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Recursion Relations for Unitary Integrals, Combinatorics and the Toeplitz Lattice

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Abstract: In a discussion in spring 2001, Alexei Borodin showed us recursion relations for the Toeplitz determinants going with the symbols $e^{t(z+z^{-1})}$ and $(1-\xi z)^{\alpha}\left(1-\xi z^{-1}\right)^{\beta}$. Borodin obtained these relations using Riemann-Hilbert methods; see the recent work of Borodin [5] and Baik [4]. The nature of Borodin's recursion relations pointed towards the Toeplitz lattice and its Virasoro algebra, introduced by us in [3]. In this paper, we take the Toeplitz lattice and Virasoro algebra approach for a fairly large class of symbols, leading to a systematic way of generating recursion relations. The latter are very naturally expressed in terms of the L-matrices appearing in the Toeplitz lattice equations. As a surprise, we find, compared to Borodin's, a different set of relations, except for the 3-step relations associated with the symbol $e^{t(z+z^{-1})}$. The Painlevé analysis of the Toeplitz lattice enables us to show the "singularity confinement" for these recursion relations.

Contents

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0. Introduction and Main Results

The weight

$$\rho(z) := e^{P_1(z)+P_2(z^{-1})}z^{\gamma}(1-d_1z)^{\gamma_1'}(1-d_2z)^{\gamma_2'}(1-d_1^{-1}z^{-1})^{\gamma_1''}(1-d_2^{-1}z^{-1})^{\gamma_2''} \tag{0.0.1}$$

with

$$P_1(z) := \sum_1^{N_1}\frac{u_iz^i}{i} \text{ and } P_2(z) := \sum_1^{N_2}\frac{u_{-i}z^i}{i}, \tag{0.0.2}$$

has a natural involution

$$\tilde{\ } : z \leftrightarrow z^{-1}, \tag{0.0.3}$$

which induces an involution on the following quantities:

$$\tilde{\ } : P_1(z) \leftrightarrow P_2(z^{-1}),\ u_i \leftrightarrow u_{-i},\ N_1 \leftrightarrow N_2,\ \gamma \leftrightarrow -\gamma, d_i \leftrightarrow d_i^{-1},\ \gamma_i' \leftrightarrow \gamma_i''. \tag{0.0.4}$$

The multiple integral below is known to be expressible, both, as the determinant of a Toeplitz matrix and as an integral over the group $U(n)$,

$$\begin{aligned} I_n^{(\varepsilon)} &:= \frac{1}{n!}\int_{(S^1)^n}|\Delta_n(z)|^2\prod_{k=1}^{n}\left(z_k^{\varepsilon}\rho(z_k)\frac{dz_k}{2\pi i z_k}\right)\\ &= \det\left(\int_{S^1} z^{\varepsilon+k-l}\rho(z)\frac{dz}{2\pi i z}\right)_{1\le k,l\le n}\\ &= \int_{U(n)}\det\left(U^{\varepsilon}\rho(U)\right)dU, \end{aligned} \tag{0.0.5}$$

which for some special choices of ρ has an interesting interpretation in terms of random permutations; for that matter, look at the examples in Sect. 4. Consider the basic variables, with $I_n := I_n^{(0)},\ I_n^{\pm} = I_n^{(\pm 1)}$,

$$x_n = (-1)^n\frac{I_n^+}{I_n},\quad y_n = (-1)^n\frac{I_n^-}{I_n} \text{ and } v_n = 1 - x_ny_n = \frac{I_{n-1}I_{n+1}}{I_n^2}. \tag{0.0.6}$$

See Sect. 1.1 for explanations. Then the basic object

$$I_n = \int_{U(n)} \det(\rho(U))\, dU = \det\left(\int_{S^1} z^{k-l}\rho(z)\frac{dz}{2\pi i z}\right)_{1\le k,l\le n},$$

which appears in several problems of random words and permutations, is obtained from the x_n, y_n and I_1, by means of the formula

$$I_n = I_1^n \prod_1^{n-1}(1 - x_i y_i)^{n-i}. \tag{0.0.7}$$

The following matrices, intimately related to the Toeplitz lattice, will play an important role in this work:

$$L_1 := \begin{pmatrix} -x_1y_0 & 1-x_1y_1 & 0 & 0 & \\ -x_2y_0 & -x_2y_1 & 1-x_2y_2 & 0 & \\ -x_3y_0 & -x_3y_1 & -x_3y_2 & 1-x_3y_3 & \\ -x_4y_0 & -x_4y_1 & -x_4y_2 & -x_4y_3 & \\ & & & & \ddots \end{pmatrix} \tag{0.0.8}$$

and

$$L_2 := \begin{pmatrix} -x_0y_1 & -x_0y_2 & -x_0y_3 & -x_0y_4 & \\ 1-x_1y_1 & -x_1y_2 & -x_1y_3 & -x_1y_4 & \\ 0 & 1-x_2y_2 & -x_2y_3 & -x_2y_4 & \\ 0 & 0 & 1-x_3y_3 & -x_3y_4 & \\ & & & & \ddots \end{pmatrix}. \tag{0.0.9}$$

The Toeplitz lattice and its relation to the 2-Toda lattice will be discussed in Sect. 1.1.

Define the matrices, depending on the positive integer $n \ge 1$, and the exponents γ, γ_i' and γ_i'' in (0.0.1),

$$\begin{aligned} \mathcal{L}_1^{(n)} &:= (aI + bL_1 + cL_1^2)P_1'(L_1) + c(n + \gamma_1' + \gamma_2' + \gamma)L_1, \\ \mathcal{L}_2^{(n)} &:= (cI + bL_2 + aL_2^2)P_2'(L_2) + a(n + \gamma_1'' + \gamma_2'' - \gamma)L_2, \end{aligned} \tag{0.0.10}$$

and depending on arbitrary parameters a, b, c. The involution $\tilde{}$, defined in (0.0.3) and (0.0.4) induces involutions

$$I_n \leftrightarrow I_n,\ I_n^+ \leftrightarrow I_n^-, x_n \leftrightarrow y_n,\ a \leftrightarrow c,\ b \leftrightarrow b,\ \text{and so}\ L_1 \leftrightarrow L_2^\top,\ \mathcal{L}_1^{(n)} \leftrightarrow \mathcal{L}_2^{(n)\top}. \tag{0.0.11}$$

Also note that (self-dual case)

$$\rho(z) = \rho(z^{-1})\ \text{implies}\ x_n = y_n,\ L_1 = L_2^\top,\ \mathcal{L}_1^{(n)} = \mathcal{L}_2^{(n)\top}. \tag{0.0.12}$$

Given a matrix $A(n)$ containing explicitly the parameter n, the "*discrete derivative*" ∂_n is defined as

$$\partial_n A(n)_{nn} := A(n+1)_{n+1,n+1} - A(n)_{nn}. \tag{0.0.13}$$

Rational relations. In Theorem 0.1, we show the polynomial relationships between consecutive (x_i, y_i)'s. When the degrees of P_1 and P_2 differ by at most one, they actually lead to inductive rational relations, as is stated in Theorem 0.2 from which I_n is rationally generated via (0.0.7). These relations are obtained by observing that the multiple integral (0.0.5) satisfy the Toeplitz lattice and an sl$(2, \mathbb{R})$-set of Virasoro relations in the u_i-variables; see [3].

Theorem 0.1. *For the weight (0.0.1), the vectors* $(x_k)_{k\geq 1}$ *and* $(y_k)_{k\geq 1}$ *satisfy two finite difference relations and their duals* $\tilde{\ }$, *involving a finite number of steps:*

- *Case 1. When* $d_1, d_2, d_1 - d_2, |\gamma_1'| + |\gamma_1''|, \ |\gamma_2'| + |\gamma_2''| \neq 0$ *in the weight (0.0.1), then the relations are*

$$\partial_n(\mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)})_{n,n} + (cL_1 - aL_2)_{nn} = 0 \tag{0.0.14}$$

$$\partial_n(v_n\mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)})_{n+1,n} + \left(cL_1^2 + bL_1\right)_{n+1,n+1} - C = 0, \tag{0.0.15}$$

for all $n \geq 1$, *and where*

$$a = (d_1\, d_2)^{-1/2}, \ b = -\left(\frac{d_1}{d_2}\right)^{1/2} - \left(\frac{d_2}{d_1}\right)^{1/2}, \ c = (d_1\, d_2)^{1/2}.$$

C is a constant independent of n, thus expressible in terms of the initial value:

$$C := (v_1\mathcal{L}_1^{(1)} - \mathcal{L}_2^{(1)})_{2,1} + \left(cL_1^2 + bL_1\right)_{1,1}. \tag{0.0.16}$$

Only the first relation is self-dual for the involution $\tilde{\ }$.

- *Case 2. When* $d_1 \neq 0, \gamma_1' \neq 0, \ \gamma_2' = \gamma_1'' = \gamma_2'' = d_2 = 0$, *we may rescale z so that* $d_1 = -1$ *and so*

$$\rho(z) = z^\gamma (1+z)^{\gamma_1'} e^{P_1(z)+P_2(z^{-1})}.$$

Then the same equations (0.0.14), (0.0.15) and their duals are satisfied, where a, b, c *can be chosen in two different ways, one being the dual of the other, namely*

$$(a, b, c) = (1, 1, 0) \ \text{ or } \ (a, b, c) = (0, 1, 1).$$

- *Case 3.* $d_1 = d_2 = \gamma_1' = \gamma_2' = \gamma_1'' = \gamma_2'' = 0$. *Then*

$$\rho(z) := z^\gamma e^{P_1(z)+P_2(z^{-1})}$$

and a, b, c *can be chosen totally arbitrary in the above relations. Here it will be more advantageous to pick different relations, both of which are polynomials, dual to each other,* $(n \geq 1)$

$$\begin{cases} \Gamma_n(x, y) := \dfrac{v_n}{y_n}\begin{pmatrix} -\left(L_1P_1'(L_1)\right)_{n+1,n+1} - \left(L_2P_2'(L_2)\right)_{n,n} \\ \\ +(P_1'(L_1))_{n+1,n} + (P_2'(L_2))_{n,n+1} \end{pmatrix} + nx_n = 0, \\ \\ \tilde{\Gamma}_n(x, y) := \dfrac{v_n}{x_n}\begin{pmatrix} -\left(L_1P_1'(L_1)\right)_{n,n} - \left(L_2P_2'(L_2)\right)_{n+1,n+1} \\ \\ +(P_1'(L_1))_{n+1,n} + (P_2'(L_2))_{n,n+1} \end{pmatrix} + ny_n = 0. \end{cases} \tag{0.0.17}$$

Theorem 0.2. *Requiring $N_1 = N_2$ or $N_2 \pm 1$ in the weight (0.0.1), the x_n and y_n's can be expressed rationally in terms of lower x's and y's, and thus, from (0.0.7), I_n can be expressed in the terms of the x's and y's. To be precise,*

- *Case 1 leads to two inductive* $\underline{\text{rational}}$ *$N_1 + N_2 + 4$-step relations,*

$$\begin{aligned} x_n &= F_n(x_{n-1}, y_{n-1}, \dots, x_{n-N_1-N_2-3}, y_{n-N_1-N_2-3}),\\ y_n &= G_n(x_{n-1}, y_{n-1}, \dots, x_{n-N_1-N_2-3}, y_{n-N_1-N_2-3}). \end{aligned}$$

- *Case 2 leads to two inductive* $\underline{\text{rational}}$ *N_1+N_2+3-step relations (0.0.14) and (0.0.15), such that*[1]

$$\begin{aligned} &\textit{when } N_1 = N_2 \textit{ or } N_1 = N_2 + 1, \textit{ use } (a, b, c) = (1, 1, 0),\\ &\textit{when } N_2 = N_1 \textit{ or } N_2 = N_1 + 1, \textit{ use } (a, b, c) = (0, 1, 1). \end{aligned}$$

Thus, we find rational functions F_n and G_n:

$$\begin{aligned} x_n &= F_n(x_{n-1}, y_{n-1}, \dots, x_{n-N_1-N_2-2}, y_{n-N_1-N_2-2}),\\ y_n &= G_n(x_{n-1}, y_{n-1}, \dots, x_{n-N_1-N_2-2}, y_{n-N_1-N_2-2}). \end{aligned}$$

- *Case 3 leads to two inductive $N_1 + N_2 + 1$-step* $\underline{\text{rational}}$ *relations,*

$$\begin{aligned} x_n &= F_n(x_{n-1}, y_{n-1}, \dots, x_{n-N_1-N_2}, y_{n-N_1-N_2}),\\ y_n &= G_n(x_{n-1}, y_{n-1}, \dots, x_{n-N_1-N_2}, y_{n-N_1-N_2}). \end{aligned}$$

Corollary 0.3. *For the self-dual weight*

$$\rho(z) = e^{\sum_1^N \frac{u_i}{i}(z^i+z^{-i})},$$

the polynomial[2] *in $x_{k-N}, x_{k-N+1}, \dots, x_k, \dots, x_{k+N}$,*

$$\begin{aligned} \Gamma_k &:= kx_k - \frac{v_k}{x_k}\left(\left(\sum_1^N u_i L_1^i\right)_{k+1,k+1} + \left(\sum_1^N u_i L_1^i\right)_{k,k} - 2\left(\sum_1^N u_i L_1^{i-1}\right)_{k+1,k}\right)\\ &= kx_k + v_k \sum_1^N u_i \left(\sum_{j=k-1}^{k+i-1} x_{j+1}(L_1^{i-1})_{k+1,j+1} + \sum_{j=k-i+1}^{k+1} x_{j-1}(L_1^{i-1})_{jk}\right) = 0 \end{aligned} \tag{0.0.18}$$

for $k \geq 1$ leads to recurrence relations

$$x_n = F_n(x_{n-1}, \dots, x_{n-2N}).$$

Remark. In the self-dual case, i.e., when $\rho(z) = \rho(z^{-1})$, the first equation (0.0.14) vanishes identically and the two equations in (0.0.17) become identical. Only one equation is required, since all $x_n = y_n$.

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[1] Both solutions can be used, when $N_1 = N_2$.

[2] The matrix L_1 appearing in (0.0.18) is the matrix (0.0.8), with $y_i = x_i$.

Remark. By the duality $a \leftrightarrow c,\ b \leftrightarrow b,\ L_1 \leftrightarrow L_2^{\top}$, Eqs. (0.0.14) and (0.0.15) map into the identities

$$\begin{cases} \partial_n(\mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)})_{n,n} + (cL_1 - aL_2)_{nn} \ = \ 0, \\ \partial_n(\mathcal{L}_1^{(n)} - v_n\mathcal{L}_2^{(n)})_{n,n+1} - \left(aL_2^2 + bL_2\right)_{n+1,n+1} = C', \end{cases} \tag{0.0.19}$$

where C' is now:

$$C' := (\mathcal{L}_1^{(1)} - v_1\mathcal{L}_2^{(1)})_{1,2} - \left(aL_2^2 + bL_2\right)_{1,1}. \tag{0.0.20}$$

An invariant manifold. The relations appearing in Theorem 0.1 for each of the cases happen to define an invariant manifold for the first Toeplitz flow. We shall do this here for Case 3, where

$$\rho(z) := z^{\gamma} e^{P_1(z)+P_2(z^{-1})}.$$

This case has the extra feature that the relations themselves (0.0.17) satisfy an interesting and simple system of differential equations with regard to the first vector field t_1, although we believe this to be true for the other vector fields as well; note that, in order to establish Theorem 0.6 on the singularity confinement, we only need the t_1-vector field. The statements on the system of differential equations will be established in Sect. 3, as an immediate consequence of the Virasoro relations satisfied by the multiple integrals.

Theorem 0.4. *Let* $x_n = x_n(t_1, s_1)$ *and* $y_n = y_n(t_1, s_1)$ *flow according to the differential equations* ($v_n := 1 - x_n y_n$)

$$\begin{aligned} \frac{\partial x_k}{\partial t_1} &= v_k x_{k+1}, & \frac{\partial y_k}{\partial t_1} &= -v_k y_{k-1}, \\ \frac{\partial x_k}{\partial s_1} &= v_k x_{k-1}, & \frac{\partial y_k}{\partial s_1} &= -v_k y_{k+1}. \end{aligned} \qquad (\textbf{Toeplitz Lattice}) \tag{0.0.21}$$

and the u_i*, appearing in the polynomials* $P_1(z)$ *and* $P_2(z)$*, according to*

$$\frac{\partial u_k}{\partial t_1} = \delta_{k,1}, \qquad \frac{\partial u_k}{\partial s_1} = -\delta_{k,-1}.$$

Then:

(i) *The polynomial recurrence relations* Γ_n *and* $\tilde{\Gamma}_n$*, defined in (0.0.17), satisfy the differential equations*

$$\begin{aligned} \frac{\partial}{\partial \left\{ \begin{smallmatrix} t_1 \\ s_1 \end{smallmatrix} \right\}} \Gamma_n &= v_n \Gamma_{n\pm1} + x_{n\pm1}\left(x_n\tilde{\Gamma}_n - y_n\Gamma_n\right), \\ \frac{\partial}{\partial \left\{ \begin{smallmatrix} t_1 \\ s_1 \end{smallmatrix} \right\}} \tilde{\Gamma}_n &= -v_n\tilde{\Gamma}_{n\mp1} + y_{n\mp1}\left(x_n\tilde{\Gamma}_n - y_n\Gamma_n\right). \end{aligned} \tag{0.0.22}$$

(ii) *The locus* $\mathfrak{M}$ *is an invariant manifold for the* t_1 *and* s_1*-flows (0.0.21) above, where*

$$\mathfrak{M} := \bigcap_{n\geq1} \left\{ (x_k, y_k)_{k\geq0},\ \textit{such that}\ \ \Gamma_n(x,y) = 0\ \textit{and}\ \tilde{\Gamma}_n(x,y) = 0 \right\}. \tag{0.0.23}$$

Corollary 0.5. *Let $x_n = x_n(t)$ flow according to the differential equations ($v_n := 1 - x_n^2$)*

$$\frac{\partial x_n}{\partial t} = v_n(x_{n+1} - x_{n-1}), \tag{0.0.24}$$

which is obtained by taking the linear combination $\frac{\partial}{\partial t} = \frac{\partial}{\partial t_1} - \frac{\partial}{\partial s_1}$ of the Toeplitz vector fields above and setting all $x_k = y_k$. Let the u_i, appearing in the self-dual weight

$$\rho(z) = e^{\sum_1^N \frac{u_i}{i}(z^i + z^{-i})},$$

flow according to

$$\frac{\partial u_k}{\partial t} = \delta_{k,1}.$$

Then:

- **(i)** *The polynomial recurrence relations Γ_n, as in (0.0.18), satisfy the differential equations*

$$\frac{\partial \Gamma_n}{\partial t} = v_n(\Gamma_{n+1} - \Gamma_{n-1}). \tag{0.0.25}$$

- **(ii)** *The locus $\mathfrak{N}$ is an invariant manifold for the t-flow (0.0.24) above, where*

$$\mathfrak{N} := \bigcap_{n \geq 1} \left\{ (x_k)_{k \geq 0}\ , \ \ \textit{such that} \ \ \Gamma_n(x, x) = 0 \right\}. \tag{0.0.26}$$

Singularity confinement. For the self-dual weight

$$\rho(z) = e^{\sum_1^N \frac{u^i}{i}(z^i + z^{-i})},$$

the polynomial relations (remember Corollary 0.3) in $x_{k-N}, x_{k-N+1}, \ldots, x_k, \ldots, x_{k+N}$,

$$0 = kx_k - \frac{v_k}{x_k}\left(\left(\sum_1^N u_i L_1^i \right)_{k+1,k+1} + \left(\sum_1^N u_i L_1^i \right)_{k,k} - 2 \left(\sum_1^N u_i L_1^{i-1} \right)_{k+1,k} \right)$$

lead, in effect, to rational recurrence relations in the x_i,

$$x_k = F_k(x_{k-1}, \ldots, x_{k-2N}; u_1, \ldots, u_N), \tag{0.0.27}$$

depending rationally on the coefficients $u_1, \ldots, u_N$ appearing in the weight $\rho(z)$.

