots) \in \mathbb{C}^\infty$, namely

$$X(t;z) = \exp\Big(\sum_1^\infty t_i z^i\Big) \exp\Big(-\sum_1^\infty \frac{z^{-i}}{i}\frac{\partial}{\partial t_i}\Big), \tag{1.21}$$

and the vector vertex operator

$$\mathbb{X}(t;z) = \Lambda^\top \exp\Big(\sum_1^\infty t_i z^i\Big) \exp\Big(-\sum_1^\infty \frac{z^{-i}}{i}\frac{\partial}{\partial t_i}\Big)\chi(z), \tag{1.22}$$

acting on vectors of functions $F = (f_0(t), f_1(t), \dots)$, with $\chi(z) := (z^i)_{i\geq 0}$. Then the composition $\mathbb{X}(t;\lambda)\mathbb{X}(t;\mu)$ is a vertex operator for the Pfaff lattice, i.e. for any τ-vector $= (\tau_0, \tau_2, \tau_4, \dots)$ of the Pfaff lattice,

$$\tau(t) + a\mathbb{X}(t;\mu)\mathbb{X}(t;\lambda)\tau(t), \quad a \in \mathbb{C}$$

is again a τ-vector of the Pfaff lattice, or coordinatewise

$$\tau_{2n} + a\left(1 - \frac{\lambda}{\mu}\right)\mu^{2n-1}\lambda^{2n-2}\exp\Big(\sum t_i(\lambda^i + \mu^i)\Big)\tau_{2n-2}(t - [\lambda^{-1}] - [\mu^{-1}])$$

provides a new sequence of Pfaff τ-functions.

In terms of the distributional weight, with the b_i as in (1.9),

$$\rho_b(x) := \begin{cases} \rho_b^{(e)}(x) = \sum_{i\geq 0} b_i(x^{-i-1} - x^i), & \text{for } N \text{ even},\\ \rho_b^{(0)}(x) = x^{-1/2}\sum_{i\geq 0} b_i(x^{-i-1} - x^{i+1}), & \text{for } N \text{ odd}. \end{cases}$$

and

$$\beta := \frac{N}{2} - \ell + 1, \tag{1.23}$$

we define the *integrated vertex operator*, in terms of the vertex operator (1.21), as

$$Y_\beta(t) := \frac{1}{(2\pi i)^2}\oint_\infty\oint_\infty X(t;y)X(t;z)\frac{\rho_b(y/z)\,dy\,dz}{z^2(yz)^\beta}$$

and the *integrated vector vertex operator*, in terms of (1.22), as

$$\mathbb{Y}_N(t) = \frac{1}{(2\pi i)^2}\oint_\infty\oint_\infty \mathbb{X}(t;y)\mathbb{X}(t;z)\frac{\rho_b(y/z)\,dy\,dz}{2(yz)^{N/2}z}. \tag{1.24}$$

In both cases, the double integral around two contours about ∞ amounts to computing the coefficient of $1/yz$.

THEOREM 1.3. *For a given set of b_i, the sequence of τ-functions $\tau_0, \tau_2, \tau_4, \dots,$ defined in (1.15), is generated by the vertex operators Y_p; to be precise, inductively*

$$Y_{(N/2)-\ell+1}\tau_{\ell-2} = \ell\tau_\ell.$$

COROLLARY 1.4. *The vector of τ-functions*

$$I = (I_0, I_2, I_4, \dots), \quad \text{with } I_\ell = \left(\frac{\ell}{2}\right)!\tau_\ell$$

is a fixed point for the vertex operator $\mathbb{Y}_N$, namely

$$(\mathbb{Y}_N I)_\ell = I_\ell, \quad \text{for } \ell \text{ even.}$$

The rational solutions to the Pfaff lattice can be q-deformed; this will be reported on at a later stage.

1.7. *Example 1: Rectangular Jack polynomials.* Jack polynomials are symmetric polynomials in the variables x_i, which are orthogonal with respect to the inner-product

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu}(1^{m_1}2^{m_2}\cdots)m_1!m_2!\cdots\alpha^{\lambda_1^\top},$$

where $m_i = m_i(\lambda)$ is the number of times that i appears in the partition λ and where

$$p_\lambda(x_1, x_2, \dots) := p_{\lambda_1}p_{\lambda_2}\cdots = \sum_i x_i^{\lambda_1}\sum_i x_i^{\lambda_2}\cdots .$$

Precise definitions and properties of Jack polynomials can be found in [**4, 6–9**].

PROPOSITION 1.5. *When*

$$b_i = \begin{cases} 2i+1, & \text{for } N \text{ even,} \\ 2i+2, & \text{for } N \text{ odd,} \end{cases}$$

then the $\tau_{2n}(t)$'s are Jack polynomials for rectangular partitions

$$\begin{aligned} \tau_{2n}(t) &= \sum_{\lambda\in\mathbb{Y}^{(2n)}_{n(N-2n)}} \prod_1^n (k_i - k_{2n+1-i})\mathbf{s}_\lambda(t), \quad \text{where } \begin{cases} k_i = \lambda_i - i + 2n \\ 0 \le 2n \le N, \end{cases} \\ &= pf\, m_{2n}(t) \\ &= \frac{1}{n!}\int_{\mathbb{R}^n} \Delta(z)^4 \prod_{k=1}^n \exp\left(\sum_1^\infty t_i z_k^i\right)\delta^{(N-2)}_{(z_k)}\, dz_k \\ &= J_\lambda^{(1/2)}(x)|_{t_i = 1/i\sum_k x_k^i} \quad \text{for } \lambda = \underbrace{(N-2n, \dots, N-2n)}_{n} \end{aligned}$$

where the $m_{2n}(t)$'s are the $(2n \times 2n)$ upper-left-hand corners of

$$m_N(t) = ((j-i)\tilde{\mathbf{s}}_{N-i-j-1})_{0\le i,j\le N-1} \tag{1.25}$$

upon setting $\tilde{\mathbf{s}}_n(t) := \mathbf{s}_n(2t)$.

1.8. *Example 2: Two-row Jack polynomials.*

PROPOSITION 1.6. *For N even, choosing*†

$$\begin{cases} b_0 = \cdots = b_{(p/2)-1} = 0 \\ b_{(p/2)+k} = \dfrac{(1-\alpha)_k (p+1)_k}{k!(\alpha+p+1)_k}, \quad \text{for } k = 0, \ldots, \dfrac{N-2-p}{2}, \end{cases} \tag{1.26}$$

one finds the most general two-row Jack polynomial for τ_2, for arbitrary α,

$$\begin{aligned} \tau_2(t) &= pf\, m_2(t) \\ &= J^{(1/\alpha)}_{((N+p-2)/2,(N-p-2)/2)}(t/\alpha) \\ &= c \oint \frac{dx}{2\pi i} \frac{dy}{2\pi i} \frac{(y-x)^{2\alpha}}{(xy)^{\alpha+(N/2)}} \\ &\quad \times \exp\left(\sum_1^\infty t_i(x^i + y^i)\right)\left(\frac{x}{y}\right)^{p/2} {}_2F_1\left(\alpha, -p; 1-\alpha-p; \frac{y}{x}\right). \end{aligned} \tag{1.27}$$

Then $\tau_\ell(t)$ for $\ell \geq 4$ is given by an integral of the same hypergeometric function in the integrand above.

2. *The vector fields $\partial m/\partial t_k = \Lambda^k m + m\Lambda^{\top k}$ and the finite Pfaff lattice*

The $(\ell \times N)$ matrix defined in (1.12) reads

$$E_{\ell,N}(t) = \left(\begin{array}{ccccc|ccc} 1 & \mathbf{s}_1(t) & \mathbf{s}_2(t) & \ldots & \mathbf{s}_{\ell-1}(t) & \mathbf{s}_\ell(t) & \ldots & \mathbf{s}_{N-1}(t) \\ 0 & 1 & \mathbf{s}_1(t) & \ldots & \mathbf{s}_{\ell-2}(t) & \mathbf{s}_{\ell-1}(t) & \ldots & \mathbf{s}_{N-2}(t) \\ \vdots & \vdots & & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & \mathbf{s}_1(t) & \mathbf{s}_2(t) & \ldots & \mathbf{s}_{N-\ell+1}(t) \\ 0 & 0 & 0 & \ldots & 1 & \mathbf{s}_1(t) & \ldots & \mathbf{s}_{N-\ell}(t) \end{array}\right).$$

The main claim of this section can be summarized in the following statement.

PROPOSITION 2.1. *The commuting equations (for the definition of Λ, see footnote on p. 5)*

$$\frac{\partial m_N}{\partial t_k} = \Lambda^k m_N + m_N \Lambda^{\top k}, \tag{2.1}$$

with $(N \times N)$ skew-symmetric initial condition $m(0)$, have the following solution:

$$m_N(t) = E_{N,N}(t) m_N(0) E^\top_{N,N}(t). \tag{2.2}$$

In particular, each $(\ell \times \ell)$ upper-left block of $m(t)$ equals

$$m_\ell(t) = E_{\ell,N}(t) m_N(0) E^\top_{\ell,N}(t). \tag{2.3}$$

Proof. Define $m_\infty(0)$ as the semi-infinite matrix formed by putting $m_N(0)$ in the upper-left corner and setting all other entries equal to zero and let Λ_∞ be the semi-infinite shift matrix. Then the solution to the differential equations

$$\frac{\partial m_\infty}{\partial t_k} = \Lambda^k_\infty m_\infty + m_\infty \Lambda^{\top k}_\infty \tag{2.4}$$

† $(a)_k = \Gamma(a+k)/\Gamma(a) = a(a+1)\cdots(a+k-1)$.

is given by

$$m_\infty(t) = \exp\left(\sum_1^\infty t_k \Lambda_\infty^k\right) m_\infty(0) \exp\left(\sum_1^\infty t_k \Lambda_\infty^{\top k}\right). \tag{2.5}$$

Result (2.1) follows from the Taylor expansion

$$\exp\left(\sum_1^\infty t_k \Lambda_\infty^k\right) = \sum_{k=0}^\infty \mathbf{s}_k(t) \Lambda_\infty^k,$$

which is an upper-triangular semi-infinite matrix, and considering only the upper-left $(\ell \times \ell)$ block. Each upper-left $(\ell \times \ell)$ block of $m_\infty(t)$ for $\ell \leq N$ equals

$$\begin{aligned} m_\ell(t) &= E_{\ell,\infty}(t) m_\infty(0) E_{\ell,\infty}^\top(t) \\ &= E_{\ell,N}(t) m_N(0) E_{\ell,N}^\top(t), \end{aligned}$$

from which (2.3) follows, as does (2.2) setting $\ell = N$. □

Remark. The flow (2.4) maintains the finite upper-left-hand corner of m_∞ and on that locus it is equivalent to the finite flow (2.1). Therefore, the whole semi-infinite theory can be applied to this case. It is possible to give a proof of Theorem 2.1 purely within finite matrices.

THEOREM 2.2. *Consider the commuting equations on the $(N \times N)$ matrix in*

$$\frac{\partial m_N}{\partial t_i} = \Lambda^i m_N + m_N \Lambda^i \tag{2.6}$$

with skew-symmetric initial condition $m_N(s)$ and its 'skew-Borel decomposition'

$$m_N = Q^{-1} J Q^{-1\top}, \quad \textit{for } Q \in G_{\mathfrak{k}}. \tag{2.7}$$

When N is odd, we further impose the differential equations for the last entry Q_{NN} of Q:

$$\frac{\partial Q_{NN}}{\partial t_i} = -\frac{1}{2} Q_{N,N-i}. \tag{2.8}$$

Then, for arbitrary $N > 0$, the matrix Q evolves according to the equations

$$\frac{\partial Q}{\partial t_i} Q^{-1} = -\pi_{\mathfrak{k}}(Q \Lambda^i Q^{-1}) \tag{2.9}$$

and the matrix $L := Q \Lambda Q^{-1}$ provides a solution to the Lax pair

$$\frac{\partial L}{\partial t_i} = [-\pi_{\mathfrak{k}} L^i, L] = [\pi_{\mathfrak{n}} L^i, L]. \tag{2.10}$$

Proof. For a matrix A, consider the projections

$$A_0 = \begin{cases} \begin{pmatrix} * & * & & & & O \\ * & * & & & & \\ & & \ddots & & & \\ & & & & * & * \\ & O & & & & \\ & & & & * & * \end{pmatrix}, & \text{for } N \text{ even}, \\ \begin{pmatrix} * & * & & & & O & \\ * & * & & & & & \\ & & \ddots & & & & \\ & & & & * & * & \\ & O & & & & & \\ & & & & * & * & \\ & & & & & & * \end{pmatrix}, & \text{for } N \text{ odd}, \end{cases}$$

and

$$A_{00} = \begin{cases} A_0, & \text{for } N \text{ even}, \\ \begin{pmatrix} * & * & & & & O & \\ * & * & & & & & \\ & & \ddots & & & & \\ & & & & * & * & \\ & O & & & & & \\ & & & & * & * & \\ & & & & & & 0 \end{pmatrix}, & \text{for } N \text{ odd}. \end{cases}$$

The main point is to prove that†

$$\begin{aligned} 0 &= \frac{\partial Q}{\partial t_i} Q^{-1} + \pi_{\mathfrak{k}} L^i \\ &= \frac{\partial Q}{\partial t_i} Q^{-1} + (L^i_- - J(L^i_+)^\top J) + \frac{1}{2}(L^i_0 - J(L^i_0)^\top J) \\ &=: A. \end{aligned}$$

Also define

$$\left(L^i + \frac{\partial Q}{\partial t_i} Q^{-1}\right) - J\left(L^i + \frac{\partial Q}{\partial t_i} Q^{-1}\right)^\top J =: B.$$

† $L^i_+ := (L^i)_+$ and $L^i_0 := (L^i)_0$.

