PDE's for the Gaussian ensemble with external source and the Pearcey distribution

Mark Adler* Pierre van Moerbeke†

Contents

*2000 *Mathematics Subject Classification.* Primary: 60J60, 60J65, 60G55; secondary: 35Q53, 35Q58. *Key words and Phrases*: Non-intersecting Brownian motions, Pearcey distribution, matrix models, random Hermitian ensembles, multi-component KP equation, Virasoro constraints.
Department of Mathematics, Brandeis University, Waltham, MA 02454, USA. E-mail: adler@brandeis.edu. The support of a National Science Foundation grant # DMS-04-06287 is gratefully acknowledged.

†Department of Mathematics, Université de Louvain, 1348 Louvain-la-Neuve, Belgium and Brandeis University, Waltham, MA 02454, USA. E-mail: vanmoerbeke@math.ucl.ac.be and @brandeis.edu. The support of a National Science Foundation grant # DMS-04-06287, a European Science Foundation grant (MISGAM), a Marie Curie Grant (ENIGMA), Nato, FNRS and Francqui Foundation grants is gratefully acknowledged.

Brézin and Hikami [12, 13, 14, 15] have considered a random Gaussian Hermitian ensemble with external source,

$$\frac{1}{Z}e^{-\frac{1}{2}\operatorname{Tr}(M-A)^2}dM,$$

where M is random and A is deterministic. Notice this matrix ensemble, which had come up in the prior literature [21, 19, 17], ceases to be unitary invariant. The matrix A is chosen so that the support of the equilibrium measure has a gap, when the size of the random matrices tends to infinity. Through a fine tuning of A, the gap can be made to close at the origin. Brézin and Hikami propose a simple model having this feature, where the matrix A is diagonal, with two eigenvalues a and $-a$ of equal multiplicity. Thus, upon letting the size of the matrix go to ∞ and after appropriate rescaling, they discover a new critical distribution, specified by a kernel involving Pearcey integrals [22] and having universality properties.

P. Zinn-Justin [24, 25] establishes the determinantal form of the correlation functions for the eigenvalues of the finite model. Then extending classical connections between random matrix theory and non-intersecting random paths, Aptekarev, Bleher and Kuijlaars in [7] give a non-intersecting Brownian motion interpretation of this Gaussian ensemble with external source. They also show that multiple orthogonal polynomials are the right tools for studying this model and its limit (see [8, 9, 10, 11])

The present paper studies the Gaussian Hermitian random matrix ensemble $\mathcal{H}_n$ with external source A, given by the diagonal matrix (set $n = k_1 + k_2$)

$$A := \begin{pmatrix} a & & & & & \\ & \ddots & & & & \mathbf{O} \\ & & a & & & \\ & & & -a & & \\ & \mathbf{O} & & & \ddots & \\ & & & & & -a \end{pmatrix} \begin{matrix} \updownarrow k_1 \\ \\ \\ \updownarrow k_2 \end{matrix}, \tag{0.1}$$

and density

$$\frac{1}{Z_n}e^{-\operatorname{Tr}(\frac{1}{2}M^2-AM)}dM. \tag{0.2}$$

Given a disjoint union of intervals $E := \bigcup_{i=1}^{r}[b_{2i-1}, b_{2i}] \subset \mathbb{R}$, define the algebra of differential operators

$$\mathcal{B}_k = \sum_1^{2r} b_i^{k+1} \frac{\partial}{\partial b_i}. \tag{0.3}$$

Consider the following probability:

$$\mathbb{P}_n(a; E) := \mathbb{P}(\text{ all eigenvalues } \in E) = \frac{1}{Z_n} \int_{\mathcal{H}_n(E)} e^{-\operatorname{Tr}(\frac{1}{2}M^2 - AM)} dM \tag{0.4}$$

In [1], we have shown that for $A = 0$, the probability for this Gaussian Hermitian ensemble (GUE) satisfies a *fourth-order PDE, with quadratic non-linearity* (reducing to Painlevé IV in the case of one boundary point) :

$$\left(\mathcal{B}_{-1}^4 + (4n + 6\mathcal{B}_{-1}^2 \log \mathbb{P}_n)\mathcal{B}_{-1}^2 + 3\mathcal{B}_0^2 - 4\mathcal{B}_{-1}\mathcal{B}_1 + 6\mathcal{B}_0\right) \log \mathbb{P}_n = 0.$$

The **first question** in this paper: *Does the integral (0.4), with A as in (0.1), satisfy a PDE?* Indeed, we prove:

Theorem 0.1 *The log of the probability $\mathbb{P}_n(a; E)$ satisfies a fourth-order PDE in a and in the endpoints $b_1, ..., b_{2r}$ of the set E, with quartic non-linearity:*

$$\begin{aligned} &\left(F^+\mathcal{B}_{-1}G^- + F^-\mathcal{B}_{-1}G^+\right)\left(F^+\mathcal{B}_{-1}F^- - F^-\mathcal{B}_{-1}F^+\right) \\ &-\left(F^+G^- + F^-G^+\right)\left(F^+\mathcal{B}_{-1}^2F^- - F^-\mathcal{B}_{-1}^2F^+\right) = 0, \end{aligned} \tag{0.5}$$

where[1]

$$F^+ := 2\mathcal{B}_{-1}\left(\frac{\partial}{\partial a} - \mathcal{B}_{-1}\right) \log \mathbb{P}_n - 4k_1, \qquad F^- = F^+\Big|_{\substack{a \to -a \\ k_1 \leftrightarrow k_2}}$$

$$2G^+ := \left\{H_1^+, F^+\right\}_{\mathcal{B}_{-1}} - \left\{H_2^+, F^+\right\}_{\partial/\partial a}, \qquad G^- = G^+\Big|_{\substack{a \to -a \\ k_1 \leftrightarrow k_2}},$$

[1]in terms of the Wronskians $\{f, g\}_X = gXf - fXg$.

with

$$
\begin{aligned}
H_1^+ &:= \frac{\partial}{\partial a}\left(\mathcal{B}_0 - a\frac{\partial}{\partial a} - a\mathcal{B}_{-1}\right)\log \mathbb{P}_n + \left(\mathcal{B}_0\mathcal{B}_{-1} + 4\frac{\partial}{\partial a}\right)\log \mathbb{P}_n \\
&\qquad\qquad + 4k_1\left(a + \frac{k_2}{a}\right) \\
H_2^+ &:= \frac{\partial}{\partial a}\left(\mathcal{B}_0 - a\frac{\partial}{\partial a} - a\mathcal{B}_{-1}\right)\log \mathbb{P}_n - \left(\mathcal{B}_0 - 2a\mathcal{B}_{-1} - 2\right)\mathcal{B}_{-1}\log \mathbb{P}_n.
\end{aligned}
\tag{0.6}
$$

Remark: : The change of variables $a \mapsto -a,\ k_1 \leftrightarrow k_2$ in the definition of F^- and G^- act at the level of the operators. In fact, later, it will be clear that $\mathbb{P}_n(a; E)$ is invariant under that change of variables.

Again here we provide a natural integrable deformation of (0.4) (section 1). As is well known, the probability for $A = 0$ relates to the standard Toda lattice and the *one-component KP* equation (see [1]), the spectrum of coupled random matrices to the 2-Toda lattice and the *two-component KP* (see [2]), whereas we show that the model (0.4) relates to the *three-component KP* equation (section 2). This deformation enjoys Virasoro constraints as well (section 3), which together with the bilinear relations arising from 3-KP leads to the PDE of Theorem 0.1 (section 4).

The **second question** concerns the *Pearcey distribution*, which we now explain. Following [7], consider $n = 2k$ non-intersecting Brownian motions on $\mathbb{R}$ (Dyson's Brownian motions), all starting at the origin, such that the k left paths end up at $-a$ and the k right paths end up at $+a$ at time $t = 1$. As observed in [7], the Karlin-McGregor formula [20] enables one to express the transition probability in terms of the Gaussian Hermitian random matrix probability $\mathbb{P}(a; E)$ with external source, as in (0.4),

$$
\begin{aligned}
\mathbb{P}_0^{\pm a}\left(\text{all } x_j(t) \in E\right) &= \lim_{\substack{\text{all } \gamma_i \to 0 \\ \delta_1,\dots,\delta_k \to -a \\ \delta_{k+1},\dots,\delta_{2k} \to a}} \int_{E^n} \\
&\quad \frac{1}{Z_n}\det(p(t;\gamma_i,x_j))_{1\le i,j\le n}\det(p(1-t;x_{i'},\delta_{j'}))_{1\le i',j'\le n}\prod_1^n dx_i, \\
&= \mathbb{P}_n\left(a\sqrt{\frac{2t}{1-t}}; E\sqrt{\frac{2}{t(1-t)}}\right)
\end{aligned}
\tag{0.7}
$$

where $p(t,x,y)$ is the Brownian transition probability

$$p(t,x,y) := \frac{1}{\sqrt{\pi t}}\, e^{-\frac{(y-x)^2}{t}}. \tag{0.8}$$

Let now the number $n = 2k$ of particles go to infinity, and let the points a and $-a$ go to $\pm\infty$. This forces the left k particles to $-\infty$ at $t = 1$ and the right k particles to $+\infty$ at $t = 1$. Since the particles all leave from the origin at $t = 0$, it is natural to believe that for small times the equilibrium measure (mean density of particles) is supported by one interval, and for times close to 1, the equilibrium measure is supported by two intervals. With a precise scaling, $t = 1/2$ is critical in the sense that for $t < 1/2$, the equilibrium measure for the particles is supported by one, and for $t > 1/2$, it is supported by two intervals. The Pearcey process $\mathcal{P}(s)$ is now defined as the motion of an infinite number of non-intersecting Brownian paths, just after time $t = 1/2$, with the precise scaling (see [7]):

$$n = 2k = \frac{2}{z^4}, \quad \pm a = \pm\frac{1}{z^2}, \quad x_i \mapsto x_i z, \quad t \mapsto \frac{1}{2} + tz^2, \quad \text{for } z \to 0. \tag{0.9}$$

Even though the pathwise interpretation of $\mathcal{P}(t)$ is unclear and still deserves investigation, it is natural to define the probability

$$\mathbb{P}(\mathcal{P}(t) \cap E = \emptyset) := \lim_{z\to 0} \mathbb{P}_0^{\pm 1/z^2}\left(\text{all } x_j\big(\frac{1}{2} + tz^2\big) \notin zE;\ 1 \le j \le n\right)\Bigg|_{n=\frac{2}{z^4}}.$$

Brézin and Hikami [12, 13, 14, 15] for the Pearcey kernel and Tracy-Widom [23] for the extended kernels show that this limit exists and equals a Fredholm determinant:

$$\mathbb{P}(\mathcal{P}(t) \cap E = \emptyset) = \det\left(I - K_t \chi_{_E}\right),$$

where $K_t(x,y)$ is the Pearcey kernel, defined as follows:

$$\begin{aligned} K_t(x,y) &:= \frac{p(x)q''(y) - p'(x)q'(y) + p''(x)q(y) - tp(x)q(y)}{x-y} \\ &= \int_0^\infty p(x+z)q(y+z)dz, \end{aligned} \tag{0.10}$$

where (note $\omega = e^{i\pi/4}$)

$$\begin{aligned} p(x) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{u^4}{4} - \frac{tu^2}{2} - iux} du \\ q(y) &:= \frac{1}{2\pi i}\int_X e^{\frac{u^4}{4} - \frac{tu^2}{2} + uy} du = \operatorname{Im}\left[\frac{\omega}{\pi}\int_0^\infty du e^{-\frac{u^4}{4} - \frac{it}{2}u^2}(e^{\omega uy} - e^{-\omega uy})\right] \end{aligned}$$

satisfy the differential equations

$$p''' - tp' - xp = 0 \text{ and } q''' - tq' + yq = 0.$$

The contour X is given by the ingoing rays from $\pm\infty e^{i\pi/4}$ to 0 and the outgoing rays from 0 to $\pm\infty e^{-i\pi/4}$, i.e., X stands for the contour

$$\begin{array}{ccc} \nwarrow & & \swarrow \\ & 0 & \\ \nearrow & & \searrow \end{array}$$

The second result of this paper[2] is to give a PDE for the Pearcey distribution below in terms of the parameter t appearing in the kernel (0.10). Since this Pearcey distribution with the parameter t can also be interpreted as the transition probability for the Pearcey process, we prove:

Theorem 0.2 *For compact* $E = \bigcup_{i=1}^{r}[x_{2i-1}, x_{2i}]$ *and* $\mathcal{B}_j = \sum_1^{2r} x_i^{j+1}\frac{\partial}{\partial x_i}$,

$$Q(t; x_1, \dots, x_{2r}) \;=\; \log \mathbb{P}\Big(\mathcal{P}(t) \cap E = \emptyset\Big) = \log\det\left(I - K_t\chi_E\right) \quad (0.11)$$

satisfies a 4th order and 3rd degree PDE, which can be written as a Wronskian[3]*:*

$$\left\{\frac{1}{2}\frac{\partial^3 Q}{\partial t^3} + (\mathcal{B}_0 - 2)\mathcal{B}_{-1}^2 Q + \frac{1}{16}\left\{\mathcal{B}_{-1}\frac{\partial Q}{\partial t}, \mathcal{B}_{-1}^2 Q\right\}_{\mathcal{B}_{-1}}, \; \mathcal{B}_{-1}^2\frac{\partial Q}{\partial t}\right\}_{\mathcal{B}_{-1}} = 0.$$

The proof of this statement, based on taking a limit on the PDE of Theorem 0.1 will be given in section 5.

