Combinatorics and integrable Geometry

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Contents

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1 Introduction

Since Russel's horse back journey along the canal from Glasgow to Edinburg in 1834, since the birth of the Korteweg-de Vries equation in 1895 and since the remarkable renaissance initiated by M. Kruskal and coworkers in the late 60's, the field of integrable systems has emerged as being at the crossroads of important new developments in the sciences.

Integrable systems typically have many different solutions. Besides the soliton and scattering solutions, other important solutions of KdV have arisen, namely rational and algebro-geometrical solutions. This was the royal road to the infinite-dimensional Grassmannian description of the KP-solutions, leading to the fundamental concept of Sato's τ-function, which enjoys Plücker relations and Hirota bilinear relations. In this way, the τ-function is a far reaching generalization of classical theta functions and is nowadays a unifying theme in mathematics: representation theory, curve theory, symmetric function theory, matrix models, random matrices, combinatorics, topological field theory, the theory of orthogonal polynomials and Painlevé theory all live under the same hat! This general field goes under the somewhat bizarre name of "integrable mathematics".

This lecture illustrates another application of integrable systems, this time, to unitary matrix integrals and ultimately to combinatorics and probability theory. Unitary matrix integrals, with an appropriate set of time parameters inserted to make it a τ function, satisfy a new lattice, the Toeplitz lattice, related to the 2d-Toda lattice for a very special type of initial condition. Besides, it also satisfies constraints, which form a very small subalgebra of the Virasoro algebra (section 2).

Along a seemingly different vein, certain unitary matrix integrals, developed in a series with respect to a parameter, have coefficients which contain information concerning random permutations, random words and random walks. Turned around, the generating function for certain probabilities turns out to be a unitary matrix integral (section 3).

The connection of these combinatorial problems with integrable systems is precious: it enables one to find differential and difference equations for these probabilities! This is explained in section 4. The purpose of this lecture is to explain these connections. For a more comprehensive account of these results, including the ones on random matrices, see [29].

2 A unitary matrix integral: Virasoro and the Toeplitz lattice

In this section, we consider integrals over the unitary group $U(n)$ with regard to the invariant measure dM. Since the spectrum $z_1, \ldots, z_n$ of M lies on the circle S^1 and since the integrand only involves traces, it is natural to integrate out the "angular part" of dM and to keep its spectral part[1] $|\Delta_n(z)|^2 dz_1 \ldots dz_n$. For $\varepsilon \in \mathbb{Z}$, define the following integrals, depending on formal time parameters $t = (t_1, t_2, \ldots)$ and $s = (s_1, s_2, \ldots)$, with $\tau_0 = 1$,

$$
\begin{aligned}
\tau_n^\varepsilon(t,s) &= \int_{U(n)} (\det M)^\varepsilon e^{\sum_1^\infty \operatorname{Tr}(t_j M^j - s_j \bar{M}^j)} dM \\
&= \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(z_k^\varepsilon e^{\sum_1^\infty (t_j z_k^j - s_j z_k^{-j})} \frac{dz_k}{2\pi i z_k} \right) \\
&= \det \left(\oint_{S^1} \frac{dz}{2\pi i z} z^{\ell - m + \varepsilon} e^{\sum_1^\infty (t_j z^j - s_j z^{-j})} \right)_{1 \le \ell, m \le n}, \qquad (2.0.1)
\end{aligned}
$$

the latter being a Toeplitz determinant. The last equality follows from the fact that the product of two Vandermonde's can be expressed as sum of determinants:

$$
\Delta_n(u)\Delta_n(v) = \sum_{\sigma \in S_n} \det \left(u_{\sigma(k)}^{\ell-1} v_{\sigma(k)}^{k-1} \right)_{1 \le \ell, k \le n}, \qquad (2.0.2)
$$

and from distributing the factors in the product (in (2.0.1)) over the columns of the matrix, appearing in the last formula of (2.0.1). Now, the main point is that the matrix integrals above satisfy **two distinct systems of equations**. These equations will be useful for the combinatorial problems discussed in section 3.

2.1 Unitary matrix integrals and the Virasoro algebra

Proposition 2.1 *(See [1]) The integrals (2.0.1) satisfy the Virasoro constraints,*

$$
\mathbb{V}_k^\varepsilon(t,s,n)\ \tau_n^\varepsilon(t,s) = 0, \quad \textit{for } k = -1, 0, 1 \qquad (2.1.1)
$$

[1] with the Vandermonde determinant $\Delta_n(z) = \prod_{1 \le i < j \le n} (z_i - z_j)$.

where $\mathbb{V}_k^\varepsilon := \mathbb{V}_k^\varepsilon(t,s,n)$ *are the operators*

$$
\begin{aligned}
\mathbb{V}_{-1}^\varepsilon &= \sum_{i\geq 1}(i+1)t_{i+1}\frac{\partial}{\partial t_i} - \sum_{i\geq 2}(i-1)s_{i-1}\frac{\partial}{\partial s_i} + nt_1 + (n-\varepsilon)\frac{\partial}{\partial s_1} \\
\mathbb{V}_0^\varepsilon &= \sum_{i\geq 1}\left(it_i\frac{\partial}{\partial t_i} - is_i\frac{\partial}{\partial s_i}\right) + \varepsilon n = 0 \qquad (2.1.2)\\
\mathbb{V}_1^\varepsilon &= -\sum_{i\geq 1}(i+1)s_{i+1}\frac{\partial}{\partial s_i} + \sum_{i\geq 2}(i-1)t_{i-1}\frac{\partial}{\partial t_i} + ns_1 + (n+\varepsilon)\frac{\partial}{\partial t_1}.
\end{aligned}
$$

<u>*Remark:*</u> Note that the generators $\mathbb{V}_k^\varepsilon$ are part of an ∞-dimensional Virasoro algebra; the claim here is that the integrals above satisfy only these three constraints, unlike the case of Hermitian matrix integrals, which satisfy a large subalgebra of constraints!

Sketch of proof: For the exponent $\varepsilon \neq 0$, the proof is a slight modification of the case $\varepsilon = 0$; so, we stick to the case $\varepsilon = 0$. The Virasoro operators $\mathbb{V}_k := \mathbb{V}_k^\varepsilon\Big|_{\varepsilon=0}$ are generated by the following *vertex operator*[2]

$$\mathbb{X}(t,s;u) := \Lambda^\top e^{\sum_1^\infty(t_iu^i - s_iu^{-i})}e^{-\sum_1^\infty\left(\frac{u^{-i}}{i}\frac{\partial}{\partial t_i} - \frac{u^i}{i}\frac{\partial}{\partial s_i}\right)}. \qquad (2.1.3)$$

This means they are a commutator realization of differentiation:

$$\frac{\partial}{\partial u}u^{k+1}\frac{\mathbb{X}(t,s;u)}{u} = \left[\mathbb{V}_k(t,s), \frac{\mathbb{X}(t,s;u)}{u}\right]. \qquad (2.1.4)$$

Then the following operator, obtained by integrating the vertex operator (2.1.3),

$$\mathbb{Y}(t,s) = \oint_{S^1}\frac{du}{2\pi iu}\mathbb{X}(t,s;u,u^{-1}) \qquad (2.1.5)$$

has, using (2.1.4), the commutation property

$$[\ \mathbb{Y}\ ,\ \mathbb{V}_k\ ] = 0.$$

Then one checks that the integrals $I_n = n!\tau_n^{(0)}$ in (2.0.1) (for $n \geq 1$) are fixed points for $\mathbb{Y}(t,s)$; namely, taking into account the shift $\Lambda^\top$ in (2.1.3), one computes

[2] The operator Λ is the semi-infinite shift matrix, with zeroes everywhere, except for 1's just above the diagonal, i.e., $(\Lambda v)_n = v_{n+1}$ and $(\Lambda^\top v)_n = v_{n-1}$.

$$
\begin{aligned}
\mathbb{Y}(t,s)I_n(t,s) &= \oint_{S^1}\frac{du}{2\pi i u}e^{\sum_1^\infty(t_iu^i-s_iu^{-i})}\ e^{-\sum_1^\infty\left(\frac{u^{-i}}{i}\frac{\partial}{\partial t_i}-\frac{u^i}{i}\frac{\partial}{\partial s_i}\right)}\\
&\qquad\int_{(S^1)^{n-1}}\Delta_{n-1}(z)\Delta_{n-1}(\bar z)\prod_{k=1}^{n-1}e^{\sum_1^\infty(t_iz_k^i-s_iz_k^{-i})}\frac{dz_k}{2\pi i z_k}\\
&= \oint_{S^1}\frac{du}{2\pi i u}e^{\sum_1^\infty(t_iu^i-s_iu^{-i})}\int_{(S^1)^{n-1}}\Delta_{n-1}(z)\Delta_{n-1}(\bar z)\\
&\quad\times\quad\prod_{k=1}^{n-1}\left(1-\frac{z_k}{u}\right)\left(1-\frac{u}{z_k}\right)e^{\sum_1^\infty(t_iz_k^i-s_iz_k^{-i})}\frac{dz_k}{2\pi i z_k}\\
&= \int_{(S^1)^n}|\Delta_n(z)|^2\prod_{k=1}^{n}\left(e^{\sum_1^\infty(t_iz_k^i-s_iz_k^{-i})}\frac{dz_k}{2\pi i z_k}\right)=I_n(t,s)
\end{aligned}
$$

Using this fixed point property and the fact that $(\Lambda^\top)^nI_n=I_0$, we have for $\mathbb{Y}:=\mathbb{Y}(t,s)$,

$$
\begin{aligned}
0 &= \left[\mathbb{V}_k,\mathbb{Y}^n\right]I_n\\
&= \mathbb{V}_k\mathbb{Y}^nI_n-\mathbb{Y}^n\mathbb{V}_kI_n\\
&= \mathbb{V}_kI_n-\mathbb{Y}^n\mathbb{V}_kI_n.\\
&= \mathbb{V}_kI_n-\oint_{S_1}\frac{du}{2\pi i u}e^{\sum_1^\infty(t_iu^i-s_iu^{-i})}e^{-\sum_1^\infty\left(\frac{u^{-i}}{i}\frac{\partial}{\partial t_i}-\frac{u^i}{i}\frac{\partial}{\partial s_i}\right)}\\
&\qquad\ldots\oint_{S_1}\frac{du}{2\pi i u}e^{\sum_1^\infty(t_iu^i-s_iu^{-i})}e^{-\sum_1^\infty\left(\frac{u^{-i}}{i}\frac{\partial}{\partial t_i}-\frac{u^i}{i}\frac{\partial}{\partial s_i}\right)}\mathbb{V}_kI_0.
\end{aligned}
$$