They now satisfy a remarkable property; Theorem 0.6 tells us -roughly speaking- that the recurrence relations (0.0.27) for a special initial condition leads to a solution, where one x_n blows up and all other x_k are finite. This is a kind of *discrete Painlevé property*, called "singularity confinement"; see Grammaticos, Nijhoff and Ramani [8], who define this to be discrete Painlevé recursion relations. For these recurrence equations (0.0.27), the precise analytical statement of this phenomenon is stated in Corollary 0.7, claiming there is a *generic* solution with the kind of singularity above. The technique used here to prove Corollary 0.7 is to deform the variables x_k and y_k by means of the Toeplitz lattice; part **(i)** of Theorem 0.6 below shows that the Toeplitz lattice has a generic solution $x_0, x_1, \ldots$, with all x_k, $k \neq n$ finite and one x_n blowing up. This is reminiscent of the Painlevé property of *algebraic integrable systems*, which originates in the work of S. Kowalewski; see [1] and references within. Part **(ii)** of Theorem 0.6 shows that these series can be made to stay within the locus $\mathfrak{N}$, by restricting the free parameters. The proof of Theorem 0.6 and Corollary 0.7, which will be given in a subsequent paper, uses heavily the ideas of Theorem 0.4 and Corollary 0.5.

Theorem 0.6. **(i)** *Consider the system of differential equations (with boundary condition* $x_0 = 1$ *and* $x_{-1} = 0$*)*

$$\frac{\partial x_k}{\partial t} = (1 - x_k^2)(x_{k+1} - x_{k-1}), \text{ for } k = 0, 1, 2, \dots, \tag{0.0.28}$$

and for a fixed, but arbitrary integer $n > 0$*, let*

$$\dots, \alpha_{n-3}, \alpha_{n-2}, c, d, \alpha_{n+2}, \alpha_{n+3}, \dots \tag{0.0.29}$$

be free parameters. Then the system (0.0.28) has a unique "formal" Laurent solution, with x_n *and only* x_n *blowing up, having the form:*

$$\begin{aligned} x_k(t) &= \alpha_k + \dots, \quad \text{for } |k - n| \geq 2, \\ x_{n-1}(t) &= \pm 1 + ct + \dots, \\ x_n(t) &= \tfrac{1}{t}(\mp \tfrac{1}{2} + \tfrac{c-d}{8} t + \dots), \\ x_{n+1}(t) &= \mp 1 + dt + \dots . \end{aligned} \tag{0.0.30}$$

The coefficients in the series (0.0.30) are polynomials in the free parameters (0.0.29). This solution is generic, since

$$\text{“ } \#\{\text{free parameters}\} + 1 = \#\{\text{variables}\} \text{ ”},$$

with the "1" accounting for the t*-parameter.*
(ii) *Given the* $2N - 1$ *free parameters* $\alpha_{n-2N}, \dots, \alpha_{n-2}$*, the series (0.0.30) above are "formal" Laurent solutions to the recurrence relations (0.0.18)*

$$x_k = F_k(x_{k-1}, \dots, x_{k-2N}; u_1 + t, \dots, u_N), \tag{0.0.31}$$

with the remaining free parameters c, d, α_i *for* $i \leq n - 2N - 1$ *or* $i \geq n - 1$*, being rational functions of* $\alpha_{n-2N}, \dots, \alpha_{n-2}$ *and the parameters* $u = (u_1, \dots, u_N)$*.*

This theorem leads to the "Painlevé singularity confinement" property for the recursive equations (0.0.27); the precise statement goes as follows:

Corollary 0.7 (Singularity confinement). *Given arbitrary initial data*

$$(x_{n-2N}, \dots, x_{n-2}) = \left(x_{n-2N}^{(0)}, \dots, x_{n-2}^{(0)}\right) =: \gamma$$

and setting

$$x_{n-1} = \pm 1 + \varepsilon, \tag{0.0.32}$$

the recurrence relations (0.0.27), namely

$$x_k = F_k(x_{k-1}, \dots, x_{k-2N}; u_1, \dots, u_N),$$

*have a "*generic*" formal series solution in ε of the form (i.e., depending on $2N-1$ degrees of freedom)*

$$\begin{aligned}
x_{n-1}(\varepsilon) &= \pm 1 + \varepsilon, \\
x_n(\varepsilon) &= \frac{1}{\varepsilon}\left(x_n^{(0)}(\gamma, u) + O(\varepsilon)\right), \\
x_{n+1}(\varepsilon) &= \mp 1 + O(\varepsilon), \\
x_k(\varepsilon) &= x_k^{(0)}(\gamma, u) + O(\varepsilon), \quad \textit{for } k \geq n+2,
\end{aligned} \tag{0.0.33}$$

with all coefficients of the ε-series depending rationally on $\gamma = (x_{n-2N}^{(0)}, \dots, x_{n-2}^{(0)})$ and $u := (u_1, \dots, u_N)$.

Remark. The initial condition (0.0.32) is the most general initial condition leading to blow-up at the n^{th} step.

Theorem 0.6 and Corollary 0.7 will be established elsewhere, as well as analogous statements that can be made for the non-symmetric weight

$$\rho(z) = e^{\sum_1^N (\frac{u_i}{i} z^i + \frac{u_{-i}}{i} z^{-i})}.$$

Examples. Several examples will be discusssed in Sect. 4. It is also interesting to point out that each of the examples discussed in that section are related to random permutations, random words and point processes. They also admit a representation as a Fredholm determinant of an interesting kernel; concerning the latter, see Borodin and Okounkov [6].

1. The Toeplitz Lattice and Its Virasoro Algebra

1.1. The Toeplitz lattice. Consider the inner-product on the circle

$$\langle f(z), g(z) \rangle_{t,s} := \oint_{S^1} \frac{dz}{2\pi i z} f(z) g(z^{-1}) e^{\sum_1^\infty (t_j z^j - s_j z^{-j})}, \tag{1.1.1}$$

the associated moments $\mu_{k-\ell}(t,s) := \langle y^k, z^\ell \rangle_{t,s}$, and the determinants ($\tau$-functions)

$$\begin{aligned}
\tau_n(t,s) &:= \det\left(\mu_{k-\ell}(t,s)\right)_{0 \leq k,\ell \leq n-1} \\
&= \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(e^{\sum_1^\infty (t_j z_k^j - s_j z_k^{-j})} \frac{dz_k}{2\pi i z_k} \right) \\
&= \int_{U(n)} e^{\sum_1^\infty \mathrm{Tr}(t_j M^j - s_j \bar{M}^j)} dM
\end{aligned} \tag{1.1.2}$$

and

$$\begin{aligned}
\tau_n^{\pm}(t,s) &= \det\left(\mu_{k-\ell \pm 1}(t,s)\right)_{0 \leq k,\ell \leq n-1}, \\
&= \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(z_k^{\pm 1}\, e^{\sum_1^\infty (t_j z_k^j - s_j z_k^{-j})} \frac{dz_k}{2\pi i z_k} \right) \\
&= \int_{U(n)} (\det M)^{\pm 1} e^{\sum_1^\infty \mathrm{Tr}(t_j M^j - s_j \bar{M}^j)} dM.
\end{aligned} \tag{1.1.3}$$

The following τ-function expressions are actually monic polynomials[3]

$$\begin{aligned} p_n^{(1)}(t,s;u) &= u^n \frac{\tau_n(t-[u^{-1}],s)}{\tau_n(t,s)} \\ &= \frac{1}{n!\tau_n}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left((u-z_k)e^{\sum_1^\infty (t_j z_k^j - s_j z_k^{-j})}\frac{dz_k}{2\pi i z_k}\right) \\ &= \frac{1}{\tau_n(t,s)}\int_{U(n)} \det(uI-M)e^{\sum_1^\infty \operatorname{Tr}(t_j M^j - s_j \bar{M}^j)} dM, \end{aligned} \tag{1.1.4}$$

$$\begin{aligned} p_n^{(2)}(t,s;u) &= u^n \frac{\tau_n(t,s+[u^{-1}])}{\tau_n(t,s)} \\ &= \frac{1}{n!\tau_n}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left((u-z_k^{-1})e^{\sum_1^\infty (t_j z_k^j - s_j z_k^{-j})}\frac{dz_k}{2\pi i z_k}\right) \\ &= \frac{1}{\tau_n(t,s)}\int_{U(n)} \det(uI-\bar{M})e^{\sum_1^\infty \operatorname{Tr}(t_j M^j - s_j \bar{M}^j)} dM, \end{aligned} \tag{1.1.5}$$

and are bi-orthogonal for the inner-product above

$$\langle p_n^{(1)}, p_m^{(2)}\rangle_{t,s} = \delta_{nm} h_n(t,s), \quad \text{with } h_n := \frac{\tau_{n+1}}{\tau_n}. \tag{1.1.6}$$

Define[4]

$$\begin{aligned} x_n(t,s) &:= p_n^{(1)}(t,s;0) \\ &= \frac{(-1)^n}{n!\tau_n}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(z_k e^{\sum_1^\infty (t_j z_k^j - s_j z_k^{-j})}\frac{dz_k}{2\pi i z_k}\right) \\ &= \frac{p_n(-\tilde{\partial}_t)\tau_n(t,s)}{\tau_n(t,s)} = (-1)^n \frac{\tau_n^+(t,s)}{\tau_n(t,s)}, \\ y_n(t,s) :&:= p_n^{(2)}(t,s;0) \\ &= \frac{(-1)^n}{n!\tau_n}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(z_k^{-1} e^{\sum_1^\infty (t_j z_k^j - s_j z_k^{-j})}\frac{dz_k}{2\pi i z_k}\right) \\ &= \frac{p_n(\tilde{\partial}_s)\tau_n(t,s)}{\tau_n(t,s)} = (-1)^n \frac{\tau_n^-(t,s)}{\tau_n(t,s)}. \end{aligned} \tag{1.1.7}$$

---

[3] For $\alpha \in \mathbb{C}$, define $[\alpha] := (\alpha, \alpha^2/2, \alpha^3/3, \dots) \in \mathbb{C}^\infty$.

[4] $\tilde{\partial} = (\partial/\partial t_1, (1/2)\partial/\partial t_2, (1/3)\partial/\partial t_3, \dots)$, and p_k are the elementary Schur functions: $\sum_{k=0}^\infty p_k(t)z^k := \exp(\sum_{i=1}^\infty t_i z^i)$.

Throughout the paper, set[5]

$$\begin{aligned} v_n := 1 - x_n y_n &= 1 - p_n^{(1)}(t,s;0)p_n^{(2)}(t,s;0) \\ &= \frac{h_n}{h_{n-1}} = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}. \end{aligned} \tag{1.1.8}$$

In [3], it was pointed out that the quantities x_n and y_n satisfy the following integrable Hamiltonian system:

$$\begin{aligned} \frac{\partial x_n}{\partial t_i} &= (1 - x_n y_n)\frac{\partial H_i^{(1)}}{\partial y_n}, & \frac{\partial y_n}{\partial t_i} &= -(1 - x_n y_n)\frac{\partial H_i^{(1)}}{\partial x_n}, \\ \frac{\partial x_n}{\partial s_i} &= (1 - x_n y_n)\frac{\partial H_i^{(2)}}{\partial y_n}, & \frac{\partial y_n}{\partial s_i} &= -(1 - x_n y_n)\frac{\partial H_i^{(2)}}{\partial x_n}, \end{aligned} \tag{1.1.9}$$

(**Toeplitz lattice**)

with initial condition $x_n(0,0) = y_n(0,0) = 0$ for $n \geq 1$ and boundary condition $x_0(t,s) = y_0(t,s) = 1$. This fact will be established in Proposition 1.1 below. The traces

$$H_i^{(k)} = -\frac{1}{i}\,\mathrm{Tr}\; L_k^i, \quad i = 1,2,3,..., \quad k = 1,2 \tag{1.1.10}$$

of the matrices L_i below are integrals in involution with regard to the symplectic structure

$$\omega := \sum_1^\infty \frac{dx_k \wedge dy_k}{1 - x_k y_k},$$

where L_1 and L_2 are given by the " rank 2" semi-infinite matrices

$$L_1 := \begin{pmatrix} -x_1y_0 & 1-x_1y_1 & 0 & 0 & \\ -x_2y_0 & -x_2y_1 & 1-x_2y_2 & 0 & \\ -x_3y_0 & -x_3y_1 & -x_3y_2 & 1-x_3y_3 & \\ -x_4y_0 & -x_4y_1 & -x_4y_2 & -x_4y_3 & \\ & & & & \ddots \end{pmatrix}$$

and

$$L_2 := \begin{pmatrix} -x_0y_1 & -x_0y_2 & -x_0y_3 & -x_0y_4 & \\ 1-x_1y_1 & -x_1y_2 & -x_1y_3 & -x_1y_4 & \\ 0 & 1-x_2y_2 & -x_2y_3 & -x_2y_4 & \\ 0 & 0 & 1-x_3y_3 & -x_3y_4 & \\ & & & & \ddots \end{pmatrix}.$$

---

[5] By computing $\langle p_{n+1}^{(1)}(u) - up_n^{(1)}(u), p_{m+1}^{(2)}(u) - up_m^{(2)}(u)\rangle$ in two different ways, in a straightforward way and in another way, using

$$\begin{aligned} p_{n+1}^{(1)}(u) - up_n^{(1)}(u) &= p_{n+1}^{(1)}(0)u^n p_n^{(2)}(u^{-1}), \\ p_{n+1}^{(2)}(u) - up_n^{(2)}(u) &= p_{n+1}^{(2)}(0)u^n p_n^{(1)}(u^{-1}). \end{aligned}$$

For details, see ([3]).

Written out, the differential equations (1.1.9) read[6]

$$\begin{aligned}
\frac{\partial x_n}{\partial t_i} &= (1 - x_n y_n)\frac{\partial H_i^{(1)}}{\partial y_n} \\
&= \frac{h_n}{h_{n-1}} \frac{\partial}{\partial y_n}\left(-\frac{1}{i} Tr L_1^i\right), \text{ using } 1 - x_n y_n = \frac{h_n}{h_{n-1}} \\
&= \frac{h_n}{h_{n-1}} \left\langle L_1^{i-1}, -\frac{\partial L_1}{\partial y_n}\right\rangle \\
&= \frac{h_n}{h_{n-1}} \left\langle L_1^{i-1}, \begin{array}{cc} \begin{array}{ccc} & \overset{n+1}{\downarrow} & \end{array} & \\ \left(\begin{array}{ccc} & 0 & \\ & \vdots & \\ O & 0 & O \\ & x_n & \\ & x_{n+1} & \\ & x_{n+2} & \\ & \vdots & \end{array}\right) & \begin{array}{c} \\ \\ \\ \leftarrow n \\ \\ \\ \\ \end{array} \end{array} \right\rangle \\
&= \frac{h_n}{h_{n-1}} \sum_{j \geq n} (L_1^{i-1})_{n+1,j} x_j, \qquad (1.1.11)
\end{aligned}$$

and similarly

$$\begin{aligned}
\frac{\partial x_n}{\partial s_i} &= (1 - x_n y_n)\frac{\partial H_i^{(2)}}{\partial y_n} \\
&= \frac{h_n}{h_{n-1}} \left\langle L_2^{i-1}, -\frac{\partial L_2}{\partial y_n}\right\rangle \\
&= \frac{h_n}{h_{n-1}} \left\langle L_2^{i-1}, \begin{array}{c} \begin{array}{ccc} & \overset{n}{\downarrow} & \end{array} \\ \left(\begin{array}{ccc} & x_0 & \\ & \vdots & \\ O & x_n & O \\ & 0 & \\ & 0 & \\ & \vdots & \end{array}\right) \end{array} \right\rangle \\
&= \frac{h_n}{h_{n-1}} \sum_{j=1}^{n+1} (L_2^{i-1})_{n,j} x_{j-1} \\
&= \frac{h_n}{h_{n-1}} \sum_{j=n-i+1}^{n+1} (L_2^{i-1})_{n,j} x_{j-1} \, , \qquad (1.1.12)
\end{aligned}$$

---

[6] Introduce the inner-product

$$\langle A, B\rangle = \mathrm{tr} A B^{\top},$$

which differentiated behaves as

$$\frac{\partial}{\partial x} \, \mathrm{tr} A^n = \left\langle n A^{n-1}, \frac{\partial A}{\partial x}\right\rangle.$$

using in the last identity the obvious fact that

$$(L_2^\alpha)_{ij} = 0, \text{ unless } j \geq i - \alpha. \tag{1.1.13}$$

By the duality, mentioned in (1.1.15) below, one reads off the differential equations for the y_n's.

Setting

$$\hat{L}_1 := hL_1h^{-1} \quad \text{and} \quad \hat{L}_2 := L_2,$$

It was shown in [3] that the "rank 2"-structure of $\hat{L}_1$ and $\hat{L}_2$ is preserved by the equations[7]

$$\frac{\partial \hat{L}_i}{\partial t_n} = \left[\left(\hat{L}_1^n\right)_+, \hat{L}_i\right] \quad \text{and} \quad \frac{\partial \hat{L}_i}{\partial s_n} = \left[\left(\hat{L}_2^n\right)_-, \hat{L}_i\right] \quad i = 1, 2 \text{ and } n = 1, 2, \ldots .$$

(Two-Toda Lattice) (1.1.14)

The Toda and Toeplitz lattices have an involution, compatible with the involution $\tilde{}$, introduced in (0.0.3),

$$x_n \leftrightarrow y_n, \quad t_n \leftrightarrow -s_n, \quad L_1 \leftrightarrow L_2^\top. \tag{1.1.15}$$

The proposition below was merely taken for granted in [3]. Here we give a complete proof.