1 An integrable deformation of Gaussian random ensemble with external source

Consider an ensemble of $n \times n$ Hermitian matrices with an external source, given by a diagonal matrix

$$A = \mathrm{diag}(a_1, \dots, a_n)$$

[2]Tracy and Widom show in [23] the existence of a large system of PDE's involving a large system of auxiliary variables for Q and also for the joint probabilities at different times.

[3]given that $\{f, g\}_X := Xf.g - f.Xg$.

and a general potential $V(z)$, with density

$$\mathbb{P}_n(M \in [M, M+dM]) = \frac{1}{Z_n} e^{-\operatorname{Tr}(V(M)-AM)} dM.$$

For the disjoint union of intervals $E = \bigcup_{i=1}^{r} [b_{2i-1}, b_{2i}]$, the following probability can be transformed by the Harrish-Chandra-Itzykson-Zuber formula, with $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$,

$$\begin{aligned}
\mathbb{P}_n(\text{spectrum } M \subset E) &= \frac{1}{Z_k} \int_{\mathcal{H}_n(E)} e^{-\operatorname{Tr}(V(M)-AM)} dM \\
&= \frac{1}{Z_n} \int_{E^n} \Delta_n^2(\lambda) \prod_1^n e^{-V(\lambda_i)} d\lambda_i \int_{U(n)} e^{\operatorname{Tr} AUDU^{-1}} dU \\
&= \frac{1}{Z_n'} \int_{E^n} \Delta_n^2(\lambda) \prod_1^n e^{-V(\lambda_i)} d\lambda_i \frac{\det[e^{a_i \lambda_j}]_{1 \le i,j \le n}}{\Delta_n(\lambda)\Delta_n(a)} \\
&= \frac{1}{Z_n''} \int_{E^n} \Delta_n(\lambda) \det[e^{-V(\lambda_j)+a_i\lambda_j}]_{1 \le i,j \le n} \prod_1^n d\lambda_i.
\end{aligned} \tag{1.1}$$

In the following Proposition, we consider a general situation, of which (1.1) with $A = \operatorname{diag}(a, \ldots, a, -a, \ldots, -a)$ is a special case, by setting $\varphi^+ = e^{az}$ and $\varphi^- = e^{-az}$. Consider the Vandermonde determinant of the variables $x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2}$, namely

$$\Delta_n(x, y) := \Delta_n(x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2}). \tag{1.2}$$

Then we have

Proposition 1.1 *Given an arbitrary potential $V(z)$ and arbitrary functions $\varphi^+(z)$ and $\varphi^-(z)$, define ($n = k_1 + k_2$)*

$$\begin{aligned}
(\rho_1, \ldots, \rho_n) \;:=\; e^{-V(z)} \big(&\varphi^+(z), z\varphi^+(z), \ldots, z^{k_1-1}\varphi^+(z), \\
&\varphi^-(z), z\varphi^-(z), \ldots, z^{k_2-1}\varphi^-(z)\big).
\end{aligned}$$

we have

$$
\begin{aligned}
&\frac{1}{n!}\int_{E^n}\Delta_n(z)\det(\rho_i(z_j))_{1\le i,j\le n}\prod_1^n dz_i\\
&= \frac{1}{k_1!k_2!}\int_{E^n}\Delta_n(x,y)\Delta_{k_1}(x)\Delta_{k_2}(y)\prod_1^{k_1}\varphi^+(x_i)e^{-V(x_i)}dx_i\prod_1^{k_2}\varphi^-(y_i)e^{-V(y_i)}dy_i\\
&= \det\begin{pmatrix}\left(\int_E z^{i+j-1}\varphi^+(z)e^{-V(z)}\right)_{\substack{1\le i\le k_1\\0\le j\le k_1+k_2-1}}\\ \\ \left(\int_E z^{i+j-1}\varphi^-(z)e^{-V(z)}\right)_{\substack{1\le i\le k_2\\0\le j\le k_1+k_2-1}}\end{pmatrix}
\end{aligned}
\tag{1.3}
$$

Proof: On the one hand, using

$$
\det\left(a_{ik}\right)_{1\le i,k\le n}\det\left(b_{ik}\right)_{1\le i,k\le n}=\sum_{\sigma\in S_n}\det\left(a_{i,\sigma(j)}\ b_{j,\sigma(j)}\right)_{1\le i,j\le n},
$$

and distributing the integration over the different columns, one computes

$$
\begin{aligned}
&\int\Delta_n(z)\det(\rho_i(z_j))_{1\le i,j\le n}\prod_1^n dz_i\\
&= \int_{E^n}\det(z_j^{i-1})_{1\le i,j\le n}\det\begin{pmatrix}\left(z_j^{i-1}\varphi^+(z_j)e^{-V(z_j)}\right)_{\substack{1\le i\le k_1\\1\le j\le k_1+k_2}}\\ \\ \left(z_j^{i-1}\varphi^-(z_j)e^{-V(z_j)}\right)_{\substack{1\le i\le k_2\\1\le j\le k_1+k_2}}\end{pmatrix}\prod_1^n dz_i\\
&= n!\det\begin{pmatrix}\left(\int_E z^{i+j-1}\varphi^+(z)e^{-V(z)}dz\right)_{\substack{1\le i\le k_1\\0\le j\le k_1+k_2-1}}\\ \\ \left(\int_E z^{i+j-1}\varphi^-(z)e^{-V(z)}dz\right)_{\substack{1\le i\le k_2\\0\le j\le k_1+k_2-1}}\end{pmatrix}
\end{aligned}
\tag{1.4}
$$

On the other hand, one computes

$$
\int_{E^n}\Delta_n(z)\det(\rho_i(z_j))_{1\le i,j\le n}\prod_1^n dz_i
$$

$$= \int_{E^n} \Delta_n(z_1, \ldots, z_n) \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n \rho_i(z_{\sigma(i)}) \prod_1^n dz_{\sigma(i)}$$

$$= \sum_\sigma (-1)^\sigma \int_{E^n} \Delta_n(z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(n)}) \prod_{i=1}^n \rho_i(z_i) \prod_1^n dz_i$$

$$= \sum_\sigma (-1)^\sigma \int_{E^n} (-1)^\sigma \Delta_n(z_1, \ldots, z_n) \prod_{i=1}^n \rho_i(z_i) dz_i$$

$$= n! \int_{E^n} \Delta_n(x, y) \prod_1^{k_1} x_i^{i-1} \varphi^+(x_i) e^{-V(x_i)} dx_i \prod_1^{k_2} y_i^{i-1} \varphi^-(y_i) e^{-V(y_i)} dy_i$$

$$= \frac{n!}{k_1! k_2!} \int_{E^n} \Delta_n(x, y) \Delta_{k_1}(x) \Delta_{k_2}(y) \prod_1^{k_1} \varphi^+(x_i) e^{-V(x_i)} dx_i \prod_1^{k_2} \varphi^-(y_i) e^{-V(y_i)} dy_i,$$

where $\Delta_n(x, y)$ is defined in (1.2). In the last identity, one uses twice the following general identity for a skew-symmetric function $F(x_1, \ldots, x_k)$ and a general measure $\mu(dx)$,

$$\int_{\mathbb{R}^k} F(x_1, \ldots, x_k) \Delta_k(x) \prod_1^k \mu(dx_i)$$

$$= \int_{\mathbb{R}^k} F(x_1, \ldots, x_k) \sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^k x_{\sigma(i)}^{i-1} \prod_1^k \mu(dx_i)$$

$$= \int_{\mathbb{R}^k} \sum_{\sigma \in S_k} (-1)^\sigma F(x_1, \ldots, x_k) \prod_{i=1}^k x_{\sigma(i)}^{i-1} \mu(dx_{\sigma(i)})$$

$$= \int_{\mathbb{R}^k} \sum_{\sigma \in S_k} (-1)^\sigma F(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(k)}) \prod_{i=1}^{k_1} x_i^{i-1} \mu(dx_i)$$

$$= k! \int_{\mathbb{R}^k} F(x_1, \ldots, x_k) \prod_{i=1}^k x_i^{i-1} \mu(dx_i).$$

This ends the proof of Proposition 1.1. ■

Add extra variables in the exponentials, one set for each Vandermonde determinant:

$$t=(t_1,t_2,\ldots),\quad s=(s_1,s_2,\ldots),\quad u=(u_1,u_2,\ldots)\ \text{ and }\ \beta$$

Then, setting $(n=k_1+k_2)$,

$$V(z):=\frac{z^2}{2}+\sum_1^\infty t_iz^i,\quad \varphi^+(z)=e^{az+\beta z^2-\sum_1^\infty s_iz^i},\quad \varphi^-(z)=e^{-az-\beta z^2-\sum_1^\infty u_iz^i},$$

Proposition 1.1 implies

$$\begin{aligned}\tau_{k_1k_2}(t,s,u;\beta;E) &:= \det m_{k_1,k_2}(t,s,u;\beta;E)\\ &= \frac{1}{k_1!k_2!}\int_{E^n}\Delta_n(x,y)\prod_{j=1}^{k_1}e^{\sum_1^\infty t_ix_j^i}\prod_{j=1}^{k_2}e^{\sum_1^\infty t_iy_j^i}\\ &\qquad\left(\Delta_{k_1}(x)\prod_{j=1}^{k_1}e^{-\frac{x_j^2}{2}+ax_j+\beta x_j^2}e^{-\sum_1^\infty s_ix_j^i}dx_j\right)\\ &\qquad\left(\Delta_{k_2}(y)\prod_{j=1}^{k_2}e^{-\frac{y_j^2}{2}-ay_j-\beta y_j^2}e^{-\sum_1^\infty u_iy_j^i}dy_j\right),\end{aligned}\tag{1.5}$$

where

$$m_{k_1,k_2}(t,s,u;\beta;E) := \left(\begin{array}{c}\left(\mu^+_{ij}(t,s;\beta,E)\right)_{\substack{1\le i\le k_1\\ 0\le j\le k_1+k_2-1}}\\ \left(\mu^-_{ij}(t,u;\beta,E)\right)_{\substack{1\le i\le k_2\\ 0\le j\le k_1+k_2-1}}\end{array}\right),$$

with

$$\begin{aligned}\mu^+_{ij}(t,s;\beta,E) &= \int_E z^{i+j-1}e^{-\frac{z^2}{2}+az+\beta z^2}e^{\sum_1^\infty(t_k-s_k)z^k}dz\\ \mu^-_{ij}(t,u;\beta,E) &= \int_E z^{i+j-1}e^{-\frac{z^2}{2}-az-\beta z^2}e^{\sum_1^\infty(t_k-u_k)z^k}dz.\end{aligned}\tag{1.6}$$