Now one checks visually that for $I_0=1$,

$$\mathbb{V}_kI_0=0\quad\text{for}\quad k=-1,0,1,$$

ending the proof of Proposition 2.1. The details of the proof can be found in Adler-van Moerbeke [1]. ∎

2.2 The Toeplitz lattice

Considering the integral $\tau_n^\varepsilon(t,s)$, as in (2.0.1), and setting, for short,

$$\tau_n:=\tau_n^{(0)},\qquad \tau_n^\pm:=\tau_n^{\pm1},$$

define the ratios

$$x_n(t,s) = (-1)^n \frac{\tau_n^+(t,s)}{\tau_n(t,s)} \quad \text{and} \quad y_n(t,s) := (-1)^n \frac{\tau_n^-(t,s)}{\tau_n(t,s)}, \tag{2.2.1}$$

and the semi-infinite matrices (they are not " rank 2", but try to be!)

$$L_1 := \begin{pmatrix} -x_1y_0 & 1-x_1y_1 & 0 & 0 & \\ -x_2y_0 & -x_2y_1 & 1-x_2y_2 & 0 & \\ -x_3y_0 & -x_3y_1 & -x_3y_2 & 1-x_3y_3 & \\ -x_4y_0 & -x_4y_1 & -x_4y_2 & -x_4y_3 & \\ & & & & \ddots \end{pmatrix}$$

and

$$L_2 := \begin{pmatrix} -x_0y_1 & -x_0y_2 & -x_0y_3 & -x_0y_4 & \\ 1-x_1y_1 & -x_1y_2 & -x_1y_3 & -x_1y_4 & \\ 0 & 1-x_2y_2 & -x_2y_3 & -x_2y_4 & \\ 0 & 0 & 1-x_3y_3 & -x_3y_4 & \\ & & & & \ddots \end{pmatrix}. \tag{2.2.2}$$

Throughout the paper, set[3]

$$h_n = \frac{\tau_{n+1}}{\tau_n} \quad \text{and} \quad v_n := 1 - x_ny_n \overset{*}{=} \frac{h_n}{h_{n-1}} = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}. \tag{2.2.3}$$

One checks that the quantities x_n and y_n satisfy the following commuting Hamiltonian vector fields, introduced by Adler and van Moerbeke in [1],

$$\begin{aligned} \frac{\partial x_n}{\partial t_i} &= (1-x_ny_n)\frac{\partial G_i}{\partial y_n} & \frac{\partial y_n}{\partial t_i} &= -(1-x_ny_n)\frac{\partial G_i}{\partial x_n} \\ \frac{\partial x_n}{\partial s_i} &= (1-x_ny_n)\frac{\partial H_i}{\partial y_n} & \frac{\partial y_n}{\partial s_i} &= -(1-x_ny_n)\frac{\partial H_i}{\partial x_n}, \end{aligned} \tag{2.2.4}$$

(**Toeplitz lattice**)

with Hamiltonians

$$G_i = -\frac{1}{i}\,\mathrm{Tr}\ L_1^i, \quad H_i = -\frac{1}{i}\,\mathrm{Tr}\ L_2^i, \quad i = 1,2,3,... \tag{2.2.5}$$

[3]The proof of equality $\overset{*}{=}$ hinges on associated bi-orthogonal polynomials on the circle, introduced later.

and symplectic structure

$$\omega := \sum_1^\infty \frac{dx_k \wedge dy_k}{1 - x_k y_k}.$$

One imposes initial conditions $x_n(0,0) = y_n(0,0) = 0$ for $n \geq 1$ and boundary conditions $x_0(t,s) = y_0(t,s) = 1$. The G_i and F_i are functions in involution with regard to the Hamiltonian vector fields (2.2.4). Setting $h := \text{diagonal}(h_0, h_1, \ldots)$, with h_i as in (2.2.3), we conjugate L_1 with a diagonal matrix so as to have 1's in the first superdiagonal:

$$\hat{L}_1 := hL_1h^{-1} \quad \text{and} \quad \hat{L}_2 := L_2.$$

The Hamiltonian vector fields (2.2.4) imply the 2-Toda lattice equations for the matrices $\hat{L}_1$ and $\hat{L}_2$,

$$\frac{\partial \hat{L}_i}{\partial t_n} = \left[\left(\hat{L}_1^n\right)_+, \hat{L}_i\right] \quad \text{and} \quad \frac{\partial \hat{L}_i}{\partial s_n} = \left[\left(\hat{L}_2^n\right)_-, \hat{L}_i\right] \quad i = 1,2 \text{ and } n = 1, 2, \ldots.$$

$$\textbf{(two-Toda Lattice)} \qquad (2.2.6)$$

Thus the particular structure of L_1 and L_2 is preserved by the 2-Toda Lattice equations. In particular, this implies that the τ_n's satisfy the KP-hierarchy.

Other equations for the τ_n's are obtained by noting that the expressions formed by means of the matrix integrals (2.0.1) above[4]

$$p_n^{(1)}(t,s;z) = z^n \frac{\tau_n(t - [z^{-1}], s)}{\tau_n(t,s)} \quad \text{and} \quad p_n^{(2)}(t,s;z) = z^n \frac{\tau_n(t, s + [z^{-1}])}{\tau_n(t,s)}$$

are actually polynomials in z, with coefficients depending on t, s; moreover, they are *bi-orthogonal polynomials* on the circle for the following (t,s)-dependent inner product[5],

$$\langle f(z), g(z) \rangle_{t,s} := \oint_{S^1} \frac{dz}{2\pi i z} f(z) g(z^{-1}) e^{\sum_1^\infty (t_i z^i - s_i z^{-i})}. \qquad (2.2.8)$$

[4]For $\alpha \in \mathbb{C}$, define $[\alpha] := (\alpha, \frac{1}{2}\alpha^2, \frac{1}{3}\alpha^3, \ldots) \in \mathbb{C}^\infty$.

[5]For this inner-product, we have $(z^k)^\top = z^{-k}$, i.e.,

$$\langle z^k f(z), g(z) \rangle_{t,s} = \langle f(z), z^{-k} g(z) \rangle_{t,s}. \qquad (2.2.7)$$

Using bi-orthogonality one shows that the variables x_n and y_n, defined in (2.2.1), equal the z^0-term of the bi-orthogonal polynomials,

$$x_n(t,s) = p_n^{(1)}(t,s;0) \text{ and } y_n(t,s) = p_n^{(2)}(t,s;0). \tag{2.2.9}$$

(i) This fact implies the following identity for the h_n's:

$$\left(1 - \frac{h_{n+1}}{h_n}\right)\left(1 - \frac{h_n}{h_{n-1}}\right) = -\frac{\partial}{\partial t_1}\log h_n \frac{\partial}{\partial s_1}\log h_n. \tag{2.2.10}$$

(ii) The mere fact that L_1 and L_2 satisfy the two-Toda lattice implies that the integrals $\tau_n(t,s)$ satisfy, besides the *KP-hierarchy* in t and s (separately), the following equations, combining (t,s)-partials and nearest neighbors $\tau_{n\pm1}$,

$$\begin{aligned}
\frac{\partial^2}{\partial s_1 \partial t_1}\log\tau_n &= -\frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}, \\
\frac{\partial^2}{\partial s_2 \partial t_1}\log\tau_n &= -2\frac{\partial}{\partial s_1}\log\frac{\tau_n}{\tau_{n-1}} \cdot \frac{\partial^2}{\partial s_1 \partial t_1}\log\tau_n - \frac{\partial^3}{\partial s_1^2 \partial t_1}\log\tau_n.
\end{aligned} \tag{2.2.11}$$

3 Matrix integrals and Combinatorics

3.1 Largest increasing sequences in Random Permutations and Words

Consider the group of *permutations of length* k

$$\begin{aligned}
S_k &= \{\text{permutations } \pi \text{ of } \{1,\ldots,k\}\} \\
&= \left\{\pi_k = \pi = \begin{pmatrix} 1 & \ldots & k \\ \pi(1) & \ldots & \pi(k)\end{pmatrix}, \text{ for distinct } 1 \le \pi(j) \le k\right\},
\end{aligned}$$

equipped with the uniform probability distribution

$$P_k(\pi_k) = 1/k!. \tag{3.1.1}$$

Also consider *words of length* k*, taken from an alphabet* $1,\ldots,p$,

$$\begin{aligned}
S_k^p &= \{\text{words } \sigma \text{ of length } k \text{ from an alphabet } \{1,\ldots,p\}\} \\
&= \left\{\sigma = \sigma_k = \begin{pmatrix} 1 & 2 & \ldots & k \\ \sigma(1) & \sigma(2) & \ldots & \sigma(k)\end{pmatrix}, \text{ for arbitrary } 1 \le \sigma(j) \le p\right\}
\end{aligned} \tag{3.1.2}$$

and uniform probability $P_k^p(\sigma) = 1/k^p$ on S_k^p.