Proposition 1.1. *The two-Toda lattice flows (1.1.14) are equivalent to the Hamiltonian Toeplitz lattice flows (1.1.9).*

Proof. From the general 2-Toda theory, as related to biorthogonal polynomials, we know that the vector (see [2])

$$\Psi_1 := (\Psi_{1,n})_{n\geq 0} = e^{\sum_1^\infty t_i z^i} \left(p_n^{(1)}(t, s; z)\right)_{n\geq 0} \tag{1.1.16}$$

with

$$\Psi_{1,n} = e^{\sum_1^\infty t_i z^i} z^n \frac{\tau_n(t - [z^{-1}], s)}{\tau_n(t, s)} = e^{\sum_1^\infty t_i z^i} p_n^{(1)}(t, s; z), \tag{1.1.17}$$

is an eigenvector for the matrix L_1,

$$L_1 \Psi_1 = z \Psi_1,$$

and (1.1.16) satisfies the differential equations

$$\frac{\partial \Psi_1}{\partial t_n} = (\hat{L}_1^n)_+ \Psi_1 \text{ and } \frac{\partial \Psi_1}{\partial s_n} = (\hat{L}_2^n)_- \Psi_1. \tag{1.1.18}$$

So, from (1.1.16), (1.1.17) and (1.1.18), it follows that the vector $p^{(1)}(z) := (p_n^{(1)}(t, s; z))_{n\geq 0}$ satisfies the differential equations

$$\begin{aligned} \frac{\partial p^{(1)}(z)}{\partial t_n} &= (\hat{L}_1^n)_+ p^{(1)}(z) - z^n p^{(1)}(z), \\ \frac{\partial p^{(1)}(z)}{\partial s_n} &= (\hat{L}_2^n)_- p^{(1)}(z). \end{aligned} \tag{1.1.19}$$

---

[7] $(A)_+$ is the upper-triangular part of A, including the diagonal, and $(A)_- := A_-(A)_+$

Since (1.1.7) implies $p^{(1)}(t,s;0) = (x_0, x_1, \dots)^\top$, the differential equations (1.1.19) evaluated at $z = 0$ read

$$\frac{\partial x}{\partial t_i} = (\hat{L}_1^i)_+ x \quad \text{and} \quad \frac{\partial x}{\partial s_i} = (\hat{L}_2^i)_- x,$$

yielding componentwise (remember $\hat{L}_1 = hL_1h^{-1}$)

$$\frac{\partial x_n}{\partial t_i} = h_n \sum_{j \geq n} (L_1^i)_{n+1,j+1} \frac{x_j}{h_j}, \tag{1.1.20}$$

$$\frac{\partial x_n}{\partial s_i} = \sum_{1 \leq j \leq n} (L_2^i)_{n+1,j} x_{j-1} = \sum_{j=n+1-i}^{n} (L_2^i)_{n+1,j} x_{j-1}. \tag{1.1.21}$$

The point of Proposition 1.1 is to show that Eqs. (1.1.21), obtained via the 2-Toda lattice, are equivalent to Eqs. (1.1.11), (1.1.12), coming from the Toeplitz lattice, i.e., we must show, for $n \geq 0$,

$$\sum_{j \geq n} (L_1^i)_{n+1,j+1} \frac{x_j}{h_j} = \frac{1}{h_{n-1}} \sum_{j \geq n} (L_1^{i-1})_{n+1,j} x_j, \tag{1.1.22}$$

$$\sum_{j=n-i+1}^{n} (L_2^i)_{n+1,j} x_{j-1} = \frac{h_n}{h_{n-1}} \sum_{j=n-i+1}^{n+1} (L_2^{i-1})_{nj} x_{j-1}, \tag{1.1.23}$$

by duality, it suffices to show equations just for the x_n-variables.

To show (1.1.22), compute, using $x_j y_j = 1 - h_j/h_{j-1}$:

$$\begin{aligned}
\sum_{j \geq n} (L_1^i)_{n+1,j+1} \frac{x_j}{h_j} &= \sum_{r \geq j \geq n} (L_1^{i-1})_{n+1,r} (L_1)_{r,j+1} \frac{x_j}{h_j} \\
&= \sum_{r \geq j \geq n} (L_1^{i-1})_{n+1,r} (\delta_{r,j} - x_r y_j) \frac{x_j}{h_j} \\
&= \sum_{j \geq n} (L_1^{i-1})_{n+1,j} \frac{x_j}{h_j} - \sum_{r \geq j \geq n} (L_1^{i-1})_{n+1,r} x_r \left(\frac{1}{h_j} - \frac{1}{h_{j-1}} \right) \\
&= \sum_{j \geq n} (L_1^{i-1})_{n+1,j} \frac{x_j}{h_j} - \sum_{r \geq n} (L_1^{i-1})_{n+1,r} x_r \left(\frac{1}{h_r} - \frac{1}{h_{n-1}} \right) \\
&= \frac{1}{h_{n-1}} \sum_{r \geq n} (L_1^{i-1})_{n+1,r} x_r.
\end{aligned}$$

Next establish (1.1.23), using

$$(L_2)_{n+1,n} = h_n/h_{n-1} = 1 - x_n y_n, \tag{1.1.24}$$

$$(L_2)_{n+1,r} x_{j-1} = x_n (L_2)_{jr} \text{ provided } r > n \text{ and } r > j - 1. \tag{1.1.25}$$

Indeed, setting $i = k + 1$, one computes, using (1.1.13),

$$\begin{aligned}
&\sum_{j=n-k}^{n} (L_2^{k+1})_{n+1,j}\, x_{j-1} - \frac{h_n}{h_{n-1}} \sum_{j=n-k}^{n+1} (L_2^k)_{nj}\, x_{j-1} \\
&= \sum_{j=n-k}^{n} \sum_{r=n}^{j+k} (L_2)_{n+1,r}(L_2^k)_{rj} x_{j-1} - \frac{h_n}{h_{n-1}} \sum_{j=n-k}^{n+1} (L_2^k)_{nj} x_{j-1} \\
&= \frac{h_n}{h_{n-1}} \sum_{j=n-k}^{n} (L_2^k)_{nj} x_{j-1} - \frac{h_n}{h_{n-1}} \sum_{j=n-k}^{n+1} (L_2^k)_{nj} x_{j-1} \\
&\quad + \sum_{j=n-k}^{n} \sum_{r=n+1}^{j+k} (L_2)_{n+1,r}(L_2^k)_{rj} x_{j-1} \qquad \text{using (1.1.24)} \\
&\overset{*}{=} x_n \left(-(L_2)_{n+1,n}(L_2^k)_{n,n+1} + \sum_{j=n-k}^{n} \sum_{r=n+1}^{j+k} (L_2)_{jr}(L_2^k)_{rj} \right) \\
&= 0, \qquad\qquad (1.1.26)
\end{aligned}$$

using in $\overset{*}{=}$ formulas (1.1.24), (1.1.25) and the inequalities $n - k - 1 \le j - 1 \le n - 1 < n + 1 \le r \le j + k$.

As a last step, we show that indeed the last expression (1.1.26) vanishes. The first expression in the bracket equals, using (1.1.13)

$$\begin{aligned}
&(L_2)_{n+1,n}(L_2^k)_{n,n+1} \\
&= (L_2)_{n+1,n} \sum_{\substack{n \le \beta_1 + 1 \le \beta_2 + 2 \le \beta_3 + 3 \le \\ \ldots \le \beta_{k-1} + k - 1 \le n + k + 1}} (L_2)_{n\beta_1}(L_2)_{\beta_1\beta_2} \ldots (L_2)_{\beta_{k-1},n+1}\ ,
\end{aligned} \tag{1.1.27}$$

whereas the second expression in the bracket of (1.1.26) equals, upon setting $\alpha_0 = r, \alpha_k = j$ and using (1.1.13):

$$\begin{aligned}
&\sum_{j=n-k}^{n} \sum_{r=n+1}^{j+k} (L_2)_{jr}(L_2^k)_{rj} \\
&= \sum_{r=n+1}^{n+k} \sum_{j=r-k}^{n} (L_2)_{jr}(L_2^k)_{rj} \\
&= \sum_{\substack{n+1 \le \alpha_0 \le \alpha_1 + 1 \le \alpha_2 + 2 \le \\ \ldots \le \alpha_{k-1} + k - 1 \le \alpha_k + k \le n + k}} (L_2)_{\alpha_0,\alpha_1}(L_2)_{\alpha_1,\alpha_2} \ldots (L_2)_{\alpha_{k-1},\alpha_k}(L_2)_{\alpha_k,\alpha_0}. \qquad (1.1.28)
\end{aligned}$$

Each term in this sum is a product of $k + 1$ entries of L_2, with nondecreasing indices $\alpha_i + i$, $0 \le i \le k$, squeezed between $n + 1$ and $n + k$. Therefore, we must have for some $1 \le j \le k$ that

$$\alpha_j + j = \alpha_{j+1} + j + 1 = n + j + 1,$$

and so for that (α_j, α_{j+1}), we have

$$(L_2)_{\alpha_j, \alpha_{j+1}} = (L_2)_{n+1,n},$$

which appears in every term of the sum (1.1.28); it can therefore be taken out, leaving sums of products of k terms. The inequalities under the summation sign of (1.1.28) can then be written as follows:

$$n+1 \le \alpha_0 \le \alpha_1+1 \le \ldots \le \alpha_{j-1}+j-1 \le \underset{\underset{\displaystyle n+j+1}{\|}}{\alpha_j+j} = \underset{\underset{\displaystyle n+j+1}{\|}}{\alpha_{j+1}+j+1} \le \alpha_{j+2}+j+2 \le \ldots \le \alpha_k+k \le n+k$$

Add $k-j$ to the sequence above, from $n+1$ up to including $\alpha_j + j = n+j+1$ and $-j-1$ to the sequence above, starting with $\alpha_{j+1}+j+1 = n+j+1$, up to $n+k$, yielding the two sequences

$$n+k-j+1 \le \alpha_0+k-j \le \ldots \le \alpha_{j-1}+k-1 \le n+k+1$$

and

$$n \le \alpha_{j+2}+1 \le \ldots \le \alpha_k+k-j-1 \le n+k-j-1.$$

Since obviously $n+k-j-1 < n+k-j+1$, we have the inequalities appearing in the summation of formula (1.1.27), but with β's replaced by α's, suitably ordered. This ends the proof of Proposition 1.1. □

According to [3], we also have $(h := \operatorname{diag}(h_0, h_1, \ldots) = \operatorname{diag}(\frac{\tau_1}{\tau_0}, \frac{\tau_2}{\tau_1}, \ldots))$

$$\begin{aligned} x_{n+1}y_{n+1} &= 1 - \tfrac{h_{n+1}}{h_n}, & y_{n+1}x_{n+1} &= 1 - \tfrac{h_{n+1}}{h_n}, \\ x_{n+1}y_n &= -\tfrac{\partial}{\partial t_1}\log h_n, & y_{n+1}x_n &= \tfrac{\partial}{\partial s_1}\log h_n, \\ x_{n+1}y_{n-1} &= -\tfrac{h_{n-1}}{h_n}\left(\tfrac{\partial}{\partial t_1}\right)^2 \log\tau_n, & y_{n+1}x_{n-1} &= -\tfrac{h_{n-1}}{h_n}\left(\tfrac{\partial}{\partial s_1}\right)^2 \log\tau_n, \end{aligned} \tag{1.1.29}$$

$$\begin{aligned} x_{n+1}y_{n-k} &= -\frac{h_{n-k}}{h_n}\frac{p_{k+1}(\tilde{\partial}_t)\tau_{n-k+1}\circ\tau_n}{\tau_{n-k+1}\tau_n}, \\ y_{n+1}x_{n-k} &= -\frac{h_{n-k}}{h_n}\frac{p_{k+1}(-\tilde{\partial}_s)\tau_{n-k+1}\circ\tau_n}{\tau_{n-k+1}\tau_n}, \; k \ge 0. \end{aligned} \tag{1.1.30}$$

Lemma 1.2. *The t_i- and s_i-derivatives of x_n and y_n can be expressed in terms of the elements on the main diagonal and one above and below the main diagonal of L_k^i and*

L_k^{i-1}:

$$\frac{x_n y_n}{v_n} \sum_{i\geq 1} \left(\alpha_i \frac{\partial}{\partial t_i} - \beta_i \frac{\partial}{\partial s_i} \right) \log x_n = y_n \frac{\partial}{\partial y_n} \sum_{i \geq 1} \left(\alpha_i H_i^{(1)} - \beta_i H_i^{(2)} \right)$$
$$= \sum_{i\geq 1} \left\{ \begin{array}{l} \alpha_i \left(-(L_1^i)_{n+1,n+1} + (L_1^{i-1})_{n+1,n} \right) \\[2ex] +\beta_i \left((L_2^i)_{nn} - (L_2^{i-1})_{n,n+1} \right) \end{array} \right\},$$

$$\frac{x_n y_n}{v_n} \sum_{i\geq 1} \left(\alpha_i \frac{\partial}{\partial t_i} - \beta_i \frac{\partial}{\partial s_i} \right) \log y_n = -x_n \frac{\partial}{\partial x_n} \sum_{i \geq 1} \left(\alpha_i H_i^{(1)} - \beta_i H_i^{(2)} \right)$$
$$= \sum_{i\geq 1} \left\{ \begin{array}{l} \alpha_i \left((L_1^i)_{nn} - (L_1^{i-1})_{n+1,n} \right) \\[2ex] -\beta_i \left((L_2^i)_{n+1,n+1} - (L_2^{i-1})_{n,n+1} \right) \end{array} \right\}. \tag{1.1.31}$$

In particular, we have

$$\begin{aligned} \frac{x_n y_n}{v_n} \left(c \frac{\partial}{\partial t_1} - a \frac{\partial}{\partial s_1} \right) \log x_n &= y_n \frac{\partial}{\partial y_n} \left(c H_1^{(1)} - a H_1^{(2)} \right) \\ &= -a x_{n-1} y_n + c x_{n+1} y_n \\ &= a(L_2)_{nn} - c(L_1)_{n+1,n+1}, \\ \frac{x_n y_n}{v_n} \left(c \frac{\partial}{\partial t_1} - a \frac{\partial}{\partial s_1} \right) \log y_n &= -x_n \frac{\partial}{\partial x_n} \left(c H_1^{(1)} - a H_1^{(2)} \right) \\ &= a x_n y_{n+1} - c x_n y_{n-1} \\ &= -a(L_2)_{n+1,n+1} + c(L_1)_{nn}. \end{aligned} \tag{1.1.32}$$

Proof. The first equality in each of the identities above follows immediately from the Hamiltonian vector fields (1.1.9), with Hamiltonians (1.1.10). Note the following:

$$\left(x_n \frac{\partial}{\partial x_n} - y_n \frac{\partial}{\partial y_n} \right) L_1 =$$

$$\begin{array}{cc} & \begin{array}{c} \qquad\qquad\qquad\qquad\qquad\qquad n+1 \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \end{array} \\ n \to & \left(\begin{array}{cccccc} & & & 0 & & \\ O & & & \vdots & & O \\ & & & 0 & & \\ -x_n y_0 & -x_n y_1 & \dots & -x_n y_{n-1} \; -x_n y_n + x_n y_n & 0 & \dots \\ & & & x_{n+1} y_n & & \\ & & & x_{n+2} y_n & & O \\ O & & & \vdots & & \end{array} \right) \end{array}$$

and

$$\left(x_n\frac{\partial}{\partial x_n}-y_n\frac{\partial}{\partial y_n}\right)L_2=$$

$$\begin{array}{cc} & \begin{array}{ccc} & \overset{n}{\downarrow} & \phantom{-x_ny_{n+1}\ -x_ny_{n+2}\ \ldots} \end{array} \\ \begin{array}{c} \\ \\ \\ n+1\rightarrow \\ \\ \\ \end{array} & \left(\begin{array}{ccc} & x_0y_n & \\ O & \vdots & O \\ & x_{n-1}y_n & \\ 0\ldots\quad\ldots 0 & x_ny_n-x_ny_n & -x_ny_{n+1}\ -x_ny_{n+2}\ \ldots \\ & 0 & \\ O & \vdots & O \end{array}\right) \end{array}. \tag{1.1.33}$$

We shall also need the following trivial identities: (see (0.0.8), (0.0.9) and (1.1.13))

$$\left(L_1^i\right)_{nn}=\begin{cases}\left(L_1L_1^{i-1}\right)_{nn}=-x_n\displaystyle\sum_{j=1}^{n+1}y_{j-1}\left(L_1^{i-1}\right)_{jn}+\left(L_1^{i-1}\right)_{n+1,n}\\[2ex] \left(L_1^{i-1}L_1\right)_{nn}=-y_{n-1}\displaystyle\sum_{j=n-2}^{n+i-2}x_{j+1}\left(L_1^{i-1}\right)_{n,j+1}+\left(L_1^{i-1}\right)_{n,n-1}\end{cases}$$

and

$$\left(L_2^i\right)_{nn}=\begin{cases}\left(L_2L_2^{i-1}\right)_{nn}=-x_{n-1}\displaystyle\sum_{j=n-2}^{n+i-2}y_{j+1}\left(L_2^{i-1}\right)_{j+1,n}+\left(L_2^{i-1}\right)_{n-1,n}\\[2ex] \left(L_2^{i-1}L_2\right)_{nn}=-y_n\displaystyle\sum_{j=1}^{n+1}x_{j-1}\left(L_2^{i-1}\right)_{n,j}+\left(L_2^{i-1}\right)_{n,n+1}\end{cases}. \tag{1.1.34}$$

We now have (see the definition (1.1.10) of $H_i^{(k)}$)

$$\begin{aligned} &y_n\frac{\partial}{\partial y_n}\sum_{i\geq 1}\left(\alpha_iH_i^{(1)}-\beta_iH_i^{(2)}\right)\\ &\quad=-y_n\frac{\partial}{\partial y_n}\sum_{i\geq 1}\operatorname{tr}\left(\frac{\alpha_i}{i}L_1^i-\frac{\beta_i}{i}L_2^i\right)\\ &\quad=\left\langle\sum_{i\geq 1}\alpha_iL_1^{i-1},-y_n\frac{\partial}{\partial y_n}L_1\right\rangle-\left\langle\sum_{i\geq 1}\beta_iL_2^{i-1},-y_n\frac{\partial}{\partial y_n}L_2\right\rangle \end{aligned}$$

$$= \sum_{i \geq 1} \alpha_i \left(y_n \sum_{j=n-1}^{n+i-1} x_{j+1} (L_1^{i-1})_{n+1,j+1} \right) + \sum_{i \geq 1} \beta_i \left(-y_n \sum_{j=1}^{n+1} x_{j-1} (L_2^{i-1})_{n,j} \right) \quad \text{using (1.1.33)}$$

$$= \sum_{i \geq 1} \alpha_i \left(-(L_1^i)_{n+1,n+1} + (L_1^{i-1})_{n+1,n} \right) + \sum_{i \geq 1} \beta_i \left((L_2^i)_{nn} - (L_2^{i-1})_{n,n+1} \right),$$

using in the last equality identities (1.1.34).

