TODA VERSUS PFAFF LATTICE AND RELATED POLYNOMIALS

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Abstract
We study the Pfaff lattice, introduced by us in the context of a Lie algebra splitting of $gl(\infty)$ into $sp(\infty)$ and lower-triangular matrices. We establish a set of bilinear identities, which we show to be equivalent to the Pfaff Lattice. In the semi-infinite case, the tau-functions are Pfaffians; interesting examples are the matrix integrals over symmetric matrices (symmetric matrix integrals) and matrix integrals over self-dual quaternionic Hermitian matrices (symplectic matrix integrals).

There is a striking parallel of the Pfaff lattice with the Toda lattice, and more so, there is a map from one to the other. In particular, we exhibit two maps, dual to each other;
(i) from the the Hermitian matrix integrals to the symmetric matrix integrals, and
(ii) from the Hermitian matrix integrals to the symplectic matrix integrals.

The map is given by the skew-Borel decomposition of a skew-symmetric operator.

We give explicit examples for the classical weights.

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Consider a weight on $\mathbb{R}$, depending on $t = (t_1, t_2, \ldots) \in \mathbb{C}^\infty$, 
\[ \rho_t(z) \, dz = e^{\sum_{i=1}^\infty t_i z^i} \rho(z) \, dz = e^{-V(z) + \sum_{i=1}^\infty t_i z^i} \, dz, \]
with rational $g$ and $f$'s, and with $\rho(z)$ decaying fast enough at $\infty$.

The Toda lattice, its $\tau$-functions and Hermitian matrix integrals (revisited)

This weight leads to a $t$-dependent moment matrix
\[ m_n(t) = (\mu_{k+\ell}(t))_{0 \leq k, \ell \leq n-1} = \left( \int_{\mathbb{R}} z^{k+\ell} \rho_t(z) \, dz \right)_{0 \leq k, \ell \leq n-1}, \]
with the semi-infinite moment matrix $m_\infty$, satisfying the commuting equations
\[ \frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty = m_\infty \Lambda^k. \]

$\Lambda$ is the customary shift matrix, with zeroes everywhere, except for 1's just above the diagonal, that is, $(\Lambda v)_n = v_{n+1}$. Consider the Borel decomposition into a lower- and an upper-triangular matrix
\[ m_\infty = S^{-1} S^\top, \]
and the following $t$-dependent matrix integrals ($n \geq 0$):
\[ \tau_n(t) := \int_{\mathcal{H}_n^t} e^{\operatorname{Tr}(-V(X) + \sum_{i} t_i X_i)} \, dX = \det m_n \quad \text{and} \quad \tau_0 = 1, \]
where $dX$ is Haar measure on the ensemble $\mathcal{H}_n^t = \{ (n \times n) - \text{Hermitian matrices} \}$. As is well known (e.g., see E. Witten [18] or M. Adler and P. van Moerbeke [6]), integral (0.4) is a solution to the following two systems.
(i) The KP-hierarchy

\[ (s_{k+4}(\tilde{\alpha}) - \frac{1}{2} \frac{\partial^2}{\partial t_1 \partial t_{k+3}}) \tau_n \circ \tau_n = 0, \quad \text{for } k, n = 0, 1, 2, \ldots \]  

(0.5)

(ii) The Toda lattice, that is, the tridiagonal matrix

\[ L(t) := S \Lambda S^{-1} = \begin{pmatrix} \frac{\partial}{\partial t_1} \log \frac{\tau_1}{\tau_0} & \frac{1}{2} \frac{\tau_0}{\tau_1} & \frac{\tau_1}{\tau_2} & 0 \\ \frac{1}{2} \frac{\tau_0}{\tau_1} & \frac{\partial}{\partial t_1} \log \frac{\tau_1}{\tau_2} & \frac{1}{2} \frac{\tau_1}{\tau_2} & \frac{\tau_2}{\tau_3} \\ 0 & \frac{1}{2} \frac{\tau_1}{\tau_2} & \frac{\partial}{\partial t_1} \log \frac{\tau_2}{\tau_3} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \]  

(0.6)

satisfies the following commuting Toda equations

\[ \frac{\partial L}{\partial t_a} = \left[ \frac{1}{2} (L^n)_{sk}, L \right], \]

where \( (A)_{sk} \) denotes the skew-part of the matrix \( A \) for the Lie algebra splitting into skew and lower-triangular matrices. Moreover, the following \( t \)-dependent polynomials in \( z \), are defined by the \( S \)-matrix obtained from the Borel decomposition (0.3); it is also given, on the one hand, in terms of the functions \( \tau_n(t) \), and, on the other hand, by a classic determinantal formula (for \( a \in \mathbb{C}, \text{define } [a] := (a, a^2/2, a^3/3, \ldots \in \mathbb{C}^\infty) \)

\[ p_n(t; z) := \sum_{i=0}^{n} S_{ni}(t) z^i = z^n \frac{\tau_n(t - [z^{-1}])}{\sqrt{\tau_n \tau_{n+1}}} \]

\[ = \frac{1}{\sqrt{\tau_n \tau_{n+1}}} \text{det} \begin{pmatrix} m_n(t) & z^n \\ \mu_n,0(t) & \mu_n,1(t) \\ \vdots & \vdots \\ \mu_n,n(t) & \mu_n,n-1(t) \end{pmatrix} \]

The \( p_n(t; z) \)'s are orthonormal with respect to the (symmetric) inner-product \( (\cdot, \cdot)_S \), defined by \( (z^i, z^j)_S = \mu_{ij} \), which is a restatement of the Borel decomposition (0.3) (see [6]). The vector \( p(t; z) = (p_n(t; z))_{n \geq 0} \) is an eigenvector of the matrix \( L(t) \) in (0.6):

\[ L(t) p(t; z) = z p(t; z). \]

*The \( s_i \)'s are the elementary Schur polynomials \( e^{\sum_{i \geq 0} t_i \bar{z}^i} := \sum_{i \geq 0} s_i(t) z^i \) and \( s_i(\tilde{\alpha}) := s_i(\bar{z}/\partial t_1, (1/2)(\partial/\partial t_2), \ldots) \). Given a polynomial \( p(t_1, t_2, \ldots) \), define the customary Hirota symbol

\[ p(\tilde{\alpha}) f \circ g := \left. \left. p(\bar{z}/\partial y_1, \partial/\partial y_2, \ldots) f(t + y) g(t - y) \right|_{y=0} \right. \]
The Pfaff lattice and its \( \tau \)-functions

For use throughout this paper, define the skew-symmetric matrix

\[
J = \begin{pmatrix}
\ddots & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1 \\
\ddots
\end{pmatrix}, \quad \text{with } J^2 = -I, \quad (0.7)
\]

and the involution on the space \( \mathcal{D} := \text{gl}_\infty \) of infinite matrices,

\[
\mathcal{J} : \mathcal{D} \rightarrow \mathcal{D} : a \mapsto \mathcal{J}(a) := J a^\top J. \quad (0.8)
\]

Also, consider the splitting of \( \mathcal{D} = \mathfrak{k} + \mathfrak{n} \) into two Lie subalgebras \( \mathfrak{k} \) and \( \mathfrak{n} \), with the corresponding projections denoted \( \pi_\mathfrak{k} \) and \( \pi_\mathfrak{n} \), where \( \mathfrak{k} \) is the Lie algebra of lower-triangular matrices with some special feature (see (1.17)) and where

\[
\mathfrak{n} := \{ a \in \mathcal{D} \text{ such that } J a^\top J = a \} = \text{sp}(\infty).
\]

Given a skew-symmetric semi-infinite matrix \( m_\infty \), consider the commuting differential equations

\[
\frac{\partial m_\infty}{\partial t_i} = \Lambda^i m_\infty + m_\infty \Lambda^i; \quad (0.9)
\]

they maintain the skew-symmetry of \( m_\infty \). The Borel decomposition of \( m_\infty \) into lower-triangular times upper-triangular matrices requires the insertion of the skew-symmetric matrix \( J \):

\[
m_\infty(t) = Q^{-1}(t) J Q^{-1\top}(t). \quad (0.10)
\]

Dressing up the shift \( \Lambda \) with the lower-triangular matrix \( Q(t) \) leads to the commuting equations (0.11) below.

**Theorem 0.1**

*The Pfaff lattice equations*

\[
\frac{\partial L}{\partial t_i} = [\pi_\mathfrak{k} L^i, L] = [\pi_\mathfrak{n} L^i, L] \quad (0.11)
\]
maintain the locus of semi-infinite matrices of the form \((a_i \neq 0)\):\[
L = \begin{bmatrix}
0 & 1 & \cdots & 0 \\
-d_i & a_1 & \cdots & 0 \\
& \ddots & \ddots & \vdots \\
& & \ddots & -d_2 & a_2 \\
& & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots
\end{bmatrix},
\]
(0.12)

The solutions \(L\) to (0.11) of the form (0.12) are given by\[
L(t) = Q(t) \Lambda Q^{-1}(t),
\]
where \(Q\) is a lower-triangular matrix, whose entries are given by the coefficients of the polynomials, obtained by the finite Taylor expansion in \(z^{-1}\) of \(\tau_{2n}(t - [z^{-1}])\) below \((h_{2n} := \tau_{2n+2}(t)/\tau_{2n}(t)):\]
\[
q_{2n}(t; z) := \sum_{j=0}^{2n} Q_{2n,j}(t) z^j = z^{2n} h_{2n}^{-1/2} \tau_{2n}(t - [z^{-1}]) \frac{\tau_{2n}(t)}{\tau_{2n}(t)},
\]
\[
q_{2n+1}(t; z) := \sum_{j=0}^{2n+1} Q_{2n+1,j}(t) z^j = z^{2n+1} h_{2n}^{-1/2} (z + \partial/\partial t_1) \tau_{2n}(t - [z^{-1}]) \frac{\tau_{2n}(t)}{\tau_{2n}(t)},
\]
(0.13)

with \(\tau_0, \tau_2, \tau_4\), a sequence of functions of \(t_1, t_2, \ldots\), characterized by the following bilinear identities for all \(n, m \geq 0\), \((t_0 = 1)\),\[
\oint_{z = \infty} \tau_{2n}(t - [z^{-1}]) \tau_{2m+2}(t' + [z^{-1}]) e^{\sum_{k=0}^{\infty} (a_i-t_i) e^{t_i} z^{2n-2m-2}} \frac{dz}{2\pi i} \\
+ \oint_{z = 0} \tau_{2n+2}(t + [z]) \tau_{2m}(t' - [z]) e^{\sum_{k=0}^{\infty} (b_j-t_j) e^{-t_j} z^{2n-2m}} \frac{dz}{2\pi i} = 0.
\]
(0.14)

Remark
Theorem 0.1 is robust and remains valid for the bi-infinite matrix \(L\). In that case, the summations in the expressions \(q_{2n}\) and \(q_{2n+1}\) run from \(j = -\infty\), instead of running from \(j = 0\).