We have, setting $\dot{} = \partial/\partial t_i$,

$$\begin{aligned}0 &= Q\left(\Lambda^i m + m\Lambda^{\top i} - \frac{\partial m}{\partial t_i}\right)Q^\top\\ &= (Q\Lambda^i Q^{-1})J + JQ^{-1\top}\Lambda^{\top i}Q^\top + (\dot{Q}Q^{-1})J + JQ^{-1\top}\dot{Q}^\top\\ &= (L^i + \dot{Q}Q^{-1})J + J(L^i + \dot{Q}Q^{-1})^\top.\end{aligned}$$

Hence†

$$\begin{aligned}0 &= \left(Q\left(\Lambda^i m + m\Lambda^{\top i} - \frac{\partial m}{\partial t_i}\right)Q^\top\right)_{-,00}\\ &= (((L^i + \dot{Q}Q^{-1}) - J(L^i + \dot{Q}Q^{-1})^\top J)J)_{-,00}\\ &= ((L^i + \dot{Q}Q^{-1}) - J(L^i + \dot{Q}Q^{-1})^\top J)_{-,00}J\\ &= B_{-,00}J.\end{aligned}$$

Therefore

$$0 = B_{-,00}J^2 = \begin{cases} B_{-,0}, & \text{for } N \text{ even},\\ B_{-,00}\begin{pmatrix} I_{N-1} & O\\ O & 0\end{pmatrix}, & \text{for } N \text{ odd},\end{cases}$$

and so

$$B_- = 0 \quad \text{and} \quad B_{00} = 0. \tag{2.11}$$

But

$$\begin{aligned}B_- &= (L^i + \dot{Q}Q^{-1} - J(L^i_+)^\top J)_-\\ &= (\dot{Q}Q^{-1})_- + ((L^i)_- - J(L^i_+)^\top J)\\ &= A_-\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}B_{00} &= 2(\dot{Q}Q^{-1})_{00} + (L^i - J(L^i)^\top J)_{00}\\ &= 2A_{00}.\end{aligned} \tag{2.13}$$

Then, by (2.12) and (2.13),

$$0 = B_- + \tfrac{1}{2}B_{00} = A_- + A_{00} = A_- + A_{00} + A_+, \quad \text{since } A_+ = 0.$$

Therefore, when N is even, $A = 0$ and the proof is finished. When N is odd, we have

$$A = 0, \quad \text{except for the } (N,N)\text{th entry}.$$

But since Q is lower-triangular, the (N,N)th entry of L^i is given by

$$(L^i)_{NN} = (Q\Lambda^i Q^{-1})_{NN} = \frac{Q_{N,N-i}}{Q_{NN}},$$

† $A_{-,00} = A_- + A_{00}$.

and thus we have, using the fact that the (N, N)th entry of $J(L^i)_0 J$ vanishes,

$$\begin{aligned} A_{NN} &= \frac{\partial}{\partial t_i} \log Q_{NN} + \frac{1}{2}(L^i)_{NN} \\ &= \frac{1}{Q_{NN}} \left(\frac{\partial Q_{NN}}{\partial t_i} + \frac{1}{2} Q_{N,N-i} \right), \\ &= 0, \quad \text{by the assumption (2.8)}, \end{aligned}$$

thus ending the proof of Theorem 2.2. □

3. *The solution to the Pfaff lattice with anti-diagonal skew-symmetric initial condition*

Consider the equations

$$\frac{\partial m_N}{\partial t_i} = \Lambda^i m_N + m_N \Lambda^{\top i}, \tag{3.1}$$

with initial condition,

$$m_N(0) = \begin{cases} \begin{pmatrix} & O & & & b_{(N-2)/2} \\ & & & \iddots & \\ & & b_0 & & \\ & -b_0 & & & \\ \iddots & & & & \\ -b_{(N-2)/2} & & & O & \end{pmatrix}, & \text{for } N \text{ even}, \\ \begin{pmatrix} & O & & & & b_{(N-3)/2} \\ & & & & \iddots & \\ & & & b_0 & & \\ & & 0 & & & \\ & -b_0 & & & & \\ \iddots & & & & & \\ -b_{(N-3)/2} & & & & O & \end{pmatrix}, & \text{for } N \text{ odd}. \end{cases} \tag{3.2}$$

PROPOSITION 3.1. *The system of equations (3.1), with initial condition (3.2), has for solution the matrix $m_N(t)$, with entries, for $0 \le \ell < k \le N$,*

$$\begin{aligned} \mu_{\ell,k}(t) = &- \sum_{j=0}^{[(N-2)/2]-k} \mathbf{s}_j \mathbf{s}_{N-\ell-k-j-1} (b_{[(N-2)/2]-k-j} - b_{[(N-2)/2]-\ell-j}) \\ &- \sum_{[(N-2)/2]-k+1}^{[(N-2)/2]-\ell} \mathbf{s}_j \mathbf{s}_{N-\ell-k-j-1} (-b_{[(N-2)/2]-\ell-j}). \end{aligned} \tag{3.3}$$

In particular,

$$\mu_{01}(t) = \begin{cases} \displaystyle\sum_{i=0}^{(N-2)/2} b_i \mathbf{s}_{[(N-2)/2]+i,[(N-2)/2]-i}(t), & \textit{for } N \textit{ even,} \\ \displaystyle\sum_{i=0}^{(N-3)/2} b_i \mathbf{s}_{[(N-1)/2]+i,[(N-3)/2]-i}(t), & \textit{for } N \textit{ odd.} \end{cases} \tag{3.4}$$

Proof. Equation (3.3) is established by explicit computation of

$$\begin{aligned} m_N(t) &= E_{N,N}(t) m_N(0) E_{N,N}(t)^\top \\ &= \left(\sum_{i,j=0}^{N-\ell-1} \mathbf{s}_i(t) \mu_{i+\ell,j+k}(0) \mathbf{s}_j(t) \right)_{0\le \ell,k\le N-1}. \end{aligned}$$

From (3.3), one computes, for N even,

$$\begin{aligned} \mu_{01}(t) &= \mathbf{s}_0\mathbf{s}_{N-2}(b_{(N/2)-1} - b_{(N/2)-2}) + \mathbf{s}_1\mathbf{s}_{N-3}(b_{(N/2)-2} - b_{(N/2)-3}) \\ &\quad + \cdots + \mathbf{s}_{(N/2)-2}\mathbf{s}_{(N/2)}(b_1 - b_0) + (\mathbf{s}_{(N/2)-1})^2 b_0 \\ &= \sum_{i=0}^{(N/2)-1} b_i (\mathbf{s}_{(N/2)-1-i}\mathbf{s}_{(N/2)-1+i} - \mathbf{s}_{(N/2)-2-i}\mathbf{s}_{(N/2)+i}) \\ &= \sum_{i=0}^{(N/2)-1} b_i \mathbf{s}_{(N/2)-1+i,(N/2)-1-i}(t), \end{aligned}$$

and for N odd,

$$\begin{aligned} \mu_{01}(t) &= \mathbf{s}_0\mathbf{s}_{N-2}(b_{(N-3)/2} - b_{(N-5)/2}) + \mathbf{s}_1\mathbf{s}_{N-3}(b_{(N-5)/2} - b_{(N-7)/2}) \\ &\quad + \cdots + \mathbf{s}_{(N-5)/2}\mathbf{s}_{(N+1)/2}(b_1 - b_0) + \mathbf{s}_{(N-3)/2}\mathbf{s}_{(N-1)/2} b_0 \\ &= \sum_{i=0}^{(N-3)/2} b_i \mathbf{s}_{[(N-1)/2]+i,[(N-3)/2]-i}(t), \end{aligned}$$

ending the proof of Proposition 3.1. □

Define†

$$\begin{aligned} m_N(0;z) &:= m_N(0), \quad \text{for } N \text{ even,} \\ m_N(0;z) &:= m_N(0) + z^2 \varepsilon_{(N+1)/2,(N+1)/2}, \quad \text{for } N \text{ odd,} \\ &= \begin{pmatrix} & & O & & & b_{(N-3)/2} \\ & & & & \iddots & \\ & & & b_0 & & \\ & & z^2 & & & \\ & -b_0 & & & & \\ \iddots & & & & & \\ -b_{(N-3)/2} & & & & O & \end{pmatrix}. \end{aligned} \tag{3.5}$$

† $\varepsilon_{i,j}$ denotes the matrix with all zero entries, except for a 1 at the (i, j)th entry.

PROPOSITION 3.2.

$$\det{}^{1/2}(E_{\ell,N}(t)m_N(0;z)E_{\ell,N}^{\top}(t)) = z^{\eta(N,\ell)} \sum_{\lambda \in \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2}} \left(\prod_1^{[\ell/2]} b_{\lambda_i - i + \ell - [(N+1)/2]} \right) \mathbf{s}_{\lambda_1 \geq \cdots \geq \lambda_\ell}(t),$$

with

$$\eta(N,\ell) = \begin{cases} 1, & \text{for } N \text{ and } \ell \text{ odd,} \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA 3.3. *Consider an arbitrary* $(N \times N)$ *matrix* $A = (A_{ij})_{1 \leq i,j \leq N}$, *with* $r = [N/2]$ *and* $A_\ell := (A_{ij})_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq N}}$ *and consider the anti-diagonal matrix*

$$m_N = \begin{cases} \begin{pmatrix} & & & & c_r \\ & & & \cdot^{\cdot^{\cdot}} & \\ & O & & & \\ & & & c_1 & \\ & & -c_1 & & \\ & \cdot^{\cdot^{\cdot}} & & O & \\ -c_r & & & & \end{pmatrix}, & \text{for } N \text{ even,} \\ \begin{pmatrix} & & & & & & c_r \\ & & & & & \cdot^{\cdot^{\cdot}} & \\ & & O & & & & \\ & & & & c_1 & & \\ & & & z^2 & & & \\ & & -c_1 & & & & \\ & \cdot^{\cdot^{\cdot}} & & & O & & \\ -c_r & & & & & & \end{pmatrix}, & \text{for } N \text{ odd.} \end{cases}$$

Setting†

$$m_\ell^A(z) := A_\ell m_N(z) A_\ell^{\top}$$

and

$$P_{N,\ell} = \sum_{1 \leq i_1 \leq \cdots \leq i_{[\ell/2]} \leq r} c_{i_1} \cdots c_{i_{[\ell/2]}} \times \begin{cases} \det(A_\ell)_{(r - i_{[\ell/2]}+1, \ldots, r - i_1 + 1, r + i_1, \ldots, r + i_{[\ell/2]})}, & \text{for } N \text{ even, } \ell \text{ even,} \\ \det(A_\ell)_{(r - i_{[\ell/2]}+1, \ldots, r - i_1 + 1, r + i_1 + 1, \ldots, r + i_{[\ell/2]}+1)}, & \text{for } N \text{ odd, } \ell \text{ even,} \\ \det(A_\ell)_{(r - i_{[\ell/2]}+1, \ldots, r - i_1 + 1, r+1, r + i_1 + 1, \ldots, r + i_{[\ell/2]}+1)}, & \text{for } N \text{ odd, } \ell \text{ odd,} \end{cases} \tag{3.6}$$

† $B_{(j_1,\ldots,j_n)}$ denotes the matrix formed with the columns $j_1, \ldots, j_n$ of B.

we have

$$\det m_\ell^A = \begin{cases} 0, & \textit{for } N \textit{ even, } \ell \textit{ odd,} \\ (pf m_\ell^A)^2 = (P_{N,\ell})^2, & \textit{for } N \textit{ even, } \ell \textit{ even,} \\ z^2 P_{N,\ell}^2, & \textit{for } N \textit{ odd, } \ell \textit{ odd,} \\ (pf m_\ell^A(0))^2 = (P_{N,\ell})^2, & \textit{for } N \textit{ odd, } \ell \textit{ even.} \end{cases}$$

Proof. Let $w_i \in \mathbb{C}^\ell$ be the columns of A_ℓ

$$A_\ell = [w_0, w_1, \dots, w_{2r}],$$

and observe that

$$\begin{aligned} m_\ell^A(z) = A_\ell m_N(z) A_\ell^\top &= A_\ell(z^2 \varepsilon_{r+1,r+1} + m_N(0)) A_\ell^\top \\ &= z^2 w_r \otimes w_r + m_\ell^A(0). \end{aligned}$$

Let U be an $(\ell \times \ell)$ matrix, rational in a_{ij}, such that

$$U w_r = \alpha e_1, \quad \det U = 1.$$

Then, using $U(x \otimes y)V = (Ux) \otimes (V^\top y)$ and setting $M := U\, m_\ell^A(0) U^\top$, which is skew-symmetric, we find

$$\begin{aligned} \det m_\ell^A(z) &= \det U m_\ell^A(z) U^\top \\ &= \det(z^2 U(w_r \otimes w_r) U^\top + U m_\ell^A(0) U^\top) \\ &= \det(z^2 \alpha^2 e_1 \otimes e_1 + U m_\ell^A(0) U^\top) \\ &= \det \left(\begin{array}{c|cccc} (z\alpha)^2 & M_{12} & M_{13} & \dots & M_{1\ell} \\ \hline -M_{12} & 0 & M_{23} & \dots & M_{2\ell} \\ -M_{13} & -M_{23} & & & \\ \vdots & \vdots & & & \vdots \\ -M_{1\ell} & -M_{2\ell} & \dots & & 0 \end{array} \right) \\ &= (z\alpha)^2 \det(M_{ij})_{2 \le i,j \le \ell} + \det(M_{ij})_{1 \le i,j \le \ell}, \end{aligned}$$

with $M_{ij} = -M_{ji}$. Therefore

$$\det m_\ell^A(z) = \begin{cases} \det m_\ell^A(0) = (pf m_\ell^A(0))^2, & \text{for } \ell \text{ even,} \\ (z\alpha)^2 \det(M_{ij})_{2 \le i,j \le \ell} = (z\alpha pf(M_{ij})_{2 \le i,j \le \ell})^2, & \text{for } \ell \text{ odd,} \end{cases} \tag{3.7}$$

the latter being the square of a polynomial in z, the c_i and the entries of the matrix A.