Similarly, one computes

$$\begin{aligned}
&-x_n \frac{\partial}{\partial x_n} \sum_{i \geq 1} \left(\alpha_i H_i^{(1)} - \beta_i H_i^{(2)} \right) \\
&\quad = x_n \frac{\partial}{\partial x_n} \sum_{i \geq 1} \operatorname{tr} \left(\frac{\alpha_i}{i} L_1^i - \frac{\beta_i}{i} L_2^i \right) \\
&\quad = \left\langle \sum_{i \geq 1} \alpha_i L_1^{i-1}, x_n \frac{\partial}{\partial x_n} L_1 \right\rangle - \left\langle \sum_{i \geq 1} \beta_i L_2^{i-1}, x_n \frac{\partial}{\partial x_n} L_2 \right\rangle \\
&\quad = \sum_{i \geq 1} \alpha_i \left(-x_n \sum_{j=1}^{n+1} y_{j-1} (L_1^{i-1})_{j,n} \right) + \sum_{i \geq 1} \beta_i \left(x_n \sum_{j=n-1}^{n+i-1} y_{j+1} (L_2^{i-1})_{j+1,n+1} \right) \\
&\quad = \sum_{i \geq 1} \alpha_i \left((L_1^i)_{nn} - (L_1^{i-1})_{n+1,n} \right) + \sum_{i \geq 1} \beta_i \left(-(L_2^i)_{n+1,n+1} + (L_2^{i-1})_{n,n+1} \right).
\end{aligned}$$

The last couple of relations (1.1.32) follow from specializing (1.1.31) to $i = 1$, thus ending the proof of Lemma 1.2. □

Lemma 1.3.

$$\begin{aligned}
\left(\frac{\partial L_1^i}{\partial t_1} \right)_{nn} &= v_n (L_1^i)_{n+1,n} - v_{n-1} (L_1^i)_{n,n-1}, \\
\left(\frac{\partial L_2^i}{\partial t_1} \right)_{nn} &= (L_2^i)_{n+1,n} - (L_2^i)_{n,n-1}, \\
\left(\frac{\partial L_1^i}{\partial s_1} \right)_{nn} &= (L_1^i)_{n-1,n} - (L_1^i)_{n,n+1}, \\
\left(\frac{\partial L_2^i}{\partial s_1} \right)_{nn} &= v_{n-1} (L_2^i)_{n-1,n} - v_n (L_2^i)_{n,n+1}.
\end{aligned}$$

Proof. From the Toda equations (1.1.14) for $\hat{L}_i$, one computes the t_1-flow for $\hat{L}_1$, where $h := \operatorname{diag}(h_0, h_1, \dots)$ and where A_+, A_{++} and A_0 denote the upper-triangular, strictly upper-triangular and diagonal part of the matrix A, (remember $L_1 = h^{-1}\hat{L}_1 h$)

$$\begin{aligned}
\frac{\partial L_1^i}{\partial t_1} &= \frac{\partial}{\partial t_1}(h^{-1}\hat{L}_1^i h) \\
&= h^{-1}\frac{\partial \hat{L}_1^i}{\partial t_1}h - \frac{\partial \log h}{\partial t_1}h^{-1}\hat{L}_1^i h + h^{-1}\hat{L}_1^i h\frac{\partial \log h}{\partial t_1} \\
&= [h^{-1}(\hat{L}_1)_+ h, h^{-1}\hat{L}_1^i h] - \left[\frac{\partial \log h}{\partial t_1}, h^{-1}\hat{L}_1^i h\right] \\
&= \left[(L_1)_+ - \frac{\partial \log h}{\partial t_1}, L_1^i\right] \\
&= [(L_1)_{++}, L_1^i], \quad \text{using (1.1.29)} \\
&= [\operatorname{diag}(v_1, v_2, \dots)\Lambda, L_1^i].
\end{aligned}$$

Hence,

$$\left(\frac{\partial L_1^i}{\partial t_1}\right)_0 = \operatorname{diag}\left(v_1(L_1^i)_{21}, v_2(L_1^i)_{32} - v_1(L_1^i)_{21}, \dots\right).$$

In particular,

$$\begin{aligned}
\left(\frac{\partial L_1^i}{\partial t_1}\right)_{nn} &= v_n(L_1^i)_{n+1,n} - v_{n-1}(L_1^i)_{n,n-1}, \\
\left(\frac{\partial L_1}{\partial t_1}\right)_{n,n} &= -v_n x_{n+1} y_{n-1} + v_{n-1} x_n y_{n-2}.
\end{aligned}$$

We also need the t_1-derivative of L_2^i,

$$\begin{aligned}
\frac{\partial L_2^i}{\partial t_1} &= [(\hat{L}_1)_+, L_2^i] \\
&= \left[\begin{pmatrix} -x_1y_0 & 1 & & & O \\ & & & & \\ & -x_2y_1 & 1 & & \\ & & -x_3y_2 & 1 & \\ & O & & \ddots & \ddots \end{pmatrix}, L_2^i\right],
\end{aligned}$$

hence

$$\left(\frac{\partial L_2^i}{\partial t_1}\right)_0 = \operatorname{diag}\left((L_2^i)_{21}, (L_2^i)_{3,2} - (L_2^i)_{2,1}, (L_2^i)_{4,3} - (L_2^i)_{3,2}, \dots\right),$$

leading to

$$\begin{aligned}
\left(\frac{\partial L_2^i}{\partial t_1}\right)_{n,n} &= (L_2^i)_{n+1,n} - (L_2^i)_{n,n-1}, \\
\left(\frac{\partial L_2}{\partial t_1}\right)_{n,n} &= x_{n-1}y_{n-1} - x_n y_n,
\end{aligned}$$

while the latter relations of Lemma 1.3 are obtained from the first two by the 2-Toda involution (1.1.15). This ends the proof of Lemma 1.3. □

1.2. Virasoro constraints. According to [3], the (vector) 2-Toda vertex operator[8]

$$\mathbb{X}_{12}(t,s;u,v) = \Lambda^{-1} e^{\sum_1^\infty (t_i u^i - s_i v^i)} e^{-\sum_1^\infty \left(\frac{u^{-i}}{i}\frac{\partial}{\partial t_i} - \frac{v^{-i}}{i}\frac{\partial}{\partial s_i}\right)} \chi(uv), \tag{1.2.1}$$

acting on vectors of functions of t and s, interacts with the operators $\mathbb{J}_k^{(i)}(t) = \left(\mathbb{J}_{k,n}^{(i)}(t,n)\right)_{n\geq 0}$, as follows: (for definitions, see Appendix 1)

$$\begin{aligned} u^k \mathbb{X}_{12}(t,s;u,v) &= \left[\mathbb{J}_k^{(1)}(t), \mathbb{X}_{12}(t,s;u,v)\right], \\ \frac{\partial}{\partial u} u^{k+1} \mathbb{X}_{12}(t,s;u,v) &= \left[\mathbb{J}_k^{(2)}(t), \mathbb{X}_{12}(t,s;u,v)\right]. \end{aligned} \tag{1.2.2}$$

A similar statement can be made, upon replacing the operators u^k and $\frac{\partial}{\partial u}u^{k+1}$ by v^k and $\frac{\partial}{\partial v}v^{k+1}$, and upon using $\tilde{\mathbb{J}}_k^{(i)}(s) = \mathbb{J}_k^{(i)}(-s)$.

Also consider the vertex operator, integrated over the unit circle and depending on an integer γ,

$$\mathbb{Y}^\gamma(t,s) = \int_{S^1} \frac{du}{2\pi i u} u^\gamma \mathbb{X}_{12}(t,s;u,u^{-1}), \tag{1.2.3}$$

and the vector Virasoro constraint $\mathbb{V}_k^\gamma(t,s) := (\mathbb{V}_{k,n}^\gamma)_{n\geq 0}$,

$$\mathbb{V}_k^\gamma := \mathbb{J}_k^{(2)}(t) - \mathbb{J}_{-k}^{(2)}(-s) - (k-\gamma)\big(\theta \mathbb{J}_k^{(1)}(t) + (1-\theta)\mathbb{J}_{-k}^{(1)}(-s)\big), \tag{1.2.4}$$

depending on a free parameter θ.

Theorem 1.4 (Adler-van Moerbeke [3]). *The multiple integrals over the unit circle S^1,*

$$\tau_n^\gamma(t,s) = \frac{1}{n!}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n z_k^\gamma e^{\sum_1^\infty (t_j z_k^j - s_j z_k^{-j})} \frac{dz_k}{2\pi i z_k}, \quad n > 0, \tag{1.2.5}$$

with $\tau_0^\gamma = 1$, satisfy an sl(2,$\mathbb{R}$)-algebra of Virasoro constraints:

$$\mathbb{V}_{k,n}^\gamma \tau_n(t,s) = 0, \quad \textit{for} \left\{\begin{array}{l} k=-1,\ \theta=0 \\ k=0,\ \theta\ arbitrary \\ k=1,\ \theta=1 \end{array}\right\} \ \textit{only.} \tag{1.2.6}$$

---

[8] For $v = (v_0, v_1, \ldots)^\top$, $(\Lambda v)_n = v_{n+1}$, $(\Lambda^\top v)_n = v_{n-1}$, and $\chi(z) := (1, z, z^2, \ldots)$.

Working out the Virasoro equations of Appendix 1 (for $\beta = 1/2$),

$$\begin{aligned} &\mathbb{V}^{\gamma}_{k,n} \\ &:= \mathbb{J}^{(2)}_{k,n}(t,n) - \mathbb{J}^{(2)}_{-k,n}(-s,n) - (k-\gamma)\left(\theta \mathbb{J}^{(1)}_{k,n}(t,n) + (1-\theta)\mathbb{J}^{(1)}_{-k,n}(-s,n)\right) \\ &= \frac{1}{2}\left(J^{(2)}_{k}(t) - J^{(2)}_{-k}(-s) + (2n+k+1)J^{(1)}_{k}(t) - (2n-k+1)J^{(1)}_{-k}(-s)\right) \\ &\quad - (k-\gamma)\left(\theta J^{(1)}_{k}(t) + (1-\theta)J^{(1)}_{-k}(-s)\right) + \gamma n \delta_{k,0}, \end{aligned} \tag{1.2.7}$$

one finds Virasoro constraints for the integral τ^{γ}_{n}:

$$\begin{aligned} \mathbb{V}^{\gamma}_{-1,n}\tau^{\gamma}_{n} &= \left(\sum_{i\geq 1}(i+1)t_{i+1}\frac{\partial}{\partial t_i} - \sum_{i\geq 2}(i-1)s_{i-1}\frac{\partial}{\partial s_i} + nt_1 + (n-\gamma)\frac{\partial}{\partial s_1}\right)\tau^{\gamma}_{n} = 0, \\ \mathbb{V}^{\gamma}_{0,n}\tau^{\gamma}_{n} &= \sum_{i\geq 1}\left(it_i\frac{\partial}{\partial t_i} - is_i\frac{\partial}{\partial s_i}\right)\tau^{\gamma}_{n} + \gamma n \tau^{\gamma}_{n} = 0, \qquad (1.2.8) \\ \mathbb{V}^{\gamma}_{1,n}\tau^{\gamma}_{n} &= \left(-\sum_{i\geq 1}(i+1)s_{i+1}\frac{\partial}{\partial s_i} + \sum_{i\geq 2}(i-1)t_{i-1}\frac{\partial}{\partial t_i} + ns_1 + (n+\gamma)\frac{\partial}{\partial t_1}\right)\tau^{\gamma}_{n} = 0. \end{aligned}$$

The theorem is based on two lemmas: (see (1.2.1), (1.2.3), (1.2.4) for the definition of $\mathbb{X}_{12}, \mathbb{Y}^{\gamma}, \mathbb{V}^{\gamma}_{k}$)

Lemma 1.5. *The following commutation relations hold:*

$$u^{-\gamma}u\frac{d}{du}u^{k+\gamma}\mathbb{X}_{12}(t,s;u,u^{-1}) = \left[\mathbb{V}^{\gamma}_{k}, \mathbb{X}_{12}(t,s;u,u^{-1})\right], \tag{1.2.9}$$

and

$$\left[\mathbb{Y}^{\gamma}, \mathbb{V}^{\gamma}_{k}\right] = 0. \tag{1.2.10}$$

Proof. Using (1.2.2), a standard computation shows

$$\begin{aligned} &u\frac{d}{du}u^{k}\mathbb{X}_{12}(t,s;u,u^{-1}) \qquad (1.2.11) \\ &= \left(u^{k+1}\frac{d}{du} + ku^{k}\right)\mathbb{X}_{12}(t,s;u,u^{-1}) \\ &= \left(u^{k+1}\frac{\partial}{\partial u} - v^{1-k}\frac{\partial}{\partial v} + ku^{k}\right)\mathbb{X}_{12}(t,s;u,v)\Big|_{v=u^{-1}} \\ &= \left(\frac{\partial}{\partial u}u^{k+1} - \frac{\partial}{\partial v}v^{1-k} - ku^{k}\right)\mathbb{X}_{12}(t,s;u,v)\Big|_{v=u^{-1}} \\ &= \left(\frac{\partial}{\partial u}u^{k+1} - \frac{\partial}{\partial v}v^{1-k} - k\theta u^{k} - k(1-\theta)v^{-k}\right)\mathbb{X}_{12}(t,s;u,v)\Big|_{v=u^{-1}} \end{aligned}$$

$$\begin{aligned} &= \Big[\mathbb{J}_k^{(2)}(t) - \mathbb{J}_{-k}^{(2)}(-s) - k\left(\theta \mathbb{J}_k^{(1)}(t) + (1-\theta)\ \mathbb{J}_{-k}^{(1)}(-s)\right), \\ &\qquad\qquad\qquad\qquad\qquad\qquad \mathbb{X}_{12}(t,s;u,u^{-1})\Big] \\ &= \left[\mathbb{V}_k^{(0)}, \mathbb{X}_{12}(t,s;u,u^{-1})\right], \end{aligned} \tag{1.2.12}$$

from which (1.2.9) follows, for $\gamma = 0$. More generally, we compute

$$u^{-\gamma} u \frac{d}{du} u^{k+\gamma} \mathbb{X}_{12}(t,s;u,u^{-1}) \tag{1.2.13}$$

$$\begin{aligned} &= \left(u^{k+1}\frac{d}{du} + ku^k\right)\mathbb{X}_{12}(t,s;u,u^{-1}) + \gamma u^k \mathbb{X}_{12}(t,s;u,u^{-1}) \\ &= \left[\mathbb{V}_k^{(0)}, \mathbb{X}_{12}(t,s;u,u^{-1})\right] + \gamma\left.\left(\theta u^k + (1-\theta)\ v^{-k}\right)\right|_{v=u^{-1}} \mathbb{X}_{12} \\ &= \left[\mathbb{V}_k^{(0)} + \gamma\left(\theta \mathbb{J}_k^{(1)}(t) + (1-\theta)\mathbb{J}_{-k}^{(1)}(-s)\right), \mathbb{X}_{12}(t,s;u,u^{-1})\right] \\ &= \left[\mathbb{V}_k^{\gamma}, \mathbb{X}_{12}(t,s;u,u^{-1})\right] \end{aligned} \tag{1.2.14}$$

from which (1.2.9) follows. We then have

$$\begin{aligned} \left[\mathbb{V}_k^{\gamma}, \mathbb{Y}^{\gamma}(t,s)\right] &= \left[\mathbb{V}_k^{\gamma}, \int_{S^1} \mathbb{X}_{12}(t,s;u,u^{-1}) u^{\gamma} \frac{du}{2\pi i u}\right] \\ &= \int_{S^1} \left[\mathbb{V}_k^{\gamma}, \mathbb{X}_{12}(t,s;u,u^{-1})\right] u^{\gamma} \frac{du}{2\pi i u} \\ &= \int_{S^1} \frac{du}{2\pi i} \frac{d}{du} u^{k+\gamma} \mathbb{X}_{12}(t,s;u,u^{-1}) \\ &= 0, \end{aligned}$$

leading to (1.2.10). □

Lemma 1.6. *The vector $I := (I_n)_{n\geq 0}$, with $I_n = n!\tau_n^{\gamma}$, is a fixed point for the vertex operator $\mathbb{Y}^{\gamma}$,*

$$\mathbb{Y}^{\gamma}(t,s) I(t,s) = I(t,s). \tag{1.2.15}$$

Proof. Setting $\rho(dz) = z^{\gamma} dz$, one computes, for $n \geq 1$,

$$\begin{aligned} I_n(t,s) &= n!\tau_n^{\gamma}(t,s) \\ &= \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n}\left(e^{\sum_1^{\infty}(t_j z_k^j - s_j z_k^{-j})} \frac{\rho(dz_k)}{2\pi i z_k}\right) \\ &= \int_{S^1} \frac{\rho(du)}{2\pi i u} e^{\sum_1^{\infty}(t_j u^j - s_j u^{-j})} u^{n-1} u^{-n+1} \\ &\quad \times \int_{(S^1)^{n-1}} \Delta_{n-1}(z)\bar{\Delta}_{n-1}(z) \\ &\quad \times \prod_{k=1}^{n-1}\left(1 - \frac{z_k}{u}\right)\left(1 - \frac{u}{z_k}\right) e^{\sum_1^{\infty}(t_j z_k^j - s_j z_k^{-j})} \frac{\rho(dz_k)}{2\pi i z_k} \end{aligned}$$

$$
\begin{aligned}
&= \int_{S^1} \frac{\rho(du)}{2\pi i u} e^{\sum_1^\infty (t_j u^j - s_j u^{-j})}\, e^{-\sum_1^\infty \left(\frac{u^{-j}}{j} \frac{\partial}{\partial t_j} - \frac{u^j}{j} \frac{\partial}{\partial s_j} \right)} \\
&\quad \times \int_{(S^1)^{n-1}} \Delta_{n-1}(z) \bar{\Delta}_{n-1}(z) \prod_{k=1}^{n-1} e^{\sum_1^\infty (t_j z_k^j - s_j z_k^{-j})} \frac{\rho(dz_k)}{2\pi i z_k} \\
&= \int_{S^1} \frac{\rho(du)}{2\pi i u} e^{\sum_1^\infty (t_j u^j - s_j u^{-j})} e^{-\sum_1^\infty \left(\frac{u^{-j}}{j} \frac{\partial}{\partial t_j} - \frac{u^j}{j} \frac{\partial}{\partial s_j} \right)} I_{n-1}(t,s) \\
&= \Big(\mathbb{Y}^\gamma(t,s) I(t,s) \Big)_n ,
\end{aligned}
\tag{1.2.16}
$$

from which (1.2.15) follows. □

Proof of Theorem 1.4. From (1.2.10), we have

$$
\begin{aligned}
0 &= \left([\mathbb{V}_k^\gamma, (\mathbb{Y}^\gamma)^n] I \right)_n \\
&= \left(\mathbb{V}_k^\gamma (\mathbb{Y}^\gamma)^n I - (\mathbb{Y}^\gamma)^n \mathbb{V}_k^\gamma I \right)_n \\
&= \left(\mathbb{V}_k^\gamma I - (\mathbb{Y}^\gamma)^n \mathbb{V}_k^\gamma I \right)_n .
\end{aligned}
$$