The \(\tau\)-functions are given by Pfaffians pf \(m_{2n}(t)\) and satisfy, as a consequence of the bilinear relations (0.14), the Pfaffian KP-hierarchy for \(k, n = 0, 1, 2, \ldots\),\[
\left( s_{k+4}(\tilde{\partial}) - \frac{1}{2} \partial^2 \partial_{t_1} \partial_{t_{k+3}} \right) \tau_{2n} \circ \tau_{2n} = s_{k}(\tilde{\partial}) \tau_{2n+2} \circ \tau_{2n-2}.
\]
(0.15)
The $t$-dependent polynomials $q_n(t; z) = (Q(t) (1, z, z^2, z^3, \ldots) \top)_n$ in $z$, obtained in (0.13) are “skew-orthonormal” with respect to the skew inner-product $\langle \cdot, \cdot \rangle_{sk}$, defined by $\langle y^i, z^j \rangle_{sk} = \mu_{ij}(t)$, namely,

$$\left( \langle q_i, q_j \rangle_{sk} \right)_{0 \leq i, j < \infty} = J,$$

and are eigenvectors for the matrix $L$:

$$L(t)q(t; z) = zq(t; z). \quad (0.16)$$

Explicit representations of $L$, in terms of the $\tau_{2n}$'s, are as follows:

$$L = Q \Lambda Q^{-1} = h^{-1/2} \begin{pmatrix} \hat{L}_{00} & \hat{L}_{01} & 0 & 0 \\ \hat{L}_{10} & \hat{L}_{11} & \hat{L}_{12} & 0 \\ \ast & \hat{L}_{21} & \hat{L}_{22} & \hat{L}_{23} \\ \ast & \ast & \hat{L}_{32} & \hat{L}_{33} \end{pmatrix} h^{1/2},$$

with the entries $\hat{L}_{ij}$ and the entries of $h$, being $(2 \times 2)$-matrices

$$h = \text{diag}(h_0 I_2, h_2 I_2, h_4 I_2, \ldots), \quad h_{2n} = \tau_{2n+2}/\tau_{2n},$$

and $\tau = \partial/\partial t_1$,

$$\hat{L}_{nn} := \begin{pmatrix} -(\log \tau_{2n}) & 1 \\ -\frac{s_2(\hat{\alpha})\tau_{2n}}{\tau_{2n}} - \frac{s_1(\hat{\alpha})\tau_{2n+2}}{\tau_{2n+2}} & (\log \tau_{2n+2}) \end{pmatrix}, \quad \hat{L}_{n,n+1} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\hat{L}_{n+1,n} := \begin{pmatrix} \ast & (\log \tau_{2n+2})^{-} \\ \ast & \ast \end{pmatrix}. \quad (0.17)$$
Also, the \( t \)-dependent polynomials \( q_n(t, z) \) in \( z \) have Pfaffian expressions, in contrast with the determinantal expressions in the Hermitian case, mentioned earlier:

\[
q_{2n}(t; z) = \frac{1}{\sqrt{\tau_{2n} \tau_{2n+2}}} \text{pf}
\begin{pmatrix}
1 & z & \cdots & z^{2n} \\
\vdots & \ddots & \vdots & \vdots \\
-1 & \cdots & -z^{2n} & 0
\end{pmatrix}
\]

\[
q_{2n+1}(t; z) = \frac{1}{\sqrt{\tau_{2n} \tau_{2n+2}}} \text{pf}
\begin{pmatrix}
1 & \mu_{0, 2n+1} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
-\mu_{0, 2n+1} & \cdots & -\mu_{1, 2n+1} & -z^{2n+1} \\
-1 & \cdots & -z & 0
\end{pmatrix}
\]

(0.18)

Theorem 0.1 and the subsequent statements are established in Sections 2 and 3. We show how a general skew-symmetric infinite matrix flowing according to (0.11) and its skew-Borel decomposition (0.10), lead to wave vectors \( \Phi_1 \), satisfying bilinear relations and differential equations. Section 3 deals with the existence, in the general setting, of a so-called Pfaffian \( \tau \)-function, satisfying bilinear equations and the so-called Pfaff-KP hierarchy. In [4], these results were obtained by embedding the system in 2-Toda theory, while in this paper, they are obtained in an intrinsic fashion.

For \( k = 0 \), the Pfaff-KP equation (0.15) has already appeared in the context of the charged BKP hierarchy, studied by V. Kac and J. van de Leur [13]; the precise relationship between the charged BKP hierarchy of Kac and van de Leur and the Pfaff Lattice, introduced here, deserves further investigation. (See the recent paper by van de Leur [16]).
Examples: Symmetric and "symplectic" matrix integrals
An important example is given by the skew-symmetric matrix \( m_{\infty} = (\mu_{ij})_{i,j \geq 0} \) of moments* defined by (\( \alpha = \pm 1 \)) (see [4])

\[
(y^i, z^j)_{sk} := \int_{\mathbb{R}^2} y^i \overline{z}^j e^{\sum_{t} \iota (y^t + z^t)} 2D^t \delta(y-z) \rho(y) \rho(z) \, dy \, dz \\
\mu_{ij}^{(1)}(t) = \int_{\mathbb{R}^2} y^i \overline{z}^j e^{\sum_{t} \iota (y^t + z^t)} \varepsilon(y-z) \rho(y) \rho(z) \, dy \, dz, \\
\mu_{ij}^{(2)}(t) = \int_{\mathbb{R}^2} \{y^i, z^j\}(y) e^{\sum_{t} 2i \iota y^t} \tilde{\rho}(y)^2 \, dy,
\]
for \( \alpha = -1 \),

\[
\mu_{ij}^{(2)}(t) = \int_{\mathbb{R}^2} \{y^i, z^j\}(y) e^{\sum_{t} 2i \iota y^t} \tilde{\rho}(y)^2 \, dy,
\]
for \( \alpha = +1 \).

The associated moment matrices \( m_{2n}^{(1)} \) and \( m_{2n}^{(2)} \) satisfy the differential equations (0.9) and lead to "symmetric" matrix integrals

\[
\tau_{2n}^{(1)}(t) := \frac{1}{(2n)!} \int_{\mathcal{S}_{2n}} e^{\text{Tr}(-V(X)+\sum_{t} \iota t X^t)} \, dX = \text{pf}(m_{2n}^{(1)}),
\]
and "symplectic" matrix integrals

\[
\tau_{2n}^{(1)}(t) := \frac{1}{n!} \int_{\mathcal{S}_{2n}} e^{2\text{Tr}(-V(X)+\sum_{t} \iota t X^t)} \, dX = \text{pf}(m_{2n}^{(2)}),
\]
both expressed in terms of the Pfaffian of the upper-left-hand principal minors of the "moment" matrix \( m_{2n}^{(l)} \), where

1. for \( i = 1 \), \( dX \) denotes Haar measure on the space \( \mathcal{S}_{2n} \) of symmetric matrices, and,
2. for \( i = 2 \), \( dX \) denotes Haar measure on the \( (2n \times 2n) \)-matrix realization \( \mathcal{S}_{2n} \) of the space of self-dual \( (n \times n) \)-Hermitian matrices, with quaternionic entries.

A remarkable map from Toda to Pfaff lattice
Remembering the notation (0.1), we act with the \( z \)-operator,

\[
\mathbf{n}_t := \sqrt{\frac{f}{\rho_t}} d_\overline{z} \sqrt{\frac{f}{\rho_t}} = e^{-(1/2) \sum u \overline{z}^2} \left( \frac{d}{dz} f(z) - \frac{f' + g}{2} (z) \right) e^{1/2 \sum u \overline{z}^2}
\]
on the \( t \)-dependent orthonormal polynomials \( p_n(t,z) \) in \( z \); in [6], we showed that the matrix \( \mathcal{N} \) defined by

\[
\mathbf{n}_t p(t,z) = \left( f(L) M - \frac{f' + g}{2} (L) \right) p(t,z) =: \mathcal{N} p(t,z)
\]

*We have \( \varepsilon(x) = 1 \), for \( x \geq 0 \), \( \varepsilon(x) = -1 \), for \( x < 0 \), and \( \{f, g\} = f' g - fg' \).
is skew-symmetric. The \( t \)-dependent matrix \( \mathcal{N} \) is expressed in terms of \( L \) and a new matrix \( M \), defined by
\[
z p = L p \quad \text{and} \quad e^{-(1/2) \sum \kappa \frac{d}{d z}} e^{(1/2) \sum \kappa z^k} p = M p. \tag{0.22}
\]
Consider now the skew-Borel decomposition of \( \mathcal{N}(2t) \) and its inverse \( \mathcal{N}(2t)^{-1} \), in terms of lower-triangular matrices \( O(\pm)(t) \) and \( O(\pm)^{-1}(t) \), respectively:
\[
\mathcal{N}(2t)^{\mp 1} = -O(\pm_1)^{-1}(t) J O(\pm)^{-1}(t). \tag{0.23}
\]
Then, the lower-triangular matrices \( O(\pm)(t) \) map orthonormal into skew-orthonormal polynomials, and the tridiagonal \( L \)-matrix into an \( \tilde{L} \)-matrix:
\[
p_n(t; z) \mapsto q_n(\pm)(t; z) = (O(\pm)(t)p(t; z))^n, \quad L(t) \mapsto \tilde{L}(t) = O(\pm)(t)L(2t)O(\pm)^{-1}(t) \quad (0.24)
\]
(Toda lattice) (Pfaff lattice).