Using the Cauchy–Bonnet formula twice, one computes, say, for N and ℓ odd,

$$\begin{aligned}
\det m_\ell^A(z) &= \det A_\ell m_N(z) A_\ell^\top \\
&= \sum_{1\le\alpha_1<\cdots<\alpha_\ell\le N} \det((A_\ell)_{i,\alpha_j})_{1\le i,j\le\ell} \det((A_\ell m^\top)_{i,\alpha_j})_{1\le i,j\le\ell} \\
&= \sum_{\substack{1\le\alpha_1<\cdots<\alpha_\ell\le N\\ 1\le\beta_1<\cdots<\beta_\ell\le N\\ \alpha_i+\beta_{\ell-i+1}=N+1}} \det((A_\ell)_{i,\alpha_j})_{1\le i,j\le\ell} \det((A_\ell)_{i,\beta_j})_{1\le i,j\le\ell} \det((m^\top)_{\beta_i,\alpha_j})_{1\le i,j\le\ell} \\
&= \sum_{\substack{1\le\alpha_1<\cdots<\alpha_\ell\le N\\ 1\le\beta_1<\cdots<\beta_\ell\le N\\ \alpha_i+\beta_{\ell-i+1}=N+1\\ \text{for } 1\le i\le\ell}} \det((A_\ell)_{i,\alpha_j})_{1\le i,j\le\ell} \det((A_\ell)_{i,\beta_j})_{1\le i,j\le\ell} \det(m_{\alpha_i,\beta_j})_{1\le i,j\le\ell} \\
&= \left(\sum_{\substack{1\le\alpha_1<\cdots<\alpha_\ell\le N\\ 1\le\beta_1<\cdots<\beta_\ell\le N\\ (\alpha_1,\dots,\alpha_\ell)=(\beta_1,\dots,\beta_\ell)\\ \alpha_i+\beta_{\ell-i+1}=N+1\\ \text{for } 1\le i\le\ell}} + \sum_{\substack{1\le\alpha_1<\cdots<\alpha_\ell\le N\\ 1\le\beta_1<\cdots<\beta_\ell\le N\\ (\alpha_1,\dots,\alpha_\ell)\ne(\beta_1,\dots,\beta_\ell)\\ \alpha_i+\beta_{\ell-i+1}=N+1\\ \text{for } 1\le i\le\ell}} \right) \det((A_\ell)_{i,\alpha_j})_{1\le i,j\le\ell} \det((A_\ell)_{i,\beta_j})_{1\le i,j\le\ell} \\
&\qquad \times \det(m_{\alpha_i,\beta_j})_{1\le i,j\le\ell} \\
&\overset{*}{=} z^2 \sum_{\substack{1\le\alpha_1<\cdots<\alpha_{(\ell-1)/2}<(N+1)/2\\ \alpha_{[(\ell+1)/2]+i}+\alpha_{[(\ell+1)/2]-i}=N+1\\ \text{for } 0\le i\le(\ell-1)/2}} c^2_{[(N+1)/2]-\alpha_1}\cdots c^2_{[(N+1)/2]-\alpha_{(\ell-1)/2}} \\
&\qquad \times \det{}^2((A_\ell)_{i,\alpha_j})_{1\le i,j\le\ell} + \cdots \\
&= \left(z \sum_{\substack{1\le\alpha_1<\cdots<\alpha_{(\ell-1)/2}<(N+1)/2\\ \alpha_{[(\ell+1)/2]+i}+\alpha_{[(\ell+1)/2]-i}=N+1}} c_{[(N+1)/2]-\alpha_1}\cdots c_{[(N+1)/2]-\alpha_{(\ell-1)/2}} \right. \\
&\qquad \left. \times \det((A_\ell)_{i,\alpha_j})_{1\le i,j\le\ell} \right)^2 \quad \text{using (3.7)} \\
&= \left(z \sum_{1\le i_1<\cdots<i_{(\ell-1)/2}\le(N-1)/2} c_{i_{(\ell-1)/2}}\cdots c_{i_1} \right. \\
&\qquad \left. \times \det(A_\ell)_{[(N+1)/2]-i_{(\ell-1)/2},\dots,[(N+1)/2]-i_1,(N+1)/2,[(N+1)/2]+i_1,\dots,[(N+1)/2]+i_{(\ell-1)/2}} \right)^2 .
\end{aligned}$$

In $\overset{*}{=}$ we have used the fact that

$$\left.\begin{aligned} (\alpha_1,\dots,\alpha_\ell)=(\beta_1,\dots,\beta_\ell)\\ \alpha_i+\beta_{\ell-i+1}=N+1 \end{aligned}\right\} \Longleftrightarrow \begin{cases} \alpha_{[(\ell+1)/2]+i}+\alpha_{[(\ell+1)/2]-i}=N+1, \\ \quad \text{for } 0\le i\le(\ell-1)/2 \\ \beta_{\ell-i+1}=N+1-\alpha_i . \end{cases} \tag{3.8}$$

Indeed, for N odd, consider sequences α_i symmetric about

$$\alpha_{(\ell+1)/2} = \frac{N+1}{2}, \tag{3.9}$$

i.e.

$$\alpha_{[(\ell+1)/2]+i} + \alpha_{[(\ell+1)/2]-i} = N+1, \quad \text{for } 0 \le i \le \frac{\ell-1}{2}. \tag{3.10}$$

Then, using (3.8) and (3.10)

$$\beta_{[(\ell+1)/2]-i} = N+1-\alpha_{[(\ell+1)/2]+i} = \alpha_{[(\ell+1)/2]-i},$$

thus implying

$$(\alpha_1, \ldots, \alpha_\ell) = (\beta_1, \ldots, \beta_\ell).$$

Vice versa, the latter implies (3.8) and thus (3.9). This establishes Lemma 3.3 for the case N and ℓ odd; for the other cases, one proceeds in a similar fashion. □

Proof of Proposition 3.2. Apply Lemma 3.3 to $A_\ell = E_{\ell,N}(t) = (\mathbf{s}_{j-i})_{\substack{1\le i\le \ell\\ 1\le j\le N}}$, with $1 \le k_1 < k_2 < \cdots < k_\ell$:

$$\begin{aligned}
\det(A_\ell)_{k_1,\ldots,k_\ell} &= \det\begin{pmatrix} \mathbf{s}_{k_1-1} & \cdots & \mathbf{s}_{k_{\ell-1}-1} & \mathbf{s}_{k_\ell-1} \\ \vdots & & \vdots & \vdots \\ \mathbf{s}_{k_1-\ell} & \cdots & \mathbf{s}_{k_{\ell-1}-\ell} & \mathbf{s}_{k_\ell-\ell} \end{pmatrix} \\
&= \det\begin{pmatrix} \mathbf{s}_{k_\ell-\ell} & \mathbf{s}_{k_\ell-\ell+1} & \cdots & \mathbf{s}_{k_\ell-1} \\ \mathbf{s}_{k_{\ell-1}-\ell} & \mathbf{s}_{k_{\ell-1}-\ell+1} & \cdots & \mathbf{s}_{k_{\ell-1}-1} \\ \vdots & & & \vdots \\ \mathbf{s}_{k_1-\ell} & & \cdots & \mathbf{s}_{k_1-1} \end{pmatrix} \\
&= \mathbf{s}_{k_\ell-\ell,k_{\ell-1}-\ell+1,\ldots,k_1-\ell+(\ell-1)} \\
&= \mathbf{s}_{\lambda_1\ge\cdots\ge\lambda_\ell} \\
&= \mathbf{s}_\lambda
\end{aligned} \tag{3.11}$$

where

$$\lambda_i = k_{\ell-i+1} - \ell + i - 1, \quad \text{for } 1 \le i \le \ell. \tag{3.12}$$

In order to apply Lemma 3.3, the k_i inherent in formula (3.6) must be as in formula (6.4), i.e. setting $r = [N/2]$, the k_j's must satisfy

$$k_j = \left[\frac{N}{2}\right] - i_{[\ell/2]-j+1} + 1 = N+1-k_{\ell-j+1}, \quad \text{for } 1 \le j \le \left[\frac{\ell+1}{2}\right] \tag{3.13}$$

and thus

$$\begin{aligned}
i_{[\ell/2]+1-j} - 1 &= k_{\ell+1-j} - \left[\frac{N+1}{2}\right] - 1 \\
&= \lambda_j + \ell - j - \left[\frac{N+1}{2}\right].
\end{aligned}$$

Therefore, formula (3.6) can be applied with

$$c_{i_{[\ell/2]-j+1}} = b_{\lambda_j+\ell-j-[(N+1)/2]}, \quad \text{for } 1 \le j \le \left[\frac{\ell}{2}\right].$$

From (3.12) and (3.13), it follows that

$$\lambda_i + \lambda_{\ell+1-i} = k_{\ell+1-i} + k_i - \ell - 1 = N + 1 - \ell - 1 = N - \ell,$$

showing that

$$\lambda \in \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2},$$

establishing Proposition 3.2. □

4. *Proof of Theorem 1.1*

Using the standard notation for the partition $1^j = \overbrace{(1, \ldots, 1)}^{j}$, we state the following.

LEMMA 4.1.

$$\left.\begin{array}{r}\mathbf{s}_i(-\tilde{\partial})\mathbf{s}_{1^j}(t)\\ \left(-\dfrac{\partial}{\partial t_i}\right)\mathbf{s}_{1^j}(t)\end{array}\right\} = (-1)^i \mathbf{s}_{1^{j-i}}(t). \tag{4.1}$$

Proof. Using the usual inner-product between symmetric functions, we have

$$\begin{aligned}\mathbf{s}_i(\tilde{\partial})\mathbf{s}_j(t) &= \langle \mathbf{s}_i(t+u)\cdot 1, \mathbf{s}_j(t+u)\rangle\\ &= \langle \mathbf{s}_j(t+u), \mathbf{s}_i(t+u)\cdot 1\rangle\\ &= \langle \mathbf{s}_{j-i}(t+u), 1\rangle\\ &= \langle 1, \mathbf{s}_{j-i}(t+u)\rangle\\ &= \mathbf{s}_{j-i}(t+u)|_{u=0}\\ &= \mathbf{s}_{j-i}(t)\end{aligned}$$

and so, changing $t \mapsto -t$,

$$\mathbf{s}_i(-\tilde{\partial})\mathbf{s}_j(-t) = \mathbf{s}_{j-i}(-t),$$

from which this first relation follows upon noticing that

$$\mathbf{s}_j(-t) = (-1)^j \mathbf{s}_{1^j}(t). \tag{4.2}$$

This last relation (4.2) also leads to the second identity (4.1), using $(\partial/\partial t_i)\mathbf{s}_j(t) = \mathbf{s}_{j-i}(t)$. □

Proof of Theorem 1.1. By Proposition 2.1, the equation for the $(N \times N)$ matrix m_N

$$\frac{\partial m_N}{\partial t_k} = \Lambda^k m_N + m_N \Lambda^k,$$

with skew-symmetric initial condition $m_N(0)$ has the following solution

$$m_N(t) = E_{\ell,N} m_N(0) E^{\top}_{\ell,N}(t),$$

which remains skew-symmetric in time. Define a t-dependent skew inner-product such that $\langle y^i, z^j\rangle_t = m_{ij}(t)$, i.e.†

$$\langle \chi_N(y)\chi(z)^\top\rangle = m_N(t).$$

Performing the skew Borel decomposition

$$m_N(t) = Q^{-1}(t)JQ^{-1\top}, \quad \text{with } Q(t) \in G_k \tag{4.3}$$

is tantamount to the process of finding a finite set of skew-orthonormal polynomials; that is, satisfying

$$(\langle q_i(t;z), q_j(t;z)\rangle)_{1\le i,j\le N} = J.$$

Indeed, the polynomials $q_i(t;z)$ in z, depending on t,

$$\begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_{N-1} \end{pmatrix} = Q \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{N-1} \end{pmatrix}$$

satisfy

$$\begin{aligned}(\langle q_i(t;y), q_i(t;z)\rangle)_{0\le i,j\le N-1} &= \langle Q(t)\chi_N(y), Q(t)\chi_N(z)\rangle \\ &= \langle Q(t)\chi_N(y)\chi_N(z)Q^\top(t)\rangle \\ &= Q(t)\langle\chi_N(y)\chi_N(z)\rangle Q^\top(t) \\ &= Q(t)m_N(t)Q^\top(t) \\ &= J.\end{aligned}$$

According to [**2**], the skew-orthogonal polynomials are related to the τ-functions ($\tau_0 = 1$, $\tau_N = c$)

$$\tau_\ell(t) = pfm_\ell(t)$$

as follows:

$$\begin{aligned} q_{2n} &= \frac{z^{2n}}{\sqrt{\tau_{2n}\tau_{2n+2}}}\tau_{2n}(t - [z^{-1}]) \\ q_{2n+1} &= \frac{z^{2n}}{\sqrt{\tau_{2n}\tau_{2n+2}}}\left(z + \frac{\partial}{\partial t_1}\right)\tau_{2n}(t - [z^{-1}]), \quad 0 \le 2n \le N-2.\end{aligned}$$

This ends the proof of Theorem 1.1 for N even. However for N odd, we must verify condition (2.8) of Theorem 2.2. This requires knowing $q_{N-1}(t;z)$ explicitly. For later purposes we shall also need $q_{N-1}(t;z)$ for N even.