In particular, by (1.3), the integral in (1.1) has the following determinantal representation in terms of moments:

$$
\begin{aligned}
& \frac{1}{n!}\int_{E^n}\Delta_n(\lambda)\det[e^{-V(\lambda_j)+a_i\lambda_j}]_{1\le i,j\le n}\prod_1^n d\lambda_i \\
&= \det\begin{pmatrix} \left(\int_E z^{i+j-1}e^{-\frac{z^2}{2}+az}dz\right)_{\substack{1\le i\le k_1\\ 0\le j\le k_1+k_2-1}} \\ \\ \left(\int_E z^{i+j-1}e^{-\frac{z^2}{2}-az}dz\right)_{\substack{1\le i\le k_2\\ 0\le j\le k_1+k_2-1}} \end{pmatrix}
\end{aligned}
\tag{1.7}
$$

and so

$$
\mathbb{P}_n(\text{spec } M\subset E)=\left.\frac{\tau_{k_1k_2}(t,s,u;\beta;E)}{\tau_{k_1k_2}(t,s,u;\beta;\mathbb{R})}\right|_{t=s=u=\beta=0}. \tag{1.8}
$$

Remark: The integral enjoys the obvious duality:

$$
x\longleftrightarrow y,\ k_1\longleftrightarrow k_2,\ t\longleftrightarrow t,\ s\longleftrightarrow u,\ a\longleftrightarrow -a,\ \beta\longleftrightarrow -\beta. \tag{1.9}
$$

2 Integrable deformations and 3-component KP

Theorem 2.1 (Adler-van Moerbeke [5]) *Given the functions τ_{n_1,n_2} as in (1.5), the wave matrix*

$$
W^{\pm}_{n_1,n_2}(\lambda;t,s,u):=\begin{pmatrix} \Psi^{(1)\pm}_{n_1,n_2} & \Psi^{(2)\pm}_{n_1\pm1,n_2} & \Psi^{(3)\pm}_{n_1,n_2\pm1} \\ \Psi^{(1)\pm}_{n_1\mp1,n_2} & \Psi^{(2)\pm}_{n_1,n_2} & \Psi^{(3)\pm}_{n_1\mp1,n_2\pm1} \\ \Psi^{(1)\pm}_{n_1,n_2\mp1} & \Psi^{(2)\pm}_{n_1\pm1,n_2\mp1} & \Psi^{(3)\pm}_{n_1,n_2} \end{pmatrix}
$$

with functions

$$
\begin{aligned}
\Psi^{(1)\pm}_{n_1,n_2}(\lambda;t,s,u) &:= \lambda^{\pm(n_1+n_2)}e^{\pm\sum t_i\lambda^i}\frac{\tau_{n_1,n_2}(t\mp[\lambda^{-1}],s,u)}{\tau_{n_1,n_2}(t,s,u)}\\
\Psi^{(2)\pm}_{n_1,n_2}(\lambda;t,s,u) &:= \lambda^{\mp n_1}e^{\pm\sum s_i\lambda^i}\frac{\tau_{n_1,n_2}(t,s\mp[\lambda^{-1}],u)}{\tau_{n_1,n_2}(t,s,u)}\\
\Psi^{(3)\pm}_{n_1,n_2}(\lambda;t,s,u) &:= \lambda^{\mp n_2}e^{\pm\sum u_i\lambda^i}\frac{\tau_{n_1,n_2}(t,s,u\mp[\lambda^{-1}])}{\tau_{n_1,n_2}(t,s,u)}
\end{aligned}
\tag{2.1}
$$

satisfies the bilinear identity

$$\oint_{\infty} W^{+}_{k_1,k_2}(\lambda;t,s,u)W^{-}_{\ell_1,\ell_2}(\lambda;t',s',u')^{\top}d\lambda = 0 \tag{2.2}$$

for all integers $k_1,k_2,\ell_1,\ell_2 \geq 0$ *and* $t,s,u,t',s',u' \in \mathbb{C}^{\infty}$.

Proof: The moments, as defined in (1.6), satisfy

$$\frac{\partial \mu^{\pm}_{ij}}{\partial t_k} = \mu^{\pm}_{i,j+k}$$

$$\begin{aligned} \frac{\partial \mu^{+}_{ij}}{\partial s_k} &= -\mu^{+}_{i+k,j}, & \frac{\partial \mu^{+}_{ij}}{\partial u_k} &= 0 \\ \frac{\partial \mu^{-}_{ij}}{\partial s_k} &= 0, & \frac{\partial \mu^{-}_{ij}}{\partial u_k} &= -\mu^{-}_{i+k,j}, \end{aligned} \tag{2.3}$$

and in matrix notation

$$\frac{\partial m_{\infty,\infty}}{\partial t_k} = m_{\infty,\infty}\Lambda^{\top k}, \quad \frac{\partial m_{\infty,\infty}}{\partial s_k} = -\Lambda^{k}_{-}m_{\infty,\infty}, \quad \frac{\partial m_{\infty,\infty}}{\partial u_k} = -\Lambda^{k}_{+}m_{\infty,\infty}, \tag{2.4}$$

where

$$\Lambda := \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & \\ & & & & \ddots \end{pmatrix}, \quad \Lambda_{-} = \begin{pmatrix} \Lambda & O \\ O & O \end{pmatrix}, \quad \Lambda_{+} = \begin{pmatrix} O & O \\ O & \Lambda \end{pmatrix}.$$

Thus the moment matrix satisfies

$$m_{\infty,\infty}(t,s,u) \;=\; e^{-\sum_1^{\infty}(s_k\Lambda^k_{-}+u_k\Lambda^k_{+})}m_{\infty,\infty}(0,0,0)e^{\sum_1^{\infty}t_k\Lambda^{\top k}},$$

which implies the bilinear identity (2.2). The details of proof can be found in [5]. ∎

Corollary 2.2 *Given the above τ-functions $\tau_{k_1k_2}(t,s,u)$, they satisfy the bilinear identities*

$$\begin{aligned}0 =& \oint_\infty d\lambda\ \lambda^{k_1+k_2-\ell_1-\ell_2} e^{\sum(t_i-t_i')\lambda^i}\tau_{k_1,k_2}\left(t-\left[\lambda^{-1}\right],s,u\right)\tau_{\ell_1,\ell_2}\left(t'+\left[\lambda^{-1}\right],s',u'\right)\\ &+\oint d\lambda\ \lambda^{\ell_1-k_1-2} e^{\sum(s_i-s_i')\lambda^i}\tau_{k_1+1,k_2}\left(t,s-\left[\lambda^{-1}\right],u\right)\tau_{\ell_1-1,\ell_2}\left(t',s'+\left[\lambda^{-1}\right],u'\right)\\ &+\oint d\lambda\ \lambda^{\ell_2-k_2-2} e^{\sum(u_i-u_i')\lambda^i}\tau_{k_1,k_2+1}\left(t,s,u-\left[\lambda^{-1}\right]\right)\tau_{\ell_1,\ell_2-1}\left(t',s',u'+\left[\lambda^{-1}\right]\right).\end{aligned} \tag{2.5}$$

Upon specializing, these identities imply PDE's expressed in terms of Hirota's symbol [4], *for $j=1,2,\ldots$:*

$$\mathbf{s}_j(\tilde\partial_t)\tau_{k_1+1,k_2}\circ\tau_{k_1-1,k_2} = \tau^2_{k_1k_2}\frac{\partial^2}{\partial s_1\partial t_{j+1}}\log\tau_{k_1k_2} \tag{2.6}$$

$$\mathbf{s}_j(\tilde\partial_s)\tau_{k_1-1,k_2}\circ\tau_{k_1+1,k_2} = \tau^2_{k_1k_2}\frac{\partial^2}{\partial t_1\partial s_{j+1}}\log\tau_{k_1k_2}, \tag{2.7}$$

yielding

$$\frac{\partial^2\log\tau_{k_1,k_2}}{\partial t_1\partial s_1} = \frac{\tau_{k_1+1,k_2}\tau_{k_1-1,k_2}}{\tau^2_{k_1,k_2}} \tag{2.8}$$

$$\frac{\partial}{\partial t_1}\log\frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} = \frac{\frac{\partial^2}{\partial t_2\partial s_1}\log\tau_{k_1,k_2}}{\frac{\partial^2}{\partial t_1\partial s_1}\log\tau_{k_1,k_2}} \tag{2.9}$$

$$-\frac{\partial}{\partial s_1}\log\frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} = \frac{\frac{\partial^2}{\partial t_1\partial s_2}\log\tau_{k_1,k_2}}{\frac{\partial^2}{\partial t_1\partial s_1}\log\tau_{k_1,k_2}} \tag{2.10}$$

Proof: The bilinear identity (2.2) yields nine identities, which are all equivalent, upon relabeling indices, to the tau-function bilinear identity (2.5).

[4]Given a polynomial $p(t_1,t_2,...)$, define the customary Hirota symbol $p(\partial_t)f\circ g :=$ $p(\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2},...)f(t+y)g(t-y)\Big|_{y=0}$. For later use, the $\mathbf{s}_\ell$'s are the elementary Schur polynomials $e^{\sum_1^\infty t_iz^i} := \sum_{i\geq 0}\mathbf{s}_i(t)z^i$ and set $\mathbf{s}_\ell(\tilde\partial) := \mathbf{s}_\ell(\frac{\partial}{\partial t_1},\frac{1}{2}\frac{\partial}{\partial t_2},\ldots)$.

Introducing standard shifts in the residue formulae

$$\begin{array}{lclcl} t & \longmapsto & t-a & t' \longmapsto t+a \\ s & \longmapsto & s-b & s' \longmapsto s+b \\ u & \longmapsto & u-c & u' \longmapsto u+c, \end{array}$$

and using Taylor's theorem, identity (2.5) is equivalent to

$$\begin{aligned} &\sum_{j=0}^{\infty} \mathbf{s}_{\ell_1+\ell_2-k_1-k_2+j-1}(-2a)\mathbf{s}_j(\tilde{\partial}_t) e^{\sum_1^\infty (a_k \frac{\partial}{\partial t_k} + b_k \frac{\partial}{\partial s_k} + c_k \frac{\partial}{\partial u_k})} \tau_{\ell_1 \ell_2} \circ \tau_{k_1 k_2} \\ + &\sum_{j=0}^{\infty} \mathbf{s}_{k_1-\ell_1+1+j}(-2b)\mathbf{s}_j(\tilde{\partial}_s) e^{\sum_1^\infty (a_k \frac{\partial}{\partial t_k} + b_k \frac{\partial}{\partial s_k} + c_k \frac{\partial}{\partial u_k})} \tau_{\ell_1-1,\ell_2} \circ \tau_{k_1+1,k_2} \\ + &\sum_{j=0}^{\infty} \mathbf{s}_{k_2-\ell_2+1+j}(-2c)\mathbf{s}_j(\tilde{\partial}_u) e^{\sum_1^\infty (a_k \frac{\partial}{\partial t_k} + b_k \frac{\partial}{\partial s_k} + c_k \frac{\partial}{\partial u_k})} \tau_{\ell_1,\ell_2-1} \circ \tau_{k_1,k_2+1} = 0. \end{aligned} \tag{2.11}$$

Taylor expanding in a, b, c and setting in equation (2.11) all $a_i, b_i, c_i = 0$, except a_{j+1}, and also setting $\ell_1 = k_1 + 2$, $\ell_2 = k_2$, equation (2.11) becomes

$$a_{j+1}\left(-2\mathbf{s}_j(\tilde{\partial}_t)\tau_{k_1+2,k_2} \circ \tau_{k_1,k_2} + \frac{\partial^2}{\partial s_1 \partial t_{j+1}} \tau_{k_1+1,k_2} \circ \tau_{k_1+1,k_2} \right) + \mathbf{O}(a_{j+1}^2) = 0,$$

and so the coefficient of a_{j+1} must vanish identically, yielding equation (2.6), setting $k_1 \to k_1 - 1$. Setting in equation (2.11) all $a_i, b_i, c_i = 0$, except b_{j+1}, and $\ell_1 = k_1$, $\ell_2 = k_2$, the coefficient of b_{j+1} in equation (2.11) yields equation (2.7). Specializing equation (2.6) to $j = 0$ and 1 respectively yields (since $\mathbf{s}_1(t) = t_1$ implies $\mathbf{s}_1(\tilde{\partial}_t) = \frac{\partial}{\partial t_1}$; also $\mathbf{s}_0 = 1$):

$$\frac{\partial^2 \log \tau_{k_1,k_2}}{\partial t_1 \partial s_1} = \frac{\tau_{k_1+1,k_2}\tau_{k_1-1,k_2}}{\tau_{k_1,k_2}^2}$$

and

$$\frac{\partial^2}{\partial s_1 \partial t_2} \log \tau_{k_1 k_2} = \frac{1}{\tau_{k_1 k_2}^2} \left[\left(\frac{\partial}{\partial t_1} \tau_{k_1+1,k_2} \right) \tau_{k_1-1,k_2} - \tau_{k_1+1,k_2} \left(\frac{\partial}{\partial t_1} \tau_{k_1-1,k_2} \right) \right].$$