It also maps the weight into a new weight
\[
\rho(z) = e^{-V(z)} \mapsto \tilde{\rho}(\pm)(z) = e^{-\tilde{V}(z)} := e^{-(1/2)(V(z) \mp \log f(z))},
\]
and the corresponding string of \( \tau \)-functions into a new string of Pfaffian \( \tau \)-functions (remember \( V_\beta(z) = V(z) - \sum_{i=\beta}^{\infty} \kappa_i z^i \)):
\[
\tau_k(t) = \int_{\mathbb{H}} e^{\text{Tr}(V(t; X))} dX \mapsto \left\{ \begin{array}{ll}
\tau^{(+)}_{2\beta}(t) := \int_{\mathbb{H}_{2\beta}} e^{\text{Tr}(\tilde{V}(t; X))} dX \quad (\beta = 4), \\
\tau^{(-)}_{2\beta}(t) := \int_{\mathbb{H}_{2\beta}} e^{\text{Tr}(\tilde{V}(t; X))} dX \quad (\beta = 1).
\end{array} \right.
\]
For the classical orthogonal polynomials \( p_n(z) \), we have shown in [6] that \( \mathcal{N}(0) \) is not only skew-symmetric but also tridiagonal; that is,
\[
L = \begin{bmatrix}
b_0 & a_0 & a_1 & \cdots \\
a_0 & b_1 & a_1 & \cdots \\
a_1 & b_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots
\end{bmatrix}, \quad -\mathcal{N} = \begin{bmatrix}
0 & c_0 & \cdots \\
c_0 & 0 & c_1 & \cdots \\
c_1 & 0 & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots
\end{bmatrix}. \tag{0.25}
\]
\[\text{See Appendix B.}\]
\[\text{The upper-signs (resp., lower-signs) correspond to each other throughout this section.}\]
\[\text{We have } p(t; z) := (p_0(t; z), p_1(t; z), \ldots)^\top.\]
In Sections 6 and 7, we show that the maps \( O(\ldots) \) and \( O(\ldots) \), as in (0.24), only involve three steps, in the following sense:

\[
q_{2n}(-)(0; z) = \sqrt{\frac{c_{2n}}{a_{2n}}} p_{2n}(0; z),
\]

\[
q_{2n+1}(-)(0; z) = \sqrt{\frac{a_{2n}}{c_{2n}}} \times \left( -c_{2n-1} p_{2n-1}(0, z) + \frac{c_{2n}}{a_{2n}} \left( \sum_{0}^{2n} b_{i} \right) p_{2n}(0; z) + c_{2n} p_{2n+1}(0; z) \right)
\]

\[ (\beta = 1), \]  

\[
p_{2n}(0; z) = -c_{2n-1} \sqrt{\frac{a_{2n-2}}{c_{2n}}} q_{2n-2}^{(+)}(0; z) + \sqrt{a_{2n} c_{2n}} q_{2n}^{(+)}(0; z)
\]

\[
p_{2n+1}(0; z) = -c_{2n} \sqrt{\frac{a_{2n-2}}{c_{2n}}} q_{2n-2}^{(+)}(0; z),
\]

\[
- \left( \sum_{0}^{2n} b_{i} \right) \sqrt{\frac{c_{2n}}{a_{2n}}} q_{2n}^{(+)}(0; z) + \sqrt{\frac{c_{2n}}{a_{2n}}} q_{2n+1}^{(+)}(0; z) \quad (\beta = 4). \]  

The abstract map \( O(\ldots) \) for \( t = 0 \) appears already in the work of E. Brézin and H. Neuberger [9]. This has been applied recently by [1] to a problem in the theory of random matrices.

1. Splitting theorems, as applied to the Toda and Pfaff lattices

In this section, we show how each of the equations

\[
\frac{\partial m_{\infty}}{\partial t_{i}} = \Lambda^{i} m_{\infty} \quad \text{and} \quad \frac{\partial m_{\infty}}{\partial t_{i}} = \Lambda^{i} m_{\infty} + m_{\infty} \Lambda^{T_{i}} \tag{1.1}
\]

lead to commuting Hamiltonian vector fields related to a Lie algebra splitting. First recall the splitting theorem due to Adler, B. Kostant, and W. Symes in [5], and later recall the R-version due to A. Reyman and M. Semenov-Tian-Shansky [15]. The R-version allows for more general initial conditions.

PROPOSITION 1.1

Let \( g = k + n \) be a (vector space) direct sum of a Lie algebra \( g \) in terms of Lie subalgebras \( k \) and \( n \), with \( g \) paired with itself via a nondegenerate ad-invariant inner product* \( \langle \cdot, \cdot \rangle \); this in turn induces a decomposition \( g = k^{\perp} + n^{\perp} \) and isomorphisms \( g \simeq g^{*}, k^{\perp} \simeq n^{*}, n^{\perp} \simeq k^{*} \). Let \( \pi_{k} \) and \( \pi_{n} \) be projections onto \( k \) and \( n \), respectively.

* \( \langle \text{Ad}_{X} Y, Z \rangle = \langle X, \text{Ad}_{X^{-1}} Z \rangle, g \in G \), and thus \( \langle [z, x], y \rangle = \langle x, [z, y] \rangle \).
Let $\mathcal{G}$, $\mathcal{G}_k$, and $\mathcal{G}_n$ be the groups associated with the Lie algebras $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{n}$. Let $\mathcal{S}(\mathfrak{g})$ be the $\text{Ad}^* \simeq \text{Ad}$-invariant functions on $\mathfrak{g}^* \simeq \mathfrak{g}$.

(i) Then, given an element $\varepsilon \in \mathfrak{g} : [\varepsilon, \mathfrak{k}] \subset \mathfrak{k}^\perp$ and $[\varepsilon, \mathfrak{n}] \subset \mathfrak{n}^\perp$, the functions

\[ \varphi(\varepsilon + \xi')|_{\mathfrak{k}^\perp}, \quad \text{with} \ \varphi \in \mathcal{S}(\mathfrak{g}) \text{ and } \xi' \in \mathfrak{k}^\perp, \]

respectively, Poisson commute for the respective Kostant-Kirillov symplectic structures of $\mathfrak{n}^* \simeq \mathfrak{k}^\perp$; the associated Hamiltonian flows are expressed in terms of the Lax pairs $\dot{\xi} = [-\pi_k \nabla \varphi(\xi), \xi] = [\pi_n \nabla \varphi(\xi), \xi]$, for $\xi \equiv \varepsilon + \xi'$, $\xi' \in \mathfrak{k}^\perp$. \hfill (1.3)

(ii) The splitting also leads to a second Lie algebra $\mathfrak{g}_R$, derived from $\mathfrak{g}$, such that $\mathfrak{g}^*_R \simeq \mathfrak{g}_R$, namely,

\[ \mathfrak{g}_R : [x, y]_R = \frac{1}{2} [Rx, y] + \frac{1}{2} [x, Ry] = [\pi_k x, \pi_k y] - [\pi_n x, \pi_n y], \]

with $R = \pi_k - \pi_n$. The functions

\[ \varphi(\xi)|_{\mathfrak{g}_R}, \quad \text{with} \ \varphi \in \mathcal{S}(\mathfrak{g}) \text{ and } \xi \in \mathfrak{g}_R, \]

respectively, Poisson commute for the respective Kostant-Kirillov symplectic structures of $\mathfrak{g}^*_R \simeq \mathfrak{g}_R$, with the same associated (Hamiltonian) Lax pairs $\dot{\xi} = [-\pi_k \nabla \varphi(\xi), \xi] = [\pi_n \nabla \varphi(\xi), \xi]$, for $\xi \in \mathfrak{g}_R$. \hfill (1.5)

Each of the equations (1.3) and (1.5) has the same solution expressible in two different ways,$^\dagger$

\[ \xi(t) = \text{Ad}_{K(t)} \xi_0 = \text{Ad}_{S^{-1}(t)} \xi_0, \]

with$^\ddagger$

\[ K(t) = \pi_{\mathfrak{g}_k} e^{\partial \varphi(\xi_0)} \quad \text{and} \quad S(t) = \pi_{\mathfrak{g}_n} e^{\partial \varphi(\xi_0)}. \]

Example 1 (The standard Toda lattice and the equations $\partial m / \partial t_i = \Lambda^i m$ for the Hänkel matrix $m_{\infty}$)

Since, in particular, the matrix $m_{\infty}$ is symmetric, the Borel decomposition into lower-times upper-triangular matrix must be done with the same lower-triangular matrix $S$:

\[ m_{\infty} = S^{-1} S^T S^{-1}. \]

$^\dagger \nabla \varphi$ is defined as the element in $\mathfrak{g}^*$ such that $d\varphi(\xi) = \langle \nabla \varphi, d\xi \rangle$, $\xi \in \mathfrak{g}$.

$^\ddagger$We naively write $\text{Ad}_{K(t)} \xi_0 = K(t) \xi_0 K(t)^{-1}$, $\text{Ad}_{S^{-1}(t)} \xi_0 = S^{-1}(t) \xi_0 S(t)$.
In turn, the matrix $S$ defines a wave vector $\Psi$, and operators $L$ and $M$, the same as the ones defined in (0.22),

$$
\Psi(t, z) := e^{(1/2) \sum t_i z_i S} \chi,

L := S \Lambda S^{-1},

M := S \left( \partial + \frac{1}{2} \sum_{i=1}^{\infty} i t_i \Lambda_i \right) S^{-1},

(1.8)
$$

satisfying the following well-known equations:†

$$
L \Psi = z \Psi, \\
M \Psi = \frac{\partial}{\partial z} \Psi, \text{ with } [L, M] = 1,

\frac{\partial S}{\partial t_n} = -\frac{1}{2} (L^n)_{bo} S, \\
\frac{\partial \Psi}{\partial t_n} = \frac{1}{2} (L^n)_{sk} \Psi,

\frac{\partial L}{\partial t_n} = \frac{1}{2} [(L^n)_{sk}, L], \\
\frac{\partial M}{\partial t_n} = \frac{1}{2} [(L^n)_{sk}, M].

(1.9)
$$

The wave vector $\Psi$ can then be expressed in terms of a sequence of $\tau$-functions $\tau_n(t) = \det m_n(t)$, but it also has a simple expression in terms of orthonormal polynomials, with respect to the moment matrix $m_\infty$:

$$
\Psi(t, z) = e^{(1/2) \sum t_i z_i} \chi_n(t - [z^{-1}]) \sqrt{\tau_n(t) \tau_{n+1}(t)} n \geq 0

= e^{(1/2) \sum t_i z_i} (p_n(t, z)) n \geq 0.

(1.10)
$$

The vector fields (1.9) on $L$ are commuting Hamiltonian vector fields, in view of the Adler-Kostant-Symes (AKS) splitting theorem (version (i)),

$$
\frac{\partial L}{\partial t_i} = -\pi_k \nabla \mathcal{H}_i, \\
L = [\pi_n \nabla \mathcal{H}_i, L], \\
\mathcal{H}_i = \frac{\operatorname{tr} L^{i+1}}{i+1}, \\
\nabla \mathcal{H}_i = L_i

(1.11)
$$

with

$$
L = \Lambda^\top a + b + a \Lambda, \\
a \text{ and } b \text{ diagonal matrices},

(1.12)
$$

for the splitting of the Lie algebra of semi-infinite matrices

$$
\mathcal{G} = \mathfrak{gl}_\infty = \mathfrak{k} + \mathfrak{n} := \{\text{skew-symmetric}\} + \{\text{lower-triangular}\}

= \mathfrak{k}^\perp + \mathfrak{n}^\perp := \{\text{symmetric}\} + \{\text{strictly upper-triangular}\},

(1.13)
$$

*In the formulas below $\chi(z) = (z^0, z, z^2, \ldots)\top$, and $\partial$ is the matrix such that $(d/dz)\chi(z) = \partial \chi(z)$.