An *increasing subsequence* of $\pi_k \in S_k$ or $\sigma_k \in S_k^p$ is a sequence[6] $1 \leq j_1 < \ldots < j_\alpha \leq k$, such that $\pi(j_1) \leq \ldots \leq \pi(j_\alpha)$. Define

$$\left.\begin{array}{l} L_k(\pi_k) \\ L_k(\sigma_k) \end{array}\right\} = \text{ length of the longest increasing subsequence of } \left\{\begin{array}{l} \pi_k \\ \sigma_k \end{array}\right. \tag{3.1.3}$$

We shall be interested in the probabilities

$$P_k(L_k(\pi) \leq n,\ \pi \in S_k) \ \text{ and } \ P_k^p(L_k(\sigma) \leq n,\ \sigma \in S_k^p).$$

$$\textit{Examples}: \left\{\begin{array}{l} \text{for } \pi_7 = (\underline{3}, 1, \underline{4}, 2, \underline{6}, \underline{7}, 5) \in S_7, \text{ we have } L_7(\pi_7) = 4. \\ \text{for } \pi_5 = (5, \underline{1}, \underline{4}, 3, 2) \in S_5, \text{ we have } L_5(\pi_5) = 2. \\ \text{for } \sigma_7 = (2, \underline{1}, 3, 2, \underline{1}, \underline{1}, \underline{2}) \in S_7^3, \text{ we have } L_7(\sigma_7) = 4. \end{array}\right.$$

In 1990, Gessel [14] considered the generating function (3.1.4) below and showed that it equals a Toeplitz determinant (determinant of a matrix, whose (i,j)th entry depends on $i-j$ only). By now, Theorem 3.1 below has many different proofs; at the end of section 3.3, we sketch a proof based on integrable ideas. See also Section 4.2.

Theorem 3.1 *(Gessel [14]) The following generating function has an expression in terms of a $U(n)$-matrix integral*[7]

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\xi^k}{k!} P_k(L_k(\pi) \leq n) &= \int_{U(n)} e^{\sqrt{\xi}\ \mathrm{Tr}(M+\bar{M})} dM \\ &= \frac{1}{n!} \oint_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left(e^{\sqrt{\xi}(z_k + \bar{z}_k)} \frac{dz_k}{2\pi i z_k} \right) \\ &= \det \left(\oint_{S^1} \frac{dz}{2\pi i z} z^{\ell - m} e^{\sqrt{\xi}(z+z^{-1})} \right)_{1 \leq \ell, k \leq n} \end{aligned} \tag{3.1.4}$$

[6]For permutations one automatically has strict inequalities $\pi(j_1) < \ldots < \pi(j_\alpha)$.

[7]The expression (3.1.4) is a determinant of Bessel functions, since $J_n(u)$ is defined by $e^{u(t-t^{-1})} = \sum_{-\infty}^{\infty} t^n J_n(2u)$ and thus

$$e^{\sqrt{\xi}(z+z^{-1})} = e^{\sqrt{-\xi}((-iz)-(-iz)^{-1})} = \sum (-iz)^n J_n(2\sqrt{-\xi}).$$

Theorem 3.2 *(Tracy-Widom [28]) We also have*[8]

$$
\begin{aligned}
\sum_{k=0}^{\infty} \frac{(p\xi)^k}{k!} P_k^p(L_k(\sigma) \leq n) &= \int_{U(n)} e^{\xi Tr \bar{M}} \det(I+M)^p dM \\
&= \det\left(\oint_{S^1} \frac{dz}{2\pi i z} z^{k-\ell} e^{\xi z^{-1}} (1+z)^p \right)_{1 \leq k, \ell \leq n}.
\end{aligned}
$$

Consider instead the subgroups of odd permutations, with $2^k k!$ elements, the hyperoctahedral group,

$$
S_{2k}^{\mathrm{odd}} = \left\{ \begin{array}{l} \pi_{2k} \in S_{2k}, \pi_{2k} : (-k, \ldots, -1, 1, \ldots, k) \circlearrowleft \\ \text{with } \pi_{2k}(-j) = -\pi_{2k}(j), \text{ for all } j \end{array} \right\} \subset S_{2k}
$$

$$
S_{2k+1}^{\mathrm{odd}} = \left\{ \begin{array}{l} \pi_{2k+1} \in S_{2k+1}, \pi_{2k} : (-k, \ldots, -1, 0, 1, \ldots, k) \circlearrowleft \\ \text{with } \pi_{2k+1}(-j) = -\pi_{2k+1}(j), \text{ for all } j \end{array} \right\} \subset S_{2k}
$$

Then, according to Rains [22] and Tracy-Widom [28], the following generating functions, again involving the length of the longest increasing sequence, are related to matrix integrals:

Theorem 3.3 *For $\pi_{2k} \in S_{2k}^{odd}$ and $\pi_{2k+1} S_{2k}^{odd}$, one has the following generating functions:*

$$
\sum_0^{\infty} \frac{(2\xi)^k}{k!} P(L(\pi_{2k}) \leq n \textit{ for } \pi_{2k} \in S_{2k}^{odd}) = \int_{U(n)} e^{\sqrt{\xi}\ \operatorname{Tr}(M^2 + \bar{M}^2)} dM
$$

$$
\begin{aligned}
&\sum_0^{\infty} \frac{(2\xi)^k}{k!} P(L(\pi_{2k+1}) \leq n \textit{ for } \pi_{2k+1} \in S_{2k}^{odd}) \\
&= \frac{1}{4} \left(\frac{\partial}{\partial t} \right)^2 \int_{U(n)} dM \left(e^{\operatorname{Tr}(t(M+\bar{M}) + \sqrt{\xi}(M^2+\bar{M}^2)} + e^{\operatorname{Tr}(t(M+\bar{M}) - \sqrt{\xi}(M^2+\bar{M}^2))} \right) \Bigg|_{t=0}
\end{aligned}
$$

Generating functions for other combinatorial quantities related to integrals over the Grassmannian $\mathrm{Gr}(p, \mathbb{R}^n)$ and $\mathrm{Gr}(p, \mathbb{C}^n)$ of p-planes in $\mathbb{R}^n$ or $\mathbb{C}^n$ have been investigated by Adler-van Moerbeke [3].

[8] The functions appearing in the contour integration are confluent hypergeometric functions ${}_1F_1$.

3.2 Combinatorial background

The reader is reminded of a few basic facts in combinatorics. Standard references to this subject are MacDonald, Sagan, Stanley, Stanton and White [19, 23, 24, 25].

• A *partition* λ of n (noted $\lambda \vdash n$) or a *Young diagram* λ of weight n is represented by a sequence of integers $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_\ell \geq 0$, such that $n = |\lambda| := \lambda_1 + ... + \lambda_\ell$; $n = |\lambda|$ is called the weight. A *dual Young diagram* $\lambda^\top = (\lambda_1^\top \geq \lambda_2^\top \geq ...)$ is the diagram obtained by flipping the diagram λ about its diagonal; clearly $|\lambda| = |\lambda^\top|$. Define $\mathbb{Y}_n := \{\text{all partitions } \lambda \text{ with } |\lambda| = n\}$.

A *skew-partition* or *skew Young diagram* $\lambda\backslash\mu$, for $\lambda \supset \mu$, is defined as the shape obtained by removing the diagram μ from λ.

• The *Schur polynomial* $\mathbf{s}_\lambda(t)$ associated with a Young diagram $\lambda \vdash n$, is defined by

$$\mathbf{s}_\lambda(t_1, t_2, ...) = \det \left(\mathbf{s}_{\lambda_i - i + j}(t)\right)_{1 \leq i,j \leq \ell}$$

in terms of elementary Schur polynomials $\mathbf{s}_i(t)$, defined by

$$e^{\sum_1^\infty t_i z^i} =: \sum_{i \geq 0} \mathbf{s}_i(t) z^i, \qquad \text{and} \quad \mathbf{s}_i(t) = 0 \text{ for } i < 0.$$

The *skew Schur polynomial* $\mathbf{s}_{\lambda\backslash\mu}(t)$, associated with a skew Young diagram $\lambda\backslash\mu$, is defined by

$$\mathbf{s}_{\lambda\backslash\mu}(t) := \det \left(\mathbf{s}_{\lambda_i - i - \mu_j + j}(t)\right)_{1 \leq i,j \leq n}. \tag{3.2.1}$$

The $\mathbf{s}_\lambda$'s form a basis of the space of symmetric functions in $x_1, x_2, \ldots$, via the map $kt_k = \sum_{i \geq 1} x_i^k$.

• A *standard Young tableau* P of shape $\lambda \vdash n$ is an array of integers $1, ..., n$ placed in the Young diagram, which are strictly increasing from left to right *and* from top to bottom. A *standard skew Young tableau* of shape $\lambda\backslash\mu \vdash n$ is defined in a similar way. Then, it is well-known that

$$\begin{aligned} f^\lambda &:= \# \left\{ \begin{array}{l} \text{standard tableaux of shape } \lambda \vdash n \\ \text{filled with integers } 1, \ldots, n \end{array} \right\} = \frac{|\lambda|!}{u^{|\lambda|}} \, \mathbf{s}_\lambda(t) \Big|_{t_i = u\delta_{i1}} \\ f^{\lambda\backslash\mu} &:= \# \left\{ \begin{array}{l} \text{standard skew tableaux of shape} \\ \lambda\backslash\mu \vdash n \text{ filled with integers } 1, \ldots, n \end{array} \right\} = \frac{|\lambda\backslash\mu|!}{u^{|\lambda\backslash\mu|}} \, \mathbf{s}_{\lambda\backslash\mu}(t) \Big|_{t_i = u\delta_{i1}}. \end{aligned} \tag{3.2.2}$$

• A *semi-standard Young tableau* of shape $\lambda \vdash n$ is an array of integers $1, \ldots, p$ placed in the Young diagram λ, which are non-decreasing from left to right *and* strictly increasing from top to bottom. The *number of semi-standard Young tableaux* of a given shape $\lambda \vdash n$, filled with integers 1 to p for $p \geq \lambda_1^\top$, has the following expression in terms of Schur polynomials:

$$\#\left\{\begin{array}{l}\text{semi-standard tableaux of shape } \lambda\\ \text{filled with numbers from 1 to } p\end{array}\right\} = \mathbf{s}_\lambda\Big(p, \frac{p}{2}, \frac{p}{3}, \ldots\Big). \tag{3.2.3}$$

• *Robinson-Schensted-Knuth (RSK) correspondence*: There is a 1-1 correspondence

$$S_k \longleftrightarrow \left\{\begin{array}{l}\text{pairs of standard Young tableaux } (P, Q),\\ \text{both of same arbitrary shape } \lambda, \text{ with}\\ |\lambda| = k, \text{ filled with integers } 1, \ldots, k\end{array}\right\}. \tag{3.2.4}$$

Given a permutation $\pi = (i_1, ..., i_k)$, the RSK correspondence constructs two standard Young tableaux P, Q having the same shape λ. This construction is inductive. Namely, having obtained two equally shaped Young diagrams P_j, Q_j from $i_1, ..., i_j$, with the numbers $(i_1, ..., i_j)$ in the boxes of P_j and the numbers $(1, ..., j)$ in the boxes of Q_j, one creates a new diagram Q_{j+1}, by putting the *next number* i_{j+1} *in the first row of* P, according to the rules:

(i) if $i_{j+1} \geq$ all numbers appearing in the first row of P_j, then one creates a new box containing i_{j+1} to the right of the first column,

(ii) if not, place i_{j+1} in the box (of the first row) with the smallest higher number. That number then gets pushed down to the second row of P_j according to the rules (i) and (ii), as if the first row had been removed.