Taking the n^{th} component and taking into account the presence of Λ^{-1} in $\mathbb{X}_{12}(t,s;u,u^{-1})$, we find

$$
\begin{aligned}
0 &= \left(\mathbb{V}_k^\gamma I - \mathbb{Y}^n \mathbb{V}_k^\gamma I \right)_n \\
&= \mathbb{V}_k^\gamma I_n - \int_{S^1} \frac{du}{2\pi i u} e^{\sum_1^\infty (t_i u^i - s_i u^{-i})} e^{-\sum_1^\infty \left(\frac{u^{-i}}{i} \frac{\partial}{\partial t_i} - \frac{u^i}{i} \frac{\partial}{\partial s_i} \right)} \\
&\qquad \cdots \int_{S^1} \frac{du}{2\pi i u} e^{\sum_1^\infty (t_i u^i - s_i u^{-i})} e^{-\sum_1^\infty \left(\frac{u^{-i}}{i} \frac{\partial}{\partial t_i} - \frac{u^i}{i} \frac{\partial}{\partial s_i} \right)} \mathbb{V}_k^\gamma I_0 .
\end{aligned}
$$

Remember from (1.2.7), $\mathbb{V}_k^\gamma(t,s)$ has the following form:

$$
\begin{aligned}
\mathbb{V}_k^\gamma(t,s) &= \frac{1}{2} \Big(J_k^{(2)}(t) - J_{-k}^{(2)}(-s) + (2n+k+1) J_k^{(1)}(t) - (2n-k+1) J_{-k}^{(1)}(-s) \Big) \\
&\quad - (k-\gamma) \Big(\theta J_k^{(1)}(t) + (1-\theta) J_{-k}^{(1)}(-s) \Big) + \gamma n \delta_{k,0},
\end{aligned}
$$

and one checks immediately that, given $\tau_0 = 1$,

$$
\mathbb{V}_k^\gamma(t,s)\tau_0 = 0 \quad \text{only for} \quad \left\{ \begin{array}{l} k=-1, \ \ \theta = 0 \\ k=0, \ \ \theta \ arbitrary \\ k=1, \ \ \theta = 1 \end{array} \right\},
$$

ending the proof of Theorem 1.4. □

2. Rational Recursion Relations

2.1. Weights.

Lemma 2.1.

$$\mathcal{L} = \left\{ \begin{array}{l} it_i = it_i^{(0)} := \begin{cases} u_i - (\gamma_1' d_1^i + \gamma_2' d_2^i), \ \textit{for}\ 1 \leq i \leq N_1 \\ -(\gamma_1' d_1^i + \gamma_2' d_2^i), \ \textit{for}\ N_1 + 1 \leq i < \infty \end{cases} \\ is_i = is_i^{(0)} := \begin{cases} -u_{-i} + (\gamma_1'' d_1^{-i} + \gamma_2'' d_2^{-i}), \ \textit{for}\ 1 \leq i \leq N_2 \\ (\gamma_1'' d_1^{-i} + \gamma_2'' d_2^{-i}), \ \textit{for}\ N_2 + 1 \leq i < \infty \end{cases} \end{array} \right\}. \tag{2.1.1}$$

Then, setting $\gamma_1 = \gamma_1' + \gamma_1''$ *and* $\gamma_2 = \gamma_2' + \gamma_2''$, *we have*

$$\begin{aligned} & e^{\sum_1^\infty (t_i z^i - s_i z^{-i})} \Big|_{\mathcal{L}} \\ & = e^{P_1(z) + P_2(z^{-1})} (1 - d_1 z)^{\gamma_1'} (1 - d_2 z)^{\gamma_2'} (1 - d_1^{-1} z^{-1})^{\gamma_1''} (1 - d_2^{-1} z^{-1})^{\gamma_2''} \\ & = k z^{-\gamma_1'' - \gamma_2''} e^{P_1(z) + P_2(z^{-1})} (1 - d_1 z)^{\gamma_1} (1 - d_2 z)^{\gamma_2} \end{aligned} \tag{2.1.2}$$

with a constant k and

$$P_1(z) := \sum_1^{N_1} \frac{u_i z^i}{i} \ \ \textit{and} \ \ P_2(z^{-1}) := \sum_1^{N_2} \frac{u_{-i} z^{-i}}{i}. \tag{2.1.3}$$

Moreover, there exist a, b, c *such that*

$$\begin{aligned} a(i+1) t_{i+1}^{(0)} + b i t_i^{(0)} + c(i-1) t_{i-1}^{(0)} &= 0 \ \ \textit{for all}\ i \geq N_1 + 2, \\ a(i-1) s_{i-1}^{(0)} + b i s_i^{(0)} + c(i+1) s_{i+1}^{(0)} &= 0 \ \ \textit{for all}\ i \geq N_2 + 2. \end{aligned} \tag{2.1.4}$$

Then

$$\begin{aligned} a(i+1) t_{i+1}^{(0)} + b i t_i^{(0)} + c(i-1) t_{i-1}^{(0)} &= a u_{i+1} + b u_i + c u_{i-1} + c \delta_{i1} (\gamma_1' + \gamma_2'), \\ & \qquad \textit{for}\ 1 \leq i \leq N_1 + 1, \\ a(i-1) s_{i-1}^{(0)} + b i s_i^{(0)} + c(i+1) s_{i+1}^{(0)} &= -c u_{-i-1} - b u_{-i} - a u_{-i+1} - a \delta_{i1} (\gamma_1'' + \gamma_2''), \\ & \qquad \textit{for}\ 1 \leq i \leq N_2 + 1, \end{aligned} \tag{2.1.5}$$

upon setting $u_0 = u_{N_1+1} = u_{N_1+2} = u_{-N_2-1} = u_{-N_2-2} = 0$.

- *Case 1.* $d_1, d_2, d_1 - d_2 \neq 0$ *and* $|\gamma_1'| + |\gamma_1''|, \ |\gamma_2'| + |\gamma_2''| \neq 0$. *Then the unique solution to (2.1.4) is given by*

$$a = 1, \ b = -d_1 - d_2, \ c = d_1 d_2.$$

- *Case 2.* $d_1 \neq 0,\ \gamma_1' \neq 0$ *arbitrary,* $d_2 = 0,\ \gamma_2' = \gamma_1'' = \gamma_2'' = 0$. *Then there exist two solutions*

  $$(a, b, c) = (1, -d_1, 0) \text{ and } (a, b, c) = (0, 1, -d_1)$$

  such that (2.1.4) holds.
- *Case 3.* $d_1 = d_2 = 0,\ \gamma_1' = \gamma_2' = \gamma_1'' = \gamma_2'' = 0$. *Then* a, b, c *may be taken arbitrary.*

Proof. Formula (2.1.2) follows immediately from $1 - x = \exp\left(-\sum_1^\infty x^i/i\right)$, while (2.1.4) and (2.1.5) are obvious. □

Remark. The locus $\mathcal{L}$, defined in (2.1.1), provides the only example where (2.1.4) holds.

2.2. Rational recursion relations. Considering the t-dependent basic variables,

$$x_n(t) = (-1)^n \frac{\tau_n^+(t)}{\tau_n(t)} \quad \text{and} \quad y_n(t) = (-1)^n \frac{\tau_n^-(t)}{\tau_n(t)}, \tag{2.2.1}$$

where τ_n is the integral

$$\tau_n^\pm = \frac{1}{n!}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(z_k^{\gamma \pm 1} e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k} \right),$$

$$\tau_n = \frac{1}{n!}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(z_k^{\gamma} e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k} \right).$$

Along the locus $\mathcal{L}$, defined in (2.1.1), these integrals and variables reduce to the original integrals and variables (0.0.5) and (0.0.6). In the statement below, we deal with the variables $x_n(t)$ and $y_n(t)$, without restricting to the locus $\mathcal{L}$.

Theorem 2.2. *Set* $v_n := 1 - x_n y_n$,

$$\alpha_i(t) := a(i+1)t_{i+1} + bit_i + c(i-1)t_{i-1} + c(n+\gamma)\delta_{i1},$$
$$\beta_i(s) := a(i-1)s_{i-1} + bis_i + c(i+1)s_{i+1} - a(n-\gamma)\delta_{i1},$$

and

$$\mathcal{L}_1^{(n)} = \sum_{i\geq 1} \alpha_i(t) L_1^i \quad \text{and} \quad \mathcal{L}_2^{(n)} = -\sum_{i\geq 1} \beta_i(t) L_2^i. \tag{2.2.2}$$

Then the following holds:

- *Cases 1 and 2.* a, c *not both* $= 0$*:*

  $$\partial_n(\mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)})_{n,n} + (cL_1 - aL_2)_{nn} = 0 \tag{2.2.3}$$

  and

  $$\partial_n^2 \left(v_{n-1}\mathcal{L}_1^{(n-1)} - \mathcal{L}_2^{(n-1)} \right)_{n,n-1} + \partial_n \left(c(L_1^2)_{nn} + b(L_1)_{nn} \right) = 0. \tag{2.2.4}$$

- *Case 3.* $a = c = 0,\ b = 1$*:*

$$\left\{\begin{array}{l} \displaystyle\sum_{i\geq 1}\left\{\alpha_i\left((L_1^i)_{n+1,n+1}-(L_1^{i-1})_{n+1,n}\right)-\beta_i\left((L_2^i)_{nn}-(L_2^{i-1})_{n,n+1}\right)\right\} \\ \hfill -n\frac{x_n y_n}{v_n}=0, \\ \displaystyle\sum_{i\geq 1}\left\{\alpha_i\left((L_1^i)_{nn}-(L_1^{i-1})_{n+1,n}\right)-\beta_i\left((L_2^i)_{n+1,n+1}-(L_2^{i-1})_{n,n+1}\right)\right\} \\ \hfill -n\frac{x_n y_n}{v_n}=0. \end{array}\right. \tag{2.2.5}$$

Remark. Written out, Eqs. (2.2.3) and (2.2.4) take on the following form:

$$(\mathcal{L}_1-\mathcal{L}_2+aL_2-cL_1)_{nn}-(\mathcal{L}_1-\mathcal{L}_2-aL_2+cL_1)_{n+1,n+1}=0 \tag{2.2.6}$$

and

$$\begin{aligned} &2(v_n\mathcal{L}_1-\mathcal{L}_2)_{n+1,n}-(v_{n-1}\mathcal{L}_1-\mathcal{L}_2)_{n,n-1}-(v_{n+1}\mathcal{L}_1-\mathcal{L}_2)_{n+2,n+1} \\ &\quad +a(v_{n+1}-v_{n-1})+b(x_{n+1}y_n-x_ny_{n-1}) \\ &\quad +c(2y_nx_{n+2}v_{n+1}-2x_ny_{n-2}v_{n-1}+x_n^2y_{n-1}^2-y_n^2x_{n+1}^2)=0. \end{aligned} \tag{2.2.7}$$

Proof. At first, compute the first and second difference in (2.2.3) and (2.2.4),

$$\begin{aligned} &\partial_n(\mathcal{L}_1^{(n)}-\mathcal{L}_2^{(n)})_{n,n} \\ &\quad =(\mathcal{L}_1^{(n)}-\mathcal{L}_2^{(n)})_{n+1,n+1}-(\mathcal{L}_1^{(n)}-\mathcal{L}_2^{(n)})_{n,n}+(cL_1-aL_2)_{n+1,n+1}. \end{aligned} \tag{2.2.8}$$

$$\begin{aligned} &\partial_n^2(v_{n-1}\mathcal{L}_1^{(n)}-\mathcal{L}_2^{(n)})_{n,n-1} \\ &\quad =(v_{n+1}\mathcal{L}_1^{(n)}-\mathcal{L}_2^{(n)})_{n+2,n+1}+(v_{n-1}\mathcal{L}_1^{(n)}-\mathcal{L}_2^{(n)})_{n,n-1}-2(v_n\mathcal{L}_1^{(n)}-\mathcal{L}_2^{(n)})_{n+1,n} \\ &\qquad +2c\left(v_{n+1}(L_1)_{n+2,n+1}-v_n(L_1)_{n+1,n}\right)-2a\big((L_2)_{n+2,n+1}-(L_2)_{n+1,n}\big) \\ &\quad =(v_{n+1}\mathcal{L}_1^{(n)}-\mathcal{L}_2^{(n)})_{n+2,n+1}+(v_{n-1}\mathcal{L}_1^{(n)}-\mathcal{L}_2^{(n)})_{n,n-1}-2(v_n\mathcal{L}_1^{(n)}-\mathcal{L}_2^{(n)})_{n+1,n} \\ &\qquad +2\partial_n\left(cv_n(L_1)_{n+1,n}-a(L_2)_{n+1,n}\right). \end{aligned} \tag{2.2.9}$$

In order to obtain the first identity, namely (2.2.3), form arbitrary linear combinations of the Virasoro equations (1.2.8),

$$\begin{aligned} &\frac{1}{\tau_n^{\gamma+\varepsilon}}\left(a\mathbb{V}_{-1,n}^{\gamma+\varepsilon}+b\mathbb{V}_{0,n}^{\gamma+\varepsilon}+c\mathbb{V}_{1,n}^{\gamma+\varepsilon}\right)\tau_n^{\gamma+\varepsilon} \\ &\quad =\left\{\begin{array}{l} \displaystyle\sum_{i\geq 1}\Big((a(i+1)t_{i+1}+bit_i+c(i-1)t_{i-1})\frac{\partial}{\partial t_i} \\ \displaystyle -(a(i-1)s_{i-1}+bis_i+c(i+1)s_{i+1})\frac{\partial}{\partial s_i}\Big) \\ \displaystyle +c(n+\gamma)\frac{\partial}{\partial t_1}+a(n-\gamma)\frac{\partial}{\partial s_1} \end{array}\right\}\log\tau_n^{\gamma+\varepsilon} \\ &\qquad +\varepsilon\left(\left(-a\frac{\partial}{\partial s_1}+c\frac{\partial}{\partial t_1}\right)\log\tau_n^{\gamma+\varepsilon}+bn\right)+n(at_1+b_\gamma+cs_1) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\geq 1}\left(\alpha_i(t)\frac{\partial}{\partial t_i} - \beta_i(s)\frac{\partial}{\partial s_i}\right)\log \tau_n^{\gamma+\varepsilon} \\
&\quad + \varepsilon\left(\left(c\frac{\partial}{\partial t_1} - a\frac{\partial}{\partial s_1}\right)\log\tau_n^{\gamma+\varepsilon} + bn\right) + n(at_1 + b_\gamma + cs_1) = 0.
\end{aligned}
$$

Subtracting the $\varepsilon = \pm 1$ contribution from the $\varepsilon = 0$ contribution and omitting the lower index n in $\mathbb{V}_{k,n}^{\gamma+\varepsilon}$, leads to

$$
\begin{aligned}
0 &= \frac{1}{\tau_n^{\gamma+\varepsilon}}\left(a\mathbb{V}_{-1}^{\gamma+\varepsilon} + b\mathbb{V}_0^{\gamma+\varepsilon} + c\mathbb{V}_1^{\gamma+\varepsilon}\right)\tau_n^{\gamma+\varepsilon} - \frac{1}{\tau_n^{\gamma}}\left(a\mathbb{V}_{-1}^{\gamma} + b\mathbb{V}_0^{\gamma} + c\mathbb{V}_1^{\gamma}\right)\tau_n^{\gamma} \\
&= \sum_{i\geq 1}\left(\alpha_i(t)\frac{\partial}{\partial t_i} - \beta_i(s)\frac{\partial}{\partial s_i}\right)\log\frac{\tau_n^{\gamma+\varepsilon}}{\tau_n^{\gamma}} + \varepsilon\left(\left(c\frac{\partial}{\partial t_1} - a\frac{\partial}{\partial s_1}\right)\log\tau_n^{\gamma+\varepsilon} + bn\right).
\end{aligned}
\tag{2.2.10}
$$

Set, for brevity, $\tau_n = \tau_n^{\gamma}$ and $\tau_n^{\pm} = \tau_n^{\gamma+\varepsilon}$, $\mathbb{V} = \mathbb{V}^{\gamma}$, $\mathbb{V}^{\pm} = \mathbb{V}^{\gamma+\varepsilon}$, with $\varepsilon = \pm 1$.

- *Case 1 and 2.* When not both $a = c = 0$, the terms $\frac{\partial}{\partial t_1}\log\tau_n^{\gamma+\varepsilon}$ and $\frac{\partial}{\partial s_1}\log\tau_n^{\gamma+\varepsilon}$ are present; they cannot be easily expressed in terms of x_n and y_n. But adding the $+$contribution to the $-$contribution eliminates those terms:

$$
\begin{aligned}
0 &= \frac{x_n y_n}{v_n}\left\{\begin{array}{c} \frac{1}{\tau_n^{+}}\left(a\mathbb{V}_{-1}^{+} + b\mathbb{V}_0^{+} + c\mathbb{V}_1^{+}\right)\tau_n^{+} + \frac{1}{\tau_n^{-}}\left(a\mathbb{V}_{-1}^{-} + b\mathbb{V}_0^{-} + c\mathbb{V}_1^{-}\right)\tau_n^{-} \\ -\frac{2}{\tau_n}\left(a\mathbb{V}_{-1} + b\mathbb{V}_0 + c\mathbb{V}_1\right)\tau_n \end{array}\right\} \\
&= \frac{x_n y_n}{v_n}\left(\sum_{i\geq 1}\left(\alpha_i(t)\frac{\partial}{\partial t_i} - \beta_i(s)\frac{\partial}{\partial s_i}\right)\log x_n y_n + \left(c\frac{\partial}{\partial t_1} - a\frac{\partial}{\partial s_1}\right)\log\frac{x_n}{y_n}\right). \\
&= \frac{x_n y_n}{v_n}\left(\sum_{i\geq 1}\left(\alpha_i\frac{\partial}{\partial t_i} - \beta_i\frac{\partial}{\partial s_i}\right)(\log x_n + \log y_n) + \left(c\frac{\partial}{\partial t_1} - a\frac{\partial}{\partial s_1}\right)(\log x_n - \log y_n)\right) \\
&= (\mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)} + aL_2 - cL_1)_{nn} - (\mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)} - aL_2 + cL_1)_{n+1,n+1}, \\
&\qquad\qquad \text{using (2.2.1), (1.1.31) and the definition (2.2.2) of } \mathcal{L}_i^{(n)}, \\
&= -\partial_n(\mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)})_{n,n} + (aL_2 - cL_1)_{nn}
\end{aligned}
$$

using (2.2.8). This establishes the first relation, namely (2.2.3).