†The notation $()_k$ and $()_{bo}$ refers to the skew-part and the lower-triangular (Borel) part, respectively, that is, projection onto $k$ and $n$, respectively.
with the form (1.12) of \( L \) being preserved in time. Note that the solution (1.6) to the differential equation (1.5) in the AKS theorem is nothing but the factorization of \( m_\infty \) followed by the dressing up of \( \Lambda_1 \).

**Example 2 (The Pfaff lattice and the equations)**

Throughout this paper the Lie algebra \( \mathcal{D} = \mathfrak{gl}_\infty \) of semi-infinite matrices is viewed as composed of \((2 \times 2)\)-blocks. It admits the natural decomposition into subalgebras:

\[
\mathcal{D} = \mathcal{D}_- \oplus \mathcal{D}_0 \oplus \mathcal{D}_+ = \mathcal{D}_- \oplus \mathcal{D}_-^0 \oplus \mathcal{D}_+^0 \oplus \mathcal{D}_+,
\]

(1.14)

where \( \mathcal{D}_0 \) has \((2 \times 2)\)-blocks along the diagonal with zeros everywhere else and where \( \mathcal{D}_+ \) (resp., \( \mathcal{D}_- \)) is the subalgebra of upper-triangular (resp., lower-triangular) matrices with \((2 \times 2)\)-zero matrices along \( \mathcal{D}_0 \) and zero below (resp., above). As pointed out in (1.14), \( \mathcal{D}_0 \) can further be decomposed into two Lie subalgebras:

\[
\mathcal{D}_-^0 = \{ \text{all} (2 \times 2)\text{-blocks} \in \mathcal{D}_0 \text{ are proportional to } \text{Id} \}, \\
\mathcal{D}_+^0 = \{ \text{all} (2 \times 2)\text{-blocks} \in \mathcal{D}_0 \text{ have trace zero} \}.
\]

(1.15)

Remember from (0.7) and (0.8) in the introduction, the matrix \( J \) and the associated Lie algebra involution \( \mathcal{J} \). The splitting into two Lie subalgebras

\[
\mathcal{D} = \mathfrak{k} + \mathfrak{n},
\]

(1.16)

with

\[
\mathfrak{k} = \mathcal{D}_- + \mathcal{D}_0^-
\]

and

\[
\mathfrak{n} = \{ a \in \mathcal{D}, \text{ such that } \mathcal{J} a = a \} = \{ b + \mathcal{J} b, \ b \in \mathcal{D} \} = \text{sp}(\infty),
\]

(1.17)

with corresponding Lie groups.\(^\dagger\) \( \mathcal{G}_k \) and \( \mathcal{G}_n = \text{Sp}(\infty) \), play a crucial role here. Let \( \pi_k \) and \( \pi_n \) be the projections onto \( \mathfrak{k} \) and \( \mathfrak{n} \). Notice that \( \mathfrak{n} = \text{sp}(\infty) \) and \( \mathfrak{g}_n = \text{Sp}(\infty) \).

\(^*\)Note \( \mathfrak{n} \) is the fixed point set of \( \mathcal{J} \).

\(^\dagger\)\( \mathcal{G}_k \) is the group of invertible elements in \( \mathfrak{k} \), that is, invertible lower-triangular matrices, with nonzero \((2 \times 2)\)-blocks proportional to \( \text{Id} \) along the diagonal.
stand for the infinite-rank affine symplectic algebra and group $\mathfrak{n}$ (e.g., see [12]). Any element $a \in \mathcal{D}$ decomposes uniquely into its projections onto $\mathfrak{k}$ and $\mathfrak{n}$, as follows:

$$a = \pi_{\mathfrak{k}} a + \pi_{\mathfrak{n}} a$$

$$= \left\{ (a_+ - \mathcal{J} a_0) + \frac{1}{2} (a_0 - \mathcal{J} a_0) \right\} + \left\{ (a_+ + \mathcal{J} a_0) + \frac{1}{2} (a_0 + \mathcal{J} a_0) \right\}.$$  

(1.18)

The following splitting, with

$$\mathfrak{k}_+ = \mathcal{D}_+ + \mathcal{D}_0^- \quad \text{and} \quad \mathfrak{n}_+ = \mathfrak{n},$$

is also used in Section 2; the projections take on the following form:

$$a = \pi_{\mathfrak{k}_+} a + \pi_{\mathfrak{n}_+} a$$

$$= \left\{ (a_+ - \mathcal{J} a_-) + \frac{1}{2} (a_0 - \mathcal{J} a_0) \right\} + \left\{ (a_- + \mathcal{J} a_-) + \frac{1}{2} (a_0 + \mathcal{J} a_0) \right\}.$$  

(1.19)

Note that $\mathcal{J}$ intertwines $\pi_{\mathfrak{k}}$ and $\pi_{\mathfrak{k}_+}$:

$$\mathcal{J} \pi_{\mathfrak{k}} = \pi_{\mathfrak{k}_+} \mathcal{J}.$$  

(1.20)

For a skew-symmetric semi-infinite matrix $m_{\infty}$, the skew-Borel decomposition

$$m_{\infty} := Q^{-1} J Q^{-1 \top}, \quad \text{with} \quad Q \in \mathcal{G}_k,$$  

(1.21)

is unique, as was shown in [2]. Here we may assume $m_{\infty}$ to be bi-infinite, as long as factorization (1.21) is unique, upon imposing a suitable normalization. Then we use $Q$ to dress up $\Lambda$:

$$L = Q \Lambda Q^{-1}.$$  

Then letting $m_{\infty}$ run according to the equations $\partial m/\partial t_i = \Lambda^i m + m \Lambda^i \top$, we show in the next proposition and corollary that $L$ evolves according to a system of commuting equations, which by virtue of the AKS theorem are Hamiltonian vector fields (for details, see [2]).

**Proposition 1.2**

For the matrices

$$m_{\infty} := Q^{-1} J Q^{-1 \top} \quad \text{and} \quad L := Q \Lambda Q^{-1}, \quad \text{with} \quad Q \in \mathcal{G}_k,$$

the following three statements are equivalent:
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(i) \( \frac{\partial Q}{\partial t_i} Q^{-1} = -\tau_k L_i \).

(ii) \( L_i + \frac{\partial Q}{\partial t_i} Q^{-1} \in \mathfrak{n} \).

(iii) \( \frac{\partial m_\infty}{\partial t_i} = \Lambda^i m_\infty + m_\infty \Lambda^{T_i} \).

Whenever the vector fields on \( Q \) or \( m \) satisfy (i), (ii), or (iii), then the matrix \( L = Q \Lambda Q^{-1} \) is a solution of the AKS-Lax pair

\[ \frac{\partial L}{\partial t_i} = [\tau_k L_i, L] = [\tau_n L_i, L]. \]

Proof

Written out and using (1.18), Proposition 1.2 amounts to showing the equivalence of the three formulas:

(I) \( \frac{\partial Q}{\partial t_i} Q^{-1} + ((L_i) - J (L_i)^T J) + \frac{1}{2} ((L_i)_0 - J (L_i)_0^T J) = 0, \)

(II) \( (L_i + \frac{\partial Q}{\partial t_i} Q^{-1}) - J (L_i + \frac{\partial Q}{\partial t_i} Q^{-1})^T J = 0, \)

(III) \( \Lambda^i m_\infty + m_\infty \Lambda^{T_i} - \frac{\partial m_\infty}{\partial t_i} = 0. \)

The point is to show that

\[ (I)_+ = 0, \quad (I)_- = (II)_- = -J (II)_+^T J, \quad (I)_0 = \frac{1}{2} (II)_0, \]

\[ Q^{-1} (II) J Q^{-1} = (III). \tag{1.22} \]

The details of this proof are found in [2].

2. Wave functions and their bilinear equations for the Pfaff lattice

Consider the commuting vector fields

\[ \frac{\partial m_\infty}{\partial t_i} = \Lambda^i m_\infty + m_\infty \Lambda^{T_i} \tag{2.1} \]

on the skew-symmetric matrix \( m_\infty(t) \) and the skew-Borel decomposition

\[ m_\infty(t) = Q^{-1}(t) J Q^{T-1}(t), \quad Q(t) \in \mathfrak{g}_k; \tag{2.2} \]

remember from (1.17) that \( Q(t) \in \mathfrak{g}_k \) means that \( Q(t) \) is lower-triangular, with along the “diagonal” \((2 \times 2)\)-matrices \( c_{2n} I \), with \( c_{2n} \neq 0 \).

In this section, we give the properties of the wave vectors and their bilinear relations. In this and the next section, the matrices are assumed to be bi-infinite; the semi-infinite case is dealt with by specialization. Upon setting

\[ Q_1 = Q(t) \quad \text{and} \quad Q_2 = J Q^{T-1}(t), \tag{2.3} \]
the matrix $Q(t)$ defines wave operators

$$W_1(t) := Q_1(t) e^{\sum_i t_i A_i}, \quad W_2(t) := Q_2(t) e^{-\sum_i t_i A^T i} = JW_1^{-1 T}(t), \quad (2.4)$$

$L$-matrices

$$L := L_1 := Q_1 \Lambda Q_1^{-1}, \quad L_2 := -\mathcal{F}(L_1) = Q_2 \Lambda^T Q_2^{-1}, \quad (2.5)$$

and wave and dual wave vectors

$$\Psi_1(t, z) = W_1(t) \chi(z) \Psi_1^\ast(t, z) = W_1^{-1 T}(t) \chi(z^{-1}) = -J \Psi_2(t, z^{-1}),$$

$$\Psi_2(t, z) = W_2(t) \chi(z) \Psi_2^\ast(t, z) = W_2^{-1 T}(t) \chi(z^{-1}) = J \Psi_1(t, z^{-1}). \quad (2.6)$$

From the definition, it follows that the wave functions $\Psi_1$ have the following asymptotics:

$$\begin{align*}
\Psi_{1,2n}(t, z) &= e^{\sum_i t_i z^{2n}_i} c_{2n}(t) \psi_{1,2n}(t, z), & \psi_{1,2n} &= 1 + O(z^{-1}), \\
\Psi_{1,2n+1}(t, z) &= e^{\sum_i t_i z^{2n+1}_i} c_{2n}(t) \psi_{1,2n+1}(t, z), & \psi_{1,2n+1} &= 1 + O(z^{-2}), \\
\Psi_{2,2n}(t, z) &= e^{-\sum_i t_i z^{-2n}_i} c_{-2n}(t) \psi_{2,2n}(t, z), & \psi_{2,2n} &= 1 + O(z), \\
\Psi_{2,2n+1}(t, z) &= e^{-\sum_i t_i z^{-2n}_i} c_{-2n}(t) \psi_{2,2n+1}(t, z), & \psi_{2,2n+1} &= 1 + O(z^2).
\end{align*} \quad (2.7)$$

where the $c_i$ are the elements of the diagonal part of $Q$.