For N even, q_{N-1} takes on the form

$$q_{N-1}(t;z) = \frac{z^{N-2}}{\sqrt{\tau_{N-2}\tau_N}}\left(z + \frac{\partial}{\partial t_1}\right)\tau_{N-2}(t - [z^{-1}]),$$

† $\chi(y) := (1, y, y^2, \dots)^\top$.

with (using Proposition 3.2)

$$\tau_{N-2}(t) = \sum_{\lambda \in \mathbb{Y}^{(N-2)}_{N-2}} \left(\prod_{1}^{(N-2)/2} b_{\lambda_i - i + (N/2) - 2} \right) \mathbf{s}_\lambda(t),$$

where

$$\mathbb{Y}^{(N-2)}_{N-2} = \{1^{N-2}, (2, 1^{N-4}), (2^2, 1^{N-6}), \ldots, (2^i, 1^{N-2i-2}), \ldots\}.$$

For N odd, q_{N-1} has the form

$$q_{N-1}(t; z) = \frac{z^{N-1}}{\sqrt{\tau_{N-1}}} \tau_{N-1}(t - [z^{-1}])$$

with

$$\tau_{N-1}(t) = b_0 \cdots b_{(N-3)/2} \mathbf{s}_{(1^{(N-1)/2})}(t). \tag{4.4}$$

Indeed, observe that the set of partitions

$$\mathbb{Y}^{\ell}_{\ell(N-\ell)/2}|_{\ell=N-1} = \mathbb{Y}^{(N-1)}_{(N-1)/2} = \left\{ \begin{array}{c} (\lambda_1, \ldots, \lambda_{N-1}) \in \mathbb{Y}_{(N-1)/2} \\ \text{with } \lambda_i + \lambda_{\ell+1-i} = 1 \end{array} \right\} = \{1^{(N-1)/2}\}$$

consists of one element $1^{(N-1)/2}$. Therefore, setting $\lambda_i = 1$ for $1 \le i \le (N-1)/2$, one finds, again by Proposition 3.2,

$$\tau_{N-1}(t) = b_0 \cdots b_{(N-3)/2} \mathbf{s}_{(1^{(N-1)/2})}(t).$$

The last row of $\tilde{Q}$ is given by

$$\begin{aligned} \sum_{0}^{N-1} \tilde{Q}_{N,j+1} z^j &= z^{N-1} \tau_{N-1}(t - [z^{-1}]) \\ &= \sum_{i=0}^{N-1} \mathbf{s}_i(-\tilde{\partial}) \tau_{N-1}(t) z^{N-1-i} \\ &= b_0 \cdots b_{(N-3)/2} \sum_{i=0}^{N-1} \mathbf{s}_i(-\tilde{\partial}) \mathbf{s}_{(1^{(N-1)/2})}(t) z^{N-1-i} \\ &= b_0 \cdots b_{(N-3)/2} \sum_{i=0}^{(N-1)/2} z^{N-1-i} (-1)^i \mathbf{s}_{(1^{[(N-1)/2]-i})}(t), \end{aligned}$$

using Lemma 4.1, and so

$$\tilde{Q}_{N,N-i} = (-1)^i \left(\prod_{0}^{(N-3)/2} b_k \right) \mathbf{s}_{(1^{[(N-1)/2]-i})}.$$

Therefore, the last row of $\tilde{Q}$ reads

$$\prod_{0}^{(N-3)/2} b_i \left(\underbrace{0, \ldots, 0}_{(N-1)/2}, (-1)^{(N-1)/2}, (-1)^{(N-3)/2} \mathbf{s}_1(t), \right.$$
$$\left. (-1)^{(N-5)/2} \mathbf{s}_{(1^2)}(t), \ldots, \mathbf{s}_{(1^{(N-1)/2})}(t) \right)$$

and the last row of $Q = D\tilde{Q}$ is

$$Q_{N,N-i} = (D\tilde{Q})_{N,N-i} = (-1)^i \prod_0^{(N-3)/2} b_k \frac{\mathbf{s}_{(1^{[(N-1)/2]-i})}(t)}{\sqrt{\tau_{N-1}}} = (-1)^i \left(\prod_0^{N-3} b_k\right)^{1/2} \frac{\mathbf{s}_{(1^{[(N-1)/2]-i})}(t)}{(\mathbf{s}_{(1^{(N-1)/2})}(t))^{1/2}},$$

and so, using Lemma 4.1,

$$\frac{\partial Q_{N,N}}{\partial t_i} = -\frac{(-1)^i}{2}\left(\prod_0^{N-3} b_k\right)^{1/2} \frac{\mathbf{s}_{(1^{[(N-1)/2]-i})}(t)}{(\mathbf{s}_{(1^{(N-1)/2})}(t))^{1/2}} = -\frac{1}{2} Q_{N,N-i}.$$

Having checked (2.6)–(2.8) (in the odd case) of Theorem 2.2, we have found a solution of the Pfaff lattice. This finally concludes the proof of Theorem 1.1. □

Proof of Theorem 1.2. According to [**2**], Pfaff τ-functions satisfy bilinear relations†: for all $t, t' \in \mathbb{C}^\infty$ and m, n positive integers,

$$\oint_{z=\infty} \tau_{2n}(t-[z^{-1}])\tau_{2m+2}(t'+[z^{-1}]) \exp\left[\sum_{i=0}^{\infty}(t_i - t_i')z^i\right] z^{2n-2m-2}\, dz + \oint_{z=0} \tau_{2n+2}(t+[z])\tau_{2m}(t'-[z]) \exp\left[\sum_{i=0}^{\infty}(t_i' - t_i)z^{-i}\right] z^{2n-2m}\, dz = 0.$$

Shifting appropriately and taking residues leads to the 'differential Fay identity'

$$\{\tau_{2n}(t-[u]), \tau_{2n}(t-[v])\} + (u^{-1}-v^{-1})(\tau_{2n}(t-[u])\tau_{2n}(t-[v]) - \tau_{2n}(t)\tau_{2n}(t-[u]-[v])) = uv(u-v)\tau_{2n-2}(t-[u]-[v])\tau_{2n+2}(t), \quad (4.5)$$

and the Hirota bilinear equations, involving nearest neighbors,

$$\left(\mathbf{s}_{k+4}(\tilde{\partial}) - \frac{1}{2}\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_{k+3}}\right)\tau_{2n}\cdot\tau_{2n} = \mathbf{s}_k(\tilde{\partial})\tau_{2n+2}\cdot\tau_{2n-2}. \quad (4.6)$$

It only remains to check the 'boundary condition':

$$\begin{cases} \tau_N = \prod_0^{(N-2)/2} b_i, & \text{for even } N, \\ \tau_{N+1} = 0, & \text{for odd } N. \end{cases} \quad (4.7)$$

Indeed, for N even, using $\det E_{NN}(t) = 1$ and the matrix (3.2), we have that

$$(pf\, m_N(t))^2 = \det(E_{N,N}(t) m_N(0) E_{N,N}^\top(t)) = \det m_N(0) = \prod_0^{(N-2)/2} b_i.$$

Moreover, for N odd, according to (4.4), τ_{N-1} is a pure Schur polynomial, which is known to satisfy the KP Fay identity, i.e. the equation (4.5), without right-hand side. This justifies setting $\tau_{N+1} = 0$ for odd N. □

† $\tilde{\partial} = (\partial/\partial t_1, (1/2)\partial/\partial t_2, (1/3)\partial/\partial t_3, \ldots)$; $\tilde{D} = (D_1, (1/2)D_2, (1/3)D_3, \ldots)$ is the corresponding Hirota symbol, $P(\tilde{D})f \cdot g := P(\partial/\partial y_1, (1/2)\partial/\partial y_2, \ldots)f(t+y)g(t-y)|_{y=0}$; and $\mathbf{s}_k$ are the previously defined elementary Schur functions, $\sum_{k=0}^{\infty}\mathbf{s}_k(t)z^k := \exp(\sum_{i=1}^{\infty} t_i z^i)$. For further notation, see Dickey [**5**].

In the next proposition, we show that the finite vectors of skew-orthogonal polynomials form an eigenvector of the matrix L, with a modified boundary condition.

PROPOSITION 4.2. *For N even, the skew-orthonormal polynomials $q = (q_0, \ldots, q_{N-1})^\top = Q \quad (1, \ldots, z^{N-1})^\top$ are eigenfunctions for L, with the boundary condition*

$$Lq = zq - (0, \ldots, 0, z^N)\sqrt{pfm_{N-2}}\left(\prod_0^{(N-2)/2} b_i\right)^{-1/2}.$$

Proof. Indeed

$$\begin{aligned} Lq &= Q\ \Lambda\ Q^{-1}Q\begin{pmatrix}1\\ \vdots\\ z^{N-1}\end{pmatrix}\\ &= Q\ \Lambda\begin{pmatrix}1\\ \vdots\\ z^{N-1}\end{pmatrix}\\ &= Q\ z\begin{pmatrix}1\\ \vdots\\ z^{N-2}\\ 0\end{pmatrix}\\ &= z\begin{pmatrix}q_0\\ q_1\\ \vdots\\ q_{N-2}\\ \bar q_{N-1}\end{pmatrix}\\ &= zq + z(0, \ldots, 0, \bar q_{N-1} - q_{N-1}),\end{aligned}$$

where $\bar q_{N-1}$ is the same as q_{N-1}, but without the leading term, i.e. $\bar q_{N-1} = q_{N-1} - Q_{NN}z^{N-1}$, where by (4.7) we have

$$Q_{NN} = \sqrt{\frac{\tau_{N-2}}{\tau_N}} = \sqrt{pfm_{N-2}}\left(\prod_0^{(N-2)/2} b_i\right)^{-1/2},$$

ending the proof of Proposition 4.2. □

5. *Vertex operators*

The purpose of this section is to prove Theorem 1.3 and Corollary 1.4. Define, as in (1.23),

$$\beta := \frac{N}{2} - \ell + 1. \tag{5.1}$$

Remembering from (1.21) the vertex operator $X(t; z)$, consider now its formal expansion in powers of z

$$X(t; z) = \exp\left(\sum_1^\infty t_i z^i\right)\exp\left(-\sum_1^\infty \frac{z^{-i}}{i}\frac{\partial}{\partial t_i}\right) =: \sum_{i\in\mathbb{Z}} B_i z^i, \tag{5.2}$$

with differential operators (see footnote on p. 22)

$$B_i := B_i^{(\alpha)}|_{\alpha=1} \quad \text{and} \quad B_i^{(\alpha)} := \sum_{j\geq 0} \mathbf{s}_{i+j}(\alpha t)\mathbf{s}_j(-\alpha\tilde{\partial}_t). \tag{5.3}$$

Also define as in (1.22) the vector vertex operator†

$$\mathbb{X}(t;z) = \Lambda^\top \exp\left(\sum_1^\infty t_i z^i\right)\exp\left(-\sum_1^\infty \frac{z^{-i}}{i}\frac{\partial}{\partial t_i}\right)\chi(z). \tag{5.4}$$

Also remember the definitions of the *integrated vertex operator*, in terms of the vertex operator (5.2) and a function ρ_b, defined in (5.8) below,

$$Y_\beta(t) := \oint_\infty \oint_\infty X(t;y)X(t;z)\frac{\rho_b(y/z)\,dy\,dz}{z^2(yz)^\beta}$$

and the *integrated vector vertex operator*, in terms of (5.4),

$$\mathbb{Y}_N(t) = \frac{1}{(2\pi i)^2}\oint_\infty \oint_\infty \mathbb{X}(t;y)\mathbb{X}(t;z)\frac{\rho_b(y/z)\,dy\,dz}{2(yz)^{N/2}z}. \tag{5.5}$$

In both cases, the double integral around the two contours about ∞ amounts to computing the coefficient of $1/yz$. The next theorem is nothing but a rephrasing of Theorem 1.3 and Corollary 1.4.

THEOREM 5.1. *For a given set of b_i, the sequence of τ-functions $\tau_0, \tau_2, \tau_4, \ldots$, defined in (1.15), is generated by the vertex operators Y_β:*

$$Y_\beta \tau_{\ell-2} = \ell \tau_\ell. \tag{5.6}$$

The vector $I = (I_0, I_2, I_4, \ldots)$, with $I_\ell = (\ell/2)!\tau_\ell$ is a fixed point for the vector vertex operator $\mathbb{Y}_N$, namely

$$(\mathbb{Y}_N I)_\ell = I_\ell, \quad \text{for } \ell \text{ even}. \tag{5.7}$$

We shall first need a few propositions.

PROPOSITION 5.2. *Defining*

$$\rho_b(x) := \begin{cases} \rho_b^{(e)}(x) := \sum_{i\geq 0} b_i(x^{-i-1} - x^i), & \text{for } N \text{ even},\\ \rho_b^{(o)}(x) := x^{-1/2}\sum_{i\geq 0} b_i(x^{-i-1} - x^{i+1}), & \text{for } N \text{ odd}, \end{cases} \tag{5.8}$$

we have

$$\begin{aligned} Y_\beta(t) &= \frac{1}{(2\pi i)^2}\oint_\infty \oint_\infty X(t;y)X(t;z)\frac{\rho_b(y/z)\,dy\,dz}{z^2(yz)^\beta}\\ &= \begin{cases} \sum_{j\geq 0} b_j(B_{\beta+j}B_{\beta-j} - B_{\beta-j-1}B_{\beta+j+1}), & \text{for } N \text{ even},\\ \sum_{j\geq 0} b_j(B_{(\beta+j+1)/2}B_{(\beta-j-1)/2} - B_{\beta-j-3/2}B_{\beta+j+3/2}), & \text{for } N \text{ odd}. \end{cases} \end{aligned} \tag{5.9}$$

† $\chi(z) := (z^i)_{i\geq 0}$.

Proof. For N even, compute

$$\begin{aligned}\frac{X(t;y)X(t;z)}{(yz)^\beta} &= \sum_{i\in\mathbb{Z}} B_i y^{i-\beta} \sum_{j\in\mathbb{Z}} B_j z^{j-\beta} \\ &= \sum_{i\in\mathbb{Z}} B_{\beta+i} y^i \cdot \sum_{j\in\mathbb{Z}} B_{\beta-j} z^{-j} \\ &= \sum_{i,j\in\mathbb{Z}} B_{\beta+i} B_{\beta-j} \frac{y^i}{z^j} \\ &= \sum_{j\in\mathbb{Z}} B_{\beta+j} B_{\beta-j} \left(\frac{y}{z}\right)^j + \sum_{i\neq j\in\mathbb{Z}} a_{ij} \frac{y^i}{z^j}\end{aligned}$$

and so

$$\begin{aligned}&\rho_b^{(\mathrm{e})}\left(\frac{y}{z}\right) \cdot \frac{X(t;y)X(t;z)}{z^2(yz)^\beta} \\ &\quad = \frac{1}{z^2}\left(\sum_{i\geq 0} b_i\left[\left(\frac{y}{z}\right)^{-(i+1)} - \left(\frac{y}{z}\right)^i\right]\right) \cdot \left(\sum_{j\in\mathbb{Z}} B_{\beta+j} B_{\beta-j}\left(\frac{y}{z}\right)^j + \sum_{i\neq j\in\mathbb{Z}} a_{ij}\frac{y^i}{z^j}\right) \\ &\quad = \frac{1}{yz}\left(\sum_{j\geq 0} b_j (B_{\beta+j}B_{\beta-j} - B_{\beta-j-1}B_{\beta+j+1})\right) + \sum_{i \text{ or } j\neq 0} c_{ij} y^{i-1} z^{j-1}.\end{aligned}$$

Therefore, upon taking the double residue,

$$\oint_\infty \oint_\infty \frac{\rho_b^{(\mathrm{e})}(y/z)X(t;y)X(t;z)}{z^2(yz)^\beta} \frac{dy\,dz}{(2\pi i)^2} = \sum_{j\geq 0} b_j (B_{\beta+j}B_{\beta-j} - B_{\beta-j-1}B_{\beta+j+1}).$$

For N odd,

$$\frac{X(t;y)X(t;z)}{(yz)^\beta (y/z)^{1/2}} = \sum_{j\in\mathbb{Z}} B_{\beta+\frac12+j} B_{\beta-\frac12-j}\left(\frac{y}{z}\right)^j + \sum_{i\neq j\in\mathbb{Z}} a_{ij}\frac{y^i}{z^j}$$

and so

$$\begin{aligned}\rho_b^{(\mathrm{o})}\left(\frac{y}{z}\right)\frac{X(t;y)X(t;z)}{z^2(yz)^\beta} &= \frac{1}{yz}\sum_{j\geq 0} b_j (B_{\beta+j+\frac12}B_{\beta-j-\frac12} - B_{\beta-j-\frac32}B_{\beta+j+\frac32}) \\ &\quad + \sum_{i \text{ or } j\neq 0} c_{ij} y^{i-1} z^{j-1}.\end{aligned}$$

Therefore,

$$\oint_\infty \oint_\infty \frac{\rho_b^{(\mathrm{o})}(y/z)X(t;y)X(t;z)}{z^2(yz)^\beta}\frac{dy\,dz}{(2\pi i)^2} = \sum_{j\geq 0} b_j (B_{\beta+j+\frac12}B_{\beta-j-\frac12} - B_{\beta-j-\frac32}B_{\beta+j+\frac32}),$$

ending the proof of Proposition 5.2. □

Defining the set

$$\mathbb{S}_N^{(\ell)} := \left\{\begin{matrix}\sigma_1 > \sigma_2 > \cdots > \sigma_{\ell/2}, \sigma_i \in \mathbb{Z} \\ \dfrac{\ell}{2} \leq \sigma_i + i \leq \left[\dfrac{N}{2}\right]\end{matrix}\right\}, \tag{5.10}$$

the map

$$\sigma : \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2} \longrightarrow \mathbb{S}^{(\ell)}_N : \lambda \longmapsto \sigma(\lambda) = \left(\lambda_i - i + \ell - \left[\frac{N+1}{2}\right]\right)_{1\le i\le n/2} \tag{5.11}$$

is a bijection.