The diagram Q is a bookkeeping device; namely, add a box (with the number $j + 1$ in it) to Q_j exactly at the place, where the new box has been added to P_j. This produces a new diagram Q_{j+1} of same shape as P_{j+1}.

The inverse of this map is constructed by reversing the steps above. The Robinson-Schensted-Knuth correspondence has the following properties:

- length (longest increasing subsequence of π) $=$ # (columns in P)
- length (longest decreasing subsequence of π) $=$ # (rows in P)
- $\pi \mapsto (P, Q)$, then $\pi^{-1} \mapsto (Q, P)$ (3.2.5)

So-called **Plancherel measure** $\tilde{P}_k$ on $\mathbb{Y}_k$ is the probability induced from the uniform probability P_k on S_k (see (3.1.1)), via the RSK map (3.2.4). For an arbitrary partition $\lambda \vdash k$, it is computed as follows:

$$\begin{aligned}
\tilde{P}_k(\lambda) &:= P_k(\text{permutations } \pi \in S_k \text{ leading to } \lambda \in \mathbb{Y}_k \text{ by RSK}) \\
&= \frac{\#\{\text{permutations leading to } \lambda \in \mathbb{Y}_k \text{ by RSK}\}}{k!} \\
&= \frac{\#\left\{\begin{array}{l}\text{pairs of standard tableaux } (P,Q)\text{, both} \\ \text{of shape } \lambda\text{, filled with numbers } 1,\dots,k\end{array}\right\}}{k!} \\
&= \frac{(f^\lambda)^2}{k!}, \qquad \text{using (3.2.2).}
\end{aligned}$$

Note that, by the first property in (3.2.5), we have

$$L_k(\pi) \le n \iff (P,Q) \text{ has shape } \lambda \text{ with } |\lambda| = k \text{ and } \lambda_1 \le n.$$

These facts prove the following Proposition:

Proposition 3.4 *Let P_k be uniform probability on the permutations in S_k and $\tilde{P}_k$ Plancherel measure on $\mathbb{Y}_k := \{$partitions $\lambda \vdash k\}$. Then:*

$$\begin{aligned}
P_k(L_k(\pi) \le n) &= \frac{1}{k!}\#\left\{\begin{array}{l}\textit{pairs of standard Young tableaux } (P,Q)\textit{, both of} \\ \textit{same arbitrary shape } \lambda\textit{, with } |\lambda| = k \textit{ and } \lambda_1 \le n\end{array}\right\} \\
&= \frac{1}{k!}\sum_{\substack{|\lambda|=k \\ \lambda_1 \le n}} (f^\lambda)^2 \\
&= \tilde{P}_k(\lambda_1 \le n). \qquad (3.2.6)
\end{aligned}$$

From a slight extension of the RSK correspondence for "words", we have

$$S_k^p \longleftrightarrow \left\{\begin{array}{c}\text{semi-standard and standard Young tableaux} \\ (P,Q) \text{ of same shape } \lambda \text{ and } |\lambda| = k\text{, filled} \\ \text{resp., with integers } (1,\dots,p) \text{ and } (1,\dots,k),\end{array}\right\},$$

and thus the uniform probability P_k^p on S_k^p induces a probability measure $\tilde{P}_k^p$ on

$$\mathbb{Y}_k^p = \{\text{partitions } \lambda \text{ such that } |\lambda| = k,\ \lambda_1^\top \le p\},$$

namely

$$\begin{aligned}\tilde{P}_k^p(\lambda) &= P_k^p\{\text{words } \sigma \in S_k^p \text{ leading to } \lambda \in \mathbb{Y}_k^p \text{ by RSK}\}\\ &= \frac{f^\lambda\ \mathbf{s}_\lambda\big(p, \frac{p}{2}, \frac{p}{3}, \ldots\big)}{p^k} \quad , \quad \lambda \in \mathbb{Y}_k^p.\end{aligned}$$

Proposition 3.5 *Let P_k be uniform probability on words in S_k^p and $\tilde{P}_k^p$ the induced measure on $\mathbb{Y}_k^p$. Then:*

$$\begin{aligned}P_k^p(L(\sigma) \le n) &= \frac{1}{p^k}\#\left\{\begin{array}{l}\textit{semi-standard and standard Young tableaux } (P,Q)\\ \textit{of same shape } \lambda, \textit{ with } |\lambda| = k \textit{ and } \lambda_1 \le n, \textit{ filled}\\ \textit{resp., with integers } (1,\ldots,p) \textit{ and } (1,\ldots,k),\end{array}\right\}\\ &= \frac{1}{p^k}\sum_{\substack{|\lambda|=k\\ \lambda_1\le n}} f^\lambda\ \mathbf{s}_\lambda\big(p, \frac{p}{2}, \frac{p}{3}, \ldots\big)\\ &= \tilde{P}_k^p(\lambda_1 \le n). \qquad (3.2.7)\end{aligned}$$

<u>*Example:*</u> For permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5\\ 5 & \underline{1} & 4 & \underline{3} & 2\end{pmatrix} \in S_5$, the RSK algorithm gives

$$P \Longrightarrow \quad \begin{matrix}5\end{matrix} \qquad \begin{matrix}1\\5\end{matrix} \qquad \begin{matrix}1 & 4\\5\end{matrix} \qquad \begin{matrix}1 & 3\\4\\5\end{matrix} \qquad \begin{matrix}1 & 2\\3\\4\\5\end{matrix}$$

$$Q \Longrightarrow \quad \begin{matrix}1\end{matrix} \qquad \begin{matrix}1\\2\end{matrix} \qquad \begin{matrix}1 & 3\\2\end{matrix} \qquad \begin{matrix}1 & 3\\2\\4\end{matrix} \qquad \begin{matrix}1 & 3\\2\\4\\5\end{matrix}$$

Hence

$$\pi \longmapsto (P,Q) = \left(\underbrace{\left(\begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}\right)}_{\text{standard}}^{\overbrace{\qquad}^{2}}, \underbrace{\left(\begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}\right)}_{\text{standard}} \right).$$

Note that the sequence $1, 3$, underlined in the permutation above is a longest increasing sequence, and so $L_5(\pi) = 2$; of course, we also have

$$L_5(\pi) = 2 = \#\ \{\text{columns of } P \text{ or } Q\}.$$

3.3 A Probability on partitions and Toeplitz determinants

Define yet another "probability measure" on the set $\mathbb{Y}$ of Young diagrams

$$\mathbb{P}(\lambda) = Z^{-1}\mathbf{s}_\lambda(t)\mathbf{s}_\lambda(s), \qquad Z = e^{\sum_{i\geq 1} it_i s_i}. \tag{3.3.1}$$

Cauchy's identity[9] guarantees that $\mathbb{P}(\lambda)$ is a probability measure, in the sense

$$\sum_{\lambda\in\mathbb{Y}} \mathbb{P}(\lambda) = 1,$$

without necessarily $0 \leq \mathbb{P}(\lambda) \leq 1$. This probability measure has been introduced and extensively studied by Borodin, Okounkov, Olshanski and others; see [11, 21] and references within. In the following Proposition, the Toeplitz determinants appearing in (2.0.1) acquire a probabilistic meaning in terms of the new probability $\mathbb{P}$:

Proposition 3.6 *Given the probability (3.3.1), the following holds*

$$\mathbb{P}\left(\lambda \textit{ with } \lambda_1 \leq n\right) = Z^{-1} \det\left(\oint_{S^1} \frac{dz}{2\pi i z} z^{k-\ell} e^{-\sum_1^\infty (t_i z^i + s_i z^{-i})}\right)_{1\leq k,\ell\leq n} \tag{3.3.2}$$

and

$$\mathbb{P}(\lambda \textit{ with } \lambda_1^\top \leq n) = Z^{-1} \det\left(\oint_{S^1} \frac{dz}{2\pi i z} z^{k-\ell} e^{\sum_1^\infty (t_i z^i + s_i z^{-i})}\right)_{1\leq k,\ell\leq n}$$

with Z given by (3.3.1).