Upon dividing the second equation by the first, we find equation (2.9) and similarly equation (2.10) follows from equation (2.7). ■

3 Virasoro constraints for the integrable deformations

Given the Heisenberg and Virasoro operators, for $m \geq -1,\ k \geq 0$:

$$\begin{aligned}
\mathbb{J}^{(1)}_{m,k}(t) &= \frac{\partial}{\partial t_m} + (-m)t_{-m} + k\delta_{0,m} \\
\mathbb{J}^{(2)}_{m,k}(t) &= \frac{1}{2}\left(\sum_{i+j=m} \frac{\partial^2}{\partial t_i \partial t_j} + 2\sum_{i\geq 1} it_i \frac{\partial}{\partial t_{i+m}} + \sum_{i+j=-m} it_i jt_j\right) \\
&\quad + (k + \frac{m+1}{2})\left(\frac{\partial}{\partial t_m} + (-m)t_{-m}\right) + \frac{k(k+1)}{2}\delta_{m0},
\end{aligned}$$

we now state:

Theorem 3.1 *The integral $\tau_{k_1k_2}(t,s,u;\beta;E)$, as defined in (1.5) satisfies*

$$\mathcal{B}_m \tau_{k_1,k_2} = \mathbb{V}^{k_1,k_2}_m \tau_{k_1,k_2} \quad for \quad m \geq -1, \tag{3.1}$$

where $\mathcal{B}_m$ and $\mathbb{V}_m$ are differential operators:

$$\mathcal{B}_m = \sum_1^{2r} b_i^{m+1} \frac{\partial}{\partial b_i}, \ for\ E = \bigcup_1^{2r} [b_{2i-1}, b_{2i}] \subset \mathbb{R}$$

and

$$\mathbb{V}^{k_1k_2}_m := \left\{ \begin{array}{l}
\mathbb{J}^{(2)}_{m,k_1+k_2}(t) - (m+1)\mathbb{J}^{(1)}_{m,k_1+k_2}(t) \\
\\
+\mathbb{J}^{(2)}_{m,k_1}(-s) + a\mathbb{J}^{(1)}_{m+1,k_1}(-s) + (1-2\beta)\mathbb{J}^{(1)}_{m+2,k_1}(-s) \\
\\
+\mathbb{J}^{(2)}_{m,k_2}(-u) - a\mathbb{J}^{(1)}_{m+1,k_2}(-u) + (1+2\beta)\mathbb{J}^{(1)}_{m+2,k_2}(-u)
\end{array} \right\}$$

We state the following lemmas:

Lemma 3.2 (Adler-van Moerbeke [3]) *Given*

$$\rho = e^{-V} \quad with \quad -\frac{\rho'}{\rho} = V' = \frac{g}{f} = \frac{\sum_0^\infty \beta_i z^i}{\sum_0^\infty \alpha_i z^i},$$

the integrand

$$dI_n(x) := \Delta_n(x) \prod_{k=1}^{n} \left(e^{\sum_1^\infty t_i x_k^i} \rho(x_k) dx_k \right),$$

satisfies the following variational formula:

$$\frac{d}{d\varepsilon} dI_n(x_i \mapsto x_i + \varepsilon f(x_i) x_i^{m+1}) \Big|_{\varepsilon=0} = \sum_{\ell=0}^{\infty} \left(\alpha_\ell \ \mathbb{J}^{(2)}_{m+\ell,n} - \beta_\ell \ \mathbb{J}^{(1)}_{m+\ell+1,n} \right) dI_n. \tag{3.2}$$

The contribution coming from $\prod_1^n dx_j$ *is given by*

$$\sum_{\ell=0}^{\infty} \alpha_\ell (\ell + m + 1) \mathbb{J}^{(1)}_{m+\ell,n} dI_n. \tag{3.3}$$

Lemma 3.3 *Setting*

$$dI_n = \Delta_n(x,y) \prod_{j=1}^{k_1} e^{\sum_1^\infty t_i x_j^i} \prod_{j=1}^{k_2} e^{\sum_1^\infty t_i y_j^i}$$
$$\left(\Delta_{k_1}(x) \prod_{j=1}^{k_1} e^{-\frac{x_j^2}{2} + a x_j + \beta x_j^2} e^{-\sum_1^\infty s_i x_j^i} dx_j \right)$$
$$\left(\Delta_{k_2}(y) \prod_{j=1}^{k_2} e^{-\frac{y_j^2}{2} - a y_j - \beta y_j^2} e^{-\sum_1^\infty u_i y_j^i} dy_j \right)$$

The following variational formula holds for $m \geq -1$*:*

$$\frac{d}{d\varepsilon} dI_n \begin{pmatrix} x_i \mapsto x_i + \varepsilon x_i^{m+1} \\ y_i \mapsto y_i + \varepsilon y_i^{m+1} \end{pmatrix} \Bigg|_{\varepsilon=0} = \mathbb{V}_m^{k_1,k_2}(dI_n). \tag{3.4}$$

Proof: The variational formula (3.4) is an immediate consequence of applying the variational formula (3.2) separately to the three factors of dI_n, and in addition applying formula (3.3) to the first factor, to account for the fact that $\prod_{j=1}^{k_1} dx_j \prod_{j=1}^{k_2} dy_j$ is missing from the first factor. ∎

Proof of Theorem 3.1: Formula (3.1) follows immediately from formula (3.4), by taking into account the variation of ∂E under the change of coordinates. ∎

Using the identity, valid when acting on $\tau_{k_1k_2}(t,s,u;\beta;E)$,

$$\frac{\partial}{\partial t_n} = -\frac{\partial}{\partial s_n} - \frac{\partial}{\partial u_n},$$

one obtains by explicit computation for $m \geq -1$,

$$\mathbb{V}_m^{k_1k_2} := \left\{ \begin{array}{l} \mathbb{J}^{(2)}_{m,k_1+k_2}(t) - (m+1)\mathbb{J}^{(1)}_{m,k_1+k_2}(t) \\ \\ +\mathbb{J}^{(2)}_{m,k_1}(-s) + a\mathbb{J}^{(1)}_{m+1,k_1}(-s) + (1-2\beta)\mathbb{J}^{(1)}_{m+2,k_1}(-s) \\ \\ +\mathbb{J}^{(2)}_{m,k_2}(-u) - a\mathbb{J}^{(1)}_{m+1,k_2}(-u) + (1+2\beta)\mathbb{J}^{(1)}_{m+2,k_2}(-u) \end{array} \right\}$$

$$\begin{aligned} = \;& \frac{1}{2}\sum_{i+j=m}\left(\frac{\partial^2}{\partial t_i\partial t_j} + \frac{\partial^2}{\partial s_i\partial s_j} + \frac{\partial^2}{\partial u_i\partial u_j}\right) + \sum_{i\geq 1}\left(it_i\frac{\partial}{\partial t_{i+m}} + is_i\frac{\partial}{\partial s_{i+m}} + iu_i\frac{\partial}{\partial u_{i+m}}\right) \\ & +(k_1+k_2)\left(\frac{\partial}{\partial t_m} + (-m)t_{-m}\right) - k_1\left(\frac{\partial}{\partial s_m} + (-m)s_{-m}\right) - k_2\left(\frac{\partial}{\partial u_m} + (-m)u_{-m}\right) \\ & +(k_1^2 + k_1k_2 + k_2^2)\delta_{m0} + a(k_1-k_2)\delta_{m+1,0} + \frac{m(m+1)}{2}(-t_{-m} + s_{-m} + u_{-m}) \\ & -\frac{\partial}{\partial t_{m+2}} + a\left(-\frac{\partial}{\partial s_{m+1}} + \frac{\partial}{\partial u_{m+1}} + (m+1)(s_{-m+1} - u_{-m+1})\right) \\ & +2\beta\left(\frac{\partial}{\partial u_{m+2}} - \frac{\partial}{\partial s_{m+2}}\right) \end{aligned}$$

The following identities, valid when acting on $\tau_{k_1k_2}(t,s,u;\beta;E)$, will also be used:

$$\frac{\partial}{\partial s_1} = -\frac{1}{2}\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial a}\right) \qquad \frac{\partial}{\partial s_2} = -\frac{1}{2}\left(\frac{\partial}{\partial t_2} + \frac{\partial}{\partial \beta}\right)$$

$$\frac{\partial}{\partial u_1} = -\frac{1}{2}\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial a}\right) \qquad \frac{\partial}{\partial u_2} = -\frac{1}{2}\left(\frac{\partial}{\partial t_2} - \frac{\partial}{\partial \beta}\right)$$

Corollary 3.4 *The tau-function $\tau = \tau_{k_1,k_2}(t,s,u;\beta;E)$ satisfies the following differential identities, with $\mathcal{B}_m = \sum_1^{2r} b_i^{m+1}\frac{\partial}{\partial b_i}$:*

$$
\begin{aligned}
-B_{-1}\tau &= \left(\frac{\partial}{\partial t_1} - 2\beta\frac{\partial}{\partial a}\right)\tau \\
&\quad -\sum_{i\geq 2}\left(it_i\frac{\partial}{\partial t_{i-1}} + is_i\frac{\partial}{\partial s_{i-1}} + iu_i\frac{\partial}{\partial u_{i-1}}\right)\tau \\
&\quad +a(k_2-k_1)\tau + (k_1s_1+k_2u_1-(k_1+k_2)t_1)\tau \\
\frac{1}{2}\left(B_{-1}-\frac{\partial}{\partial a}\right)\tau &= \left(\frac{\partial}{\partial s_1} + \beta\frac{\partial}{\partial a}\right)\tau \\
&\quad +\frac{1}{2}\sum_{i\geq 2}\left(it_i\frac{\partial}{\partial t_{i-1}} + is_i\frac{\partial}{\partial s_{i-1}} + iu_i\frac{\partial}{\partial u_{i-1}}\right)\tau \\
&\quad +\frac{a}{2}(k_1-k_2)\tau + \frac{1}{2}((k_1+k_2)t_1-k_1s_1-k_2u_1)\tau \\
-\left(B_0 - a\frac{\partial}{\partial a}\right)\tau &= \frac{\partial\tau}{\partial t_2} - (k_1^2+k_2^2+k_1k_2)\tau \\
&\quad -2\beta\frac{\partial\tau}{\partial\beta} - \sum_{i\geq 1}\left(it_i\frac{\partial}{\partial t_i} + is_i\frac{\partial}{\partial s_i} + iu_i\frac{\partial}{\partial u_i}\right)\tau \\
\frac{1}{2}\left(B_0 - a\frac{\partial}{\partial a} - \frac{\partial}{\partial\beta}\right)\tau &= \frac{\partial\tau}{\partial s_2} + \frac{1}{2}(k_1^2+k_2^2+k_1k_2)\tau \\
&\quad +\beta\frac{\partial\tau}{\partial\beta} + \frac{1}{2}\sum_{i\geq 1}\left(it_i\frac{\partial}{\partial t_i} + is_i\frac{\partial}{\partial s_i} + iu_i\frac{\partial}{\partial u_i}\right)\tau
\end{aligned}
\tag{3.5}
$$