To prove the second relation (2.2.4), we take the t_1-derivative of the first relation. Using

$$
\begin{aligned}
\frac{\partial\alpha_i}{\partial t_1} &= 0 \text{ for } i \geq 3 \\
&= c \text{ for } i = 2 \\
&= b \text{ for } i = 1, \\
\frac{\partial\beta_i}{\partial t_1} &= 0 \text{ for } i \geq 1,
\end{aligned}
$$

we obtain

$$\frac{\partial(\mathcal{L}_1-\mathcal{L}_2)}{\partial t_1}=\sum_{i\geq 1}\alpha_i(t)\frac{\partial L_1^i}{\partial t_1}+\sum_{i\geq 1}\beta_i(t)\frac{\partial L_2^i}{\partial t_1}+cL_1^2+bL_1.$$

Using the above and

$$\begin{aligned}
(L_1^2)_{n,n} &= x_n^2y_{n-1}^2-x_ny_{n-2}v_{n-1}-x_{n+1}y_{n-1}v_n,\\
(L_2^2)_{n,n} &= y_n^2x_{n-1}^2-y_nx_{n-2}v_{n-1}-y_{n+1}x_{n-1}v_n,\\
(L_2^2)_{n,n-1} &= -v_{n-1}(x_{n-1}y_n+x_{n-2}y_{n-1}),
\end{aligned}$$

one computes, using (2.2.2) and Lemma 1.3,

$$\begin{aligned}
0 &= \frac{\partial}{\partial t_1}(\mathcal{L}_1-\mathcal{L}_2+aL_2-cL_1)_{n,n}-\frac{\partial}{\partial t_1}(\mathcal{L}_1-\mathcal{L}_2-aL_2+cL_1)_{n+1,n+1}\\
&= \left(\frac{\partial(\mathcal{L}_1-\mathcal{L}_2)}{\partial t_1}\right)_{n,n}-\left(\frac{\partial(\mathcal{L}_1-\mathcal{L}_2)}{\partial t_1}\right)_{n+1,n+1}\\
&\qquad +a\left(\left(\frac{\partial L_2}{\partial t_1}\right)_{n,n}+\left(\frac{\partial L_2}{\partial t_1}\right)_{n+1,n+1}\right)-c\left(\left(\frac{\partial L_1}{\partial t_1}\right)_{n,n}+\left(\frac{\partial L_1}{\partial t_1}\right)_{n+1,n+1}\right)\\
&= \sum_{i\geq 1}\alpha_i(t)\left(2v_n(L_1^i)_{n+1,n}-v_{n-1}(L_1^i)_{n,n-1}-v_{n+1}(L_1^i)_{n+2,n+1}\right)\\
&\quad +\sum_{i\geq 1}\beta_i(t)\left(2(L_2^i)_{n+1,n}-(L_2^i)_{n,n-1}-(L_2^i)_{n+2,n+1}\right)\\
&\quad +c\left((L_1^2)_{nn}-(L_1^2)_{n+1,n+1}-v_{n+1}(L_1)_{n+2,n+1}+v_{n-1}(L_1)_{n,n-1}\right)\\
&\quad +a\left((L_2)_{n+2,n+1}-(L_2)_{n,n-1}\right)+b\left((L_1)_{nn}-(L_1)_{n+1,n+1}\right)\\
&= 2v_n(\mathcal{L}_1)_{n+1,n}-v_{n-1}(\mathcal{L}_1)_{n,n-1}-v_{n+1}(\mathcal{L}_1)_{n+2,n+1}\\
&\quad -2(\mathcal{L}_2)_{n+1,n}+(\mathcal{L}_2)_{n,n-1}+(\mathcal{L}_2)_{n+2,n+1}\\
&\quad +c\left((L_1^2)_{nn}-(L_1^2)_{n+1,n+1}-v_{n+1}(L_1)_{n+2,n+1}+v_{n-1}(L_1)_{n,n-1}\right)\\
&\quad +a\left((L_2)_{n+2,n+1}-(L_2)_{n,n-1}\right)+b\left((L_1)_{nn}-(L_1)_{n+1,n+1}\right)\\
&\overset{*}{=} 2(v_n\mathcal{L}_1-\mathcal{L}_2)_{n+1,n}-(v_{n-1}\mathcal{L}_1-\mathcal{L}_2)_{n,n-1}-(v_{n+1}\mathcal{L}_1-\mathcal{L}_2)_{n+2,n+1}\\
&\quad +a(v_{n+1}-v_{n-1})+b(x_{n+1}y_n-x_ny_{n-1})\\
&\quad +c(2y_nx_{n+2}v_{n+1}-2x_ny_{n-2}v_{n-1}+x_n^2y_{n-1}^2-y_n^2x_{n+1}^2)\\
&= 2(v_n\mathcal{L}_1-\mathcal{L}_2)_{n+1,n}-(v_{n-1}\mathcal{L}_1-\mathcal{L}_2)_{n,n-1}-(v_{n+1}\mathcal{L}_1-\mathcal{L}_2)_{n+2,n+1}\\
&\quad +\partial_n\left(\begin{array}{l}
a\left((L_2)_{n,n-1}+(L_2)_{n+1,n}\right)\\
-\left(c(L_1^2)_{nn}+b(L_1)_{nn}\right)\\
-c\left(v_n(L_1)_{n+1,n}+v_{n-1}(L_1)_{n,n-1}\right)
\end{array}\right)
\end{aligned}$$

$$\begin{aligned}
&\overset{**}{=} -\partial_n^2(v_{n-1}\mathcal{L}_1 - \mathcal{L}_2)_{n,n-1} + 2\partial_n\left(cv_n(L_1)_{n+1,n} - a(L_2)_{n+1,n}\right)\\
&\quad + \partial_n \left(\begin{array}{l} a\left((L_2)_{n,n-1} + (L_2)_{n+1,n}\right)\\ -\left(c(L_1^2)_{nn} + b(L_1)_{nn}\right)\\ -c\left(v_n(L_1)_{n+1,n} + v_{n-1}(L_1)_{n,n-1}\right)\end{array}\right)\\
&= -\partial_n^2(v_{n-1}\mathcal{L}_1 - \mathcal{L}_2)_{n,n-1}\\
&\quad + \partial_n \left(\begin{array}{l} a\left((L_2)_{n,n-1} - (L_2)_{n+1,n}\right)\\ -\left(c(L_1^2)_{nn} + b(L_1)_{nn}\right)\\ +c\left(v_n(L_1)_{n+1,n} - v_{n-1}(L_1)_{n,n-1}\right)\end{array}\right)\\
&= -\partial_n^2(v_{n-1}\mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)} + aL_2 - cv_{n-1}L_1)_{n,n-1}\\
&\quad - \partial_n\left(c(L_1^2)_{nn} + b(L_1)_{nn}\right)\\
&= -\partial_n^2(v_{n-1}\mathcal{L}_1^{(n-1)} - \mathcal{L}_2^{(n-1)})_{n,n-1} - \partial_n\left(c(L_1^2)_{nn} + b(L_1)_{nn}\right).\\
&= -\partial_n\left(\partial_n(v_{n-1}\mathcal{L}_1^{(n-1)} - \mathcal{L}_2^{(n-1)})_{n,n-1} + \left(c(L_1^2)_{nn} + b(L_1)_{nn}\right)\right),
\end{aligned}$$

using (2.2.9) in $\overset{**}{=}$, ending the proof of identity (2.2.4). Equality $\overset{*}{=}$ leads to the expression in the remark after the statement of the theorem.

• *Case 3.* When both $a = c = 0$, the terms $\frac{\partial}{\partial t_1}\log\tau_n^{\gamma+\varepsilon}$ and $\frac{\partial}{\partial s_1}\log\tau_n^{\gamma+\varepsilon}$ are absent in (2.2.10). So, using again (1.1.31), setting $\alpha_i(t) = it_i$, $\beta_i(s) = is_i$ and $b = 1$, leads to the polynomials

$$\begin{aligned}
0 &= -x_n\left(\sum_{i\geq 1}\left(\alpha_i(t)\frac{\partial}{\partial t_i} - \beta_i(s)\frac{\partial}{\partial s_i}\right)\log x_n + bn\right)\\
&= \frac{v_n}{y_n}\sum_{i\geq 1}\left\{\alpha_i\left((L_1^i)_{n+1,n+1} - (L_1^{i-1})_{n+1,n}\right) - \beta_i\left((L_2^i)_{nn} - (L_2^{i-1})_{n,n+1}\right)\right\} - nx_n,
\end{aligned}$$

$$\begin{aligned}
0 &= y_n\left(\sum_{i\geq 1}\left(\alpha_i(t)\frac{\partial}{\partial t_i} - \beta_i(s)\frac{\partial}{\partial s_i}\right)\log y_n - bn\right)\\
&= \frac{v_n}{x_n}\sum_{i\geq 1}\left\{\alpha_i\left((L_1^i)_{nn} - (L_1^{i-1})_{n+1,n}\right) - \beta_i\left((L_2^i)_{n+1,n+1} - (L_2^{i-1})_{n,n+1}\right)\right\} - ny_n,
\end{aligned}\tag{2.2.11}$$

ending the proof of Theorem 2.2. □

2.3. Proof of main Theorem.

Proof of Theorem 0.1. Remember the locus, defined in (2.1.1),

$$\mathcal{L} = \left\{ \begin{array}{l} it_i = it_i^{(0)} := \begin{cases} u_i - (\gamma_1' d_1^i + \gamma_2' d_2^i), & \text{for } 1 \le i \le N_1 \\ -(\gamma_1' d_1^i + \gamma_2' d_2^i), & \text{for } N_1 + 1 \le i < \infty \end{cases} \\ is_i = is_i^{(0)} := \begin{cases} -u_{-i} + (\gamma_1'' d_1^{-i} + \gamma_2'' d_2^{-i}), & \text{for } 1 \le i \le N_2 \\ (\gamma_1'' d_1^{-i} + \gamma_2'' d_2^{-i}), & \text{for } N_2 + 1 \le i < \infty \end{cases} \end{array} \right\}. \tag{2.3.1}$$

From (2.1.5) in Lemma 2.1, for all $i \ge 1$,

$$\begin{aligned} \alpha_i(t^{(0)}) &:= au_{i+1} + bu_i + cu_{i-1} + c(n + \gamma_1' + \gamma_2' + \gamma)\delta_{i1}, \\ \beta_i(s^{(0)}) &:= -au_{-i+1} - bu_{-i} - cu_{-i-1} - a(n + \gamma_1'' + \gamma_2'' - \gamma)\delta_{i1}. \end{aligned}$$

Then the locus (2.3.1) can equally be described by

$$\mathcal{L} = \left\{ \begin{array}{l} \text{all } \alpha_i(t) = 0, \text{ for } i \ge N_1 + 2 \text{ and all } \beta_i(t) = 0, \text{ for } i \ge N_2 + 2, \\ \alpha_i(t) = au_{i+1} + bu_i + c\big(u_{i-1} + \delta_{i1}(n + \gamma_1' + \gamma_2' + \gamma)\big) \text{ and} \\ \beta_i(s) = -cu_{-i-1} - bu_{-i} - a\big(u_{-i+1} + \delta_{i1}(n + \gamma_1'' + \gamma_2'' - \gamma)\big), \\ \hfill \text{otherwise} \end{array} \right\}. \tag{2.3.2}$$

The $\mathcal{L}_i$-matrices (2.2.2) now and only now are finite sums and so have the form, setting $u_0 = u_{N_1+1} = u_{N_1+2} = u_{-N_2-1} = u_{-N_2-2} = 0$,

$$\begin{aligned} \mathcal{L}_1 &= \sum_1^{N_1+1} \alpha_i(t^{(0)}) L_1^i \\ &= \sum_1^{N_1+1} (au_{i+1} + bu_i + cu_{i-1})\, L_1^i + c\big(n + \gamma_1' + \gamma_2' + \gamma\big) L_1 \\ &= \Big(aI + bL_1 + cL_1^2\Big) \sum_1^{N_1} u_i L_1^{i-1} + c\big(n + \gamma_1' + \gamma_2' + \gamma\big) L_1 - au_1 I \\ &= \Big(aI + bL_1 + cL_1^2\Big)\, P_1'(L_1) + c\big(n + \gamma_1' + \gamma_2' + \gamma\big) L_1 - au_1 I, \end{aligned}$$

$$\begin{aligned} \mathcal{L}_2 &= -\sum_1^{N_2+1} \beta_i(s^{(0)}) L_2^i \\ &= \sum_1^{N_2+1} (cu_{-i-1} + bu_{-i} + au_{-i+1})\, L_2^i + a\big(n + \gamma_1'' + \gamma_2'' - \gamma\big) L_2 \\ &= \Big(cI + bL_2 + aL_2^2\Big) \sum_1^{N_2} u_{-i} L_2^{i-1} + a\big(n + \gamma_1'' + \gamma_2'' - \gamma\big) L_2 - cu_{-1} I \\ &= \Big(cI + bL_2 + aL_2^2\Big)\, P_2'(L_2) + a\big(n + \gamma_1'' + \gamma_2'' - \gamma\big) L_2 - cu_{-1} I, \end{aligned}$$

which are precisely the expressions $\mathcal{L}_i^{(n)}$, introduced in (0.0.10), in the introduction, but modulo the identity pieces. Still, the identities (2.2.3), (2.2.4) and (2.2.5) remain valid, upon evaluating along the locus $\mathcal{L}$; i.e., with $\alpha_i(t^{(0)})$ and $\beta_i(s^{(0)})$ as in (2.3.2) and t and s replaced by $t^{(0)}$ and $s^{(0)}$ in the variables $x_n(t,s)$ and $y_n(t,s)$.

- *Case 1 and 2*. not both $a = c = 0$. Thus, the first identity (2.2.3) holds and the second identity (2.2.4) expresses the fact that a difference ∂_n of an expression vanishes; therefore the expression equals that same expression at the origin. This ends the proof of identities (0.0.14) and (0.0.15) in Theorem 0.1, upon observing the identity pieces in $\mathcal{L}_1$ and $\mathcal{L}_2$ above make no contribution, and the involution $\tilde{}$ yields the dual relations.
- *Case 3*. both $a = c = 0$ and $b = 1$. From (2.1.1) or (2.3.1), setting $a = c = 0$ and $b = 1$, we have

$$\alpha_i(t^{(0)}) = u_i \quad \text{and} \quad \beta_i(t^{(0)}) = -u_{-i}.$$

Remember from (0.0.2), $P_1(z) := \sum_1^{N_1} \frac{u_i z^i}{i}$ and $P_2(z) := \sum_1^{N_2} \frac{u_{-i} z^i}{i}$, and so we have

$$\sum_{i\geq 1} \alpha_i(t^{(0)}) L_1^{i-1} = \sum_{i\geq 1} u_i L_1^{i-1} = P_1'(L_1),$$
$$\sum_{i\geq 1} \beta_i(t^{(0)}) L_2^{i-1} = -\sum_{i\geq 1} u_{-i} L_2^{i-1} = -P_2'(L_2),$$

and so (2.2.5) leads immediately to (0.0.17), ending the proof of Theorem 0.1. □

3. Invariant Manifolds for the First Toeplitz Flow

Proof of Theorem 0.4. The weight here is

$$\rho(z) := z^\gamma e^{P_1(z)+P_2(z^{-1})},$$

corresponding to Case 3 of Theorem 0.1. From the latter, the variables x_n, y_n satisfy recurrence relations

$$\begin{cases} \Gamma_n(x,y) := \dfrac{v_n}{y_n}\begin{pmatrix} -\left(L_1P_1'(L_1)\right)_{n+1,n+1} - \left(L_2P_2'(L_2)\right)_{n,n} \\ +\,(P_1'(L_1))_{n+1,n} + (P_2'(L_2))_{n,n+1} \end{pmatrix} + nx_n = 0, \\ \tilde{\Gamma}_n(x,y) := \dfrac{v_n}{x_n}\begin{pmatrix} -\left(L_1P_1'(L_1)\right)_{n,n} - \left(L_2P_2'(L_2)\right)_{n+1,n+1} \\ +\,(P_1'(L_1))_{n+1,n} + (P_2'(L_2))_{n,n+1} \end{pmatrix} + ny_n = 0, \end{cases} \tag{3.0.1}$$

which by virtue of (2.2.11) and the nature of the locus (2.3.1), can be written

$$\begin{aligned} \Gamma_n &= \mathcal{V}_0 x_n + nx_n, \\ \tilde{\Gamma}_n &= -\mathcal{V}_0 y_n + ny_n, \end{aligned} \tag{3.0.2}$$

where

$$\mathcal{V}_0 := \sum_{i\geq 1}\left(u_i\frac{\partial}{\partial t_i} + u_{-i}\frac{\partial}{\partial s_i}\right),$$

in terms of the Toeplitz vector fields. Recall from Sect. 6 (Appendix 2) the form of the first vector field: $(v_n := 1 - x_n y_n)$

$$\frac{\partial x_n}{\partial \begin{Bmatrix} t_1 \\ s_1 \end{Bmatrix}} = v_n x_{n\pm 1}, \qquad \frac{\partial y_n}{\partial \begin{Bmatrix} t_1 \\ s_1 \end{Bmatrix}} = -v_n y_{n\mp 1}.$$

Also in the statement of Theorem 0.4, we assume the u_i, appearing in the polynomials $P_1(z)$ and $P_2(z)$, flow according to

$$\frac{\partial u_k}{\partial t_1} = \delta_{k,1}, \qquad \frac{\partial u_k}{\partial s_1} = -\delta_{k,-1}. \tag{3.0.3}$$

Noticing that, from (3.0.3),

$$\left[\frac{\partial}{\partial t_1}, \mathcal{V}_0\right] = \frac{\partial}{\partial t_1} \ , \quad \left[\frac{\partial}{\partial s_1}, \mathcal{V}_0\right] = -\frac{\partial}{\partial s_1},$$

we compute

$$\begin{aligned} \frac{\partial}{\partial t_1}\Gamma_n &= \mathcal{V}_0 \frac{\partial x_n}{\partial t_1} + \frac{\partial x_n}{\partial t_1} + n\frac{\partial x_n}{\partial t_1} \\ &= \mathcal{V}_0(v_n x_{n+1}) + (n+1) v_n x_{n+1} \\ &= v_n \left(\mathcal{V}_0 x_{n+1} + (n+1)x_{n+1}\right) + x_{n+1}\mathcal{V}_0(v_n) \\ &= v_n \Gamma_{n+1} - x_{n+1}\left(x_n \mathcal{V}_0(y_n) + y_n \mathcal{V}_0(x_n)\right) \\ &= v_n \Gamma_{n+1} - x_{n+1}\left(x_n(\mathcal{V}_0(y_n) - n y_n) + y_n(\mathcal{V}_0(x_n) + n x_n)\right) \\ &= v_n \Gamma_{n+1} + x_{n+1}\left(x_n \tilde{\Gamma}_n - y_n \Gamma_n\right). \end{aligned} \tag{3.0.4}$$

Similarly, one shows

$$\begin{aligned} \frac{\partial}{\partial s_1}\Gamma_n &= \mathcal{V}_0 \frac{\partial x_n}{\partial s_1} - \frac{\partial x_n}{\partial s_1} + n\frac{\partial x_n}{\partial s_1} \\ &= \mathcal{V}_0(v_n x_{n-1}) + (n-1) v_n x_{n-1} \\ &= v_n \Gamma_{n-1} + x_{n-1}\left(x_n \tilde{\Gamma}_n - y_n \Gamma_n\right), \end{aligned} \tag{3.0.5}$$

and so by the duality $\tilde{}$

$$\begin{aligned} \frac{\partial}{\partial t_1}\tilde{\Gamma}_n &= -v_n \tilde{\Gamma}_{n-1} + y_{n-1}\left(x_n \tilde{\Gamma}_n - y_n \Gamma_n\right), \\ \frac{\partial}{\partial s_1}\tilde{\Gamma}_n &= -v_n \tilde{\Gamma}_{n+1} + y_{n+1}\left(x_n \tilde{\Gamma}_n - y_n \Gamma_n\right), \end{aligned} \tag{3.0.6}$$

thus establishing (0.0.22). Setting as in (0.0.23),

$$\mathfrak{M} := \bigcap_{n \geq 1} \left\{(x_k, y_k)_{k \geq 0}, \text{ such that } \ \Gamma_n(x,y) = 0 \text{ and } \tilde{\Gamma}_n(x,y) = 0\right\},$$

the differential equations (3.0.4), (3.0.5) and (3.0.6) imply at once that, along the locus $\mathfrak{M}$, defined in (0.0.23),

$$\left.\frac{\partial}{\partial t_1}\Gamma_n\right|_{\mathfrak{M}} = \left.\frac{\partial}{\partial t_1}\tilde{\Gamma}_n\right|_{\mathfrak{M}} = \left.\frac{\partial}{\partial s_1}\Gamma_n\right|_{\mathfrak{M}} = \left.\frac{\partial}{\partial s_1}\tilde{\Gamma}_n\right|_{\mathfrak{M}} = 0,$$

showing the locus $\mathfrak{M}$ is an invariant manifold for these flows. This ends the proof of Theorem 0.4. □

Proof of Corollary 0.5. In the self-dual case,

$$\Gamma_n = \tilde{\Gamma}_n,$$

since in (3.0.2) all $u_n = u_{-n}$, $x_n = y_n$ and so $\partial x_n/\partial t_i = -\partial y_n/\partial s_i$, $\partial x_n/\partial s_i = -\partial y_n/\partial t_i$ for all n, i.