Indeed, $\lambda_1 \ge \lambda_2 \ge \cdots$ implies at once the strict inequalities $\sigma_1 > \sigma_2 > \cdots$ and also implies, together with the fact that for $\lambda \in \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2}$ and $1 \le i \le \ell/2$, $2\lambda_i \ge \lambda_i + \lambda_{\ell+1-i} = N - \ell$ and, clearly $\lambda_i \le N - \ell$.

Conversely, every $\sigma \in \mathbb{S}^{(\ell)}_N$ comes from a $\lambda \in \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2}$.

LEMMA 5.3. *For a given partition*

$$\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{\ell-2}) \in \mathbb{Y}^{(\ell-2)}_{(\ell-2)(N-\ell+2)/2}$$

and $j \ge 0$, the following holds:

$$B_{\beta+j}B_{\beta-j}\mathbf{s}_\lambda = -B_{\beta-j-1}B_{\beta+j+1}\mathbf{s}_\lambda = \begin{cases} 0, & \textit{if } \beta + j = \textit{some } \lambda_\nu - \nu - 1 \\ & \textit{for } 1 \le \nu \le \ell/2 - 1, \textit{ or if } j \ge N/2, \\ \mathbf{s}_{\lambda'}, & \textit{if } \beta + j \ne \textit{every } \lambda_\nu - \nu - 1 \\ & \textit{for } 1 \le \nu \le \ell/2 - 1, \end{cases} \tag{5.12}$$

where

$$\begin{aligned}\lambda' = (&\lambda_1 - 2 \ge \cdots \ge \lambda_\nu - 2 \ge \beta + j + \nu \ge \lambda_{\nu+1} - 1 \ge \cdots \ge \lambda_{(\ell/2)-1} - 1 \\ &\ge \lambda_{\ell/2} - 1 \ge \cdots \ge \lambda_{\ell-2-\nu} - 1 \ge (N-\ell) - (\beta + j + \nu) \ge \lambda_{\ell-1-\nu} \ge \cdots \ge \lambda_{\ell-2}) \\ &\in \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2}.\end{aligned} \tag{5.13}$$

Moreover, for j's such that $\beta + j \ne$ every $\lambda_\nu - \nu - 1$, the maps $B_{\beta+j}B_{\beta-j}$ induce maps

$$B_{\beta+j}B_{\beta-j} : \mathbb{Y}^{(\ell-2)}_{(\ell-2)(N-\ell+2)/2} \longrightarrow \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2} : \lambda \longmapsto \lambda' \tag{5.14}$$

having, as a whole, a 'surjectivity property', meaning that to each $\lambda' \in \mathbb{Y}^{\ell}_{(\ell-2)(N-\ell)/2}$ there are $\ell/2$ choices of $j \ge 0$ and $\lambda \in \mathbb{Y}^{(\ell-2)}_{(\ell-2)(N-\ell+2)/2}$ mapping to λ', by means of the map $B_{\beta+j}B_{\beta-j}$, as in (5.12).

At the level of the $\mathbb{S}$-spaces, the maps $B_{\beta+j}B_{\beta-j}$ induce maps

$$\mathbb{S}^{(\ell-2)}_N \longrightarrow \mathbb{S}^{(\ell)}_N : \sigma = (\sigma_1, \ldots, \sigma_{(\ell-2)/2}) \longmapsto \sigma' = (\sigma_1, \ldots, \sigma_\nu, j, \sigma_{\nu+1}, \ldots, \sigma_{(\ell-2)/2}), \tag{5.15}$$

having the same 'surjectivity property' as above.

For N odd, all the formulae above remain the same, except for the substitution $j \mapsto j + \frac{1}{2}$ in (5.12) and (5.13).

Proof. Extending a classic identity (see MacDonald [**8**]) to arbitrary sequences $(\lambda_1, \ldots, \lambda_n)$, we have

$$B_{\lambda_1}, \ldots, B_{\lambda_n}(1) = (\lambda_1, \ldots, \lambda_n) := \det(\mathbf{s}_{\lambda_i+j-i}(t))_{1\le i,j\le n}$$

and, in particular, for a partition $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$, we have, for an arbitrary choice of $j \geq 0$,

$$B_{\beta+j}B_{\beta-j}\mathbf{s}_{(\lambda_1,\ldots,\lambda_{\ell-2})} = \mathbf{s}_{(\beta+j,\beta-j,\lambda_1,\ldots,\lambda_{\ell-2})} = \det\begin{pmatrix} \mathbf{s}_{\beta+j} & \mathbf{s}_{\beta+j+1} & \mathbf{s}_{\beta+j+2} & \cdots & \mathbf{s}_{\beta+j+\ell-1} \\ \mathbf{s}_{\beta-j-1} & \mathbf{s}_{\beta-j} & \mathbf{s}_{\beta-j+1} & \cdots & \mathbf{s}_{\beta-j+\ell-2} \\ \mathbf{s}_{\lambda_1-2} & \mathbf{s}_{\lambda_1-1} & \mathbf{s}_{\lambda_1} & \cdots & \mathbf{s}_{\lambda_1+\ell-3} \\ \vdots & & & \ddots & \vdots \\ \mathbf{s}_{\lambda_{\ell-2}-\ell+1} & \cdots & \cdots & \cdots & \mathbf{s}_{\lambda_{\ell-2}} \end{pmatrix}. \quad (5.16)$$

Using the value (5.1) of β, it is immediately clear, from the matrix (5.16), that for $j \geq N/2$ the second row of the matrix (5.16) vanishes and therefore the determinant. Therefore, we assume $0 \leq j \leq (N/2) - 1$. We give the proof for N even; for N odd, it is identical with $j \mapsto j + 1/2$.

The first column of the matrix above involves the indices

$$\frac{N}{2} - \ell + 1 + j, \frac{N}{2} - \ell - j, \lambda_1 - 2, \lambda_2 - 3, \ldots, \lambda_{\ell/2} - \frac{\ell}{2} - 1, \ldots, \lambda_{\ell-2} - \ell + 1. \quad (5.17)$$

Consider now an arbitrary integer $j \geq 0$ and an arbitrary partition

$$\lambda \in \mathbb{Y}^{(\ell-2)}_{(\ell-2)(N-\ell+2)/2};$$

it has the property that

$$\lambda_i + \lambda_{\ell-1-i} = N - \ell + 2 \quad \text{for } 1 \leq i \leq \frac{\ell-2}{2}.$$

Hence, for $i = (\ell - 2)/2$

$$2\lambda_{\ell/2} \leq \lambda_{(\ell/2)-1} + \lambda_{\ell/2} = N - \ell + 2,$$

and so

$$\lambda_{\ell/2} \leq \frac{N - \ell + 2}{2};$$

thus, for the arbitrary $j \geq 0$ chosen above

$$\lambda_{\ell/2} - \ell/2 - 1 \leq \frac{N}{2} - \ell < \frac{N}{2} - \ell + j + 1.$$

The partition $\lambda_1 \geq \lambda_2 \geq \cdots$ implies the strict inequalities

$$\lambda_1 - 1 - 1 > \lambda_2 - 2 - 1 > \lambda_3 - 3 - 1 > \cdots > \lambda_{\nu+1} - (\nu + 1) - 1 > \cdots > \lambda_{\ell/2} - \ell/2 - 1$$

and, therefore, there exist $0 \leq \nu \leq (\ell/2) - 1$ such that

$$\lambda_\nu - \nu - 1 \geq \frac{N}{2} - \ell + j + 1 \geq \lambda_{\nu+1} - \nu - 2.$$

These inequalities together with the fact that

$$\lambda_\nu + \lambda_{\ell-1-\nu} = N - \ell + 2, \quad \lambda_{\nu+1} + \lambda_{\ell-2-\nu} = N - \ell + 2$$

also imply

$$\lambda_{\ell-2-\nu} - (\ell - 1 - \nu) \geq \frac{N}{2} - \ell - j \geq \lambda_{\ell-1-\nu} - (\ell - \nu).$$

Therefore, the indices (5.17) of the first column of the matrix (5.16) are now rearranged by order, as follows:

$$\begin{aligned}
&\lambda_1 - 2 > \lambda_2 - 3 > \cdots > \lambda_\nu - \nu - 1 \\
&\quad \geq \frac{N}{2} - \ell + 1 + j \geq \lambda_{\nu+1} - \nu - 2 > \cdots > \lambda_{(\ell/2)-1} - \frac{\ell}{2} > \lambda_{\ell/2} - \frac{\ell}{2} - 1 \\
&\qquad > \cdots > \lambda_{\ell-2-\nu} - (\ell - 1 - \nu) \\
&\quad \geq \frac{N}{2} - \ell - j \\
&\quad \geq \lambda_{\ell-1-\nu} - (\ell - \nu) > \cdots > \lambda_{\ell-2} - \ell + 1.
\end{aligned} \tag{5.18}$$

Notice that the determinant (5.16) vanishes if any of the equalities hold in (5.18) above. Therefore, we may assume strict inequalities. Upon rearranging the rows of the matrix (5.16) according to the order in (5.18), we now list the corresponding partitions by looking at the indices on the diagonal. This amounts to adding $i - 1$ to the ith entry of (5.18), thus leading to

$$\begin{aligned}
\lambda' = (&\underset{\substack{\uparrow\\1}}{\lambda_1 - 2} \geq \underset{\substack{\uparrow\\2}}{\lambda_2 - 2} \geq \cdots \geq \underset{\substack{\uparrow\\\nu}}{\lambda_\nu - 2} \geq \underset{\substack{\uparrow\\\nu+1}}{\frac{N}{2} - \ell + 1 + j + \nu} \geq \underset{\substack{\uparrow\\\nu+2}}{\lambda_{\nu+1} - 1} \geq \cdots \\
&\geq \underset{\substack{\uparrow\\\ell/2}}{\lambda_{(\ell/2)-1} - 1} \geq \underset{\substack{\uparrow\\\ell/2+1}}{\lambda_{\ell/2} - 1} \geq \cdots \geq \underset{\substack{\uparrow\\\ell-2-\nu+1}}{\lambda_{\ell-2-\nu} - 1} \geq \underset{\substack{\uparrow\\\ell-2-\nu+2}}{\frac{N}{2} - j - \nu - 1} \geq \underset{\substack{\uparrow\\\ell-2-\nu+3}}{\lambda_{\ell-1-\nu}} \geq \cdots \geq \underset{\substack{\uparrow\\\ell}}{\lambda_{\ell-2}}).
\end{aligned} \tag{5.19}$$

The rearrangement does not change the sign of the determinant (5.16). Knowing that $\lambda \in \mathbb{Y}^{(\ell-2)}_{(\ell-2)(N-\ell+2)/2}$, we now prove that λ' is the new partition (obtained in (5.19)),

$$\lambda' \in \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2}$$

i.e. where we prove

(i)

$$\begin{aligned}
\sum_i^\ell \lambda'_i &= \sum_1^{\ell-2} \lambda_i - 2\nu - (\ell - 2 - 2\nu) + \left(\frac{N}{2} - \ell + 1 + j + \nu\right) + \left(\frac{N}{2} - j - \nu - 1\right) \\
&= \frac{\ell(N - \ell)}{2},
\end{aligned}$$

and

(ii)

$$\lambda'_i + \lambda'_{\ell+1-i} = N - \ell \quad \text{for all } 1 \leq i \leq \frac{\ell}{2},$$

e.g.

$$\lambda'_i + \lambda'_{\ell+1-i} = \lambda_i - 2 + \lambda_{\ell-1-i} = N - \ell \quad \text{for } 1 \le i \le \nu$$
$$\lambda'_{\nu+1} + \lambda'_{\ell-\nu} = \left(\frac{N}{2} - \ell + 1 + j + \nu\right) + \left(\frac{N}{2} - j - \nu - 1\right) = N - \ell$$
$$\lambda'_i + \lambda'_{\ell+1-i} = \lambda_{i-1} - 1 + \lambda_{\ell-i} - 1 = N - \ell \quad \text{for } \nu + 2 \le i \le \ell/2.$$

So far, we have shown that to an arbitrary integer $j \ge 0$ and a partition

$$\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{\ell-2}) \in \mathbb{Y}^{(\ell-2)}_{(\ell-2)(N-\ell+2)/2},$$

such that the inequalities in (5.19) are strict, there corresponds a new partition

$$\lambda' = (\lambda'_1 \ge \cdots \ge \lambda'_\ell) \in \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2},$$

with λ' totally determined by (5.19). Then $\ell/2$ different choices of $\lambda \in \mathbb{Y}^{(\ell-2)}_{(\ell-2)(N-\ell+2)/2}$ and $j \ge 0$ will lead to the same sequence of numbers (5.19), as appears from the next argument.