**THEOREM 2.1**

The following statements are equivalent:

(i) $m_\infty$ satisfies $\partial m_\infty/\partial t_i = \Lambda_i m_\infty + m_\infty \Lambda^T_i$.

(ii) $Q_1$ satisfies the hierarchy of equations (with $L_i$ defined in (2.5))

$$\frac{\partial Q_1}{\partial t_i} = -(\pi_k L_1^i) Q_1, \quad (2.8)$$

(iii) $Q_2 = JL_1^{-1}$ satisfies

$$\frac{\partial Q_2}{\partial t_i} = -(\mathcal{F}(\pi_k L_1^i)) Q_2 = (\pi_k, L_2^i) Q_2.$$

(iv) $\Psi_1 = e^{\sum_i t_i z^i} Q_1(t) \chi(z)$ satisfies

$$\frac{\partial \Psi_1}{\partial t_i} = (\pi_n L_1^i) \Psi_1, \quad (2.9)$$

(v) $\Psi_2 = e^{-\sum_i t_i z^{-i}} JL_1^{-1}(t) \chi(z)$ satisfies

$$\frac{\partial \Psi_2}{\partial t_i} = -(L_2^i - \pi_{k_i} L_1^i) \Psi_2 = -(\pi_n, L_2^i) \Psi_2.$$

(vi) \(\Psi_1 \) and \(\Psi_2\) satisfy the bilinear identity for all \(n, m \in \mathbb{Z}\),

\[
\oint_\infty \psi_{1,n}(t, z) \psi_{2,m}(t', z^{-1}) \frac{dz}{2\pi i z} + \oint_0 \psi_{2,n}(t, z) \psi_{1,m}(t', z^{-1}) \frac{dz}{2\pi i z} = 0.
\]

(2.10)

If any one of these six conditions is satisfied, then

\[
\frac{\partial L_1}{\partial t_1} = [-\pi_k L_1^0, L_1], \quad \frac{\partial L_2}{\partial t_1} = [\pi_k, L_2^1, L_2].
\]

(2.11)

and

\[
L_1 \psi_1 = z \psi_1, \quad L_2 \psi_2 = z^{-1} \psi_2,
\]

\[
L_1 \tilde{\psi}_1^* = z \tilde{\psi}_1^*, \quad L_2 \tilde{\psi}_2^* = z^{-1} \tilde{\psi}_2^*.
\]

(2.12)

For later use, we also consider the “monic” wave functions, with the factors \(c_{2n}(t)\) removed; that is,

\[
\tilde{\psi}_1(t, z) := Q_0^{-1} \psi_1 \quad \text{and} \quad \tilde{\psi}_2(t, z) := Q_0 \psi_2
\]

(2.13)

and the matrix \(\hat{L}_1\), normalized so as to have 1’s above the main diagonal, with \(\hat{Q} := Q_0^{-1} Q\),

\[
\hat{L}_1 = Q_0^{-1} L_1 Q_0 = (Q_0^{-1} Q) \Lambda (Q_0^{-1} Q)^{-1} = \hat{Q} \Lambda \hat{Q}^{-1},
\]

\[
\hat{L}_2 = Q_0 L_2 Q_0^{-1} = -Q_0 \mathcal{J} (L_1) Q_0^{-1} = -\mathcal{J} (\hat{L}_1)
\]

(2.14)

Then, in terms of the elements \(\hat{q}_{ij}\) of the matrix \(\hat{Q} := Q_0^{-1} Q\), one easily computes by conjugation that \(\hat{L}_1\) has the following block structure:

\[
\hat{L}_1 = Q_0^{-1} L_1 Q_0 = (Q_0^{-1} Q) \Lambda (Q_0^{-1} Q)^{-1}
\]

\[
= \begin{pmatrix}
\vdots \\
\ldots & \hat{L}_{00} & \hat{L}_{01} & 0 & 0 \\
\hat{L}_{10} & \hat{L}_{11} & \hat{L}_{12} & 0 \\
* & \hat{L}_{21} & \hat{L}_{22} & \hat{L}_{23} \\
* & * & \hat{L}_{32} & \hat{L}_{33} & \ldots \\
\vdots
\end{pmatrix},
\]

with

\[
\hat{L}_{ij} := \begin{pmatrix}
\hat{q}_{2i+1, 2i-1} & 1 \\
\hat{q}_{2i+1, 2i-1} & -\hat{q}_{2i+2, 2i+1}
\end{pmatrix}, \quad \hat{L}_{i,i+1} := \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix},
\]

(2.15)

\[
\hat{L}_{i+1,i} := \begin{pmatrix}
* & -\hat{q}_{2i+2, 2i+1} & \hat{q}_{2i+3, 2i+1} + \hat{q}_{2i+2, 2i} \\
* & * & *
\end{pmatrix}.
\]

These definitions lead to a new statement that is equivalent to Theorem 2.1.
Theorem 2.2
\( \hat{L}_i, \dot{\hat{Q}}, \hat{\Psi}_1, \hat{\Psi}_2 \) satisfy the following equations:

\[
\frac{\partial \hat{Q}}{\partial t_n} = -((\hat{L}_1^n)_- - Q_0^{-2} \cdot J((\hat{L}_1^n)_+) \cdot J \cdot Q_0^2) \cdot \hat{Q},
\]

and

\[
\hat{L}_1 \hat{\Psi}_1 = z \hat{\Psi}_1, \quad \hat{L}_2 \hat{\Psi}_2 = z^{-1} \hat{\Psi}_2,
\]

with

\[
\frac{\partial}{\partial t_n} \hat{\Psi}_1(t, z) = ((\hat{L}_1^n)_+ + (\hat{L}_1^n)_0 + Q_0^{-2} J((\hat{L}_1^n)_+) \cdot Q_0^2) \cdot \hat{\Psi}_1(t, z),
\]

\[
\frac{\partial}{\partial t_n} \hat{\Psi}_2(t, z) = J((\hat{L}_1^n)_+ + (\hat{L}_1^n)_0 + Q_0^{-2} J((\hat{L}_1^n)_+) \cdot Q_0^2) \cdot \hat{\Psi}_2(t, z)
\]

\[
= -(\hat{L}_2^n)_+ + (\hat{L}_2^n)_0 + Q_0^2 J((\hat{L}_2^n)_-) \cdot Q_0^{-2} \cdot \hat{\Psi}_2(t, z).
\]

The proof of Theorem 2.1 hinges on the following proposition.

Proposition 2.3
The following three statements are equivalent:

(i) \( \partial m_\infty / \partial t_i = \Lambda^i m_\infty + m_\infty \Lambda^{T_i} \),

(ii) the matrices \( W_1(t) \) and \( W_2(t) \) (defined in (2.4)) satisfy

\[
W_1(t) W_1(t')^{-1} = W_2(t) W_2(t')^{-1},
\]

(iii) the \( \Psi_i(t, z) = W_i(t) \chi(z) \) satisfy the bilinear identity

\[
\oint \Psi_{1,n}(t, z) \Psi_{2,m}(t', z') \frac{dz}{2\pi i z} + \oint \Psi_{2,n}(t, z) \Psi_{1,m}(t', z') \frac{dz}{2\pi i z} = 0.
\]

Proof
The solution to (2.1) is given by

\[
m_\infty(t) = \sum_{\Lambda^k} m_\infty(0) e^{\sum_{t_i \Lambda^{T_i}}(t)}.
\]

Therefore, skew-Borel decomposing \( m_\infty(t) \) and \( m_\infty(0) \), we find

\[
Q^{-1}(0) J Q^{-1}(t) = e^{-\sum_{t_i \Lambda^i}} Q^{-1}(0) J Q^{-1}(t) e^{-\sum_{t_i \Lambda^i}},
\]

and so, from the definition of \( W_1 \) and \( W_2 \),

\[
W_1^{-1}(0) W_2(0) = Q^{-1}(0) J Q^{-1}(t) J Q^{-1}(0) = Q^{-1}(0) J Q^{-1}(t) J Q^{-1}(t) e^{-\sum_{t_i \Lambda^i}} \] (using (2.19))

\[
= W_1(t)^{-1} J W_1(t)^{-1}
\]

\[
= W_1(t)^{-1} W_2(t).
\]

(2.20)
implying the independence in $t$ of the right-hand side of (2.20). Therefore, we have

$$W_1(t)^{-1}W_2(t) = W_1(t')^{-1}W_2(t'),$$

for all $t, t' \in \mathbb{C}^{\infty}$, and so

$$W_1(t)W_1^{-1}(t') = W_2(t)W_2^{-1}(t'),$$

thus yielding (ii). Reversing the steps yields the differential equation (i).

Finally, the proof of the bilinear identity (iii) proceeds as follows. Using the well-known formula (see [3, Prop. 4.1]),

$$W_1(t)W_2(t')^{-1} = \oint_0^{\infty} \Psi_1(t, z) \otimes \Psi_2^*(t', z) \frac{dz}{2\pi i z},$$

statement (ii) becomes

$$\oint_0^{\infty} \Psi_1(t, z) \otimes \Psi_1^*(t', z) \frac{dz}{2\pi i z} = \oint_0^{\infty} \Psi_2(t, z) \otimes \Psi_2^*(t', z) \frac{dz}{2\pi i z},$$

whose $(m, n)$th component is

$$\oint_0^{\infty} \Psi_{1,n}(t, z) \Psi_{1,n}^*(t', z) \frac{dz}{2\pi i z} - \oint_0^{\infty} \Psi_{2,n}(t, z) \Psi_{2,n}^*(t', z) \frac{dz}{2\pi i z} = 0.$$

Next we use the relations $\Psi_1^*(t, z) = -J\Psi_2(t, z^{-1})$ and $\Psi_2^*(t, z) = J\Psi_1(t, z^{-1})$ to yield

$$\oint_0^{\infty} \Psi_1(t, z) \otimes J\Psi_2(t', z^{-1}) \frac{dz}{2\pi i z} + \oint_0^{\infty} \Psi_2(t, z) \otimes J\Psi_1(t', z^{-1}) \frac{dz}{2\pi i z} = 0,$$

which again leads to (iii). That (iii) implies (ii) is obtained by reversing the arguments.

\[\square\]

Proof of Theorem 2.1

The proof of statement (ii) for $Q_1$, namely,

$$\frac{\partial Q_1}{\partial t_i} = -(\pi_k L_1^1) Q_1,$$

follows at once from Proposition 1.2.