Proof: Consider the semi-infinite Toeplitz matrix

$$m_\infty(t,s) = (\mu_{k\ell})_{k,\ell\geq 0}, \text{ with } \mu_{k\ell}(t,s) = \oint_{S^1} z^{k-\ell} e^{\sum_1^\infty (t_j z^j - s_j z^{-j})} \frac{dz}{2\pi i z}.$$

[9] Cauchy's identity takes on the following form in the t and s variables:

$$\sum_{\lambda\in\mathbb{Y}} \mathbf{s}_\lambda(t)\mathbf{s}_\lambda(s) = e^{\sum_1^\infty it_i s_i}.$$

Note that

$$\begin{aligned}\frac{\partial\mu_{k\ell}}{\partial t_i} &= \oint_{S^1} z^{k-\ell+i}e^{\sum_1^\infty(t_jz^j-s_jz^{-j})}\frac{dz}{2\pi iz}=\mu_{k+i,\ell}\\ \frac{\partial\mu_{k\ell}}{\partial s_i} &= -\oint_{S^1} z^{k-\ell-i}e^{\sum(t_jz^j-s_jz^{-j})}\frac{dz}{2\pi iz}=-\mu_{k,\ell+i}\end{aligned}\tag{3.3.3}$$

with initial condition $\mu_{k\ell}(0,0)=\delta_{k\ell}$. In matrix notation, this amounts to the system of differential equations[10]

$$\frac{\partial m_\infty}{\partial t_i}=\Lambda^i m_\infty \text{ and } \frac{\partial m_\infty}{\partial s_i}=-m_\infty(\Lambda^\top)^i, \text{ with initial condition } m_\infty(0,0)=I_\infty. \tag{3.3.4}$$

The solution to this initial value problem is given by the following two expressions:

(i)
$$m_\infty(t,s)=(\mu_{k\ell}(t,s))_{k,\ell\geq 0}, \tag{3.3.5}$$

as follows from the differential equation (3.3.3), and

(ii)
$$m_\infty(t,s)=e^{\sum_1^\infty t_i\Lambda^i}m_\infty(0,0)e^{-\sum_1^\infty s_i\Lambda^{\top i}}, \tag{3.3.6}$$

upon using $(\partial/\partial t_k)e^{\sum_1^\infty t_i\Lambda^i}=\Lambda^k e^{\sum_1^\infty t_i\Lambda^i}$. Then, by the uniqueness of solutions of ode's, the two solutions coincide, and in particular the $n\times n$ upper-left blocks of (3.3.5) and (3.3.6), namely

$$m_n(t,s)=E_n(t)m_\infty(0,0)E_n^\top(-s), \tag{3.3.7}$$

where

$$E_n(t) = \begin{pmatrix} 1 & \mathbf{s}_1(t) & \mathbf{s}_2(t) & \mathbf{s}_3(t) & \dots & \mathbf{s}_{n-1}(t) & \dots\\ 0 & 1 & \mathbf{s}_1(t) & \mathbf{s}_2(t) & \dots & \mathbf{s}_{n-2}(t) & \dots\\ \vdots & & & & & & \\ & & & & & \mathbf{s}_1(t) & \dots\\ 0 & & \dots & & 0 & 1 & \dots \end{pmatrix} = (\mathbf{s}_{j-i}(t))_{\substack{1\leq i<n\\ 1\leq j<\infty}}$$

is the $n\times n$ upper-left blocks of

$$e^{\sum_1^\infty t_i\Lambda^i}=\sum_0^\infty \mathbf{s}_i(t)\Lambda^i = \begin{pmatrix} 1 & \mathbf{s}_1(t) & \mathbf{s}_2(t) & \mathbf{s}_3(t) & \dots\\ 0 & 1 & \mathbf{s}_1(t) & \mathbf{s}_2(t) & \dots\\ 0 & 0 & 1 & \mathbf{s}_1(t) & \dots\\ 0 & 0 & 0 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} = (\mathbf{s}_{j-i}(t))_{\substack{1\leq i<\infty\\ 1\leq j<\infty}}.$$

[10]The operator Λ is the semi-infinite shift matrix defined in footnote 2. Also I_∞ is the semi-infinite identity matrix.

Therefore the determinants of the matrices (3.3.7) coincide:

$$\det m_n(t,s) = \det(E_n(t) m_\infty(0,0) E_n^\top(-s)). \tag{3.3.8}$$

Moreover, from the **Cauchy-Binet formula**[11], applied twice, one proves the following: given an arbitrary semi-infinite initial condition $m_\infty(0,0)$, the expression below admits an expansion in Schur polynomials,

$$\det(E_n(t) m_\infty(0,0) E_n^\top(-s)) = \sum_{\substack{\lambda,\ \nu \\ \lambda_1^\top,\ \nu_1^\top \leq n}} \det(m^{\lambda,\nu}) \mathbf{s}_\lambda(t) \mathbf{s}_\nu(-s), \quad \text{for } n > 0, \tag{3.3.9}$$

where the sum is taken over all Young diagrams λ and ν, with first columns $\leq n$ (i.e., $\lambda_1^\top$ and $\nu_1^\top \leq n$) and where $m^{\lambda,\nu}$ is the matrix

$$m^{\lambda,\nu} := \left(\mu_{\lambda_i - i + n, \nu_j - j + n}(0,0)\right)_{1 \leq i,j \leq n}. \tag{3.3.10}$$

Applying formula (3.3.10) to $m_\infty(0,0) = I_\infty$, we have

$$\det m^{\lambda,\nu} = \det(\mu_{\lambda_i - i + n, \nu_j - j + n})_{1 \leq i,j \leq n} \neq 0 \quad \text{if and only if } \lambda = \nu, \tag{3.3.11}$$

in which case $\det m^{\lambda,\lambda} = 1$. Therefore,

$$\sum_{\substack{\lambda \in \mathbb{Y} \\ \lambda_1^\top \leq n}} s_\lambda(t) s_\lambda(-s) = \det \left(\oint_{S^1} \frac{dz}{2\pi i z} z^{k-\ell} e^{\sum_1^\infty (t_i z^i - s_i z^{-i})} \right)_{1 \leq k,\ell \leq n}. \tag{3.3.12}$$

But, we also have, using the probability $\mathbb{P}$, defined in (3.3.1), that

$$\mathbb{P}\left(\lambda \text{ with } \lambda_1^\top \leq n\right) = Z^{-1} \sum_{\substack{\lambda \in \mathbb{Y} \\ \lambda_1^\top \leq n}} \mathbf{s}_\lambda(t) \mathbf{s}_\lambda(s) \tag{3.3.13}$$

[11] Given two matrices $\underset{(m,n)}{A}$, $\underset{(n,m)}{B}$, for n large $\geq m$

$$\begin{aligned} \det(AB) &= \det\left(\sum_i a_{\ell i} b_{ik}\right)_{1 \leq k,\ell \leq m} \\ &= \sum_{1 \leq i_1 < \ldots < i_m \leq n} \det(a_{k,i_\ell})_{1 \leq k,\ell \leq m} \det(b_{i_k,\ell})_{1 \leq k,\ell \leq m}. \end{aligned}$$

Comparing the two formulas (3.3.12) and (3.3.13) and changing $s \mapsto -s$ in (3.3.12), yield

$$\begin{aligned} \mathbb{P}\left(\lambda \text{ with } \lambda_1^\top \leq n\right) &= Z^{-1} \det \left(\oint_{S^1} \frac{dz}{2\pi i z} z^{k-\ell} e^{\sum_1^\infty (t_i z^i + s_i z^{-i})} \right)_{1 \leq k, \ell \leq n} \\ &= Z^{-1} \sum_{\substack{\lambda \in \mathbb{Y} \\ \lambda_1^\top \leq n}} \mathbf{s}_\lambda(t) \mathbf{s}_\lambda(s). \qquad (3.3.14) \end{aligned}$$

Using $s_\lambda(-t) = (-1)^{|\lambda|} s_{\lambda^\top}(t)$, one easily checks

$$\begin{aligned} \mathbb{P}\left(\lambda \text{ with } \lambda_1 \leq n\right) &= Z^{-1} \sum_{\substack{\lambda \in \mathbb{Y} \\ \lambda_1 \leq n}} \mathbf{s}_\lambda(t) \mathbf{s}_\lambda(s) \ , \quad \text{by definition} \\ &= Z^{-1} \sum_{\substack{\lambda \in \mathbb{Y} \\ \lambda_1^\top \leq n}} \mathbf{s}_{\lambda^\top}(t) \mathbf{s}_{\lambda^\top}(s) \\ &= Z^{-1} \sum_{\substack{\lambda \in \mathbb{Y} \\ \lambda_1^\top \leq n}} \mathbf{s}_\lambda(-t) \mathbf{s}_\lambda(-s) \\ &= Z^{-1} \det \left(\oint_{S^1} \frac{dz}{2\pi i z} z^{k-\ell} e^{-\sum_1^\infty (t_i z^i + s_i z^{-i})} \right)_{1 \leq k, \ell \leq n}, \end{aligned}$$

using (3.3.14) in the last equality, with Z as in (3.3.1). This establishes Proposition 3.6. ∎

Proof of Theorem 3.1: For real $\xi > 0$, consider the locus

$$\mathcal{L}_1 = \{\text{all } s_k = t_k = 0, \text{ except } t_1 = s_1 = \sqrt{\xi}\} \qquad (3.3.15)$$

Indeed, for an arbitrary $\lambda \in \mathbb{Y}$, the probability (3.3.1) evaluated along $\mathcal{L}_1$ reads:

$$\begin{aligned} \mathbb{P}(\lambda)\Big|_{\mathcal{L}_1} &= e^{-\sum_{k \geq 1} k t_k s_k} \mathbf{s}_\lambda(t) \mathbf{s}_\lambda(s) \Big|_{\substack{t_i = \sqrt{\xi}\delta_{i1} \\ s_i = \sqrt{\xi}\delta_{i1}}} \\ &= e^{-\xi} \xi^{|\lambda|/2} \frac{f^\lambda}{|\lambda|!} \xi^{|\lambda|/2} \frac{f^\lambda}{|\lambda|!}, \quad \text{using (3.2.2)}, \\ &= e^{-\xi} \frac{\xi^{|\lambda|}}{|\lambda|!} \frac{(f^\lambda)^2}{|\lambda|!}. \end{aligned}$$

Therefore

$$
\begin{aligned}
\mathbb{P}(\lambda_1 \le n)\Big|_{\mathcal{L}_1} &= \sum_{\substack{\lambda\in\mathbb{Y}\\ \lambda_1\le n}} e^{-\xi}\frac{\xi^{|\lambda|}}{|\lambda|!}\frac{(f^\lambda)^2}{|\lambda|!} \\
&= e^{-\xi}\sum_0^\infty \frac{\xi^k}{k!}\sum_{\substack{|\lambda|=k\\ \lambda_1\le n}}\frac{(f^\lambda)^2}{k!} \\
&= e^{-\xi}\sum_0^\infty \frac{\xi^k}{k!}P_k(L_k(\pi)\le n), \qquad \text{by Proposition 3.4.}
\end{aligned}
\tag{3.3.16}
$$