In view of the σ-map in (5.11), it is obvious that the $(\nu+1)$th number in λ' of (5.19) gets mapped by σ into j, namely

$$\frac{N}{2} - \ell + 1 + j + \nu \longmapsto j,$$

and, in general, (5.15) holds. The 'surjectivity property' is straightforward in this description, since given a sequence $\sigma' \in \mathbb{S}^{(\ell)}_N$ you may choose j to be any of the $\ell/2$ numbers appearing in σ'; then σ is the sequence formed by the remaining numbers in order. This establishes Lemma 5.3. □

PROPOSITION 5.4. *Given positive integers N and ℓ with ℓ even and the operator*

$$Y_\beta = \begin{cases} \sum_{j\ge 0} b_j (B_{\beta+j} B_{\beta-j} - B_{\beta-j-1} B_{\beta+j+1}), & N \text{ even}, \\ \sum_{j\ge 0} b_j (B_{\beta+j+\frac12} B_{\beta-j-\frac12} - B_{\beta-j-\frac32} B_{\beta+j+\frac32}), & N \text{ odd}, \end{cases}$$

we have

$$Y_{(N/2)-\ell+1}\tau_{\ell-2} = \ell\tau_\ell.$$

Proof. The indices of b_i in the ℓth τ-function can now be expressed in terms of the σ-map as follows:

$$\tau_\ell(t) = \sum_{\lambda \in \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2}} \left(\prod_1^{\ell/2} b_{\lambda_i - i + \ell - [(N+1)/2]}\right) \mathbf{s}_\lambda(t) = \sum_{\lambda \in \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2}} \left(\prod_1^{\ell/2} b_{\sigma_i(\lambda)}\right) \mathbf{s}_\lambda(t).$$

We give the proof for N even. From (5.15), it follows at once that

$$b_j \prod_1^{(\ell-2)/2} b_{\sigma_i(\lambda)} = \prod_1^{\ell/2} b_{\sigma_i(\lambda')}. \tag{5.20}$$

Setting $Y_\beta = \sum_{i\geq 0} b_i\Gamma_i$, one computes, using Lemma 5.3, (5.20) and in $\stackrel{*}{=}$ the $\ell/2$-to-1 'surjectivity' of the maps (5.14) or (5.15),

$$\begin{aligned}
Y_{(N/2)-\ell+1}\tau_{\ell-2}(t) &= \sum_{\lambda\in\mathbb{Y}^{(\ell-2)}_{(\ell-2)(N-\ell-2)/2}} \left(\prod_1^{(\ell-2)/2} b_{\sigma_i(\lambda)}\right) Y_\beta(s_\lambda(t)) \\
&= \sum_{\lambda\in\mathbb{Y}^{(\ell-2)}_{(\ell-2)(N-\ell-2)/2}} \sum_{j\geq 0}\left(\prod_1^{(\ell-2)/2} b_{\sigma_i(\lambda)}\right) b_j\Gamma_j(s_\lambda(t)) \\
&\stackrel{*}{=} \frac{\ell}{2}\sum_{\lambda'\in\mathbb{Y}^{\ell}_{\ell(N-\ell)/2}} \prod_1^{\ell/2} b_{\sigma_i(\lambda')} 2 s_{\lambda'}(t) \\
&= \ell\tau_\ell(t),
\end{aligned}$$

ending the proof of Proposition 5.4. □

Proof of Theorem 5.1. Formula (5.6) follows at once from Propositions 5.2 and 5.4. To prove (5.7), first notice that, upon setting $I_\ell := (\ell/2)!\tau_\ell$,

$$(\mathbb{X}(t;y)\mathbb{X}(t;z)I)_\ell = y^{\ell-1}z^{\ell-2}X(t;y)X(t;z)I_{\ell-2}.$$

Then

$$\begin{aligned}
(\mathbb{Y}(t)I)_\ell &= \left(\frac{1}{(2\pi i)^2}\oint_\infty\oint_\infty \mathbb{X}(t;y)\mathbb{X}(t;z)\frac{\rho_b(y/z)\,dy\,dz}{2z(yz)^{N/2}} I\right)_\ell \\
&= \frac{1}{(2\pi i)^2}\oint_\infty\oint_\infty \frac{dy\,dz\,\rho_b(y/z)}{2z^2(yz)^{N/2-\ell+1}} X(t;y)X(t;z)I_{\ell-2} \\
&= \frac{1}{2}Y_{(N/2)-\ell+1}I_{\ell-2}, \quad \text{by definition (5.5) of } Y_\beta, \\
&= \frac{1}{2}Y_{(N/2)-\ell+1}\left(\frac{\ell-2}{2}\right)!\tau_{\ell-2} \\
&= \left(\frac{\ell}{2}\right)!\tau_\ell, \quad \text{using (5.6)} \\
&= I_\ell,
\end{aligned}$$

ending the proof of Theorem 5.1. □

EXAMPLE. *For $b_i = 2i+1$ and N even, the function $\rho_b(x)$, defined in (5.8), equals*†

$$\rho_b(x) = \sum_{i\geq 0} b_i(x^{-i-1} - x^i) = -\frac{1+x}{(1-x)^2} + x^{-1}\frac{1+x^{-1}}{(1-x^{-1})^2}. \tag{5.21}$$

The corresponding vertex operator (5.9) takes on a particularly simple form

$$Y_{(N/2)-\ell+1} = 2B^{(2)}_{N-2\ell+2} = 2\int_{\mathbb{R}} du\,\delta^{(N-2)}(u)u^{2\ell-4}X^{(2)}(u), \tag{5.22}$$

† $\rho_b(x)$ is actually a distribution!

where $\delta^{(N-2)}$ *is the* $(N-2)$*th derivative of the customary* δ*-function and where the* $B_i^{(2)}$ *are the differential operators (5.3) in* t_i,

$$B_i^{(2)} := \sum_{j\geq 0} \mathbf{s}_{i+j}(2t)\mathbf{s}_j(-2\tilde{\partial}_t),$$

given by the coefficients of the expansion in powers of z *of the vertex operator*

$$X^{(2)}(z) := \exp\left(2\sum_1^\infty t_i z^i\right)\exp\left(-2\sum_1^\infty \frac{z^{-i}}{i}\frac{\partial}{\partial t_i}\right) = \sum_{i\in\mathbb{Z}} B_i^{(2)} z^i.$$

Proof. Formula (5.21) follows immediately from the series

$$\frac{1+x}{(1-x)^2} = 1 + 3x + 5x^2 + 7x^3 + \cdots .$$

Setting, for convenience,

$$X(t; y, z) := \exp\left[\sum_1^\infty t_i(y^i + z^i)\right]\exp\left[-\sum_1^\infty \left(\frac{y^{-i}+z^{-i}}{i}\right)\frac{\partial}{\partial t_i}\right]$$

and using $X(t;y)X(t;z) = (1-(z/y))X(t;y,z)$ and $X(t;z,z) = X^{(2)}(t;z)$, one computes ($\beta = (N/2) - \ell + 1$)

$$\begin{aligned}
Y_\beta &= \frac{1}{(2\pi i)^2}\oint_\infty\oint_\infty \frac{\rho^{(e)}(y/z)}{(yz)^\beta z^2} X(t;y)X(t;z)\,dy\,dz \\
&= \frac{1}{(2\pi i)^2}\oint_\infty\oint_\infty \left(\frac{z(1+z/y)}{y(1-z/y)^2} - \frac{(1+y/z)}{(1-y/z)^2}\right)\frac{(1-z/y)}{z^2(zy)^\beta} X(t;y,z)\,dy\,dz \\
&= \frac{1}{(2\pi i)^2}\oint_\infty\oint_\infty \frac{z}{y}\frac{(1+z/y)}{(1-z/y)^2}\frac{(1-z/y)}{z^2(zy)^\beta} X(t;y,z)\,dy\,dz \\
&= \oint_\infty\left(\oint_\infty \frac{(1+z/y)}{(y-z)z(zy)^\beta} X(t;y,z)\frac{dy}{2\pi i}\right)\frac{dz}{2\pi i} \\
&= 2\oint_\infty \frac{X(t;z,z)}{z^{2\beta+1}}\frac{dz}{2\pi i} \\
&= 2\oint_\infty \frac{X^{(2)}(t;z)}{z^{2\beta+1}}\frac{dz}{2\pi i} \\
&= 2B_{2\beta}^{(2)} = 2B_{N-2\ell+2}^{(2)} = 2\int_{\mathbb{R}} du\,\delta^{(N-2)}(u)u^{2\ell-4}X^{(2)}(t;u),
\end{aligned}$$

establishing (5.22). □

6. *Duality*

PROPOSITION 6.1. *For N odd and ℓ odd, the following holds:*

$$\tilde{\tau}_\ell(t) := z^{-1} \det{}^{1/2}\left(E_{\ell,N}(t)(m_N(0) + z^2 \varepsilon_{(N+1)/2,(N+1)/2})E^\top_{\ell,N}(t)\right)$$
$$= \sum_{\lambda \in \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2}} \left(\prod_1^{[\ell/2]} b_{\lambda_i - i + \ell - [(N+1)/2]} \right) \mathbf{s}_{\lambda_1 \geq \cdots \geq \lambda_\ell}(t). \tag{6.1}$$

Then the functions

$$\tilde{\tau}_\ell(t) = (-1)^{\ell(N-\ell)/2} \left(\prod_0^{(N-3)/2} b_i \right) (\tau_{N-\ell}(-t)|_{b_i \to b_i^{-1}}), \quad \textit{for } \ell \textit{ odd}, \tag{6.2}$$

are the τ-functions $\tau_k(t)$ (in reverse order and modulo a multiplicative factor) of the Pfaff lattice for N odd and k even, with $t \mapsto -t$, and with initial condition

$$\begin{pmatrix} & O & & & b^{-1}_{(N-3)/2} \\ & & & \iddots & \\ & & b_0^{-1} & & \\ & 0 & & & \\ & -b_0^{-1} & & & \\ \iddots & & & & \\ -b^{-1}_{(N-3)/2} & & & O & \end{pmatrix}. \tag{6.3}$$

Proof. Defining k_i and $k_i^\top$ by

$$\lambda_i = k_i - \ell + i, \quad \lambda_i^\top = k_i^\top - (N-\ell) + i, \tag{6.4}$$

it is easy to see the one-to-one correspondence between

$$\mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2} \longleftrightarrow \left\{ \begin{array}{l} N-1 \geq k_1 > k_2 > \cdots > k_\ell \geq 0 \\ \quad \text{with } k_i + k_{\ell+1-i} = N-1 \text{ for } 1 \leq i \leq (\ell+1)/2 \end{array} \right\}$$

and also between

$$\mathbb{Y}^{(N-\ell)}_{\ell(N-\ell)/2} \longleftrightarrow \left\{ \begin{array}{l} N-1 \geq k_1^\top > k_2^\top > \cdots > k^\top_{N-\ell} \geq 0 \\ \quad \text{with } k_i^\top + k^\top_{N-\ell+1-i} = N-1 \text{ for } 1 \leq i \leq (N-\ell)/2 \end{array} \right\}. \tag{6.5}$$

LEMMA 6.2.

(1) *The following correspondence holds:*

$$\lambda \in \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2} \longleftrightarrow \lambda^\top \in \mathbb{Y}^{(N-\ell)}_{\ell(N-\ell)/2}. \tag{6.6}$$

(2) *For λ and $\lambda^\top$, we have the following disjoint union:*

$$\{k_1 > \cdots > k_\ell\} \cup \{k_1^\top > \cdots > k^\top_{N-\ell}\} = \{0, 1, \ldots, N-1\}. \tag{6.7}$$

Proof. Considering

$$(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell) \in \mathbb{Y}^{\ell}_{\ell(N-\ell)/2},$$

we have

$$\begin{aligned} \lambda_1^\top = \cdots = \lambda_{\lambda_\ell}^\top &= \ell \\ \lambda_{\lambda_\ell+1}^\top = \cdots = \lambda_{\lambda_{\ell-1}}^\top &= \ell - 1 \\ \lambda_{\lambda_{\ell-1}+1}^\top = \cdots = \lambda_{\lambda_{\ell-2}}^\top &= \ell - 2 \\ &\vdots \end{aligned} \tag{6.8}$$

and so, since

$$\begin{aligned} k_i = N - 1 - k_{\ell+1-i} &= N - 1 - (\lambda_{\ell+1-i} + \ell - (\ell + 1 - i)) \\ &= N - (\lambda_{\ell+1-i} + i), \end{aligned}$$

we have, on the one hand,

$$k_1 = N - \lambda_\ell - 1 > k_2 = N - \lambda_{\ell-1} - 2 > k_3 = N - \lambda_{\ell-2} - 3, \tag{6.9}$$

and, on the other hand, using (6.4) and (6.8),

$$\begin{aligned} &k_1^\top = N - 1 > k_2^\top = N - 2 > \cdots > k_\alpha^\top = N - \alpha > \cdots > k_{\lambda_\ell}^\top = N - \lambda_\ell > \\ &k_{\lambda_\ell+1}^\top = N - \lambda_\ell - 2 > \cdots > k_\beta^\top = N - \beta - 1 > \cdots > k_{\lambda_{\ell-1}}^\top = N - \lambda_{\ell-1} - 1 > \\ &k_{\lambda_{\ell-1}+1}^\top = N - \lambda_{\ell-1} - 3 > \cdots > k_\gamma^\top = N - \gamma - 2 > \cdots > k_{\lambda_{\ell-2}}^\top = N - \lambda_{\ell-2} - 2 > \cdots . \end{aligned} \tag{6.10}$$

So the gaps in (6.10) coincide with the sequence (6.9). This ends the proof of Lemma 6.2. □