The proof of (iii) for $Q_2 = JQ_1^{-1}$ is based on the identity $\mathcal{J}\pi_k a = \pi_k \mathcal{J} a$. 
Indeed, we compute
\[
\frac{\partial Q_2}{\partial t_i} Q_2^{-1} = -J Q_1^{-1} \frac{\partial Q_1^T}{\partial t_i} Q_1^{-1} Q_2^{-1} = -J Q_1^{-1} (\pi_k L_1^i)^T Q_1^{-1} J
\]
\[
= -J (\pi_k L_1^i)^T J
\]
\[
= -\mathcal{J} (\pi_k L_1^i)
\]
\[
= -\pi_k i \mathcal{J} L_1^i
\]
\[
= -\pi_k i \mathcal{J} (\mathcal{J} L_2)^i \quad \text{(using (2.5))}
\]
\[
= -\pi_k i (-1)^i (\mathcal{J} L_2)^i
\]
\[
= -\pi_k i (-1)^i (\mathcal{J} L_2)^i = -\pi_k i L_2^i.
\]

Statements (iv) and (v) for \( \Psi_1, \Psi_2 \) are straightforwardly equivalent to (ii) and (iii), respectively. According to Propositions 1.2 and 2.3 combined, the bilinear identity (2.10) in (vi) is equivalent to statement (i), (ii), or (iii). The hierarchy concerning the \( L_i \)'s follows at once from (ii) and (iii), thus ending the proof of Theorem 2.1. \( \Box \)

**Proof of Theorem 2.2**

To prove (2.16), remember from Theorem 2.1 that
\[
\frac{\partial Q}{\partial t_n} Q^{-1} = -\pi_k L^n = -(L^n)_- - J (L^n_+)^T - \frac{1}{2} ((L^n)_0 - J (L^n_0)^T J);
\]
hence, taking the (0)-part of this expression yields
\[
\frac{\partial \log Q_0}{\partial t_n} = (\frac{\partial Q}{\partial t_n} Q^{-1})_0 = -\pi_k (L^n)_0 = -\frac{1}{2} (L^n)_0 + \frac{1}{2} J (L^n_0)^T J.
\]

Using the fact that \( Q_0, Q_0^{-1}, \dot{Q}_0 \in G_k \cap \mathcal{D}_0 \) commute among themselves and commute with \( J \) and the fact that \( \mathcal{D}_0 \mathcal{D}_\pm, \mathcal{D}_\pm \mathcal{D}_0 \subset \mathcal{D}_\pm \), we compute for \( \dot{Q} = Q_0^{-1} \dot{Q}, \dot{L}_1 = Q_0^{-1} L_1 Q_0 \) (see (2.14))
\[
\frac{\partial \dot{Q}}{\partial t_n} \dot{Q}^{-1} = -Q_0^{-1} \dot{Q}_0 Q_0^{-1} Q \dot{Q}^{-1} Q_0 + Q_0^{-1} \dot{Q} Q^{-1} Q_0
\]
\[
= -Q_0^{-1} \dot{Q}_0 + Q_0^{-1} \dot{Q} Q^{-1} Q_0
\]
\[
= Q_0^{-1} (-Q_0 \dot{Q}_0 + \dot{Q} Q^{-1}) Q_0
\]
\[
= Q_0^{-1} (-(L_1^n)_- + J (L_1^n_+)^T J) Q_0
\]
\[
= -(Q_0^{-1} L_1^n Q_0)_- + Q_0^{-1} J (Q_0 (Q_0^{-1} L_1^n Q_0) + Q_0^{-1})^T J Q_0
\]
\[
= -(\dot{L}_1^n)_- + Q_0^{-2} J (\dot{L}_1^n_+)^T J Q_0^2.
\]
Using this result and \( \hat{L}_1 \hat{\Psi}_1(t, z) = z \hat{\Psi}_1(t, z) \), we find

\[
\frac{\partial \hat{\Psi}_1(t, z)}{\partial t_n} = \frac{\partial}{\partial t_n} e^{z \sum t^i \hat{\mathcal{Q}}(z)} = z^n e^{z \sum \tau_k t^i} \hat{\mathcal{Q}}(z) + e^{z \sum \tau_k t^i} \big( - (\hat{L}_1^n) - Q_0^{-2} J Q_0^n (\hat{L}_1^n) + J Q_0^n \hat{\mathcal{Q}} \big) \hat{\mathcal{Q}}(z)
\]

\[
= \big( \hat{L}_1^n - (\hat{L}_1^n)^{-} + Q_0^{-2} J (\hat{L}_1^n)^{+} J Q_0^{2} \big) \hat{\Psi}_1(t, z)
\]

\[
= ((\hat{L}_1^n)_{+} + (\hat{L}_1^n)^{+})_{0} + Q_0^{-2} (J (\hat{L}_1^n)^{+} Q_0^{2} \hat{\mathcal{Q}}) \hat{\Psi}_1(t, z).
\]  

(2.21)

But, we also have that \( \hat{\Psi}_1 = Q_0^{-1} \hat{\Psi}_1(t, z) \) and \( \hat{\Psi}_2 = Q_0 \hat{\Psi}_2(t, z) \) satisfy, using \( W_2 = JW_1^{-1} \tau \),

\[
\frac{\partial \hat{\Psi}_1(t, z)}{\partial t_n} = (Q_0^{-1} W_1) \hat{\mathcal{Q}}(z) = (Q_0^{-1} W_1) (Q_0^{-1} W_1) \hat{\Psi}_1(t, z),
\]

(2.22)

\[
\frac{\partial \hat{\Psi}_2(t, z)}{\partial t_n} = (Q_0 W_2) \hat{\mathcal{Q}}(z) = (Q_0 W_2) (Q_0 W_2) \hat{\Psi}_2(t, z).
\]

(2.23)

Comparing (2.21), (2.22), and (2.23), and invoking (2.14),

\[
- \mathcal{J}(\hat{L}_2^n) = \hat{L}_2^n,
\]

and so, in particular,

\[
- \mathcal{J}(\hat{L}_1^n) = \hat{L}_2^n
\]

and

\[
- \mathcal{J}(\hat{L}_1^n) = (\hat{L}_2^n)_{+} \quad \text{and} \quad - \mathcal{J}(\hat{L}_1^n) = (\hat{L}_2^n)_{-}
\]

\[
\frac{\partial \hat{\Psi}_2(t, z)}{\partial t_n} = -((\hat{L}_2^n)_{+} + (\hat{L}_2^n)_{-} + Q_0^{-2} (J (\hat{L}_2^n)_{-} Q_0^{2} \hat{\mathcal{Q}}) \hat{\Psi}_2(t, z),
\]

which establishes Theorem 2.2.

\[\square\]

3. **Existence of the Pfaff \( \tau \)-function**

The point of this section is to show that the solution of the Pfaff lattice can be expressed in terms of a sequence of functions \( \tau \), which are not \( \tau \)-functions in the usual sense but which enjoy a different set of bilinear identities and partial differential equations.
PROPOSITION 3.1
There exist functions $\tau_{2n}(t)$ such that

$$
\psi_{1,2n}(t, z) = \frac{\tau_{2n}(t - [z^{-1}])}{\tau_{2n}(t)} \quad \text{and} \quad \psi_{2,2n}(t, z) = \frac{\tau_{2n+2}(t + [z])}{\tau_{2n+2}(t)}.
$$

(3.1)

The proof of Proposition 3.1 is postponed until later. For future use, we define the diagonal matrix

$$
h = \text{diag}(\ldots, h_{-2}, h_{-2}, h_0, h_0, h_2, h_2, \ldots) \in \mathcal{D}_0^-,
\quad \text{with} \quad h_{2n} = \frac{\tau_{2n+2}}{\tau_{2n}}.
$$

(3.2)

THEOREM 3.2
$\Psi_1$ and $\Psi_2$ have the following $\tau$-function representation:

$$
\begin{align*}
\Psi_{1,2n}(t, z) &= e^{\sum_{i \leq j} z_{2n}^{-1} h_{2n}^{-1/2} \tau_{2n}(t - [z^{-1}])} 
\tau_{2n}(t), \\
\Psi_{1,2n+1}(t, z) &= e^{\sum_{i \leq j} z_{2n}^{-1} h_{2n}^{-1/2} (z + \partial/\partial t_1) \tau_{2n}(t - [z^{-1}])} 
\tau_{2n}(t), \\
\Psi_{2,2n}(t, z) &= e^{-\sum_{i \leq j} h_{2n}^{1/2} \tau_{2n+2}(t + [z])} 
\tau_{2n}(t), \\
\Psi_{2,2n+1}(t, z) &= e^{-\sum_{i \leq j} h_{2n+2}^{1/2} \tau_{2n+2}(t + [z])} 
\tau_{2n}(t).
\end{align*}
$$

with the $\tau_{2n}(t)$ satisfying the following bilinear identity for all $n, m \in \mathbb{Z}$:

$$
\oint_{z=\infty} \tau_{2n}(t - [z^{-1}]) \tau_{2m+2}(t' + [z^{-1}]) e^{\sum_{i \leq j} z_{2n-2m-2}^{-1} \tau_{2n-2m-2} d\tau_{2n-2m-2}} dz + \oint_{z=0} \tau_{2n+2}(t + [z]) \tau_{2m}(t' - [z]) e^{\sum_{i \leq j} z_{2n-2m}^{-1} \tau_{2n-2m} d\tau_{2n-2m}} dz = 0.
$$

(3.4)

Conversely, this bilinear relation characterizes the $\tau$-function for the Pfaff lattice.

Remark
Then $L$ has the following representation in terms of the Pfaffian $\tau$-functions:

$$
h^{1/2} L h^{-1/2} = \begin{pmatrix}
\vdots \\
\cdots & \hat{L}_{00} & \hat{L}_{01} & 0 & 0 \\
\hat{L}_{10} & \hat{L}_{11} & \hat{L}_{12} & 0 \\
* & \hat{L}_{21} & \hat{L}_{22} & \hat{L}_{23} \\
* & * & \hat{L}_{32} & \hat{L}_{33} & \cdots \\
\vdots
\end{pmatrix}.
$$
with \( (\, = \partial/\partial t_1) \),
\[
\hat{L}_{nn} := \begin{pmatrix} -(\log \tau_{2n}) & 1 \\ -s_2(\tilde{\partial})\tau_{2n} & -s_2(-\tilde{\partial})\tau_{2n+2} \end{pmatrix}, \quad \hat{L}_{n,n+1} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]
\[
\hat{L}_{n+1,n} := \begin{pmatrix} * & (\log \tau_{2n+2})^* \\ * & * \end{pmatrix}.
\]
(3.5)

The following bilinear relations follow from (3.4) and are due to [4].