The next step is to evaluate (3.3.2) in Proposition 3.5 along the locus $\mathcal{L}_1$,

$$
\begin{aligned}
\mathbb{P}(\lambda_1 \le n)\Big|_{\mathcal{L}_1} &= e^{-\sum_{i\ge1} it_is_i}\det\left(\oint_{S^1}\frac{dz}{2\pi iz}z^{k-\ell}e^{-\sum_1^\infty(t_iz^i+s_iz^{-i})}\right)_{1\le k,\ell\le n}\Bigg|_{\mathcal{L}_1} \\
&= e^{-\xi}\det\left(\oint_{S^1}\frac{dz}{2\pi iz}z^{k-\ell}e^{-\sqrt{\xi}(z+z^{-1})}\right)_{1\le k,\ell\le n} \\
&= e^{-\xi}\det\left(\oint_{S^1}\frac{dz}{2\pi iz}z^{k-\ell}e^{\sqrt{\xi}(z+z^{-1})}\right)_{1\le k,\ell\le n},
\end{aligned}
\tag{3.3.17}
$$

by changing $z \mapsto -z$. Finally, comparing (3.3.16) and (3.3.17) yields (3.1.4), ending the proof of Theorem 3.1. ■

Proof of Theorem 3.2: The proof of this theorem goes along the same lines, except one uses Proposition 3.4 and one evaluates (3.3.2) along the locus

$$\mathcal{L}_2 = \{t_k = \delta_{k1}\xi \ \text{ and } \ ks_k = p\},$$

instead of $\mathcal{L}_1$; then one makes the change of variable $z \mapsto -z^{-1}$ in the integral. ■

3.4 Non-intersecting random walks

Consider n walkers in $\mathbb{Z}$, walking from $x = (x_1 < x_2 < ... < x_n)$ to $y = (y_1 < y_2 < ... < y_n)$, such that, at each moment, only one walker moves either one step to the left, or one step to the right, with all possible moves equally likely. This section deals with a generating function for the probability

$$P(k,x,y) := P\left(\begin{array}{c}\text{that } n \text{ walkers in } \mathbb{Z} \text{ go from } x_1,\dots,x_n \text{ to}\\ y_1,\dots,y_n \text{ in } k \text{ steps, and do not intersect}\end{array}\right) = \frac{b_{xy}^{(k)}}{(2n)^k}$$

We now state a Theorem which generalizes Theorem 3.1; the latter can be recovered by assuming close packing $x = y = (0,1,\dots,n-1)$. In Section 4.3 discrete equations will be found for $P(k;x,y)$.

Theorem 3.7 *(Adler-van Moerbeke [4]) The generating function for the $P(k;x,y)$ above has the following matrix integral representation:*

$$\begin{aligned}\sum_{k\geq 0}\frac{(2nz)^k}{k!}P(k;x,y) &= \int_{U(n)} \mathbf{s}_\lambda(M)\mathbf{s}_\mu(\bar{M})e^{z\,\mathrm{Tr}(M+\bar{M})}dM =: a_{\lambda\mu}(z)\\ &= \det\left(\oint_{S^1}\frac{du}{2\pi iu}u^{\lambda_\ell-\ell-\mu_k+k}e^{z(u+u^{-1})}\right)_{1\leq k,\ell\leq n},\end{aligned}$$

where $\mathbf{s}_\lambda$ and $\mathbf{s}_\mu$ are Schur polynomials[12] *with regard to the partitions λ and μ, themselves determined by the initial and final positions x and y,*

$$\lambda_{n-i+1} := x_i - i + 1\ , \quad \mu_{n-i+1} := y_i - i + 1. \text{ for } i = 1,\dots,n. \tag{3.4.1}$$

<u>Remark:</u> The partitions λ and μ measure the discrepancy of x and y from close packing $0,1,\dots,n-1$!

<u>Remark:</u> Connections of random walks with Young diagrams have been known in various situations in the combinatorial literature; see R. Stanley [24] (p. 313), P. Forrester [13], D. Grabiner & P. Magyar [16, 17] and J. Baik [5].

Proof: Consider the locus

$$\mathcal{L}_1 = \{\text{all } t_k = s_k = 0, \text{ except } t_1 = z,\ s_1 = -z\}.$$

Then, since

$$e^{\sum_1^\infty (t_i u^i - s_i u^{-i})}\Big|_{\mathcal{L}_1} = e^{z(u+u^{-1})},$$

[12]Given a unitary matrix M, the notation $\mathbf{s}_\lambda(M)$ denotes a symmetric function of the eigenvalues $x_1,\dots,x_n$ of the unitary matrix M and thus in the notation of the present paper $\mathbf{s}_\lambda(M) := \mathbf{s}_\lambda\big(\mathrm{Tr}\,M, \frac{1}{2}\mathrm{Tr}\,M^2, \frac{1}{3}\mathrm{Tr}\,M^3, ...\big)$.

we have, combining (3.3.9) and (3.3.8),

$$\int_{U(n)} e^{z\,\mathrm{Tr}(M+\bar M)} e^{\sum_1^\infty \mathrm{Tr}(t_i M^i - s_i \bar M^i)} dM = \sum_{\substack{\lambda,\mu \text{ such that} \\ \lambda_1^\top,\mu_1^\top \le n}} a_{\lambda\mu}(z) s_\lambda(t) s_\mu(-s), \quad (3.4.2)$$

with (for definitions and formulas for skew Schur polynomials and tableaux, see (3.2.1) and (3.2.2))

$$\begin{aligned}
a_{\lambda\mu}(z) &\overset{(i)}{=} \det\left(\oint_{S^1} u^{\lambda_\ell-\ell-\mu_k+k} e^{z(u+u^{-1})}\frac{du}{2\pi i u}\right)_{1\le \ell,k\le n} \\
&\overset{(ii)}{=} \int_{U(n)} s_\lambda(M) s_\mu(\bar M) e^{z\,\mathrm{Tr}(M+\bar M)} dM \\
&\overset{(iii)}{=} \sum_{\substack{\nu \text{ with} \\ \nu_1^\top \le n}} s_{\nu\backslash\lambda}(t) s_{\nu\backslash\mu}(-s)\Big|_{\mathcal{L}_1} \\
&\overset{(iv)}{=} \sum_{\substack{\nu \text{ with } \nu\supset\lambda,\mu \\ \nu_1^\top\le n}} \frac{z^{|\nu\backslash\lambda|}}{|\nu\backslash\lambda|!} f^{\nu\backslash\lambda} \frac{z^{|\nu\backslash\mu|}}{|\nu\backslash\mu|!} f^{\nu\backslash\mu} \\
&\overset{(v)}{=} \sum_{k=0}^\infty \frac{z^k}{k!}\frac{k!}{k_1!\,k_2!} \sum_{\substack{\nu \text{ with } \nu\supset\lambda,\mu \\ |\nu\backslash\lambda|=k_1 \\ |\nu\backslash\mu|=k_2 \\ \nu_1^\top\le n}} f^{\nu\backslash\lambda} f^{\nu\backslash\mu}, \qquad \text{where } k_{\left\{ {1 \atop 2} \right\}} = \tfrac{1}{2}(k \mp |\lambda| \pm |\mu|), \\
&= \sum_{k\ge 0} \frac{z^k}{k!} \# \left\{ \begin{array}{l} \text{ways that } n \text{ non-intersecting} \\ \text{walkers in } \mathbb{Z} \text{ move in } k \text{ steps} \\ \text{from } x_1 < x_2 < \ldots < x_n \\ \text{to } y_1 < y_2 < \ldots < y_n \end{array} \right\} = \sum_{k\ge 0} \frac{(2nz)^k}{k!} P(k;x,y).
\end{aligned}$$

Equality (i) follows from (3.3.11) and (3.3.9). The Fourier coefficients $a_{\lambda\mu}(z)$ of (3.4.2) can be obtained by taking the inner-product[13] of the sum (3.4.2) with $\mathbf{s}_\alpha(t)\mathbf{s}_\beta(-s)$. Equality (iii) is the analogue of (3.3.14) for skew-partitions and also follows from the Cauchy-Binet formula. Equality (iv) follows from formula (3.2.2) for skew-partitions. Equality (v) follows immediately from (iv), whereas the last equality follows from an analogue of RSK as is now explained.

Consider, as in the picture below, the two skew-tableaux P and Q of shapes $\nu\backslash\lambda$ and $\mu\backslash\lambda$, with integers $1,\ldots,|\nu\backslash\lambda|$ and $1,\ldots,|\nu\backslash\mu|$ inserted respectively (strictly increasing from left to right and from top to bottom).

[13] $\langle \mathbf{s}_\alpha, \mathbf{s}_\lambda\rangle := \mathbf{s}_\alpha(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \ldots)\mathbf{s}_\lambda(t)\Big|_{t=0}$

The integers c_{ij} in the tableau P provide the instants of left move for the corresponding walker (indicated on the left), assuming they all depart from $(x_1, \ldots, x_n)$, which itself is specified by ν. This construction implies that, at each instant, only one walker moves and they never intersect. That takes an amount of time $|\nu\backslash\lambda| = \frac{1}{2}(k - |\lambda| + |\mu|) = k_1$, at which they end up at a position specified by ν. At the next stage and from that position, they start moving right at the instants $k - c'_{ij}$, where the c'_{ij} are given by the second skew tableau and forced to end up at positions $(y_1, \ldots, y_n)$, itself specified by μ; see (3.4.1). Again the construction implies here that they never intersect and only one walker moves at the time. The time (of right move) elapsed is $|\nu\backslash\mu| = \frac{1}{2}(k + |\lambda| - |\mu|) = k_2$. So, the total time elapsed is $k_1 + k_2 = k$.