Corollary 3.5 *On the locus* $\mathcal{L} = \{t = s = u = 0, \beta = 0\}$*, the function* $f = \log\tau_{k_1k_2}(t,s,u;\beta;E)$ *satisfies the following differential identities:*

$$
\begin{aligned}
\frac{\partial f}{\partial t_1} &= -\mathcal{B}_{-1}f + a(k_1-k_2) \\
\frac{\partial f}{\partial s_1} &= \frac{1}{2}\left(\mathcal{B}_{-1} - \frac{\partial}{\partial a}\right)f + \frac{a}{2}(k_2-k_1) \\
\frac{\partial f}{\partial t_2} &= \left(-\mathcal{B}_0 + a\frac{\partial}{\partial a}\right)f + k_1^2 + k_1k_2 + k_2^2
\end{aligned}
$$

$$\begin{aligned}\frac{\partial f}{\partial s_2} &= \frac{1}{2}\left(\mathcal{B}_0 - a\frac{\partial}{\partial a} - \frac{\partial}{\partial \beta}\right) f - \frac{1}{2}(k_1^2 + k_2^2 + k_1k_2) \qquad (3.6)\\
2\frac{\partial^2 f}{\partial t_1 \partial s_1} &= \mathcal{B}_{-1}\left(\frac{\partial}{\partial a} - \mathcal{B}_{-1}\right) f - 2k_1\\
2\frac{\partial^2 f}{\partial t_1 \partial s_2} &= \left(a\frac{\partial}{\partial a} + \frac{\partial}{\partial \beta} - \mathcal{B}_0 + 1\right)\mathcal{B}_{-1} f - 2\frac{\partial f}{\partial a} - 2a(k_1 - k_2)\\
2\frac{\partial^2 f}{\partial t_2 \partial s_1} &= \frac{\partial}{\partial a}(\mathcal{B}_0 - a\frac{\partial}{\partial a} + a\mathcal{B}_{-1})f - \mathcal{B}_{-1}(\mathcal{B}_0 - 1)f - 2a(k_1 - k_2) \quad (3.7)\end{aligned}$$

Proof: Upon dividing equations (3.5) by τ and restricting to the locus $\mathcal{L}$, equations (3.6) follow immediately. The essence of deriving (3.7) is that the Virasoro operators $\mathbb{V}_n$ and the boundary operators $\mathcal{B}_m$ commute. To derive, say, the first equation in the list (3.7), rewrite the two first equations of (3.5) as

$$\begin{aligned}-\mathcal{B}_{-1} f &= \frac{\partial f}{\partial t_1} + a(k_2 - k_1) + L_1(f) + \ell_1\\
\frac{1}{2}\left(\mathcal{B}_{-1} - \frac{\partial}{\partial a}\right) f &= \frac{\partial f}{\partial s_1} + \frac{1}{2}a(k_1 - k_2) + L_2(f) + \ell_2\end{aligned}$$

where L_i are linear operators vanishing on $\mathcal{L}$ and the ℓ_i are functions vanishing on $\mathcal{L}$. This yields:

$$\begin{aligned}&(-\mathcal{B}_{-1})\frac{1}{2}\left(\mathcal{B}_{-1} - \frac{\partial}{\partial a}\right) f\Big|_{\mathcal{L}}\\
&= \left(\frac{\partial}{\partial s_1} + \beta\frac{\partial}{\partial a}\right)(-\mathcal{B}_{-1} f)\Big|_{\mathcal{L}}\\
&\qquad + \frac{1}{2}\sum_{i\geq 2}\left(it_i\frac{\partial}{\partial t_{i-1}} + is_i\frac{\partial}{\partial s_{i-1}} + iu_i\frac{\partial}{\partial u_{i-1}}\right)(-\mathcal{B}_{-1} f)\Big|_{\mathcal{L}}\\
&= \frac{\partial}{\partial s_1}(-\mathcal{B}_{-1} f)\Big|_{\mathcal{L}}\\
&= \frac{\partial}{\partial s_1}\left(\begin{array}{l}\left(\frac{\partial}{\partial t_1} - 2\beta\frac{\partial}{\partial a}\right) f + a(k_2 - k_1)\\ -\sum_{i\geq 2}\left(it_i\frac{\partial}{\partial t_{i-1}} + is_i\frac{\partial}{\partial s_{i-1}} + iu_i\frac{\partial}{\partial u_{i-1}}\right) f\\ +\ (k_1s_1 + k_2u_1 - (k_1 + k_2)t_1)\end{array}\right)\Bigg|_{\mathcal{L}} = \frac{\partial^2}{\partial s_1 \partial t_1} f + k_1.\end{aligned}$$

The other identities (3.7) can be obtained in a similar way. ■

4 A PDE for the Gaussian ensemble with external source

Proof of Theorem 0.1: First observe that, with $n = k_1 + k_2$,

$$\mathbb{P}_n(a;E) = \frac{1}{Z_n}\int_{\mathcal{H}_n(E)} e^{-\operatorname{Tr}(\frac{1}{2}M^2 - AM)}dM = \frac{\tau_{k_1k_2}(t,s,u;\beta;E)}{\tau_{k_1k_2}(t,s,u;\beta;\mathbb{R})}\Big|_{t=s=u=\beta=0}$$

An explicit computation over the whole range yields:

$$\begin{aligned}
&\tau_{k_1k_2}(t,s,u;\beta;\mathbb{R})\Big|_{t=s=u=\beta=0}\\
&= \frac{1}{k_1!k_2!}\int_{\mathbb{R}^n}\Delta_n(x,y)\left(\Delta_{k_1}(x)\prod_{i=1}^{k_1}e^{-\frac{x_i^2}{2}+ax_i}dx_i\right)\\
&\qquad\qquad\left(\Delta_{k_2}(y)\prod_{i=1}^{k_2}e^{-\frac{y_i^2}{2}-ay_i}dy_i\right)\\
&= c_{k_1k_2}a^{k_1k_2}e^{(k_1+k_2)a^2/2}.
\end{aligned}$$

This is obtained from the representation (1.5) in terms of moments, which themselves are Gaussian integrals,as shown in Appendix 1. From this formula, it follows that

$$\log\tau_{k_1k_2}(t,s,u;\beta;\mathbb{R})\Big|_{t=s=u=\beta=0} = \frac{k_1+k_2}{2}a^2 + k_1k_2\log a + C_{k_1k_2},$$

where $c_{k_1k_2}$ and $C_{k_1k_2}$ are constants depending on k_1, k_2 only. It follows that

$$\log\mathbb{P}_n(a;E) = \log\tau_{k_1k_2}(0,0,0;0;E) - \frac{k_1+k_2}{2}a^2 - k_1k_2\log a - C_{k_1k_2} \quad (4.1)$$

Thus we need to concentrate on $\tau_{k_1k_2}(t,s,u;\beta;E)$, which, by Theorem 2.1, satisfies the bilinear identity (2.2) and thus the identities (2.9) and (2.10) of Corollary 2.2:

$$\begin{aligned}
\frac{\partial}{\partial t_1}\log\frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} &= \frac{\frac{\partial^2}{\partial t_2\partial s_1}\log\tau_{k_1k_2}}{\frac{\partial^2}{\partial t_1\partial s_1}\log\tau_{k_1k_2}}\\
\frac{\partial}{\partial s_1}\log\frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} &= -\frac{\frac{\partial^2}{\partial t_1\partial s_2}\log\tau_{k_1k_2}}{\frac{\partial^2}{\partial t_1\partial s_1}\log\tau_{k_1k_2}},
\end{aligned}\quad (4.2)$$

whereas the first two Virasoro equations (3.6) yield, specializing to the locus $\mathcal{L} = \{t = s = u = 0, \beta = 0\}$ and the indices $k_1 \pm 1, k_2$,

$$\begin{aligned} \frac{\partial}{\partial t_1} \log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} &= -\mathcal{B}_{-1} \log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} + 2a \\ \frac{\partial}{\partial s_1} \log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} &= \frac{1}{2}\left(\mathcal{B}_{-1} - \frac{\partial}{\partial a}\right) \log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} - a \end{aligned} \tag{4.3}$$

From these three equations, the expression $\log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}}$ can be eliminated, by first subtracting the first equations in (4.2) and (4.3) and then the second equations in (4.2) and (4.3). Subsequently one eliminates $\log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}}$ from the equations thus obtained, yielding

$$\frac{1}{2}\left(\mathcal{B}_{-1} - \frac{\partial}{\partial a}\right)\left(\frac{\frac{\partial^2}{\partial t_2 \partial s_1} \log \tau_{k_1 k_2}}{\frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{k_1 k_2}} - 2a\right) = \mathcal{B}_{-1}\left(\frac{\frac{\partial^2}{\partial t_1 \partial s_2} \log \tau_{k_1 k_2}}{\frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{k_1 k_2}} - a\right)$$

or equivalently

$$\mathcal{B}_{-1}\left(\frac{\left(\frac{\partial^2}{\partial t_2 \partial s_1} - 2\frac{\partial^2}{\partial t_1 \partial s_2}\right) \log \tau_{k_1 k_2}}{\frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{k_1 k_2}}\right) - \frac{\partial}{\partial a}\left(\frac{\left(\frac{\partial^2}{\partial t_2 \partial s_1} - 2a\frac{\partial^2}{\partial t_1 \partial s_1}\right) \log \tau_{k_1 k_2}}{\frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{k_1 k_2}}\right) = 0. \tag{4.4}$$

Using the Virasoro relations (3.7), one obtains along the locus $\mathcal{L} = \{t = s = u = 0, \beta = 0\}$:

$$\begin{aligned} 4\frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{k_1 k_2} &=: F^+ \\ 2\left(\frac{\partial^2}{\partial t_2 \partial s_1} - 2\frac{\partial^2}{\partial t_1 \partial s_2}\right) \log \tau_{k_1 k_2} &= H_1^+ - 2\mathcal{B}_{-1}\frac{\partial}{\partial \beta} \log \tau_{k_1 k_2} \\ 2\left(\frac{\partial^2}{\partial t_2 \partial s_1} - 2a\frac{\partial^2}{\partial t_1 \partial s_1}\right) \log \tau_{kk} &= H_2^+ \end{aligned} \tag{4.5}$$

where, using the identity (4.1), along the locus $\mathcal{L}=\{t=s=u=0,\beta=0\}$,

$$
\begin{aligned}
F^+ &:= 2\mathcal{B}_{-1}(\frac{\partial}{\partial a}-\mathcal{B}_{-1})\log\tau_{k_1k_2}-4k_1 = 2\mathcal{B}_{-1}(\frac{\partial}{\partial a}-\mathcal{B}_{-1})\log\mathbb{P}_n-4k_1\\
H_1^+ &:= \frac{\partial}{\partial a}\left(\mathcal{B}_0-a\frac{\partial}{\partial a}-a\mathcal{B}_{-1}\right)\log\tau_{k_1k_2}+\left(\mathcal{B}_0\mathcal{B}_{-1}+4\frac{\partial}{\partial a}\right)\log\tau_{k_1k_2}+2a(k_1-k_2)\\
&= \frac{\partial}{\partial a}\left(\mathcal{B}_0-a\frac{\partial}{\partial a}-a\mathcal{B}_{-1}\right)\log\mathbb{P}_n+\left(\mathcal{B}_0\mathcal{B}_{-1}+4\frac{\partial}{\partial a}\right)\log\mathbb{P}_n+4ak_1+\frac{4k_1k_2}{a}\\
H_2^+ &:= \frac{\partial}{\partial a}\left(\mathcal{B}_0-a\frac{\partial}{\partial a}-a\mathcal{B}_{-1}\right)\log\tau_{k_1k_2}+(2a\mathcal{B}_{-1}-\mathcal{B}_0+2)\mathcal{B}_{-1}\log\tau_{k_1k_2}+2a(k_1+k_2)\\
&= \frac{\partial}{\partial a}\left(\mathcal{B}_0-a\frac{\partial}{\partial a}-a\mathcal{B}_{-1}\right)\log\mathbb{P}_n+(2a\mathcal{B}_{-1}-\mathcal{B}_0+2)\mathcal{B}_{-1}\log\mathbb{P}_n,
\end{aligned}
$$

confirming (0.6). Notice that the expressions above do not contain partials in β, except for the β-partial appearing in the second expression of (4.5). Putting these expressions (4.5) into (4.4) yields

$$
\begin{aligned}
\left\{\mathcal{B}_{-1}\frac{\partial}{\partial\beta}\log\tau_{k_1k_2}\Big|_{\mathcal{L}},F^+\right\}_{\mathcal{B}_{-1}} &= \left\{H_1^+,\frac{1}{2}F^+\right\}_{\mathcal{B}_{-1}}-\left\{H_2^+,\frac{1}{2}F^+\right\}_{\partial/\partial a}\\
&=: G^+ \qquad (4.6)
\end{aligned}
$$

and by involution $a\mapsto -a,\ \beta\mapsto-\beta,\ k_1\longleftrightarrow k_2$:

$$
\begin{aligned}
-\left\{\mathcal{B}_{-1}\frac{\partial}{\partial\beta}\log\tau_{k_1k_2}\Big|_{\mathcal{L}},F^-\right\}_{\mathcal{B}_{-1}} &= \left\{H_1^-,\frac{1}{2}F^-\right\}_{\mathcal{B}_{-1}}-\left\{H_2^-,\frac{1}{2}F^-\right\}_{-\partial/\partial a}\\
&=: G^- \qquad (4.7)
\end{aligned}
$$

where

$$
F^-=F^+\Big|_{\substack{a\to -a\\ k_1\leftrightarrow k_2}},\ H_i^-=H_i^+\Big|_{\substack{a\to -a\\ k_1\leftrightarrow k_2}}.
$$