One checks, using Proposition 3.2, that

$$\begin{aligned} \tilde{\tau}_\ell(t) &= \sum_{\lambda \in \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2}} \prod_1^{(\ell-1)/2} b_{\lambda_i - i + \ell - (N+1)/2}\, \mathbf{s}_\lambda(t) \\ &= (-1)^{|\lambda|} \sum_{\lambda \in \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2}} \prod_1^{(\ell-1)/2} b_{k_i - (N+1)/2} \mathbf{s}_{\lambda^\top}(-t) \\ &= (-1)^{|\lambda|} \prod_0^{(N-3)/2} b_i \sum_{\lambda \in \mathbb{Y}^{(\ell)}_{\ell(N-\ell)/2}} \left(\prod_1^{(N-\ell)/2} b_{k_i^\top - [(N+1)/2]} \right)^{-1} \mathbf{s}_{\lambda^\top}(-t), \\ &\qquad \text{using Lemma 6.2,} \\ &= (-1)^{|\lambda|} \prod_0^{(N-3)/2} b_i \sum_{\lambda^\top \in \mathbb{Y}^{(N-\ell)}_{\ell(N-\ell)/2}} \prod_1^{(N-\ell)/2} b^{-1}_{\lambda_i^\top - i + \ell - [(N+1)/2]} \mathbf{s}_{\lambda^\top}(-t) \\ &= (-1)^{\ell(N-\ell)/2} \prod_0^{(N-3)/2} b_i (\tau_{N-\ell}(-t)\big|_{b_i \to b_i^{-1}}), \end{aligned}$$

which is, using Theorem 1.1, the τ-function (modulo a constant) for the case where N is odd and $N - \ell$ even, concluding the proof of the proposition. □

7. *Examples*

7.1. *Example 1: Rectangular Jack polynomials*

PROPOSITION 7.1. *When*

$$b_i = \begin{cases} 2i+1, & \textit{for } N \textit{ even},\\ 2i+2, & \textit{for } N \textit{ odd},\end{cases} \tag{7.1}$$

then the $\tau_{2n}(t)$*'s are Jack polynomials for rectangular partitions, with* $n \le [N/2]$,

$$\begin{aligned}\tau_{2n}(t) &= pf\, m_{2n}(t)\\ &= \sum_{\lambda\in\mathbb{Y}^{(2n)}_{n(N-2n)}} \prod_1^n (k_i - k_{2n+1-i})\mathbf{s}_\lambda(t), \quad \textit{where } k_i = \lambda_i - i + 2n\\ &= J^{(1/2)}_\lambda(x)|_{t_i = 1/i\sum_k x_k^i}, \quad \textit{for the partition } \lambda = (N-2n)^n\\ &= \frac{1}{n!}\int_{\mathbb{R}^n} \Delta(z)^4 \prod_{k=1}^n \exp\left(2\sum_1^\infty t_i z_k^i\right)\delta^{(N-2)}(z_k)\,dz_k.\end{aligned} \tag{7.2}$$

Then

$$m_\ell(t) = E_{\ell,N}(t) m_N(0) E^\top_{\ell,N}(t),$$

with

$$m_N(0) = \begin{cases} \begin{pmatrix} & & & O & & & N-1\\ & & & & & N-3 & \\ & & & & \cdot^{\cdot^{\cdot}} & & \\ & & & 1 & & & \\ & & -1 & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & & \\ & -N+3 & & & & & \\ -N+1 & & & & & O & \end{pmatrix}, & \textit{for } N \textit{ even},\\[2ex] \begin{pmatrix} & & & & O & & & N-1\\ & & & & & & N-3 & \\ & & & & & \cdot^{\cdot^{\cdot}} & & \\ & & & & 2 & & & \\ & & & 0 & & & & \\ & & -2 & & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & & & \\ & -N+3 & & & & & & \\ -N+1 & & & & & & O & \end{pmatrix}, & \textit{for } N \textit{ odd},\end{cases}$$

where (setting $\tilde{\mathbf{s}}_n(t) = \mathbf{s}_n(2t)$*)*

$$
\begin{aligned}
m_N(t) &= ((j-i)\tilde{\mathbf{s}}_{N-i-j-1})_{0\le i,j\le N-1} \\
&= \begin{pmatrix}
0 & \tilde{\mathbf{s}}_{N-2} & 2\tilde{\mathbf{s}}_{N-3} & \dots & (N-2)\tilde{\mathbf{s}}_1 & N-1 \\
-\tilde{\mathbf{s}}_{N-2} & 0 & \tilde{\mathbf{s}}_{N-4} & \dots & N-3 & \\
-2\tilde{\mathbf{s}}_{N-3} & -\tilde{\mathbf{s}}_{N-4} & 0 & \iddots & & \\
 & & & 1 & & \\
\vdots & \vdots & -1 & & & \\
 & & \iddots & & & \\
-(N-2)\tilde{\mathbf{s}}_1 & -N+3 & & & & \\
 & & & & O & \\
-N+1 & & & & &
\end{pmatrix}, \quad \textit{for } N \textit{ even,} \\
&= \begin{pmatrix}
0 & \tilde{\mathbf{s}}_{N-2} & 2\tilde{\mathbf{s}}_{N-3} & \dots & (N-2)\tilde{\mathbf{s}}_1 & N-1 \\
-\tilde{\mathbf{s}}_{N-2} & 0 & \tilde{\mathbf{s}}_{N-4} & \dots & N-3 & \\
-2\tilde{\mathbf{s}}_{N-3} & -\tilde{\mathbf{s}}_{N-4} & 0 & \iddots & & \\
 & & & 2 & & \\
 & & 0 & & & \\
\vdots & \vdots & -2 & & & \\
 & \iddots & & & & \\
-(N-2)\tilde{\mathbf{s}}_1 & -N+3 & & & & \\
 & & & & O & \\
-N+1 & & & & &
\end{pmatrix}, \quad \textit{for } N \textit{ odd.}
\end{aligned}
\tag{7.3}
$$

Proof. Setting

$$t_k = \frac{1}{k}\sum_{i=1}^{\ell} x_i^k$$

we have

$$
\begin{aligned}
\exp\left(\beta \sum_{1}^{\infty} t_k z^k\right) &= \exp\left(\beta \sum_{i=1}^{\ell} \sum_{k=1}^{\infty} \frac{1}{k}(x_i z)^k\right) \\
&= \prod_{i=1}^{\ell}\left(\exp\left(\sum_{k=1}^{\infty}\frac{1}{k}(x_i z)^k\right)\right)^{\beta} \\
&= \prod_{i=1}^{\ell}(1-x_i z)^{-\beta}.
\end{aligned}
$$

According to Awata *et al* [**4**], the Jack polynomials for rectangular partitions s^n have the following integral representation (for connections with random matrix theory, see [**10**]):

$$\begin{aligned} cJ^{1/\beta}_{s^n} &= \oint_{z_1=\cdots=z_n=0} |\Delta(z)|^{2\beta} \prod_{j=1}^n z_j^{-(n-1)\beta-s} \prod_{i=1}^{\ell}(1-x_i z_j)^{-\beta} \frac{dz_j}{2\pi i z_j} \\ &= \oint_{z_1=\cdots=z_n=0} |\Delta(z)|^{2\beta} \prod_{j=1}^n z_j^{-(n-1)\beta-s} \exp\left(\beta \sum_{k=1}^{\infty} t_k z_j^k\right) \frac{dz_j}{2\pi i z_j} \\ &= c_n \int_{\mathbb{R}^n} |\Delta_n(z)|^{2\beta} \prod_{j=1}^n \exp\left(\beta \sum_{k=1}^{\infty} t_k z_j^k\right) \delta^{s+(n-1)\beta}(z_j)\, dz_j. \end{aligned}$$

Setting $\beta = 2,\ s = N - 2n$ and $2 \le 2n \le N$ in the last integral, we have, using the standard derivation of the 'symplectic' matrix integral (see [**2**]),

$$\begin{aligned} &\frac{1}{n!}\int_{\mathbb{R}^n} \Delta_n^4(z) \prod_{k=1}^n \exp\left(2\sum_{k=1}^{\infty} t_k z_j^k\right) \delta^{N-2}(z_j)\, dz_j \\ &\quad = pf\left(\int_{\mathbb{R}} \{y^k, y^\ell\} \exp\left(2\sum_{i=1}^{\infty} t_i y^i\right) \delta^{(N-2)}(y)\, dy\right)_{0\le k,\ell\le 2n-1} \\ &\quad = pf\left((k-\ell)\int_{\mathbb{R}} y^{k+\ell-1} \exp\left(2\sum_{i=1}^{\infty} t_i y^i\right) \delta^{(N-2)}(y)\, dy\right)_{0\le k,\ell\le 2n-1} \\ &\quad = pf\left((k-\ell)\sum_{i=0}^{\infty} \tilde{\mathbf{s}}_i(t) \int_{\mathbb{R}} y^{i+k+\ell-1} \delta^{(N-2)}(y)\, dy\right) \\ &\quad = pf((-1)^{N-2}(N-2)!(k-\ell)\tilde{\mathbf{s}}_{N-1-k-\ell}(t))_{0\le k,\ell\le 2n-1} \\ &\quad = c_{N,n}\, pf((\ell-k)\tilde{\mathbf{s}}_{N-1-k-\ell}(t))_{0\le k,\ell\le 2n-1}. \end{aligned} \tag{7.4}$$

In order to find the initial condition $m_N(0)$, one sets $t = 0$ in the last matrix appearing in (7.3), to yield

$$((\ell-k)\tilde{\mathbf{s}}_{N-1-k-\ell}(0))_{0\le k,\ell\le N-1}.$$

All entries of this matrix vanish, except the antidiagonal, from which one reads off the b_i's.

For N even, we have $b_i = 2i + 1$ and thus

$$\begin{aligned} b_{\lambda_i-i+\ell-N/2} &= 2\left(\lambda_i - i + \ell - \frac{N}{2}\right) + 1 \\ &= \lambda_i - \lambda_{\ell+1-i} - 2i + \ell + 1 \quad \text{using } \lambda_i + \lambda_{\ell+1-i} = N - \ell \\ &= k_i - k_{\ell+1-i} \quad \text{using } k_i = \lambda_i - i + 2n. \end{aligned}$$

For N odd, we have $b_i = 2i + 2$ and thus

$$\begin{aligned} b_{\lambda_i-i+\ell-(N+1)/2} &= 2\left(\lambda_i - i + \ell - \frac{N+1}{2}\right) + 2 \\ &= \lambda_i - \lambda_{\ell+1-i} - 2i + \ell + 1 \quad \text{using } \lambda_i + \lambda_{\ell+1-i} = N - \ell \\ &= k_i - k_{\ell+1-i}, \end{aligned}$$

ending the proof of Proposition 7.1. □

Example. For $n = 4$ and $b_0 = 1,\ b_1 = 3$, the solution to the system (1.8) is given by

$$L = \frac{1}{(t_2 + t_1^2)^2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ t_1 & 2(t_2 - t_1^2) & -\sqrt{3}t_1 & 0 \\ \frac{2}{\sqrt{3}}(t_2 - t_1^2) & -\frac{16}{\sqrt{3}}t_1 t_2 & -2(t_2 - t_1^2) & 1 \\ -\sqrt{3}t_1 & -2\sqrt{3}(t_2 - t_1^2) & 3t_1 & 0 \end{pmatrix}. \tag{7.5}$$

Indeed

$$m_4 = \begin{pmatrix} 0 & -\tilde{\mathbf{s}}_2 & -2\tilde{\mathbf{s}}_1 & -3 \\ \tilde{\mathbf{s}}_2 & 0 & -1 & 0 \\ 2\tilde{\mathbf{s}}_1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix} = Q^{-1}\ J\ Q^{\top -1},$$

with

$$Q = D \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -2\tilde{\mathbf{s}}_1 & \tilde{\mathbf{s}}_2 & 0 \\ 0 & -3 & 0 & \tilde{\mathbf{s}}_2 \end{pmatrix}$$

where

$$D = \operatorname{diag}\left(\frac{1}{\sqrt{\tilde{\mathbf{s}}_2}}, \frac{1}{\sqrt{\tilde{\mathbf{s}}_2}}, \frac{1}{\sqrt{3\tilde{\mathbf{s}}_2}}, \frac{1}{\sqrt{3\tilde{\mathbf{s}}_2}}\right).$$

Therefore

$$\begin{aligned} L &= Q\Lambda Q^{-1} \\ &= \frac{1}{\tilde{\mathbf{s}}_2^2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2\tilde{\mathbf{s}}_1 & 4(\tilde{\mathbf{s}}_2 - \tilde{\mathbf{s}}_1^2) & -2\sqrt{3}\tilde{\mathbf{s}}_1 & 0 \\ \frac{4}{\sqrt{3}}(\tilde{\mathbf{s}}_2 - \tilde{\mathbf{s}}_1^2) & -\frac{8\tilde{\mathbf{s}}_1}{\sqrt{3}}(2\tilde{\mathbf{s}}_2 - \tilde{\mathbf{s}}_1^2) & -4(\tilde{\mathbf{s}}_2 - \tilde{\mathbf{s}}_1^2) & 1 \\ -\frac{6}{\sqrt{3}}\tilde{\mathbf{s}}_1 & -\frac{12}{\sqrt{3}}(\tilde{\mathbf{s}}_2 - \tilde{\mathbf{s}}_1^2) & 6\tilde{\mathbf{s}}_1 & 0 \end{pmatrix} \end{aligned}$$

leads to formula (7.5).