The final argument hinges on the fact that any motion, where exactly one walker moves either left or right during time k can be transformed (in a canonical way) into a motion where the walkers first move left during time k_1 and then move right during time k_2. The precise construction is based on an idea of Forrester [13].This map is many-to-one: there are precisely $\frac{k!}{k_1!k_2!}$ walks leading to a walk where walkers first move left and then right.

$$
\begin{array}{l}
\text{1}^{\text{st}}\text{ walker} \\
\text{2}^{\text{nd}}\text{ walker} \\
\text{3}^{\text{rd}}\text{ walker} \\
\text{4}^{\text{th}}\text{ walker}
\end{array}
\quad
\overbrace{\begin{array}{|c|c|c|c|c|c|}
\hline
 & & & & c_{11} & c_{12} \\
\hline
 & & c_{21} & c_{22} & \multicolumn{2}{c}{} \\
\cline{1-4}
 & c_{31} & c_{32} & c_{33} & \multicolumn{2}{c}{} \\
\cline{1-4}
 & c_{41} & \multicolumn{4}{c}{} \\
\cline{1-2}
\end{array}}^{\text{instants of left move}}
\qquad
\overbrace{\begin{array}{|c|c|c|c|c|c|}
\hline
 & & & c'_{11} & c'_{12} & c'_{13} \\
\hline
 & c'_{21} & c'_{22} & c'_{23} & \multicolumn{2}{c}{} \\
\cline{1-4}
 & c'_{31} & c'_{32} & c'_{33} & \multicolumn{2}{c}{} \\
\cline{1-4}
c'_{41} & c'_{42} & \multicolumn{4}{c}{} \\
\cline{1-2}
\end{array}}^{\text{instants of right move}}
$$

P of shape $\nu\backslash\lambda$ $\qquad$ Q of shape $\nu\backslash\mu$

This sketches the proof of Theorem 3.7. ■

4 What do integrable systems tell us about combinatorics?

The fact that the matrix integrals are related to the Virasoro constraints and the Toeplitz lattice will lead to various statements about the various combinatorial problems considered in section 3.

4.1 Recursion relations for Unitary matrix integrals

Motivated by the integrals appearing in Theorems 3.1, 3.2 and 3.3, consider the integrals, for $\varepsilon = 0, \pm$, (different from the I_n introduced before)

$$I_n^\varepsilon := \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n z_k^\varepsilon e^{\sum_{j=1}^N \frac{u_j}{j}(z_k^j + z_k^{-j})} \frac{dz_k}{2\pi i z_k}. \tag{4.1.1}$$

They enjoy the following property:

Theorem 4.1 *The integral $I_n := I_n^0$ can be expressed as a polynomial in I_1 and the expressions $x_1, \ldots, x_{n-1}$,*

$$I_n = (I_1)^n \prod_1^{n-1} (1 - x_k^2)^{n-k}, \tag{4.1.2}$$

with the x_k's satisfying rational $2N+1$-step recursion relations in terms of prior x_i's; to be precise

$$\left(\left(\sum_1^N u_i L_1^i \right)_{k+1,k+1} + \left(\sum_1^N u_i L_1^i \right)_{k,k} - 2 \left(\sum_1^N u_i L_1^{i-1} \right)_{k+1,k} \right) = \frac{k x_k^2}{1 - x_k^2}, \tag{4.1.3}$$

where L_1 is the matrix[14] *defined in (2.2.2) and the u_i's appear in the integral (4.1.1). The left hand side of this expression is polynomial in the $x_{k-N}, \ldots, x_k, \ldots, x_{k+N}$ and linear in x_{k+N} and the parameters $u_1, \ldots, u_N$ .This implies the recursion relation*

$$x_{k+N} = F(x_{k+N-1}, \ldots, x_k, \ldots, x_{k-N}; u_1, \ldots, u_N),$$

with F rational in all arguments.

Remark: Note the x_n's are the same ratios as in (2.2.1) but for the integrals (4.1.1), i.e.,

$$x_n = (-1)^n \frac{I_n^+}{I_n} \text{, with } I_n := I_n^\varepsilon \Big|_{\varepsilon=0} \quad \text{and} \quad I_n^+ := I_n^\varepsilon \Big|_{\varepsilon=+1},$$

Example 1: *Symbol $e^{t(z+z^{-1})}$.*

[14] Note in the case of an integral the type (4.1.1), we have $x_n = y_n$, and thus $L_2 = L_1^\top$.

This concerns the integral in Theorem 3.1, expressing the generating function for the *probabilities of the length of longest increasing sequences in random permutations*. Setting $u_1 = u$, $u_i = 0$ for $i \geq 2$ in the equation (4.1.3), one finds that

$$x_n = \frac{\displaystyle\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n z_k e^{u(z_k+z_k^{-1})} \frac{dz_k}{2\pi i z_k}}{\displaystyle\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n e^{u(z_k+z_k^{-1})} \frac{dz_k}{2\pi i z_k}} \tag{4.1.4}$$

satisfies the simple three-step rational relation,

$$u(x_{k+1} + x_{k-1}) = \frac{k x_k}{x_k^2 - 1}. \tag{4.1.5}$$

This so-called MacMillan equation [20] for x_n was first derived by Borodin [7] and Baik [6], using Riemann-Hilbert methods. In [4], we show this is part of the much larger system of equations (4.1.3), closely related to the Toeplitz lattice. This map (4.1.5) is the simplest instance of a family of area-preserving maps of the plane, having an invariant, as found by McMillan, and extended by Suris [26] to maps of the form $\partial_n^2 x(n) = f(x(n))$, having an analytic invariant of two variables $\Phi(\beta, \gamma)$. The invariant in the case of the maps (4.1.5) is

$$\Phi(\beta, \gamma) = t\left(1 - \beta^2\right)\left(1 - \gamma^2\right) - n\beta\gamma,$$

which means that for all n,

$$\Phi(x_{n+1}, x_n) = \Phi(x_n, x_{n-1}).$$

For more on this matter, see the review by B. Grammaticos, F. Nijhoff, A. Ramani [18].

Example 2: *Symbol* $e^{t(z+z^{-1})+u(z^2+z^{-2})}$.

These symbols appear in the longest increasing sequence problem for the *hyperoctahedral group* ; see Theorem 3.3. Here we set $u_1 = t$, $u_2 = u$, $u_i = 0$ for $i \geq 3$ in the equation (4.1.3); one finds

$$x_n = \frac{\displaystyle\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n z_k e^{t(z_k+z_k^{-1})+u(z^2+z^{-2})} \frac{dz_k}{2\pi i z_k}}{\displaystyle\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n e^{t(z_k+z_k^{-1})+u(z^2+z^{-2})} \frac{dz_k}{2\pi i z_k}} \tag{4.1.6}$$

satisfies the five-step rational relation, $(v_n := 1 - x_n^2)$

$$0 = nx_n + tv_n(x_{n-1}+x_{n+1}) + 2uv_n\left(x_{n+2}v_{n+1} + x_{n-2}v_{n-1} - x_n(x_{n+1}+x_{n-1})^2\right). \tag{4.1.7}$$

Also here the map has a polynomial invariant

$$\Phi(\alpha,\beta,\gamma,\delta) = \Big(t + 2u(\alpha(\delta-\beta) - \gamma(\delta+\beta))\Big)(1-\beta^2)(1-\gamma^2) - n\beta\gamma;$$

that is for all n,

$$\Phi(x_{n-1},x_n,x_{n+1},x_{n+2}) = \Phi(x_{n-2},x_{n-1},x_n,x_{n+1}).$$

Proof of Theorem 4.1: Formula (4.1.2) follows straightforwardly from the identity (2.2.3). Moreover Proposition 2.1 implies the integrals

$$\tau_n^{\varepsilon}(t,s) = \frac{1}{n!}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} z_k^{\varepsilon} e^{\sum_1^{\infty}(t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k} \tag{4.1.8}$$

satisfy the Virasoro constraints (2.1.1). Thus, setting $\mathbb{V}_n := \mathbb{V}_n^{\varepsilon}\Big|_{\varepsilon=0}$ and $\mathbb{V}_n^{+} := \mathbb{V}_n^{\varepsilon}\Big|_{\varepsilon=1}$, we have

$$\begin{aligned}
0 &= \frac{\mathbb{V}_0^{+}\tau_n^{+}}{\tau_n^{\varepsilon}} - \frac{\mathbb{V}_0\tau_n}{\tau_n}\\
&= \sum_{i\geq 1}\left(it_i\frac{\partial}{\partial t_i} - is_i\frac{\partial}{\partial s_i}\right)\log x_n + n, \qquad \text{where } x_n = (-1)^n\frac{\tau_n^{+}}{\tau_n}\\
&= \frac{1-x_n^2}{x_n}\frac{\partial}{\partial x_n}\sum_{i\geq 1}(it_iG_i - is_iH_i) + n, \qquad \text{using (2.2.4)}\\
&= \frac{1-x_n^2}{x_n^2}\sum_{i\geq 1}\left\{\begin{array}{l} it_i\left(-(L_1^i)_{n+1,n+1} + (L_1^{i-1})_{n+1,n}\right)\\ +is_i\left((L_2^i)_{nn} - (L_2^{i-1})_{n,n+1}\right)\end{array}\right\} + n.
\end{aligned}$$

Setting

$$it_i = -is_i = \begin{cases} u_i & \text{for } 1\leq i\leq N\\ 0 & \text{for } i > N,\end{cases}$$

leads to the claim (4.1.3). Relations (4.1.5) and (4.1.7) are obtained by specialization. ∎

4.2 The Painlevé V equation for the longest increasing sequence problem

The statement of Theorem 3.1 can now be completed by the following Theorem, due to Tracy-Widom [27]. The integrable method explained below captures many other situations, like longest increasing sequences in involutions and words; see Adler-van Moerbeke [1].