Remember the change of variables $a\mapsto -a,\ \beta\mapsto-\beta,\ k_1\longleftrightarrow k_2$ acts on the operators, since $\tau_{k_1k_2}$ is invariant under this change; see (1.6).
Equations (4.6) and (4.7) yield a linear system of equations in

$$
\mathcal{B}_{-1}\frac{\partial\log\tau_{k_1k_2}}{\partial\beta}\quad\text{and}\quad\mathcal{B}_{-1}^2\frac{\partial\log\tau_{k_1k_2}}{\partial\beta}
$$

from which

$$
\begin{aligned}
\mathcal{B}_{-1}\frac{\partial\log\tau_{k_1k_2}}{\partial\beta} &= \frac{G^-F^++G^+F^-}{-F^-(\mathcal{B}_{-1}F^+)+F^+(\mathcal{B}_{-1}F^-)}\\
\mathcal{B}_{-1}^2\frac{\partial\log\tau_{k_1k_2}}{\partial\beta} &= \frac{G^-(\mathcal{B}_{-1}F^+)+G^+(\mathcal{B}_{-1}F^-)}{-F^-(\mathcal{B}_{-1}F^+)+F^+(\mathcal{B}_{-1}F^-)}
\end{aligned}
$$

Subtracting the second equation from $\mathcal{B}_{-1}$ of the first equation yields the following:

$$\begin{aligned}&\Big(F^+\mathcal{B}_{-1}G^- + F^-\mathcal{B}_{-1}G^+\Big)\Big(F^+\mathcal{B}_{-1}F^- - F^-\mathcal{B}_{-1}F^+\Big)\\ &-\Big(F^+G^- + F^-G^+\Big)\Big(F^+\mathcal{B}_{-1}^2F^- - F^-\mathcal{B}_{-1}^2F^+\Big) = 0,\end{aligned}$$

establishing Theorem 0.1. ■

5 A PDE for the Pearcey transition probability

From the Karlin-McGregor formula for non-intersecting Brownian motions $x_j(t)$, we have:

$$\begin{aligned}&\mathbb{P}\left(\text{all } x_i(t)\in E,\ 1\le i\le n \;\middle|\; \begin{array}{l}\text{given } x_i(0)=\gamma_i\\ \text{given } x_i(1)=\delta_i\end{array}\right)\\ &= \int_{E^n}\frac{1}{Z_n}\det(p(t;\gamma_i,x_j))_{1\le i,j\le n}\det(p(1-t;x_{i'},\delta_{j'}))_{1\le i',j'\le n}\prod_1^n dx_i\end{aligned}$$

for the Brownian motion kernel

$$p(t,x,y) := \frac{1}{\sqrt{\pi t}}\, e^{-\frac{(y-x)^2}{t}}.$$

Aptekarev, Bleher and Kuijlaars introduce in [7] a change of variables transforming the Brownian motion problem into the Gaussian random ensemble with external source. For $E := \bigcup_{i=1}^r [b_{2i-1}, b_{2i}]$, we have, using this change of variables,

$$x_i = x_i'\sqrt{\frac{t(1-t)}{2}} \quad \text{and} \quad y_i = y_i'\sqrt{\frac{t(1-t)}{2}},$$

in equality $\overset{*}{=}$,

$$\begin{aligned}&\mathbb{P}_0^{\pm a}(\text{all } x_j(t)\in E)\\ &:= \ \mathbb{P}\left(\text{all } x_j(t)\in E \;\middle|\; \begin{array}{c}\text{all } x_j(0)=0\\ k \text{ left paths end up at } -a \text{ at time } t=1,\\ k \text{ right paths end up at } +a \text{ at time } t=1\end{array}\right)\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\substack{\text{all } \gamma_i \to 0 \\ \delta_1,\dots,\delta_k \to -a \\ \delta_{k+1},\dots,\delta_{2k} \to a}} \int_{E^n} \frac{1}{Z_n} \det(p(t;\gamma_i,x_j))_{1\le i,j\le n} \det(p(1-t;x_{i'},\delta_{j'}))_{1\le i',j'\le n} \prod_1^n dx_i \\
&= \frac{1}{Z_n}\int_{E^n} \Delta_n(x,y) \left(\Delta_k(x)\prod_{i=1}^k e^{-\frac{x_i^2}{t(1-t)}+\frac{2ax_i}{1-t}}dx_i\right)\left(\Delta_k(y)\prod_{i=1}^k e^{-\frac{y_i^2}{t(1-t)}-\frac{2ay_i}{1-t}}dy_i\right) \\
&\overset{*}{=} \frac{1}{Z_n'}\int_{\left(E\sqrt{\frac{2}{t(1-t)}}\right)^n} \Delta_n(x',y') \left(\Delta_k(x')\prod_{i=1}^k e^{-\frac{x_i'^2}{2}+a\sqrt{\frac{2t}{1-t}}x_i'}dx_i'\right) \\
&\qquad\qquad \left(\Delta_k(y')\prod_{i=1}^k e^{-\frac{y_i'^2}{2}-a\sqrt{\frac{2t}{1-t}}y_i'}dy_i'\right) \\
&= \mathbb{P}_n\left(a\sqrt{\frac{2t}{1-t}};b_1\sqrt{\frac{2}{t(1-t)}},\dots,b_{2r}\sqrt{\frac{2}{t(1-t)}}\right)
\end{aligned}
$$

with $\mathbb{P}_n$ of Theorem 0.1, using (1.1) and (1.3), with $k=k_1=k_2$. Setting

$$
e^{g(t)} := \sqrt{\frac{2t}{1-t}} \text{ and } e^{h(t)} := \sqrt{\frac{2}{t(1-t)}} \tag{5.1}
$$

$$
\tilde{\mathcal{B}}_k = \sum_1^{2r} v_i^{k+1}\frac{\partial}{\partial v_i}, \qquad \mathcal{B}_k = \sum_1^{2r} b_i^{k+1}\frac{\partial}{\partial b_i}, \tag{5.2}
$$

we find

$$
\mathbb{P}_0^{\pm a}(t;b_1,\dots,b_{2r}) = \mathbb{P}_n(ae^{g(t)};b_1e^{h(t)},\dots,b_{2r}e^{h(t)}) = \mathbb{P}_n(u;v_1,\dots,v_{2r})\Big|_{\substack{u=ae^{g(t)}\\ v=be^{h(t)}}}. \tag{5.3}
$$

From Theorem 0.1, it follows that $\mathbb{P}_n(u;v_1,\dots,v_{2r})$ satisfies the non-linear equation (0.5), with a and all b_i's replaced by u and v_i respectively. In order to find the equation for $\mathbb{P}_0^{\pm a}(t;b_1,\dots,b_{2r})$, one needs to compute the partial derivatives in t_i and b_i in terms of partials in u and v_i, appearing in equation (0.5) and use the relationship (5.3). To be precise, compute

$$
\left(\frac{\partial}{\partial t}\right)^i (\mathcal{B}_0)^j(\mathcal{B}_{-1})^\ell\, \mathbb{P}_0^{\pm a} \text{ with } i+j+\ell\le 4 \text{ and } i,j,\ell\ge 0, \tag{5.4}
$$

yielding a system of 34 linear equations in 34 unknowns

$$\left(\frac{\partial}{\partial u}\right)^i (\tilde{\mathcal{B}}_0)^j (\tilde{\mathcal{B}}_{-1})^\ell \, \mathbb{P}_n \text{ with } i+j+\ell \le 4 \text{ and } i,j,\ell \ge 0, \tag{5.5}$$

which one solves. Notice, one always writes $(\tilde{\mathcal{B}}_0)^j(\tilde{\mathcal{B}}_{-1})^\ell$ in that order, using the commutation relation $[\tilde{\mathcal{B}}_{-1}, \tilde{\mathcal{B}}_0] = \tilde{\mathcal{B}}_{-1}$. For instance,

$$\begin{aligned}
(\mathcal{B}_{-1})^j \mathbb{P}_0^{\pm a} &= e^{jh(t)} (\tilde{\mathcal{B}}_{-1})^j \mathbb{P}_n, \quad (\mathcal{B}_0)^j \mathbb{P}_0^{\pm a} = (\tilde{\mathcal{B}}_0)^j \mathbb{P}_n \qquad j = 1, \ldots, 4. \\
\frac{\partial \mathbb{P}_0^{\pm a}}{\partial t} &= \left(g'(t) u \frac{\partial}{\partial u} + h'(t) \tilde{\mathcal{B}}_0 \right) \mathbb{P}_n \\
\frac{\partial^2 \mathbb{P}_0^{\pm a}}{\partial t^2} &= \left(g'(t) u \frac{\partial}{\partial u} + h'(t) \tilde{\mathcal{B}}_0 \right) \left(g'(t) u \frac{\partial}{\partial u} + h'(t) \tilde{\mathcal{B}}_0 \right) \mathbb{P}_n \\
&\quad + \left(g''(t) u \frac{\partial}{\partial u} + h''(t) \tilde{\mathcal{B}}_0 \right) \mathbb{P}_n \\
&\quad \vdots
\end{aligned}$$

The partials (5.5) thus obtained are now being substituted into the 4th order equation (0.5), with a and b_i replaced by u and v_i, and thus the $\mathcal{B}_j$ by $\tilde{\mathcal{B}}_j$, yielding a new 4th order equation involving the partials (5.4).

Let now the number of particles n go to infinity, together with the corresponding scaling (see [7, 23])

$$n = 2k = \frac{2}{z^4}, \quad \pm a = \pm \frac{1}{z^2}, \quad b_i = x_i z, \quad t = \frac{1}{2} + sz^2, \quad \text{for } z \to 0. \tag{5.6}$$

It is convenient to replace the $\pm$ in (5.6) by the variable ε, which one keeps in the computation as a variable. The scaling combined with the change of variables (5.3) leads to the following expressions u and v_i in terms of z:

$$\begin{aligned}
2k &= \frac{2}{z^4} \\
u &= a e^{g(t)} = a \sqrt{\frac{2t}{1-t}} = \frac{\varepsilon \sqrt{2}}{z^2} \sqrt{\frac{\frac{1}{2} + sz^2}{\frac{1}{2} - sz^2}} \\
v_i &= b_i e^{h(t)} = b_i \sqrt{\frac{2}{t(1-t)}} = x_i \frac{z\sqrt{2}}{\sqrt{\frac{1}{4} - s^2 z^4}}
\end{aligned} \tag{5.7}$$

So, the question now is to estimate:

$$\left\{ \begin{array}{l} \Big(F^+\tilde{\mathcal{B}}_{-1}G^- + F^-\tilde{\mathcal{B}}_{-1}G^+\Big)\Big(F^+\tilde{\mathcal{B}}_{-1}F^- - F^-\tilde{\mathcal{B}}_{-1}F^+\Big) \\ -\Big(F^+G^- + F^-G^+\Big)\Big(F^+\tilde{\mathcal{B}}^2_{-1}F^- - F^-\tilde{\mathcal{B}}^2_{-1}F^+\Big) \end{array} \right\}\Bigg|_{\substack{u \mapsto \frac{\varepsilon\sqrt{2}}{z^2}\sqrt{\frac{\frac12+sz^2}{\frac12-sz^2}} \\ v_i \mapsto x_i\frac{z\sqrt{2}}{\sqrt{\frac14-s^2z^4}} \\ n\mapsto \frac{2}{z^4}}} \tag{5.8}$$