7.2. *Example 2: Two-column Jack polynomials*

PROPOSITION 7.2. *For N even, choosing*

$$\begin{cases} b_0 = \cdots = b_{(p/2)-1} = 0 \\ b_{(p/2)+k} = \dfrac{(1-\alpha)_k (p+1)_k}{k!(\alpha + p + 1)_k}, \quad \text{for } k = 0, \ldots, \dfrac{N-2-p}{2}, \end{cases} \tag{7.6}$$

one finds the most general two-row Jack polynomial for τ_2, for arbitrary α,

$$\begin{aligned}\tau_2(t) &= pfm_2(t)\\ &= J^{(1/\alpha)}_{((N+p-2)/2,(N-p-2)/2)}(t/\alpha)\\ &= c\oint \frac{dx}{2\pi i}\frac{dy}{2\pi i}\frac{(y-x)^{2\alpha}}{(xy)^{\alpha+(N/2)}}\\ &\quad\times\exp\left[\sum_1^\infty t_i(x^i+y^i)\right]\left(\frac{x}{y}\right)^{p/2}{}_2F_1\left(\alpha,-p;1-\alpha-p;\frac{y}{x}\right)\end{aligned} \tag{7.7}$$

and, for general $\ell \geq 2$,

$$\begin{aligned}\tau_\ell(t) &= \frac{2c}{\ell!!}\oint \frac{(z_2-z_1)^{2\alpha-1}}{z_2(z_1z_2)^{\alpha-1}}\left(\frac{z_1}{z_2}\right)^{p/2}{}_2F_1\left(\alpha,-p;1-\alpha-p;\frac{z_2}{z_1}\right)\\ &\quad\times\frac{\prod_{i=2}^{\ell/2}\rho(z_{2i}/z_{2i-1})}{\prod_{i=1}^{\ell/2} z_{2i-1}^{(N/2)-2i+3}z_{2i}^{(N/2)-2i+1}}\prod_{1\leq i<j\leq\ell}\left(1-\frac{z_i}{z_j}\right)\prod_{j=1}^{\ell}\exp\left(\sum_{k=1}^\infty t_k z_j^k\right)\frac{dz_j}{2\pi i},\end{aligned} \tag{7.8}$$

where

$$\rho(x) = \sum_{i=0}^{(N-2)/2} b_i(x^{-i-1}-x^i). \tag{7.9}$$

Proof. According to a formula by Stanley [**9**], two-column Jack polynomials can be expressed as a linear combination of two-column Schur polynomials. So, setting in the end $2s = N-2-p$, we have

$$\begin{aligned}\tau_2(t) &= \sum_{k=0}^{(N-2)/2} b_k\mathbf{s}_{[(N-2)/2]+k,[(N-2)/2]-k}(t), \quad \text{with } b_k \text{ as in (7.6)},\\ &= \sum_{k=p/2}^{(N-2)/2}\frac{(1-\alpha)_{k-p/2}(p+1)_{k-p/2}(-1)^{N-2}}{(k-p/2)!(\alpha+p+1)_{k-p/2}}\mathbf{s}_{[(N-2)/2]+k,[(N-2)/2]-k}(t)\\ &= \sum_{k=p/2}^{(N-2)/2}\frac{(1-\alpha)_{k-p/2}(p+1)_{k-p/2}}{(k-p/2)!(\alpha+p+1)_{k-p/2}}\mathbf{s}_{2^{[(N-2)/2-k]}1^{2k}}(-t)\\ &= \sum_{k=0}^{(N-p-2)/2}\frac{(1-\alpha)_k(p+1)_k}{k!(\alpha+p+1)_k}\mathbf{s}_{2^{[(N-2)/2-k-p/2]}1^{2k+p}}(-t)\\ &= \sum_{k=0}^{s}\frac{(1-\alpha)_k(p+1)_k}{k!(\alpha+p+1)_k}\mathbf{s}_{2^{s-k}1^{2k+p}}(-t)\\ &= J^{(\alpha)}_{2^s1^p}(-t) \quad \text{(Stanley's formula)}\\ &= J^{(1/\alpha)}_{(p+s,s)}(t/\alpha) \quad \text{(using duality)},\end{aligned}$$

showing that any two-row Jack polynomial can serve as the Pfaff τ-function τ_2.

According to [**4**], Jack polynomials also have an integral representation, and so $\tau_2(t)$ can also be expressed as

$$\begin{aligned}
\tau_2(t) &= J^{(1/\alpha)}_{(p+s,s)}(t/\alpha)\\
&= c' \oint \frac{dx}{2\pi i x}\frac{dy}{2\pi i y}\frac{dz}{2\pi i z}\frac{(x-y)^{2\alpha}(xy)^{-s}z^{-p}}{((x-z)(y-z))^{\alpha}}\exp\left[\sum_1^\infty t_i(x^i+y^i)\right]\\
&= c' \oint \frac{dx}{2\pi i x}\frac{dy}{2\pi i y}(x-y)^{2\alpha}(xy)^{-s}\\
&\quad \times \exp\left[\sum_1^\infty t_i(x^i+y^i)\right] D_z^p((x-z)(y-z))^{-\alpha}\Big|_{z=0}\\
&= c'(\alpha)_p \oint \frac{dx}{2\pi i x}\frac{dy}{2\pi i y}\frac{(x-y)^{2\alpha}}{(xy)^{\alpha+s}y^p}\\
&\quad \times \exp\left[\sum_1^\infty t_i(x^i+y^i)\right] {}_2F_1\left(\alpha,-p;1-\alpha-p;\frac{y}{x}\right),
\end{aligned}$$

where we used the identity

$$\begin{aligned}
&D_z^p((x-z)(y-z))^{-\alpha}|_{z=0}\\
&\quad = (xy)^{-\alpha} D_z^p\left(\left(1-\frac{z}{x}\right)^{-\alpha}\left(1-\frac{z}{y}\right)^{-\alpha}\right)\Big|_{z=0}\\
&\quad = (xy)^{-\alpha} D_z^p\left(\sum_{k,\ell=0}^\infty \frac{(\alpha)_k(\alpha)_\ell}{k!\,\ell!}\frac{z^{k+\ell}}{x^k y^\ell}\right)\Big|_{z=0}\\
&\quad = p!(xy)^{-\alpha}\sum_{k+\ell=p}\frac{(\alpha)_k(\alpha)_\ell}{k!\,\ell!}x^{-k}y^{-\ell}\\
&\quad = p!(xy)^{-\alpha}y^{-p}\sum_{k=0}^p \frac{(\alpha)_k(\alpha)_{p-k}}{k!(p-k)!}\left(\frac{y}{x}\right)^k\\
&\quad = (\alpha)_p(xy)^{-\alpha}y^{-p}\sum_{k=0}^p \frac{(\alpha)_k(-p)_k}{k!(1-\alpha-p)_k}\left(\frac{y}{x}\right)^k\\
&\qquad \text{using } \frac{p!(\alpha)_{p-k}}{(p-k)!(\alpha)_p} = \frac{(-p)_k}{(1-\alpha-p)_k}\\
&\quad = (\alpha)_p(xy)^{-\alpha}y^{-p}\,{}_2F_1\left(\alpha,-p;1-\alpha-p;\frac{y}{x}\right).
\end{aligned}$$

This proves identity (7.7).

Applying Theorem 1.3, we find the higher τ_ℓ's, by applying the integrated vertex operator

$$\begin{aligned}
&Y_{[(N-2)/2]-2j}(t)\\
&\quad = \frac{1}{(2\pi i)^2}\oint_\infty\oint_\infty X(t;z_{2j+2})X(t;z_{2j+1})\frac{\rho_b(z_{2j+2}/z_{2j+1})\,dz_{2j+2}\,dz_{2j+1}}{z_{2j+1}^2(z_{2j+1}z_{2j+2})^{[(N-2)/2]-2j}}, \quad (7.10)
\end{aligned}$$

for $j = 1, 2, \ldots, (\ell - 2)/2$ to τ_2 (see formula (7.7)); so, one finds†

$$\tau_\ell = \frac{2}{\ell!!} Y_{(N/2)-\ell+1} \cdots Y_{(N/2)-5} Y_{(N/2)-3} \tau_2$$

$$= \frac{2c'(\alpha)_p}{\ell!!} \oint \frac{(z_2 - z_1)^{2\alpha}}{(z_1 z_2)^{\alpha+(N/2)}} \left(\frac{z_1}{z_2}\right)^{p/2} {}_2F_1\left(\alpha, -p; 1-\alpha-p; \frac{z_2}{z_1}\right)$$

$$\times \frac{\rho(z_\ell/z_{\ell-1}) \cdots \rho(z_4/z_3)}{(z_3 z_5 \cdots z_{\ell-1})^2 (z_3 z_4)^{(N/2)-3} (z_5 z_6)^{(N/2)-5} \cdots (z_{\ell-1} z_\ell)^{(N/2)-\ell+1}}$$

$$\times X(t; z_\ell) X(t; z_{\ell-1}) \cdots X(t; z_4) X(t; z_3) \exp\left[\sum_1^\infty t_k (z_1^k + z_2^k)\right] \prod_{j=1}^{\ell} \frac{dz_j}{2\pi i}$$

$$= \frac{2c'(\alpha)_p}{\ell!!} \oint \frac{(z_2 - z_1)^{2\alpha} z_1^2 (z_1 z_2)^{N/2-1}}{(z_1 z_2)^{\alpha+(N/2)}} \left(\frac{z_1}{z_2}\right)^{p/2} {}_2F_1\left(\alpha, -p; 1-\alpha-p; \frac{z_2}{z_1}\right)$$

$$\times \left(1 - \frac{z_1}{z_2}\right)^{-1} \frac{\rho(z_\ell/z_{\ell-1}) \cdots \rho(z_4/z_3)}{\prod_1^{\ell/2} z_{2i-1}^2 (z_{2i-1} z_{2i})^{(N/2)-2i+1}} \prod_{1 \le i < j \le \ell} \left(1 - \frac{z_i}{z_j}\right)$$

$$\times \prod_{j=1}^{\ell} \exp\left(\sum_1^\infty t_k z_j^k\right) \frac{dz_j}{2\pi i}$$

$$= \frac{2c'(\alpha)_p}{\ell!!} \oint \frac{(z_2 - z_1)^{2\alpha-1}}{z_2 (z_1 z_2)^{\alpha-1}} \left(\frac{z_1}{z_2}\right)^{p/2} {}_2F_1\left(\alpha, -p; 1-\alpha-p; \frac{z_2}{z_1}\right)$$

$$\times \frac{\prod_{i=2}^{\ell/2} \rho(z_{2i}/z_{2i-1})}{\prod_{i=1}^{\ell/2} z_{2i-1}^{(N/2)-2i+3} z_{2i}^{(N/2)-2i+1}} \prod_{1 \le i < j \le \ell} \left(1 - \frac{z_i}{z_j}\right) \prod_{j=1}^{\ell} \exp\left(\sum_{k=1}^\infty t_k z_j^k\right) \frac{dz_j}{2\pi i},$$

establishing formula (7.8). □

7.3. *Alternative formula.* The following formula has the advantage of being more symmetric, but the disadvantage of having many more integrations:

$$\tau_\ell(t) = \oint \prod_{i=1}^{\ell} \prod_{j=1}^{i} \frac{dz_j^{(i)}}{z_j^{(i)}}$$

$$\times \prod_{i=1}^{\ell} \exp\left[\sum_1^\infty t_k (z_i^{(\ell)})^{-k}\right] \frac{\prod_{k=1}^{\ell} \prod_{\substack{1 \le i, j \le k \\ i \ne j}} (1 - (z_i^{(k)}/z_j^{(k)}))}{\prod_{k=1}^{\ell-1} \prod_{\substack{1 \le i \le k+1 \\ 1 \le j \le k}} (1 - (z_i^{(k+1)}/z_j^{(k)}))} K_{N,p,\ell}(Z)$$

† Replacing x, y in τ_2 with z_1, z_2.

with

$$K_{N,p,\ell} = \frac{\left(\prod_{j=1}^{\ell} z_j^{(\ell)}\right)^{(N-p)/2-1} \left(\prod_{j=1}^{\ell/2} z_j^{(\ell/2)}\right)^{p+1}}{\prod_{i=1}^{\ell-1}\prod_1^i z_j^{(i)}}$$

$$\times \prod_{i=1}^{\ell/2} {}_2F_1\left(1-\alpha,\, p+1;\, 1+\alpha+p;\, \frac{\prod_{j=1}^{i} z_j^{(i)} \prod_{j=1}^{\ell-i} z_j^{(\ell-i)}}{\prod_{j=1}^{i-1} z_j^{(i-1)} \prod_{j=1}^{\ell+i-i} z_j^{(\ell+1-i)}}\right).$$

Acknowledgements. MA acknowledges the support of grant #DMS-01-00782 from the National Science Foundation. VBK acknowledges grants from FSR, Université de Louvain, Belgium and EPSRC, UK. PVM acknowledges grants from NATO, FNRS and the Francqui Foundation. This work was carried out whilst VBK was visiting the University of Louvain and Brandeis University. We thank Michael Kleber for useful discussions and insights.

References

[1] M. Adler, T, Shiota and P. van Moerbeke. Pfaff τ-functions. *Math. Annal.* **322** (2002), 423–476 (arXiv: solv-int/9909010).

[2] M. Adler and P. van Moerbeke. Toda versus Pfaff lattice and related polynomials. *Duke Math. J.* **112** (2002) (arXiv: solv-int/9912008).

[3] M. Adler, E. Horozov and P. van Moerbeke. The Pfaff lattice and skew-orthogonal polynomials. *Int. Math. Res. Notices* **11** (1999), 569–588.

[4] H. Awata, Y. Matsuo, S. Odake and J. Shiraishi. Excited states of the Calogero-Sutherland model and singular vectors of the W_N algebra. *Nucl. Phys.* B **449** (1995), 347–374.

[5] L. Dickey. *Soliton Equations and Integrable Systems*. World Scientific, Singapore, 1991.

[6] V. B. Kuznetsov and E. K. Sklyanin. Separation of variables in A_2 type Jack polynomials. *RIMS Kokyuroku* **919** (1995), 27–34.

[7] V. B. Kuznetsov and E. K. Sklyanin. Factorisation of Macdonald polynomials. *Symmetries and Integrability of Difference Equations (London Mathematical Society Lecture Note Series, 255)*. Eds. P. A. Clarkson and F. W. Nijhoff. Cambridge University Press, Cambridge, 1999, pp. 370–384.

[8] I. G. MacDonald. *Symmetric Functions and Hall Polynomials*. Clarendon Press, Oxford, 1995.

[9] R. P. Stanley. Some combinatorial properties of Jack symmetric functions. *Adv. Math.* **77** (1989), 76–115.

[10] P. van Moerbeke. Integrable lattices: random matrices and random permutations. *Random Matrices and Their Applications (Mathematical Sciences Research Institute Publications, 40)*. Cambridge University Press, Cambridge, 2001, pp. 321–406.

Summary of Comments on 24611e.dvi

Page: 14

Sequence number: 1
Author: Author Query
Date: 19/06/02 16:14:10
Type: Note
Au: equation (3.5).
Is this part for N odd only? Please clarify.

Page: 22

Sequence number: 1
Author: Author Query
Date: 19/06/02 16:15:40
Type: Note
Au: Please define `KP'.

Page: 23

Sequence number: 1
Author: Author Query
Date: 19/06/02 16:15:12
Type: Note
Au: Is `vectors' OK?

Page: 26

Sequence number: 1
Author: Author Query
Date: 19/06/02 16:16:25
Type: Note
Au: Is `mapping' OK, or do you mean `maps'?

Page: 29

Sequence number: 1
Author: Author Query
Date: 19/06/02 16:17:26
Type: Note
Au: We have ended the list here. Is this OK?

Page: 30

Sequence number: 1
Author: Author Query
Date: 19/06/02 16:18:12
Type: Note
Au: Is Example OK in `theorem' style?

Page: 41

Sequence number: 1
Author: Author Query
Date: 19/06/02 16:19:06
Type: Note
Au: ref. [2].
Please give page numbers.