**Corollary 3.3**
The functions \( \tau_{2n}(t) \) satisfy the following “differential Fay identity”:
\[
\{ \tau_{2n}(t - [u]), \tau_{2n}(t - [v]) \} + (u^{-1} - v^{-1})(\tau_{2n}(t - [u])\tau_{2n}(t - [v]) - \tau_{2n}(t)\tau_{2n}(t - [u] - [v])) = uv(u - v)\tau_{2n-2}(t - [u] - [v])\tau_{2n+2}(t),
\]
(3.6)

and Hirota-type bilinear equations, always involving nearest neighbours:
\[
\left( p_{k+4}(\tilde{\partial}) - \frac{1}{2} \frac{\partial^2}{\partial t_1 \partial t_{k+3}} \right) \tau_{2n} \circ \tau_{2n} = p_k(\tilde{\partial}) \tau_{2n+2} \circ \tau_{2n-2}, \quad k, n = 0, 1, 2, \ldots.
\]
(3.7)

**Lemma 3.4**
Consider an arbitrary function \( \varphi(t, z) \) depending on \( t \in \mathbb{C}^\infty, z \in \mathbb{C} \), having the asymptotics \( \varphi(t, z) = 1 + O(1/z) \) for \( z \not\to \infty \) and satisfying the functional relation
\[
\frac{\varphi(t - [z_2^{-1}], z_1)}{\varphi(t, z_1)} = \frac{\varphi(t - [z_1^{-1}], z_2)}{\varphi(t, z_2)}, \quad t \in \mathbb{C}^\infty, \quad z \in \mathbb{C}.
\]
Then there exists a function \( \tau(t) \) such that
\[
\varphi(t, z) = \frac{\tau(t - [z^{-1}])}{\tau(t)}.
\]

**Proof**
See Appendix C.

*We define \( \{ f, g \} := f'g - fg' \), where \( ' = \partial/\partial t_1 \).
Lemma 3.5
The following holds for the Pfaffian wave functions $\Psi_1$ and $\Psi_2$, as in (2.7),

$$\frac{\psi_{1,2n}(t - [z_{2}^{-1}], z_1)}{\psi_{1,2n}(t, z_1)} = \frac{\psi_{1,2n}(t - [z_{1}^{-1}], z_2)}{\psi_{1,2n}(t, z_2)}$$

(3.8)

and

$$\psi_{2,2n-2}(t - [z_{-1}], z^{-1})\psi_{1,2n}(t, z) = 1.$$  

(3.9)

Proof
Setting (2.7) in the bilinear equation (2.12), with $n \mapsto 2n$, $m \mapsto 2n - 2$, yields

$$\frac{c_{2n}(t)}{c_{2n-2}(t)} \int_{\infty}^{t} e^{\sum (t_i - t_i')z_i} \psi_{1,2n}(t, z)\psi_{2,2n-2}(t', z^{-1}) \frac{dz}{2\pi i}$$

$$+ \frac{c_{2n-2}(t)}{c_{2n}(t)} \int_{0}^{t} e^{\sum (t_i' - t_i)z^{-1}} \psi_{2,2n}(t, z)\psi_{1,2n-2}(t', z^{-1}) \frac{z^2dz}{2\pi i} = 0.$$

Setting $t - t' = [z_{1}^{-1}] + [z_{2}^{-1}]$

in the above and using $e^{\sum x^i} = 1/(1 - x)$ yields

$$\frac{c_{2n}}{c_{2n-2}} \int_{\infty}^{t} \psi_{1,2n}(t, z)\psi_{2,2n-2}(t', z^{-1}) \frac{dz}{(1 - z/z_1)(1 - z/z_2)}$$

$$= \frac{c_{2n-2}}{c_{2n}} \int_{0}^{t} z^2 \left(1 - \frac{1}{zz_1}\right) \left(1 - \frac{1}{zz_2}\right) \psi_{2,2n}(t, z)\psi_{1,2n-2}(t', z^{-1}) \frac{dz}{2\pi i} = 0,$$

the latter being equal to zero, because the integrand on the right-hand side is holomorphic. The integral on the left-hand side can be viewed as an integral along a contour encompassing $\infty$ and the points $z_1$ and $z_2$, thus leading to

$$\psi_{1,2n}(t, z_1)\psi_{2,2n-2}(t - [z_{1}^{-1}] - [z_{2}^{-1}], z_1^{-1})$$

$$= \psi_{1,2n}(t, z_2)\psi_{2,2n-2}(t - [z_1^{-1}] - [z_2^{-1}], z_2^{-1})$$  

(3.10)

with

$$\psi_{1,2n}(t, z) = 1 + O(z^{-1}), \quad \psi_{2,2n-2}(t - [z_{-1}] - [z_{-1}], z^{-1}) = 1 + O(z^{-1}).$$

Therefore, letting $z_2 \not\to \infty$, one finds

$$\psi_{1,2n}(t, z_1)\psi_{2,2n-2}(t - [z_{1}^{-1}], z_1^{-1}) = 1,$$

(3.11)

yielding (3.9), and so, upon shifting $t \mapsto t - [z_{2}^{-1}]$,

$$\psi_{2,2n-2}(t - [z_{-1}] - [z_{2}^{-1}], z^{-1}) = \frac{1}{\psi_{1,2n}(t - [z_{2}^{-1}], z_1)};$$
similarly,
\[ \psi_{2,2n-2}(t - [z_1^{-1}] - [z_2^{-1}], z_2^{-1}) = \frac{1}{\psi_{1,2n}(t - [z_1^{-1}], z_2)}. \]  
(3.12)

Setting the two expressions (3.12) in (3.10) yields
\[ \frac{\psi_{1,2n}(t - [z_2^{-1}], z_1)}{\psi_{1,2n}(t, z_1)} = \frac{\psi_{1,2n}(t - [z_1^{-1}], z_2)}{\psi_{1,2n}(t, z_2)}. \]

**Proof of Proposition 3.1**
From Lemmas 3.4 and 3.5, there exists, for each \(2n\), a function \(\tau_{2n}\) such that the first relation of (3.1) is satisfied; that is,
\[ \psi_{1,2n}(t, z) = \frac{\tau_{2n}(t - [z^{-1}])}{\tau_{2n}(t)}, \]
and so from (3.9)
\[ \psi_{2,2n-2}(t - [z^{-1}], z_1^{-1}) = \frac{1}{\psi_{1,2n}(t, z)} = \frac{\tau_{2n}(t)}{\tau_{2n}(t - [z^{-1}])),} \]
thus leading to
\[ \psi_{2,2n-2}(t, z) = \frac{\tau_{2n}(t + [z])}{\tau_{2n}(t)}, \]
which is the second relation of (3.1).  
\(\square\)

**Proof of Theorem 3.2**
At first, remembering that \(\hat{Q} = Q_0^{-1} Q\), observe that
\[
e^{\sum h_{i,j}} ((\hat{Q} \chi(z))_{2n} = (Q_0^{-1}\psi_1(t, z))_{2n} \\
e^{\sum h_{i,j}} z^{2n} \psi_{1,2n}(t, z) \\
e^{\sum h_{i,j}} z^{2n} \frac{\tau_{2n}(t - [z^{-1}])}{\tau_{2n}(t)} \\
e^{\sum h_{i,j}} z^{2n} \left( 1 + \sum_{n=1}^{\infty} \frac{s_k(-\tilde{d}) \tau_{2n}(t)}{\tau_{2n}(t)} \right). \]
showing that a few subdiagonals of the matrix \( \hat{Q} \) are given by

\[
\hat{Q} = \begin{pmatrix}
  \ddots & & & \\
  1 & 0 & & \\
  0 & 1 & & \\
  \hat{q}_{2n,2n-2} & \hat{q}_{2n,2n-1} & 1 & 0 \\
  \hat{q}_{2n+1,2n-2} & \hat{q}_{2n+1,2n-1} & 0 & 1 \\
  \ddots & & & \\
\end{pmatrix}
\]

with

\[
\hat{q}_{2n,2n-1} = -\frac{\partial}{\partial t_1} \log \tau_{2n}, \quad \hat{q}_{2n,2n-2} = \frac{s_2(-\bar{\partial}) \tau_{2n}}{\tau_{2n}}. \tag{3.13}
\]

Remembering that (2.7), normalized, becomes

\[
\begin{align*}
\hat{\Psi}_{1,2n}(t, z) &= e^{\sum_t t_i z_i^{2n} \psi_{1,2n}(t, z)}, \quad \psi_{1,2n} = 1 + O(z^{-1}), \\
\hat{\Psi}_{1,2n+1}(t, z) &= e^{\sum_t t_i z_i^{2n+1} \psi_{1,2n+1}(t, z)}, \quad \psi_{1,2n+1} = 1 + O(z^{-2}), \tag{3.14}
\end{align*}
\]

\[
\begin{align*}
\hat{\Psi}_{2,2n}(t, z) &= e^{-\sum_t t_i z_i^{2n} \psi_{2,2n}(t, z)}, \quad \psi_{2,2n} = 1 + O(z), \\
\hat{\Psi}_{2,2n+1}(t, z) &= -e^{-\sum_t t_i z_i^{2n} \psi_{2,2n+1}(t, z)}, \quad \psi_{2,2n+1} = 1 + O(z^2), \tag{3.15}
\end{align*}
\]

we now show (3.3). Compute, using Theorem 2.2,

\[
e^{\sum_t t_i z_i^{2n} \psi_{1,2n}(t, z)} = \left( \frac{\partial}{\partial t_1} \hat{\Psi}_1(t, z) \right)_{2n} \\
= \left( (\hat{L}_1)_0 + (\hat{L}_1)^{-1} J (\hat{Q}^2_0)^{-1} \right)_{2n} \tag{3.16}
\]

and

\[
e^{-\sum_t t_i z_i^{2n} \psi_{2,2n}(t, z)} = \left( \frac{\partial}{\partial t_1} \hat{\Psi}_2(t, z) \right)_{2n} \\
= \left( (\mathcal{J} (\hat{L}_1)_0 + (\hat{L}_1)^{-1} J (\hat{Q}^2_0)^{-1} ) \right)_{2n}. \tag{3.17}
\]
In this expression, the matrix equals, according to (2.14),

\[
(\hat{L}_1)_+ + (\hat{L}_1)_0 + Q_0^{-2} J (\hat{L}_1)_+^T J Q_0^2
\]

\[
= \begin{pmatrix}
\vdots \\
\hat{q}_{0,-1} & 1 & 0 & 0 & 0 \\
\hat{q}_{1,-1} - \hat{q}_{20} & -\hat{q}_{21} & 1 & 0 & 0 \\
c_0^2/c_2^2 & 0 & \hat{q}_{21} & 1 & 0 \\
0 & 0 & 0 & \hat{q}_{43} & 1 \\
0 & 0 & c_2^2/c_4^2 & 0 & \hat{q}_{53} - \hat{q}_{64} - \hat{q}_{65} \\
\vdots 
\end{pmatrix},
\]

and, acting with \( J \) on this matrix,

\[
J (\hat{L}_1)_+ + (\hat{L}_1)_0 + Q_0^{-2} J (\hat{L}_1)_+^T J Q_0^2
\]

\[
= \begin{pmatrix}
\vdots \\
\hat{q}_{21} & 1 & 0 & 0 & 0 \\
\hat{q}_{1,-1} - \hat{q}_{20} & -\hat{q}_{0,-1} & 0 & 0 & 0 \\
c_0^2/c_2^2 & 0 & \hat{q}_{43} & 1 & 0 \\
1 & 0 & \hat{q}_{31} - \hat{q}_{42} - \hat{q}_{21} & c_2^2/c_4^2 & 0 \\
0 & 0 & 0 & 1 & \hat{q}_{53} - \hat{q}_{64} - \hat{q}_{65} \\
\vdots 
\end{pmatrix},
\]

using the fact that

\[
J \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}.
\]