For this, we need to compute the expressions $F^\pm, \tilde{\mathcal{B}}_{-1}F^\pm, \tilde{\mathcal{B}}^2_{-1}F^\pm, G^\pm$ and $\tilde{\mathcal{B}}_{-1}G^\pm$ appearing in (5.8) in terms of

$$\begin{aligned} & Q_z(s;x_1,\dots,x_{2r}) \\ :=\ & \log \mathbb{P}_{2/z^4}\left(\frac{\varepsilon\sqrt{2}}{z^2}\sqrt{\frac{\frac12+sz^2}{\frac12-sz^2}}\ ;\ x_1\frac{z\sqrt{2}}{\sqrt{\frac14-s^2z^4}},\dots,x_{2r}\frac{z\sqrt{2}}{\sqrt{\frac14-s^2z^4}}\right) \\ =\ & Q(s;x_1,\dots,x_{2r})+O(z), \end{aligned} \tag{5.9}$$

with

$$Q(s;x_1,\dots,x_{2r}) = \log\det\left(I-K_s\chi_{E^c}\right), \tag{5.10}$$

as shown in [23]. Without taking a limit on $Q_z(s;x_1,\dots,x_{2r})$ yet, one computes

$$\begin{aligned} F^\varepsilon &= -\frac{4}{z^4}-\frac{1}{4z^2}\mathcal{B}^2_{-1}Q_z+\frac{\varepsilon}{4z}\mathcal{B}_{-1}\frac{\partial Q_z}{\partial s}+O(z) \\ \frac{1}{\sqrt{2}}\tilde{\mathcal{B}}_{-1}F^\varepsilon &= -\frac{1}{16z^3}\mathcal{B}^3_{-1}Q_z+\frac{\varepsilon}{16z^2}\mathcal{B}^2_{-1}\frac{\partial Q_z}{\partial s}-\frac{\varepsilon s}{8}\mathcal{B}^2_{-1}\frac{\partial Q_z}{\partial s}+O(z) \\ \tilde{\mathcal{B}}^2_{-1}F^\varepsilon &= -\frac{1}{32z^4}\mathcal{B}^4_{-1}Q_z+\frac{\varepsilon}{32z^3}\mathcal{B}^3_{-1}\frac{\partial Q_z}{\partial s}-\frac{\varepsilon s}{16z}\mathcal{B}^3_{-1}\frac{\partial Q_z}{\partial s}+O(1) \\ G^\varepsilon &= \frac{3\varepsilon}{8z^9}\mathcal{B}^3_{-1}Q_z+\frac{\varepsilon s}{4z^7}\mathcal{B}^3_{-1}Q_z \\ &\quad -\frac{1}{128z^6}\left[(\mathcal{B}_{-1}\frac{\partial Q_z}{\partial s})(\mathcal{B}^3_{-1}Q_z)+32\mathcal{B}_0\mathcal{B}^2_{-1}Q_z-\left(\mathcal{B}^2_{-1}Q_z+64s\right)\mathcal{B}^2_{-1}\frac{\partial Q_z}{\partial s}\right. \\ &\qquad\qquad \left. -64\mathcal{B}^2_{-1}Q_z+16\frac{\partial^3 Q_z}{\partial s^3}\right]+O(\frac{1}{z^5}) \\ \frac{1}{\sqrt{2}}\tilde{\mathcal{B}}_{-1}G^\varepsilon &= \frac{3\varepsilon}{32z^{10}}\mathcal{B}^4_{-1}Q_z+\frac{\varepsilon s}{16z^8}\mathcal{B}^4_{-1}Q_z \\ &\quad +\frac{1}{512z^7}\left[-(\mathcal{B}_{-1}\frac{\partial Q_z}{\partial s})(\mathcal{B}^4_{-1}Q_z)-32\mathcal{B}_0\mathcal{B}^3_{-1}Q_z+\left(\mathcal{B}^2_{-1}Q+64s\right)\mathcal{B}^3_{-1}\frac{\partial Q_z}{\partial s}\right. \\ &\qquad\qquad \left. +32\mathcal{B}^3_{-1}Q_z-16\mathcal{B}_{-1}\frac{\partial^3 Q_z}{\partial s^3}\right]+O(\frac{1}{z^6}). \end{aligned} \tag{5.11}$$

The formulae needed to obtain the expansions above for G^{ε} and $\tilde{\mathcal{B}}_{-1}G^{\varepsilon}$ are given in Appendix 2. From the expressions above one readily deduces

$$\begin{aligned}
F^+\tilde{\mathcal{B}}_{-1}G^- + F^-\tilde{\mathcal{B}}_{-1}G^+ &= -\frac{\sqrt{2}}{64z^{11}}\begin{pmatrix} 2(\mathcal{B}_{-1}\frac{\partial Q_z}{\partial s})(\mathcal{B}_{-1}^4 Q_z) \\ -32(\mathcal{B}_0 - 2s\frac{\partial}{\partial s} - 1)\mathcal{B}_{-1}^3 Q_z \\ +(\mathcal{B}_{-1}^2\frac{\partial Q_z}{\partial s})(\mathcal{B}_{-1}^3 Q_z) - 16\mathcal{B}_{-1}\frac{\partial^3 Q_z}{\partial s^3}\end{pmatrix} + O(\frac{1}{z^9}) \\
F^+\tilde{\mathcal{B}}_{-1}F^- - F^-\tilde{\mathcal{B}}_{-1}F^+ &= \varepsilon\frac{\frac{\partial}{\partial s}\mathcal{B}_{-1}^2 Q_z}{\sqrt{2}z^6} + O(\frac{1}{z^4}) \\
F^+G^- + F^-G^+ &= -\frac{1}{16z^{10}}\begin{pmatrix} 2(\mathcal{B}_{-1}\frac{\partial Q_z}{\partial s})(\mathcal{B}_{-1}^3 Q_z) \\ -32(\mathcal{B}_0 - 2s\frac{\partial}{\partial s} - 2)\mathcal{B}_{-1}^2 Q_z \\ +(\mathcal{B}_{-1}^2\frac{\partial Q_z}{\partial s})(\mathcal{B}_{-1}^2 Q_z) - 16\frac{\partial^3 Q_z}{\partial s^3}\end{pmatrix} + O(\frac{1}{z^8}) \\
F^+\tilde{\mathcal{B}}_{-1}^2F^- - F^-\tilde{\mathcal{B}}_{-1}^2F^+ &= \varepsilon\frac{\frac{\partial}{\partial s}\mathcal{B}_{-1}^3 Q_z}{4z^7} + O(\frac{1}{z^5}).
\end{aligned}$$

Using this expressions, one easily deduces for small z,

$$\begin{aligned}
0 &= \left\{\begin{array}{l} \left(F^+\tilde{\mathcal{B}}_{-1}G^- + F^-\tilde{\mathcal{B}}_{-1}G^+\right)\left(F^+\tilde{\mathcal{B}}_{-1}F^- - F^-\tilde{\mathcal{B}}_{-1}F^+\right) \\ -\left(F^+G^- + F^-G^+\right)\left(F^+\tilde{\mathcal{B}}_{-1}^2F^- - F^-\tilde{\mathcal{B}}_{-1}^2F^+\right)\end{array}\right\}\Bigg|_{\substack{u\mapsto\frac{\sqrt{2}}{z^2}\sqrt{\frac{\frac{1}{2}+sz^2}{\frac{1}{2}-sz^2}} \\ v_i\mapsto x_i\frac{z\sqrt{2}}{\sqrt{\frac{1}{4}-s^2z^4}} \\ n\mapsto\frac{2}{z^4}}} \\
&= -\frac{\varepsilon}{2z^{17}}\begin{pmatrix}\left\{\mathcal{B}_{-1}^2\frac{\partial Q}{\partial s}, \frac{1}{2}\frac{\partial^3 Q_z}{\partial s^3} + (\mathcal{B}_0 - 2)\mathcal{B}_{-1}^2 Q_z\right\}_{\mathcal{B}_{-1}} \\ +\frac{1}{16}\mathcal{B}_{-1}\frac{\partial Q_z}{\partial s}\left\{\mathcal{B}_{-1}^3 Q_z, \mathcal{B}_{-1}^2\frac{\partial Q_z}{\partial s}\right\}_{\mathcal{B}_{-1}}\end{pmatrix} + O(\frac{1}{z^{15}}) \\
&= -\frac{\varepsilon}{2z^{17}}\left(\text{ the same expression for } Q(s; x_1, \ldots, x_{2r})\ \right) + O(\frac{1}{z^{16}}),
\end{aligned}$$

using (5.10) in the last equality. Taking the limit when $z \to 0$ yields equation (0.5) of Theorem 0.2. ■

6 Appendix 1

Setting

$$\mu_{i+j-1}(\pm a) := \mu_{ij}^{\pm}(t, s, u; \beta, \mathbb{R})\big|_{t=s=u=\beta=0} = \int_{\mathbb{R}} z^{i+j-1}e^{-\frac{z^2}{2}\pm az}dz,$$

one computes[5]:

Lemma 6.1

$$\tau_{k_1k_2}(t,s,u;\beta,\mathbb{R})\Big|_{t=s=u=\beta=0} = \det\begin{pmatrix} (\mu_{i+j}(a))_{\substack{0\le i\le k_1-1\\0\le j\le n-1}} \\ \\ (\mu_{i+j}(-a))_{\substack{0\le i\le k_2-1\\0\le j\le n-1}}\end{pmatrix}$$

$$= c_{k_1k_2}a^{k_1k_2}e^{\frac{(k_1+k_2)}{2}}a^2.$$

with

$$c_{k_1k_2} = (-2)^{k_1k_2}(2\pi)^{\frac{k_1+k_2}{2}}\prod_0^{k_1-1} j!\prod_0^{k_2-1} j!\ .$$

Proof: By explicit integration, one computes

$$\mu_0(a)=\sqrt{2\pi}\ e^{\frac{a^2}{2}} \text{ and } \mu_i(\pm a)=\sqrt{2\pi}\left(\pm\frac{d}{da}\right)^i e^{\frac{a^2}{2}}.$$

Define the Hermite polynomials (except for a minor change of variables)

$$p_i(a) := e^{-\frac{a^2}{2}}\left(\frac{d}{da}\right)^i e^{\frac{a^2}{2}} = \left(\frac{d}{da}+a\right)p_{i-1}(a).$$

The following holds:

$$p_{2i}(a) = \text{even polynomial}, \qquad p_{2i+1}(a) = \text{odd polynomial of } a,$$

which is used in equality $\overset{**}{=}$ below, and

$$p_{k+n}(a) = p_k^{(n)} + \beta_1(a)p_k^{(n-1)} + \beta_2(a)p_k^{(n-2)} + \ldots + \beta_n p_k,$$

[5]Remember $n = k_1 + k_2$.