Theorem 4.2 *For every $n \geq 0$, the generating function (3.1.4) for the probability of the longest increasing sequence can be expressed in terms of a specific solution of the Painlevé V equation :*

$$\sum_{k=0}^{\infty} \frac{\xi^k}{k!} P_k(L_k(\pi) \leq n) = \exp \int_0^{\xi} \log \left(\frac{\xi}{u} \right) g_n(u) du; \tag{4.2.1}$$

the function $g_n = g$ is the unique solution to the **Painlevé V equation***, with the following initial condition:*

$$\begin{cases} g'' - \dfrac{g'^2}{2} \left(\dfrac{1}{g-1} + \dfrac{1}{g} \right) + \dfrac{g'}{u} + \dfrac{2}{u} g(g-1) - \dfrac{n^2}{2u^2} \dfrac{g-1}{g} = 0 \\ \\ \text{with } \; g_n(u) = 1 - \dfrac{u^n}{(n!)^2} + O(u^{n+1}), \; \text{ near } u = 0. \end{cases} \tag{4.2.2}$$

Proof: For the sake of this proof, consider the locus

$$\mathcal{L} = \{ \text{ all } t_i = s_i = 0, \text{ except } t_1, s_1 \neq 0 \}.$$

From (2.1.2), we have on $\mathcal{L}$,

$$\begin{aligned} 0 = \frac{\mathbb{V}_0 \tau_n}{\tau_n} \Big|_{\mathcal{L}} &= \left(t_1 \frac{\partial}{\partial t_1} - s_1 \frac{\partial}{\partial s_1} \right) \log \tau_n \Big|_{\mathcal{L}} \\ 0 = \frac{\mathbb{V}_0 \tau_n}{\tau_n} - \frac{\mathbb{V}_0 \tau_{n-1}}{\tau_{n-1}} \Big|_{\mathcal{L}} &= \left(t_1 \frac{\partial}{\partial t_1} - s_1 \frac{\partial}{\partial s_1} \right) \log \frac{\tau_n}{\tau_{n-1}} \Big|_{\mathcal{L}} \\ 0 = \frac{\partial}{\partial t_1} \frac{\mathbb{V}_{-1} \tau_n}{\tau_n} \Big|_{\mathcal{L}} &= \left(-s_1 \frac{\partial^2}{\partial s_2 \partial t_1} + n \frac{\partial^2}{\partial t_1 \partial s_1} \right) \log \tau_n \Big|_{\mathcal{L}} + n. \end{aligned}$$

Then combining with identities (2.2.11) and (2.2.10), one finds after some computations that

$$g_n(x) = -\frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_n(t,s)\Big|_{\mathcal{L}} = \frac{d}{dx} x \frac{d}{dx} \left(\log \tau_n(t,s)\Big|_{\substack{t_i=\delta_{i0}\sqrt{x} \\ s_i=-\delta_{i0}\sqrt{x}}} \right) \tag{4.2.3}$$

satisfies equation (4.2.2). The initial condition follows from the combinatorics. ■

4.3 Backward and forward equation for non-intersecting random walks

Consider the n random walkers, walking in k steps from $x = (x_1 < x_2 < ... < x_n)$ to $y = (y_1 < y_2 < ... < y_n)$, as introduced in section 3.4. These data define difference operators [15] for $k, n \in \mathbb{Z}_+,\ x, y \in \mathbb{Z}$,

$$\begin{aligned} \mathcal{A}_1 \quad &:= \quad \sum_{i=1}^{n} \left(\frac{k}{2n} \Lambda_k^{-1} \partial_{2y_i}^{+} + x_i \partial_{x_i}^{-} + \partial_{y_i}^{+} y_i - (x_i - y_i) \right) \\ \mathcal{A}_2 \quad &:= \quad \sum_{i=1}^{n} \left(\frac{k}{2n} \Lambda_k^{-1} \partial_{2x_i}^{+} + y_i \partial_{y_i}^{-} + \partial_{x_i}^{+} x_i - (y_i - x_i) \right) \end{aligned} \tag{4.3.2}$$

With these definitions, we have

Theorem 4.3 *[4] The probability*

$$P(k;x,y) = \frac{b_{xy}^{(k)}}{(2n)^k} = P \left(\begin{array}{l} \textit{that } n \textit{ non-intersecting walkers in } \mathbb{Z} \textit{ move during } k \\ \textit{instants from } x_1 < x_2 < ... < x_n \textit{ to } y_1 < y_2 < ... < y_n, \\ \textit{where at each instant exactly one walker moves} \\ \textit{either one step to the left, or one step to the right} \end{array} \right) \tag{4.3.3}$$

satisfies both a <u>forward and backward</u> random walk equation,

$$\mathcal{A}_i P(k,x,y) = 0, \tag{4.3.4}$$

[15] in terms of difference operators, acting on functions $f(k,x,y)$, with $k \in \mathbb{Z}_+,\ x, y \in \mathbb{Z}$:

$$\begin{aligned} \partial_{\alpha x_i}^{+} f \quad &:= \quad f(k, x + \alpha e_i, y) - f(k,x,y) \\ \partial_{\alpha x_i}^{-} f \quad &:= \quad f(k,x,y) - f(k, x - \alpha e_i, y) \\ \Lambda_k^{-1} f \quad &:= \quad f(k-1, x, y). \end{aligned} \tag{4.3.1}$$

Remark: “Forward and backward”, because $\mathcal{A}_1$ essentially involves the end points y, whereas $\mathcal{A}_2$ involves the initial points x.

Proof: The unitary integral below is obtained from the integral $\tau_n^0(t,s)$, appearing in (2.0.1), by means of the shifts $t_1 \mapsto t_1+z$, $s_1 \mapsto s_1-z$. Thus it satisfies the Virasoro constraints for $k=-1,0,1$, with the same shifts inserted. This integral has a double Fourier expansion in Schur polynomials; see (3.4.2). So we have, with $\mathbb{V}_k$ defined in (2.1.2),

$$\begin{aligned} 0 &= \mathbb{V}_k\Big|_{\substack{t_1\mapsto t_1+z\\ s_1\mapsto s_1-z}} \int_{U(n)} e^{z\,\mathrm{Tr}(M+\bar{M})} e^{\sum_1^\infty \mathrm{Tr}(t_iM^i - s_i\bar{M}^i)} dM \\ &= \mathbb{V}_k\Big|_{\substack{t_1\mapsto t_1+z\\ s_1\mapsto s_1-z}} \sum_{\substack{\lambda,\mu \text{ such that}\\ \lambda_1^\top,\mu_1^\top\leq n}} a_{\lambda\mu}(z)\mathbf{s}_\lambda(t)\mathbf{s}_\mu(-s) \\ &\overset{*}{=} \sum_{\substack{\lambda_1^\top\leq n\\ \mu_1^\top\leq n}} \mathbf{s}_\lambda(t)\mathbf{s}_\mu(-s)\mathcal{L}(a_{\lambda\mu}(z)), \end{aligned}$$

To explain the equality $\overset{*}{=}$ above, notice the Virasoro constraints $\mathbb{V}_k$ act on the terms $\mathbf{s}_\lambda(t)\mathbf{s}_\mu(-s)$ in the expansion. Since the constraints (2.1.2) decouple as a sum of a t-part and an s-part, it suffices to show $V_k(t)\mathbf{s}_\lambda(t)$ can be expanded in a Fourier series in $\mathbf{s}_\mu(t)$'s; this is done below. Therefore $\mathbb{V}_k\mathbf{s}_\lambda(t)\mathbf{s}_\mu(-s)$ can again be expanded in double Fourier series, yielding new coefficients $\mathcal{L}(a_{\lambda\mu}(z))$, depending linearly on the old ones $a_{\lambda\mu}(z)$. Thus we must compute $V_k(t)\mathbf{s}_\lambda(t)$ for

$$V_k(t) = \frac{1}{2}\sum_{i+j=k}\frac{\partial^2}{\partial t_i\partial t_j} + \sum_{-i+j=k} it_i\frac{\partial}{\partial t_j} + \frac{1}{2}\sum_{-i-j=k}(it_i)(jt_j). \tag{4.3.5}$$

This will generalize the Murnaghan-Nakayama rules,

$$\begin{aligned} nt_n\ \mathbf{s}_\lambda(t) &= \sum_{\substack{\mu\\ \mu\backslash\lambda\in B(n)}} (-1)^{\mathrm{ht}(\mu\backslash\lambda)}\mathbf{s}_\mu(t) \\ \frac{\partial}{\partial t_n}\mathbf{s}_\lambda(t) &= \sum_{\substack{\mu\\ \lambda\backslash\mu\in B(n)}} (-1)^{\mathrm{ht}(\lambda\backslash\mu)}\mathbf{s}_\mu(t). \end{aligned} \tag{4.3.6}$$

To explain the notation, $b\in B(i)$ denotes a border-strip (i.e., a connected skew-shape $\lambda\backslash\mu$ containing i boxes, with no 2×2 square) and the height ht b

of a border strip b is defined as

$$\text{ht } b := \#\{\text{rows in } b\} - 1. \tag{4.3.7}$$

Indeed in [4] it is shown that

$$V_{-n}\mathbf{s}_\lambda(t) = \sum_{\substack{\mu \\ \mu\backslash\lambda \in B(n)}} d^{(n)}_{\lambda\mu}\mathbf{s}_\mu(t) \quad \text{and} \quad V_n\mathbf{s}_\lambda(t) = \sum_{\substack{\mu \\ \lambda\backslash\mu \in B(n)}} d^{(n)}_{\mu\lambda}\mathbf{s}_\mu(t) \tag{4.3.8}$$

with the same precise sum, except the coefficients are different: $(n \geq 1)$

$$\begin{aligned} d^{(n)}_{\lambda\mu} &= \sum_{i\geq 1} \sum_{\left\{\substack{\nu \quad \text{such that} \\ \lambda\backslash\nu \in B(i) \\ \mu\backslash\nu \in B(n+i) \\ \lambda\backslash\nu \subset \mu\backslash\nu}\right\}} (-1)^{\text{ht}(\lambda\backslash\nu)+\text{ht}(\mu\backslash\nu)} \\ &\quad + \frac{1}{2}\sum_{i=1}^{n} \sum_{\left\{\substack{\nu \quad \text{such that} \\ \nu\backslash\lambda \in B(i) \\ \mu\backslash\nu \in B(n-i)}\right\}} (-1)^{\text{ht}(\nu\backslash\lambda)+\text{ht}(\mu\backslash\nu)}. \end{aligned} \tag{4.3.9}$$

In view of the infinite sum in the Virasoro generators (4.3.5), one would expect $V_n\mathbf{s}_\lambda$ to be expressible as an *infinite sum of Schur polynomials*. This is not so: acting with Virasoro V_n leads to the same precise sum as acting with nt_n (resp. $\partial/\partial t_n$), except the coefficients in (4.3.8) are different from the ones in (4.3.6). This is to say *the two operators have the same band structure or locality*! Then setting

$$a_{\lambda\mu}(z) = \sum_{k\geq 0} b^{(k)}_{xy}\frac{z^k}{k!},$$

leads to the result (4.3.4), upon remembering the relation (3.4.1) between the λ, μ's and the x, y's. ■

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