From the differential equations (3.0.4) and (3.0.5), it follows that

$$\frac{\partial}{\partial t_1}\Gamma_n = v_n \Gamma_{n+1} \quad \text{and} \quad \frac{\partial}{\partial s_1}\Gamma_n = v_n \Gamma_{n-1},$$

and so for $\frac{\partial}{\partial t} = \frac{\partial}{\partial t_1} - \frac{\partial}{\partial s_1}$,

$$\frac{\partial \Gamma_n}{\partial t} = v_n(\Gamma_{n+1} - \Gamma_{n-1})$$

establishing Eq. (0.0.25). Therefore, along the locus $\mathfrak{N}$, defined in (0.0.26),

$$\left.\frac{\partial \Gamma_n}{\partial t}\right|_{\mathfrak{N}} = 0,$$

and so the locus $\mathfrak{N}$ is invariant with respect to the $\partial/\partial t$ vector field, ending the proof of Corollary 0.5. □

4. Rational Relations for Special Weights

4.1. Weight $e^{t(z+z^{-1})}$. This weight comes up by considering the uniform probability P on the group S_k of permutations π_k and

$$L(\pi_k) = \text{ length of the longest (strictly) increasing subsequence of } \pi_k\,. \tag{4.1.1}$$

Then, according to Gessel [9], the generating function below can be expressed as the determinant of a Toeplitz matrix and thus as a unitary matrix integral:

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{t^{2k}}{k!} P(L(\pi_k) \le n) &= E_{U(n)} e^{t\,\mathrm{Tr}(M+\bar{M})} \qquad (4.1.2)\\ &= \frac{1}{n!}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left(e^{t(z_k+\bar{z}_k)} \frac{dz_k}{2\pi i z_k}\right).\end{aligned}$$

Then

$$\begin{aligned}x_n &= (-1)^n \frac{E_{U(n)}(\det M) e^{t\,\mathrm{Tr}(M+\bar{M})}}{E_{U(n)} e^{t\,\mathrm{Tr}(M+\bar{M})}}\\ &= (-1)^n \frac{\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left(z_k e^{t(z_k+\bar{z}_k)} \frac{dz_k}{2\pi i z_k}\right)}{\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left(e^{t(z_k+\bar{z}_k)} \frac{dz_k}{2\pi i z_k}\right)}.\end{aligned}$$

This weight is a special case of the self-dual weight of Corollary 0.3, which lead to the relation (0.0.18). Thus we find

$$\begin{aligned} 0 &= nx_n - \frac{v_n}{x_n}\left(t(L_1)_{n+1,n+1} + t(L_1)_{nn}\right) \\ &= nx_n - \frac{v_n}{x_n}\left(-tx_{n+1}x_n - tx_nx_{n-1}\right), \end{aligned}$$

yielding the 3-step relation, found by Borodin [5], with highest (respectively, lowest) terms doubly (respectively, simply) underlined,

$$nx_n + t(1-x_n^2)(\underline{\underline{x_{n+1}}} + \underline{x_{n-1}}) = 0. \tag{4.1.3}$$

It is interesting to point out that this map (4.1.3) is the simplest instance of a family of area-preserving maps of the plane, having an invariant, as found by McMillan [7], and extended by Suris [11] to maps of the form $\partial_n^2 x(n) = f(x(n))$, having an analytic invariant of two variables $\Phi(y,z)$, i.e.,

$$\Phi(x_{n+1}, x_n) = \Phi(x_n, x_{n-1}).$$

The invariant in the case of the maps (4.1.3) is

$$\left(1-y^2\right)\left(1-z^2\right) + ayz, \quad \text{with } a = -\frac{n}{t}.$$

For more on this matter, see the review by B. Grammaticos, F. Nijhoff, A. Ramani [8].

4.2. Weight $e^{t(z+z^{-1})+s(z^2+z^{-2})}$. Consider instead the subgroups of odd permutations, with $2^k k!$ elements

$$S_{2k}^{\text{odd}} = \left\{ \begin{array}{l} \pi_{2k} \in S_{2k}, \pi_{2k} : (-k, \ldots, -1, 1, \ldots, k) \circlearrowleft \\ \text{with } \pi_{2k}(-j) = -\pi_{2k}(j), \text{ for all } j \end{array} \right\} \subset S_{2k},$$

$$S_{2k+1}^{\text{odd}} = \left\{ \begin{array}{l} \pi_{2k+1} \in S_{2k+1}, \pi_{2k} : (-k, \ldots, -1, 0, 1, \ldots, k) \circlearrowleft \\ \text{with } \pi_{2k+1}(-j) = -\pi_{2k+1}(j), \text{ for all } j \end{array} \right\} \subset S_{2k}.$$

Then, according to Rains [10] and Tracy-Widom [12], the following generating functions, again involving the length of the longest increasing sequence, are related to matrix integrals:

$$\begin{aligned} \sum_0^\infty \frac{(\sqrt{2}\,s)^{2k}}{k!} P(L(\pi_{2k}) \le n) &= E_{U(n)} e^{sTr(M^2+\bar{M}^2)} \\ &= \frac{1}{n!}\int_{(S')^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(e^{s(z_k^2+z_k^{-2})} \frac{dz_k}{2\pi z_k} \right), \end{aligned}$$

$$\begin{aligned} &\sum_0^\infty \frac{(\sqrt{2}\,s)^{2k}}{k!} P(L(\pi_{2k+1}) \le n) \\ &\quad = \frac{1}{4}\frac{\partial}{\partial t^2}\left(E_{U(n)} e^{Tr(t(M+\bar{M})+s(M^2+\bar{M}^2)} + E_{U(n)} e^{Tr(t(M+\bar{M})-s(M^2+\bar{M}^2))} \right)\Big|_{t=0}. \end{aligned}$$

The weight $e^{t(z+z^{-1})+s(z^2+z^{-2})}$ is a special case of the one in Corollary 0.3, thus leading to (0.0.17). So, we find

$$x_n = (-1)^n \frac{E_{U(n)}(\det M)e^{tTr(M+\bar{M})+sTr(M^2+\bar{M}^2)}}{E_{U(n)}e^{tTr(M+\bar{M})+sTr(M^2+\bar{M}^2)}}$$

satisfies a 5-step relation, with highest (respectively, lowest) terms doubly (respectively, simply) underlined,

$$0 = nx_n + tv_n(x_{n-1} + x_{n+1}) + 2sv_n \left(\underline{\underline{x_{n+2}}}v_{n+1} + \underline{x_{n-2}}v_{n-1} - x_n(x_{n+1} + x_{n-1})^2\right). \tag{4.2.1}$$

Also here the map has a polynomial invariant

$$\Phi(x, y, z, u) = nyz - (1 - y^2)(1 - z^2)\Big(t + 2s(x(u - y) - z(u + y))\Big).$$

So we have for all n,

$$\Phi(x_{n-1}, x_n, x_{n+1}, x_{n+2}) = \Phi(x_n, x_{n+1}, x_{n+2}, x_{n+3}).$$

4.3. Weight $(1 + z)^\alpha e^{-sz^{-1}}$. Let P be the uniform probability on

$$S_{k,\alpha} = \{\text{words } \pi_k \text{ of length } k \text{ from an alphabet of } \alpha \text{ letters}\}.$$

Then, if L denotes the same as in (4.1.1), but without "strictly", we have, according to Tracy-Widom [13] (see also [3]),

$$\sum_{k=0}^{\infty} \frac{(-\alpha s)^k}{k!} P\left\{\pi_k \in S_{k,\alpha} \;\middle|\; L(\pi_k) \le n\right\} = E_{U(n)} \det(I + M)^\alpha e^{-s\, tr\bar{M}}.$$

For this weight, $P_1(z) = 0$ and $P_2(z) = -sz$, so that $N_1 = 0,\ N_2 = 1,\ u_{-1} = -s$, and all other $u_i = 0$. Also $\gamma_1' = \alpha,\ \ \gamma = \gamma_2' = \gamma_1'' = \gamma_2'' = 0$. This is a special case of Case 2 of Theorems 0.1 and 0.2. Hence, we choose

$$a = 0,\ \ b = c = 1,$$

for which one computes

$$\begin{aligned}\mathcal{L}_1^{(n)} &= (n + \alpha)L_1,\\ \mathcal{L}_2^{(n)} &= s(I + L_2),\end{aligned}$$

and so (0.0.14) and (0.0.15) become

$$\begin{aligned}&\partial_n((n + \alpha)L_1 + sL_2)_{nn} + (L_1)_{nn} = 0,\\ &\partial_n((n - 1 + \alpha)v_{n-1}L_1 + sL_2)_{n,n-1} + \left(L_1^2 + L_1\right)_{n,n} = \text{same}\Big|_{n=1}.\end{aligned}$$

Thus spelled out, the variables

$$\begin{matrix} x_n \\ y_n \end{matrix} = (-1)^n \frac{E_{U(n)}\ (\det M)^{\pm 1} \det(I + M)^\alpha e^{-s\, tr\bar{M}}}{E_{U(n)}\ \det(I + M)^\alpha e^{-s\, tr\bar{M}}}$$

satisfy a 3-step and a 4-step relation, linear in x_{n+1} and y_{n+1}:

$$-(n+\alpha+1)\underline{\underline{x_{n+1}}}y_n - sx_n\underline{\underline{y_{n+1}}} + (n+\alpha-1)x_n y_{n-1} + sx_{n-1}y_n = 0,$$

$$\begin{aligned}&-v_n((n+\alpha+1)\underline{\underline{x_{n+1}}}y_{n-1} - s) + v_{n-1}((n+\alpha-2)x_n\underline{y_{n-2}} - s)\\ &\quad + x_n y_{n-1}(x_n y_{n-1} - 1) = v_1(s-(2+\alpha)x_2) + x_1(x_1 - 1).\end{aligned}$$

4.4. Weight $(1-\xi z)^\alpha\left(1-\xi z^{-1}\right)^\beta$. This weight, considered by Borodin [5] and coming up in point processes, is obtained by setting

$$\gamma_1' = \alpha,\ \ \gamma_2'' = \beta\ ,\ \ \gamma = \gamma_2' = \gamma_1'' = 0,\ \ d_1 = \xi,\ d_2 = \xi^{-1}, \text{all } u_i = 0,\ N_1 = N_2 = 0.$$

$$a = c = 1,\ \ b = -\xi - \xi^{-1}$$

in the weight (0.0.1). We have that

$$\alpha_i = (n+\alpha)\delta_{i1} \ \text{ and } \ \beta_i = -(n+\beta)\delta_{i1}$$

and so

$$\mathcal{L}_1^{(n)} = (n+\alpha)L_1 \ \text{ and } \ \mathcal{L}_2^{(n)} = -(n+\beta)L_2.$$

Therefore (0.0.14) and (0.0.15) read

$$((n+\alpha+1)L_1 - (n+\beta+1)L_2)_{n+1,n+1} - ((n+\alpha-1)L_1 - (n+\beta-1)L_2)_{nn} = 0,$$

$$\begin{aligned}&(v_n(n+\alpha)L_1 - (n+\beta)L_2)_{n+1,n} - (v_{n-1}(n-1+\alpha)L_1 - (n-1+\beta)L_2)_{n,n-1}\\ &\quad + \left(L_1^2 + bL_1\right)_{n,n} = \left(v_1(1+\alpha)L_1 - (1+\beta)L_2\right)_{21} + \left(L_1^2 + bL_1\right)_{11},\end{aligned}$$

leading to a 3-step relation and a 4-step relation in x_{n+1} and y_{n+1},

$$-(n+\alpha+1)\underline{\underline{x_{n+1}}}y_n + (n+\beta+1)\underline{\underline{y_{n+1}}}x_n + (n+\alpha-1)y_{n-1}x_n - (n+\beta-1)x_{n-1}y_n = 0,$$

and

$$\begin{aligned}&-v_n((n+\alpha+1)\underline{\underline{x_{n+1}}}y_{n-1} + n + \beta) + v_{n-1}((n+\alpha-2)x_n\underline{y_{n-2}} + n + \beta - 1)\\ &\quad + x_n y_{n-1}(x_n y_{n-1} + \xi + \xi^{-1}) = -v_1\,(x_2(\alpha+2) + \beta + 1) + x_1(x_1 + \xi + \xi^{-1}).\end{aligned}$$

So, all x_n and y_n are rational expressions in terms of x_1, x_2, y_1, y_2. Note these relations are different from those found by Borodin [5].

5. Appendix 1: Virasoro Algebras

In [2], we defined a Heisenberg and Virasoro algebra of vector operators ${}^\beta\mathbb{J}_k^{(i)}$, depending on a parameter $\beta > 0$:

$$\left({}^\beta\mathbb{J}_k^{(1)}\right)_n = {}^\beta J_k^{(1)} + nJ_k^{(0)} \text{ and } \left(\mathbb{J}_k^{(0)}\right)_n = nJ_k^{(0)} = n\delta_{0k},$$

and

$$\begin{aligned}{}^{\beta}\mathbb{J}_k^{(2)} &= \beta \sum_{i+j=k} :{}^{\beta}\mathbb{J}_i^{(1)}{}^{\beta}\mathbb{J}_j^{(1)}: + (1-\beta)\left((k+1)\ {}^{\beta}\mathbb{J}_k^{(1)} - k\ \mathbb{J}_k^{(0)}\right) \\ &= \left(\beta.\ {}^{\beta}J_k^{(2)} + (2n\beta + (k+1)(1-\beta))\ .\ {}^{\beta}J_k^{(1)} + n(n\beta+1-\beta)J_k^{(0)}\right)_{n\in\mathbb{Z}}.\end{aligned} \tag{5.0.1}$$

The ${}^{\beta}\mathbb{J}_k^{(i)}$'s for $i = 1, 2$ satisfy the commutation relations: (see [2])

$$\begin{aligned}\left[{}^{\beta}\mathbb{J}_k^{(1)}, {}^{\beta}\mathbb{J}_\ell^{(1)}\right] &= \frac{k}{2\beta}\delta_{k,-\ell}, \\ \left[{}^{\beta}\mathbb{J}_k^{(2)},\ {}^{\beta}\mathbb{J}_\ell^{(1)}\right] &= -\ell\ {}^{\beta}\mathbb{J}_{k+\ell}^{(1)} + \frac{k(k+1)}{2}\left(\frac{1}{\beta}-1\right)\delta_{k,-\ell}, \\ \left[{}^{\beta}\mathbb{J}_k^{(2)},\ {}^{\beta}\mathbb{J}_\ell^{(2)}\right] &= (k-\ell)\ {}^{\beta}\mathbb{J}_{k+\ell}^{(2)} + c\left(\frac{k^3-k}{12}\right)\delta_{k,-\ell},\end{aligned} \tag{5.0.2}$$

with central charge

$$c = 1 - 6\left(\beta^{1/2} - \beta^{-1/2}\right)^2.$$

In the expressions above,

$$\begin{aligned}{}^{\beta}J_k^{(1)} &= \frac{\partial}{\partial t_k} \text{ for } k > 0 \\ &= \frac{1}{2\beta}(-k)t_{-k} \text{ for } k < 0 \\ &= 0 \text{ for } k = 0,\end{aligned}$$

$${}^{\beta}J_k^{(2)} = \sum_{i+j=k}\frac{\partial^2}{\partial t_i \partial t_j} + \frac{1}{\beta}\sum_{-i+j=k} it_i\frac{\partial}{\partial t_j} + \frac{1}{4\beta^2}\sum_{-i-j=k} it_i\, jt_j. \tag{5.0.3}$$

In particular, for $\beta = 1/2$ and 1, the ${}^{\beta}\mathbb{J}_k^{(2)}$ take on the form:

$${}^{\beta}\mathbb{J}_k^{(2)}(t)\Big|_{\beta=1/2} = \frac{1}{2}\left({}^{\beta}J_k^{(2)} + (2n+k+1)\ {}^{\beta}J_k^{(1)} + n(n+1)J_k^{(0)}\right)_{n\in\mathbb{Z}}\Bigg|_{\beta=1/2}, \tag{5.0.4}$$

$${}^{\beta}\mathbb{J}_k^{(2)}(t)\Big|_{\beta=1} = \left({}^{\beta}J_k^{(2)} + 2n\ {}^{\beta}J_k^{(1)} + n^2J_k^{(0)}\right)_{n\in\mathbb{Z}}\Bigg|_{\beta=1}. \tag{5.0.5}$$

6. Appendix 2: Useful Formulae About the Toeplitz Lattice

The first Toeplitz vector field (1.1.9), corresponding to the Hamiltonians

$$H_1^{(1)} = -\operatorname{Tr}\ L_1 = \sum_0^\infty x_{i+1}y_i,\ H_1^{(2)} = -\operatorname{Tr}\ L_2 = \sum_0^\infty x_i y_{i+1},$$

and

$$H_2^{(1)} = -\frac{1}{2}\,\mathrm{Tr}\ L_1^2 = -\frac{1}{2}\sum_0^\infty x_{i+1}^2 y_i^2 + \sum_0^\infty y_i x_{i+2} v_{i+1},$$
$$H_2^{(2)} = -\frac{1}{2}\,\mathrm{Tr}\ L_2^2 = -\frac{1}{2}\sum_0^\infty x_i^2 y_{i+1}^2 + \sum_0^\infty x_i y_{i+2} v_{i+1}$$

reads:

$$\frac{\partial x_n}{\partial t_1} = v_n x_{n+1} \qquad \frac{\partial y_n}{\partial t_1} = -v_n y_{n-1},$$
$$\frac{\partial x_n}{\partial s_1} = v_n x_{n-1} \qquad \frac{\partial y_n}{\partial s_1} = -v_n y_{n+1},$$

and

$$\frac{\partial x_n}{\partial t_2} = v_n \frac{\partial H_2^{(1)}}{\partial y_n} = -v_n\Big(x_{n+1}^2 y_n - x_{n+2} v_{n+1} + x_n x_{n+1} y_{n-1}\Big),$$
$$\frac{\partial y_n}{\partial t_2} = -v_n \frac{\partial H_2^{(1)}}{\partial x_n} = v_n\Big(x_n y_{n-1}^2 - y_{n-2} v_{n-1} + y_n y_{n-1} x_{n+1}\Big),$$
$$\frac{\partial x_n}{\partial s_2} = v_n \frac{\partial H_2^{(2)}}{\partial y_n} = -v_n\Big(x_{n-1}^2 y_n - x_{n-2} v_{n-1} + x_n x_{n-1} y_{n+1}\Big),$$
$$\frac{\partial y_n}{\partial s_2} = -v_n \frac{\partial H_2^{(2)}}{\partial x_n} = v_n\Big(x_n y_{n+1}^2 - y_{n+2} v_{n+1} + y_n y_{n+1} x_{n-1}\Big).$$