where $p_k^{(n)} := \left(\frac{d}{da}\right)^n p_k$ and where $\beta_i(a)$ are polynomials in a, independent of k; this feature is used in equality $\stackrel{*}{=}$ below. Then we compute:

$$
\begin{aligned}
&\tau_{k_1k_2}(t,s,u;\beta,\mathbb{R})\Big|_{t=s=u=\beta=0}\\
&= (\sqrt{2\pi})^n\, e^{\frac{na^2}{2}} \det\begin{pmatrix} \left(p_{i+j}\right)_{\substack{0\le i\le k_1-1\\ 0\le j\le n-1}} \\ \\ \left((-1)^{i+j}p_{i+j}\right)_{\substack{0\le i\le k_2-1\\ 0\le j\le n-1}}\end{pmatrix}\\
&= (\sqrt{2\pi})^n(-1)^{\frac{k_2(k_2-1)}{2}} e^{\frac{na^2}{2}} \det\begin{pmatrix} \left(p_{i+j}\right)_{\substack{0\le i\le k_1-1\\ 0\le j\le n-1}} \\ \\ \left((-1)^{j}p_{i+j}\right)_{\substack{0\le i\le k_2-1\\ 0\le j\le n-1}}\end{pmatrix}\\
&\stackrel{*}{=} (\sqrt{2\pi})^n(-1)^{\frac{k_2(k_2-1)}{2}} e^{\frac{na^2}{2}} \det\begin{pmatrix} \left(p_j^{(i)}\right)_{\substack{0\le i\le k_1-1\\ 0\le j\le n-1}} \\ \\ \left((-1)^{j}p_j^{(i)}\right)_{\substack{0\le i\le k_2-1\\ 0\le j\le n-1}}\end{pmatrix}\\
&\stackrel{**}{=} c_{k_1k_2} e^{\frac{na^2}{2}} \det\begin{pmatrix} \left((a^{j-1})^{(i)}\right)_{\substack{0\le i\le k_1-1\\ 1\le j\le n}} \\ \\ \left(((-a)^{j-1})^{(i)}\right)_{\substack{0\le i\le k_2-1\\ 1\le j\le n}}\end{pmatrix}\\
&= c_{k_1k_2} e^{\frac{na^2}{2}} \det\begin{pmatrix} \left(\alpha_{ij}a^{j-i}\right)_{\substack{1\le i\le k_1\\ 1\le j\le n}} \\ \\ \left(\alpha_{ij}a^{j-i+k_1}\right)_{\substack{k_1+1\le i\le n\\ 1\le j\le n}}\end{pmatrix}\\
&= c_{k_1k_2} e^{\frac{na^2}{2}} \sum_{\sigma\in S_n}(-1)^\sigma \prod_{1\le i\le k_1}\alpha_{i\sigma(i)}a^{\sigma(i)-i}\prod_{k_1+1\le i\le n}\alpha_{i\sigma(i)}a^{\sigma(i)-i+k_1}\\
&= c_{k_1k_2} e^{\frac{na^2}{2}} \sum(-1)^\sigma a^{\sum_1^n(\sigma(i)-i)}\left(a^{k_1}\right)^{k_2}\prod_{1\le i\le n}\alpha_{i\sigma(i)}\\
&= c'_{k_1k_2} e^{\frac{(k_1+k_2)}{2}a^2} a^{k_1k_2},
\end{aligned}
$$

where the α_{ij} are coefficients, some of which vanish. Indeed, each of the blocks in the matrix above is upper-triangular. To evaluate $c'_{k_1k_2}$, observe, upon completing the squares in the exponentials and setting $x_j \mapsto x_j -$

$a,\ y_j \mapsto y_j + a$ in the integral,

$$\begin{aligned}
&\tau_{k_1k_2}(t,s,u;\beta;\mathbb{R})\Big|_{t=s=u=\beta=0} \\
&= \frac{1}{k_1!k_2!}\int_{\mathbb{R}^{k_1+k_2}} \Delta_{k_1+k_2}(x,y)\left(\Delta_{k_1}(x)\prod_{j=1}^{k_1} e^{-\frac{x_j^2}{2}+ax_j}dx_j\right) \\
&\qquad \left(\Delta_{k_2}(y)\prod_{j=1}^{k_2} e^{-\frac{y_j^2}{2}-ay_j}dy_j\right).
\end{aligned}$$

This integral equals

$$\begin{aligned}
&= e^{(k_1+k_2)a^2/2}\left((-2a)^{k_1k_2}c_{k_1,0}c_{0,k_2} + \text{lower order terms in } a\right) \\
&= e^{(k_1+k_2)a^2/2}\left((-2a)^{k_1k_2}(2\pi)^{\frac{k_1+k_2}{2}}\prod_0^{k_1-1} j!\prod_0^{k_2-1} j! + \text{lower order terms in } a\right)
\end{aligned}$$

The result in the first part of this proof implies the absence of the lower terms and thus Lemma 6.1. ■

7 Appendix 2

In order to compute the asymptotics (5.11) for the expression G^ε and $\mathcal{B}_{-1}G^\varepsilon$, as defined in (0.6), one needs the following asymptotics:

$$\begin{aligned}
F^\varepsilon &= -\frac{4}{z^4} - \frac{1}{4z^2}\mathcal{B}_{-1}^2 Q_z + \frac{\varepsilon}{4z}\mathcal{B}_{-1}\frac{\partial Q_z}{\partial s} + O(z) \\
\frac{1}{\sqrt{2}}\tilde{\mathcal{B}}_{-1}F^\varepsilon &= -\frac{1}{16z^3}\mathcal{B}_{-1}^3 Q_z + \frac{\varepsilon}{16z^2}\mathcal{B}_{-1}^2\frac{\partial Q_z}{\partial s} - \frac{\varepsilon s}{8}\mathcal{B}_{-1}^2\frac{\partial Q_z}{\partial s} + O(z) \\
\tilde{\mathcal{B}}_{-1}^2 F^\varepsilon &= -\frac{1}{32z^4}\mathcal{B}_{-1}^4 Q_z + \frac{\varepsilon}{32z^3}\mathcal{B}_{-1}^3\frac{\partial Q_z}{\partial s} - \frac{\varepsilon s}{16z}\mathcal{B}_{-1}^3\frac{\partial Q_z}{\partial s} + O(1) \\
\frac{1}{\sqrt{2}}\frac{\partial}{\partial a}F^\varepsilon &= -\frac{\varepsilon}{16z^2}\mathcal{B}_{-1}^2\frac{\partial Q_z}{\partial s} + O(\frac{1}{z}) \\
\frac{\partial}{\partial a}\tilde{\mathcal{B}}_{-1}F^\varepsilon &= -\frac{\varepsilon}{32z^3}\mathcal{B}_{-1}^3\frac{\partial Q_z}{\partial s} + O(\frac{1}{z^2}) \\
\frac{1}{\sqrt{2}}H_1^\varepsilon &= \frac{6\varepsilon}{z^6} + \frac{4\varepsilon s}{z^4} - \frac{1}{8z^3}\mathcal{B}_{-1}\frac{\partial Q_z}{\partial s} + O(\frac{1}{z^2})
\end{aligned}$$

$$
\begin{aligned}
\tilde{\mathcal{B}}_{-1}H_1^{\varepsilon} &= -\frac{1}{16z^4}\mathcal{B}_{-1}^2\frac{\partial Q_z}{\partial s} - \frac{\varepsilon}{16z^3}\mathcal{B}_{-1}\frac{\partial^2 Q_z}{\partial s^2} + \frac{1}{8z^2}\mathcal{B}_0\mathcal{B}_{-1}^2 Q_z + O(\frac{1}{z}) \\
\frac{1}{\sqrt{2}}\tilde{\mathcal{B}}_{-1}^2 H_1^{\varepsilon} &= -\frac{1}{64z^5}\mathcal{B}_{-1}^3\frac{\partial Q_z}{\partial s} - \frac{\varepsilon}{64z^4}\mathcal{B}_{-1}^2\frac{\partial^2 Q_z}{\partial s^2} + \frac{1}{32z^3}(\mathcal{B}_0+1)\mathcal{B}_{-1}^3 Q_z + O(\frac{1}{z^2}) \\
\frac{1}{\sqrt{2}}H_2^{\varepsilon} &= \frac{\varepsilon}{4z^4}\mathcal{B}_{-1}^2 Q_z + O(\frac{1}{z^3}) \\
\frac{\partial}{\partial a}H_2^{\varepsilon} &= \frac{1}{8z^4}\mathcal{B}_{-1}^2\frac{\partial Q_z}{\partial s} - \frac{1}{16z^3}\mathcal{B}_{-1}\frac{\partial^2 Q_z}{\partial s^2} - \frac{1}{16z^2}(\frac{\partial^3}{\partial s^3} - 4\mathcal{B}_{-1}^2)Q_z + O(\frac{1}{z}) \\
\tilde{\mathcal{B}}_{-1}H_2^{\varepsilon} &= \frac{\varepsilon}{8z^5}\mathcal{B}_{-1}^3 Q_z + O(\frac{1}{z^4}) \\
\frac{1}{\sqrt{2}}\frac{\partial}{\partial a}\tilde{\mathcal{B}}_{-1}H_2^{\varepsilon} &= \frac{1}{32z^5}\mathcal{B}_{-1}^3\frac{\partial Q_z}{\partial s} - \frac{\varepsilon}{64z^4}\mathcal{B}_{-1}^2\frac{\partial^2 Q_z}{\partial s^2} - \frac{1}{64z^3}(\frac{\partial^3}{\partial s^3} - 4\mathcal{B}_{-1}^2)\mathcal{B}_{-1}Q_z + O(\frac{1}{z^2}).
\end{aligned}
$$

References

[1] M. Adler, T. Shiota and P. van Moerbeke: *Random matrices, vertex operators and the Virasoro algebra*, Phys. Lett. **A 208**, 67-78, (1995).

[2] M. Adler and P. van Moerbeke: *The spectrum of coupled random matrices*, Annals of Math., **149**, 921–976 (1999).

[3] M. Adler and P. van Moerbeke: *Hermitian, symmetric and symplectic random ensembles: PDE's for the distribution of the spectrum*, Annals of Math., **153**, 149–189 (2001).

[4] M. Adler and P. van Moerbeke: *PDE's for the joint distributions of the Dyson, Airy and Sine processes*, The Annals of Probability, **33**, 1326-1361 (2005). (arXiv:math.PR/0302329)

[5] M. Adler and P. van Moerbeke: to appear.

[6] J. Ambjorn and Yu. Makeenko: *Properties of loop equations for the Hermitian matrix model and for two-dimensional quantum gravity*, Modern Physics letters **A 5** 1753–1763 (1990)

[7] A. Aptekarev, P. Bleher and A. Kuijlaars: *Large n limit of Gaussian random matrices with external source, Part II*, arXiv:math-ph/0408041.

[8] A. Aptekarev, A. Branquinho and W. Van Assche: *Multiple orthogonal polynomials for classical weights*, Trans. Amer. Math. Soc., **355**, 3887–3914 (2003)

[9] P. Bleher and A. Kuijlaars: *Random matrices with external source and multiple orthogonal polynomials process*, Internat. Math. Research Notices **3**, 109–129 (2004) (arXiv:math-ph/0307055).

[10] P. Bleher and A. Kuijlaars: *Large n limit of Gaussian random matrices with external source, Part I*, Comm. Math. Phys., **252**, 43–76 (2004).

[11] P. Bleher and A. Kuijlaars: *Large n limit of Gaussian random matrices with external source, Part II*, arXiv: math-ph/0408041.

[12] E. Brézin and S. Hikami: *Correlations of nearby levels induced by a random potential*, Nuclear Physics **B 479**, 697–706 (1996).

[13] E. Brézin and S. Hikami: *Extension of level spacing universality*, Phys. Rev., **E 56**, 264–269 (1997).

[14] E. Brézin and S. Hikami: *Universal singularity at the closure of a gap in a random matrix theory*, Phys. Rev., **E 57**, 4140–4149 (1998).

[15] E. Brézin and S. Hikami: *Level spacing of random matrices in an external source*, Phys. Rev., **E 58**, 7176–7185 (1998).

[16] F.J. Dyson: *A Brownian-Motion Model for the Eigenvalues of a Random Matrix*, Journal of Math. Phys. **3**, 1191–1198 (1962)

[17] D. J. Gross and M.J. Newman: *Phys. Letters*, **B 266** 291–299 (1991)

[18] A. Okounkov and N. Reshitikhin: *Random skew plane partitions and the Pearcey process*, math.CO/0503508 (2005).

[19] V.A. Kazakov: Nucl. Phys. B (Proc. Suppl.)**4**, 93 (1988).

[20] S. Karlin and J. McGregor: *Coincidence probabilities*, Pacific J. Math. **9**, 1141–1164 (1959).

[21] L.A. Pastur: *The spectrum of random matrices (Russian)*, Teoret. Mat. Fiz. **10**, 102–112 (1972).

[22] T. Pearcey: *The structure of an electromagnetic field in the neighborhood of a cusp of a caustic*, Philos. Mag. **37**, 311–317 (1946).

[23] C. Tracy and H. Widom: *The Pearcey Process*, arXiv:math. PR /0412005.

[24] P. Zinn-Justin: *Random Hermitian matrices in an external field*, Nuclear Physics **B 497**, 725–732 (1997).

[25] P. Zinn-Justin: *Universality of correlation functions in Hermitian random matrices in an external field*, Comm. Math. Phys. **194**, 631–650 (1998).