7. Appendix 3: Proof of Theorem 0.2

Before giving the proof of Theorem 0.2, we need

Lemma 7.1.

$$\left(L_1^{N+1}\right)_{nn} = -x_{n+N} y_{n-1} \prod_1^N v_{n+i-1} + \ldots - x_n y_{n-N-1} \prod_1^N v_{n-i},$$
$$\left(L_2^{N+1}\right)_{nn} = -y_{n+N} x_{n-1} \prod_1^N v_{n+i-1} + \ldots - y_n x_{n-N-1} \prod_1^N v_{n-i},$$
$$v_n \left(L_1^{N+1}\right)_{n+1,n} = -x_{n+N+1} y_{n-1} \prod_0^N v_{n+i} + \ldots - x_{n+1} y_{n-N-1} \prod_0^N v_{n-i},$$
$$\left(L_2^{N+1}\right)_{n+1,n} = -y_{n+N} x_n \prod_0^{N-1} v_{n+i} + \ldots - x_{n-N} y_n \prod_1^N v_{n-i+1}.$$

The two highest and two lowest terms in

$$\mathcal{L}_1^{(n)} = \sum_1^{N_1+1} \alpha_i(t) L_1^i \quad \textit{and} \quad \mathcal{L}_2^{(n)} = -\sum_1^{N_2+1} \beta_i(t) L_2^i,$$

are the following:

$$\begin{aligned}
\left(\mathcal{L}_1^{(n)}\right)_{nn} &= -\alpha_{N_1+1}\left(x_{n+N_1}\; y_{n-1}\prod_1^{N_1} v_{n+i-1} + \ldots + x_n y_{n-N_1-1}\prod_1^{N_1} v_{n-i}\right)\\
&\quad -\alpha_{N_1}\left(x_{n+N_1-1}\; y_{n-1}\prod_1^{N_1-1} v_{n+i-1} + \ldots + x_n y_{n-N_1}\prod_1^{N_1-1} v_{n-i}\right) + \ldots,\\
\left(\mathcal{L}_2^{(n)}\right)_{nn} &= \beta_{N_2+1}\left(y_{n+N_2}\; x_{n-1}\prod_1^{N_2} v_{n+i-1} + \ldots + y_n x_{n-N_2-1}\prod_1^{N_2} v_{n-i}\right)\\
&\quad +\beta_{N_2}\left(y_{n+N_2-1}\; x_{n-1}\prod_1^{N_2-1} v_{n+i-1} + \ldots + y_n x_{n-N_2}\prod_1^{N_2-1} v_{n-i}\right) + \ldots,
\end{aligned}$$

$$\begin{aligned}
v_n\left(\mathcal{L}_1^{(n)}\right)_{n+1,n} &= -\alpha_{N_1+1}\left(x_{n+N_1+1}\; y_{n-1}\prod_0^{N_1} v_{n+i} + \ldots + x_{n+1} y_{n-N_1-1}\prod_0^{N_1} v_{n-i}\right)\\
&\quad -\alpha_{N_1}\left(x_{n+N_1}\; y_{n-1}\prod_0^{N_1-1} v_{n+i} + \ldots + x_{n+1} y_{n-N_1}\prod_0^{N_1-1} v_{n-i}\right) + \ldots,\\
\left(\mathcal{L}_2^{(n)}\right)_{n+1,n} &= \beta_{N_2+1}\left(y_{n+N_2}\; x_n\prod_0^{N_2-1} v_{n+i} + \ldots + x_{n-N_2} y_n\prod_1^{N_2} v_{n-i+1}\right)\\
&\quad +\beta_{N_2}\left(y_{n+N_2-1}\; x_n\prod_0^{N_2-2} v_{n+i} + \ldots + x_{n-N_2+1} y_n\prod_1^{N_2-1} v_{n-i+1}\right) + \ldots,
\end{aligned}$$

with

$$\begin{aligned}
\alpha_{N_1+1} &= c\left(u_{N_1} + \delta_{N_1,0}(n + \gamma_1' + \gamma_2' + \gamma)\right),\\
\alpha_{N_1} &= c\left(u_{N_1-1} + \delta_{N_1,1}(n + \gamma_1' + \gamma_2' + \gamma)\right) + b u_{N_1},\\
-\beta_{N_2+1} &= a\left(u_{-N_2} + \delta_{N_2,0}(n + \gamma_1'' + \gamma_2'' - \gamma)\right),\\
-\beta_{N_2} &= a\left(u_{-N_2+1} + \delta_{N_2,1}(n + \gamma_1'' + \gamma_2'' - \gamma)\right) + b u_{-N_2}.
\end{aligned}$$

Proof of Theorem 0.2.
Case 1: $a, c \neq 0$: Equations (0.0.14) and (0.0.15) are two inductive equations, one having $(N_1 + N_2 + 4)$ steps and the other having $(N_1 + N_2 + 3)$ steps. In the equations below, we underline twice the highest terms and once the lowest ones. The exact equations (0.0.14) and (0.0.15) are denoted by $(0.0.14)_n$ and $(0.0.15)_n$. Remember the dual relations also hold and so we shall employ them, when necessary.

- $N_1 = N_2 = N$: The two Eqs. $(0.0.14)_{n+1}$ and $(0.0.15)_n$, form a system of two equations in the unknowns x_{n+N+2} and y_{n+N+2}, the variable with the lowest index being y_{n-N-1}:

$$(0.0.14)_{n+1} = -\left(\alpha_{N+1}\underline{\underline{x_{n+N+2}}}\; y_{n+1} + \beta_{N+1}\underline{\underline{y_{n+N+2}}}\; x_{n+1}\right)\prod_{1}^{N} v_{n+i+1} + \dots ,$$

$$(0.0.15)_n = -\alpha_{N+1}\underline{\underline{x_{n+N+2}}}\; y_n \prod_{1}^{N} v_{n+i+1} + \dots + \alpha_{N+1}\underline{y_{n-N-1}}\; x_{n+1}\prod_{0}^{N} v_{n-i}.$$

- $N_1 = N,\ \ N_2 = N+1$: The two Eqs. $(0.0.14)_n$ and $(0.0.15)_n$ form a system of two equations in the unknowns x_{n+N+2} and y_{n+N+2}, the variable with the lowest index being x_{n-N-2}:

$$(0.0.14)_n = -\beta_{N+2}\underline{\underline{y_{n+N+2}}}x_n \prod_{1}^{N+1} v_{n+i} + \dots + \beta_{N+2}y_n\underline{x_{n-N-2}}\prod_{1}^{N+1} v_{n-i},$$

$$(0.0.15)_n = -\left(\alpha_{N+1}\underline{\underline{x_{n+N+2}}}y_n + \beta_{N+2}\underline{\underline{y_{n+N+2}}}x_{n+1}\right)\prod_{0}^{N} v_{n+i+1} + \dots .$$

- $N_1 = N,\ \ N_2 = N-1$: The two Eqs. $(0.0.14)_n$ and the dual equation $(0.0.15)\tilde{}_n$, via the involution $\tilde{}$, form a system of two equations in the unknowns x_{n+N+1} and y_{n+N+1}, the variable with the lowest index being y_{n-N-1}:

$$(0.0.14)_n = -\alpha_{N+1}\underline{\underline{x_{n+N+1}}}y_n \prod_{1}^{N} v_{n+i} + \dots + \alpha_{N+1}\underline{y_{n-N-1}}x_n \prod_{1}^{N} v_{n-i},$$

$$(0.0.15)\tilde{}_n = -\left(\alpha_{N+1}\underline{\underline{x_{n+N+1}}}y_{n+1} + \beta_N\underline{\underline{y_{n+N+1}}}x_n\right)\prod_{0}^{N-1} v_{n+i+1} + \dots .$$

Case 2. All cases below lead to two inductive equations, one having $(N_1 + N_2 + 3)$ steps and the other having $(N_1 + N_2 + 2)$ steps:

- $N_1 = N,\ \ N_2 = N, a = 0,\ b = c = 1$: The two Eqs. $(0.0.14)_n$ and $(0.0.15)\tilde{}_n$, form a system of two equations in the unknowns x_{n+N+1} and y_{n+N+1}, the variable with the lowest index being y_{n-N-1}:

$$(0.0.14)_n = -\alpha_{N+1}\underline{\underline{x_{n+N+1}}}y_n \prod_{1}^{N} v_{n+i} + \dots + \alpha_{N+1}\underline{y_{n-N-1}}x_n \prod_{1}^{N} v_{n-i},$$

$$(0.0.15)\tilde{}_n = -\left(\alpha_{N+1}\underline{\underline{x_{n+N+1}}}y_{n+1} + \beta_N\underline{\underline{y_{n+N+1}}}x_n\right)\prod_{0}^{N-1} v_{n+i+1} + \dots .$$

- $N_1 = N,\ \ N_2 = N+1, a = 0,\ b = c = 1$: The two Eqs. $(0.0.14)_{n+1}$ and $(0.0.15)_n$ form a system of two equations in the unknowns x_{n+N+2} and y_{n+N+2}, the variable with the lowest index being y_{n-N-1}:

$$(0.0.14)_{n+1} = -\left(\alpha_{N+1}\underline{\underline{x_{n+N+2}}}y_{n+1} + \beta_{N+1}\underline{\underline{y_{n+N+2}}}x_{n+1}\right)\prod_{1}^{N} v_{n+i+1} + \dots ,$$

$$(0.0.15)_n = -\alpha_{N+1}\underline{\underline{x_{n+N+2}}}y_n \prod_{0}^{N} v_{n+i+1} + \dots + \alpha_{N+1}\underline{y_{n-N-1}}x_{n+1}\prod_{0}^{N} v_{n-i}.$$

- $N_1 = N, \quad N_2 = N - 1 \; c = 0, \; a = b = 1$: The two Eqs. $(0.0.14)_{n+1}$ and Eq. $\widetilde{(0.0.15)}_n$ form a system of two equations in the unknowns x_{n+N+1} and y_{n+N+1}, the variable with the lowest index being x_{n-N}:

$$(0.0.14)_{n+1} = -\left(\alpha_N \underline{\underline{x_{n+N+1}}} y_{n+1} + \beta_N \underline{\underline{y_{n+N+1}}} x_{n+1}\right) \prod_1^{N-1} v_{n+i+1} + \dots ,$$

$$\widetilde{(0.0.15)}_n = -\beta_N \underline{\underline{y_{n+N+1}}} x_n \prod_0^{N-1} v_{n+i+1} + \dots + \beta_N \underline{x_{n-N}} y_{n+1} \prod_0^{N-1} v_{n-i} .$$

Case 3. All cases below lead to two inductive equations, both having $N_1 + N_2 + 1$ steps. Using again Lemma 7.1, one searches for the highest and lowest terms in the relations (0.0.17):

$$\begin{aligned}
0 &= nx_n + \frac{v_n}{y_n} \begin{pmatrix} -\left(L_1 P_1'(L_1)\right)_{n+1,n+1} - \left(L_2 P_2'(L_2)\right)_{n,n} \\ +(P_1'(L_1))_{n+1,n} + (P_2'(L_2))_{n,n+1} \end{pmatrix} \\
&= \frac{v_n}{y_n} \begin{pmatrix} -u_{N_1} (L_1^{N_1})_{n+1,n+1} - u_{-N_2} (L_2^{N_2})_{nn} + \dots \\ +u_{N_1} (L_1^{N_1-1})_{n+1,n} + u_{-N_2} (L_2^{N_2-1})_{n,n+1} + \dots \end{pmatrix} \\
&= u_{N_1} \frac{v_n}{y_n} \left(x_{n+N_1} y_n \prod_1^{N_1-1} v_{n+i} + \dots + x_{n+1} y_{n-N_1+1} \prod_1^{N_1-1} v_{n+1-i} \right) \\
&\quad - u_{N_1} \frac{v_n}{y_n} \left(\dots + x_{n+1} y_{n-N_1+1} \prod_1^{N_1-2} v_{n-i} \right) \\
&\quad + u_{-N_2} \frac{v_n}{y_n} \left(y_{n+N_2-1} x_{n-1} \prod_1^{N_2-1} v_{n+i-1} + \dots + y_n x_{n-N_2} \prod_1^{N_2-1} v_{n-i} \right) \\
&\quad - u_{-N_2} \frac{v_n}{y_n} \left(y_{n+N_2-1} x_{n-1} \prod_1^{N_2-2} v_{n+i} + \dots \right) + \dots \\
&= u_{N_1} \left(x_{n+N_1} \prod_0^{N_1-1} v_{n+i} + \dots - x_{n+1} y_{n-N_1+1} x_n \prod_0^{N_1-2} v_{n-i} \right) \\
&\quad - u_{-N_2} \left(y_{n+N_2-1} x_{n-1} x_n \prod_0^{N_2-2} v_{n+i} + \dots - x_{n-N_2} \prod_0^{N_2-1} v_{n-i} \right) + \dots ,
\end{aligned}$$

and, by duality,

$$
\begin{aligned}
0 &= ny_n + \frac{v_n}{x_n}\begin{pmatrix} -\big(L_1P_1'(L_1)\big)_{n,n} - \big(L_2P_2'(L_2)\big)_{n+1,n+1} \\ +(P_1'(L_1))_{n+1,n} + (P_2'(L_2))_{n,n+1} \end{pmatrix} \\
&= -u_{N_1}\left(x_{n+N_1-1}y_{n-1}y_n \prod_0^{N_1-2} v_{n+i} + \ldots - y_{n-N_1}\prod_0^{N_1-1} v_{n-i}\right) \\
&\quad + u_{-N_2}\left(y_{n+N_2}\prod_0^{N_2-1} v_{n+i} + \ldots - y_{n+1}x_{n-N_2+1}y_n\prod_0^{N_2-2} v_{n-i}\right) + \ldots .
\end{aligned}
$$

Here again, one uses different indices n for each of the cases:

- $N_1 = N, \quad N_2 = N$

$$
\begin{aligned}
(0.0.17)_n &= u_N\underline{\underline{x_{n+N}}}\prod_0^{N-1} v_{n+i} + \ldots + u_{-N}\underline{x_{n-N}}\prod_0^{N-1} v_{n-i} + \ldots , \\
\widetilde{(0.0.17)}_n &= u_{-N}\underline{\underline{y_{n+N}}}\prod_0^{N-1} v_{n+i} + \ldots + u_N\underline{y_{n-N}}\prod_0^{N-1} v_{n-i} + \ldots .
\end{aligned}
$$

- $N_1 = N, \quad N_2 = N+1$

$$
\begin{aligned}
(0.0.17)_{n+1} &= \Big(u_N\underline{\underline{x_{n+N+1}}} - u_{-N-1}\underline{\underline{y_{n+N+1}}}x_nx_{n+1}\Big)\prod_0^{N-1} v_{n+i+1} + \ldots \\
&\quad + u_{-N-1}\underline{x_{n-N}}\prod_0^{N} v_{n-i+1} + \ldots , \\
\widetilde{(0.0.17)}_n &= u_{-N-1}\underline{\underline{y_{n+N+1}}}\prod_0^{N} v_{n+i} + \ldots \\
&\quad - \Big(u_{-N-1}y_{n+1}\underline{x_{n-N}}y_n - u_N\underline{y_{n-N}}\Big)\prod_0^{N-1} v_{n-i} + \ldots .
\end{aligned}
$$

- $N_1 = N, \quad N_2 = N-1,$

$$
\begin{aligned}
(0.0.17)_n &= u_N\underline{\underline{x_{n+N}}}\prod_0^{N-1} v_{n+i} + \ldots \\
&\quad - \Big(u_Nx_{n+1}\underline{y_{n-N+1}}x_n - u_{-N+1}\underline{x_{n-N+1}}\Big)\prod_0^{N} v_{n-i} + \ldots ,
\end{aligned}
$$

$$
(0.0.17)\tilde{y}_{n+1} = \Big(u_{-N+1}\underline{\underline{y_{n+N}}} - u_N \underline{\underline{x_{n+N}}}\, y_n y_{n+1}\Big) \prod_0^{N-2} v_{n+i+1} + \dots
$$

$$
+ u_N \underline{y_{n-N+1}} \prod_0^{N-1} v_{n-i+1} + \dots .
$$

This ends the proof of Theorem 0.2. □

References

1. Adler, M., van Moerbeke, P.: The complex geometry of the Kowalewski-Painlevé analysis. Inv. Math. **97**, 3–51 (1989)
2. Adler, M., van Moerbeke, P.: The spectrum of coupled random matrices. Ann. Math. **149**, 921–976 (1999)
3. Adler, M., van Moerbeke, P.: Integrals over classical groups, random permutations, Toda and Toeplitz lattices. Commun. Pure Appl. Math. **54**, 153–205 (2001) (arXiv: math.CO/9912143)
4. Baik, J.: *Riemann-Hilbert problems for last passage percolation* (arXiv:math.PR/0107079)
5. Borodin, A.: *Discrete gap probabilities and discrete Painlevé equations* (arXiv:math-ph/0111008)
6. Borodin, A., Okounkov, A.: A Fredholm determinant formula for Toeplitz determinants. Integral equations operator theory **37**, 386–396 (2000) (arXiv:math-ph/9907165)
7. McMillan, E.M.: A problem in the stability of periodic systems. In: *Topics in Modern Physics*, A tribute to Condon, E.U., Brittin, W.E., Odabasi, H. (eds.) (Colorado Ass. Univ. Press, Boulder), 1971, pp. 219–244
8. Grammaticos, B., Nijhoff, F., Ramani, A.: *Discrete Painlevé equations*. In: *The Painlevé property, CRM series in Math. Phys.*, Chapter 7, New York: Springer, 1999, pp. 413–516
9. Gessel, I.M.: Symmetric functions and P-recursiveness. J. of Comb. Theory, Ser A **53**, 257–285 (1990)
10. Rains, E.M.: Increasing subsequences and the classical groups. Elect. J. of Combinatorics **5**, R12 (1998)
11. Suris, Yu.B.: Integrable mappings of standard type. Funct. Anal. Appl. **23**, 74–76 (1987)
12. Tracy, C.A., Widom, H.: *Random unitary matrices, permutations and Painlevé*, Commun. Math. Phys. **207**, 665–685 (1999) (arXiv:math.CO /9811154)
13. Tracy, C.A., Widom, H.: On the distributions of the lengths of the longest monotone subsequences in random words. Probab. Theory Relat. Fields **119**, 350–380 (2001) (arXiv: math.CO/9904042)

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