Therefore the 2\( n \)th rows of both matrices, respectively, have the form

\[
(0, \ldots, 0, \hat{q}_{2n,2n-1}(t), 1, 0, 0, \ldots),
\]

\[
\uparrow_{2n}
\]

\[
(0, \ldots, 0, \hat{q}_{2n+2,2n+1}(t), 1, 0, 0, \ldots),
\]

\[
\uparrow_{2n}
\]
and thus from (3.16) and (3.17), and expansions (3.14) and (3.15), we have
\[
\left( \frac{\partial}{\partial t_1} + z \right) z^{2n} \psi_{1,2n}(t, z) = \hat{q}_{2n,2n-1}(t) z^{2n} \psi_{1,2n} + z^{2n+1} \psi_{1,2n+1},
\]
\[
\left( \frac{\partial}{\partial t_1} - z^{-1} \right) z^{2n+1} \psi_{2,2n}(t, z) = \hat{q}_{2n+2,2n+1}(t) z^{2n+1} \psi_{2,2n} + z^{2n} \psi_{2,2n+1}. \tag{3.18}
\]
So, using the expression (3.13) for \( \hat{q}_{2n,2n-1} \) and the first expression of (3.1),
\[
z^{2n+1} \psi_{1,2n+1}(t, z)
= \left( z + \frac{\partial}{\partial t_1} \right) z^{2n} \psi_{1,2n}(t, z) - \hat{q}_{2n,2n-1}(t) z^{2n} \psi_{1,2n}(t, z)
= \left( z + \frac{\partial}{\partial t_1} \right) z^{2n} \psi_{1,2n}(t, z) + \left( \frac{\partial}{\partial t_1} \log \tau_{2n}(t) \right) z^{2n} \psi_{1,2n}(t, z)
= \left( z + \frac{\partial}{\partial t_1} \right) z^{2n} \psi_{1,2n}(t, z) + \left( \frac{\partial}{\partial t_1} \right) z^{2n} \frac{\tau_{2n}(t - \{z^{-1}\})}{\tau_{2n}(t)}
= z^{2n} \left( z + \frac{\partial}{\partial t_1} \right) \frac{\tau_{2n}(t - \{z^{-1}\})}{\tau_{2n}(t)}, \tag{3.19}
\]
and similarly, using the second relation (3.18),
\[
z^{2n} \psi_{2,2n+1}(t, z) = \frac{z^{2n+1} (-z^{-1} + \partial / \partial t_1) \tau_{2n+2}(t + \{z\})}{\tau_{2n+2}(t)}. \tag{3.20}
\]
This establishes (3.3) modulo the denominators. Therefore, we also have
\[
\psi_{1,2n+1}(t, z) = \frac{1}{\tau_{2n}(t)} \left( z + \frac{\partial}{\partial t_1} \right) \frac{\tau_{2n}(t) - \frac{\partial}{\partial t_1} \tau_{2n} z^{-1} + s_2(\tilde{z}) \tau_{2n} z^{-2} + \cdots}{\frac{\partial^2}{\partial t_1^2} + s_2(\tilde{z})} \tau_{2n} z^{-2} + O(z^{-3});
\]
thus, referring to the matrix \( \tilde{Q} \) just preceding (3.13),
\[
\hat{q}_{2n,2n} = 0, \quad \hat{q}_{2n+1,2n-1} = \frac{1}{\tau_{2n}} \left( s_2(\tilde{z}) - \frac{\partial^2}{\partial t_1^2} \right) \frac{\tau_{2n}}{\tau_{2n}}. \tag{3.21}
\]
To show (3.4), setting \( n \mapsto 2n \) and \( m \mapsto 2n \) in bilinear relation (2.10) and substituting, using (2.7) and the expressions for \( \psi_{1,2n}(t, z) \) and \( \psi_{2,2n}(t, z) \) in the proof of Proposition 3.1,
\[
\psi_{1,2n}(t, z) = e^{\sum \alpha_j z^j} z^{2n} c_{2n}(t) \frac{\tau_{2n}(t - \{z^{-1}\})}{\tau_{2n}(t)}
\]
and
\[
\psi_{2,2n}(t', z) = e^{\sum \alpha_j z^j} z^{2n+1} c_{2n-1}(t') \frac{\tau_{2n+2}(t' + \{z\})}{\tau_{2n+2}(t')}
\]
Setting yields the following relation, which involves a constant (3.1), (3.19), and (3.20) yield (3.3); substituting (3.3) into (2.10) yields (3.4).

Rescaling so that the first integral has a simple pole at \( z = \alpha \) amounts to replacing the exponential:

\[
e^{\sum (\alpha - t_i) z^k} = 1 - \alpha z, \quad e^{\sum (\alpha - t_i) z^k} = \frac{1}{1 - \alpha / z},
\]

so that the first integral has a simple pole at \( z = \infty \) and the second integral has one at \( z = \alpha \). Evaluating the integrals yields

\[
-\alpha c_{2n}(t') \frac{\tau_{2n+2}(t') / \tau_{2n}(t')} {\tau_{2n+2}(t) / \tau_{2n}(t')} + \alpha = 0;
\]

that is,

\[
(e^{\sum (\alpha - t_i) (\partial / \partial t_i)} - 1)c_n^2(t) \frac{\tau_{2n+2}(t)} {\tau_{2n}(t)} = 0
\]

yields the following relation, which involves a constant \( c_n \), independent of time,

\[
c_{2n}(t) = c_n \frac{\tau_{2n}(t)} {\tau_{2n+2}(t)} = c_n \cdot h_{2n}^{-1}(t).
\] (3.22)

Rescaling \( \tau_{2n} \mapsto \tau_{2n}/(c_1 c_2 \cdots c_{n-1}) \), in effect, sets \( c_n = 1 \), and then (3.22), (2.7), (3.1), (3.19), and (3.20) yield (3.3); substituting (3.3) into (2.10) yields (3.4).

Finally, identity (3.22) actually says \( Q_0 = h^{-1/2} \). To derive the form (3.5) of the matrix \( L \), set (3.13) and (3.21) in the elements just below the main diagonal of matrix (2.15), to yield (\( c = \partial / \partial t_1 \))

\[
-\dot{q}_{2n,2n-1}^2 - \dot{q}_{2n+1,2n-1} + \dot{q}_{2n,2n-2} = \left( \frac{\dot{\tau}_{2n}} {\tau_{2n}} \right)^2 - \frac{(s_2(-\dot{\vartheta}) - (\dot{\vartheta}^2/\tau_{2n}^2)) \tau_{2n}} {\tau_{2n}}
\]

\[
+ \frac{s_2(-\dot{\vartheta}) \tau_{2n}} {\tau_{2n}}
\]

\[
= \frac{\dot{\tau}_{2n}} {\tau_{2n}} - \left( \frac{\dot{\tau}_{2n}} {\tau_{2n}} \right)^2
\]

\[
= (\log \tau_{2n})^{-1}.
\]
and
\[
\hat{q}_{2n+1,2n-1} - \hat{q}_{2n+2,2n} = \frac{(s_2(-\tilde{\alpha}) - \tilde{\alpha}^2/\tilde{\alpha}^2)\tau_{2n}}{\tau_{2n}} - \frac{s_2(-\tilde{\alpha})\tau_{2n+2}}{\tau_{2n+2}}.
\]

concluding the proof of Theorem 3.2, upon substituting the two relations (3.13) and (3.14) and also \(Q_0 = h^{-1/2}\) into (2.15). \(\square\)

4. Semi-infinite matrices \(m_\infty\), (skew-)orthogonal polynomials, and matrix integrals

In this section, consider the following inner-product* for \(\alpha = 0\), \(\mp 1\):

\[
(f, g)_t = \iint_{\mathbb{R}^2} f(y)g(z)e^{\sum h(y')}\Sigma \rho(y)2D\delta(y-z)\rho(y)dydz
\]

\[
= \begin{cases} 
\iint_{\mathbb{R}} f(y)g(y)e^{\sum h(y')}2\rho(y)^2dy, & \text{for } \alpha = 0, \\
\iint_{\mathbb{R}^2} f(y)g(z)e^{\sum h(y')}\Sigma \rho(y)\rho(z)dydz, & \text{for } \alpha = -1, \\
\iint_{\mathbb{R}} f(y)g(y)e^{\sum h(y')}\Sigma \rho(y)^2dy, & \text{for } \alpha = +1.
\end{cases}
\]

(4.1)

Each type of inner-product is discussed in Sections 4.1 and 4.2.

4.1. \(\partial m_\infty/\partial t_k = \Lambda k m_\infty\), orthogonal polynomials, and Hermitian matrix integrals

(\(\alpha = 0\))

The inner-product above, with \(\alpha = 0\), corresponds to Hermitian matrix integrals; this theory is sketched here for the sake of completeness and analogy; it mainly summarizes [6]. Consider a \(t\)-dependent weight

\[
\rho_t(dz) := e^{\sum h_i(z_i)}\rho(dz) = e^{-V(z)+\sum h_i(z_i)}dz
\]

on \(\mathbb{R}\), as in (0.1) and the induced \(t\)-dependent measure

\[
e^{Tr(-V(X)+\sum h_iX_i)}dX,
\]

(4.2)

on the ensemble \(\mathcal{H}_n\) of Hermitian matrices, with Haar measure \(dX\); the latter can be decomposed into a spectral part (radial part) and an angular part:

\[
dX := \prod_{1}^{n} dX_{ii} \prod_{1 \leq i < j \leq n} (d\delta X_{ij} d\Theta X_{ij}) = \Delta^2(z)dz_1 \cdots dz_n dU,
\]

(4.3)

*We have \(\varepsilon(x) = \text{sign } x\), having the property \(\varepsilon' = 2\delta(x)\). Also, consider the Wronskian \((f, g) := (\partial f/\partial y)g - f(\partial g/\partial y)\).
where $\Delta(z) = \prod_{1 \leq i < j \leq n} (z_i - z_j)$ is the Vandermonde determinant. Here we form the following matrix integral

$$
\int \mathcal{H}_n e^{Tr(-V(X)+\sum_i u_i X_i)} dX = c_n \int \mathbb{R}^n \Delta^2(z) \prod_{k=1}^n e^{\sum_\infty u_k z^*_k} \rho(dz_k).
$$

(4.4)

The weight $\rho_t(dz)$ defines a (symmetric) $t$-dependent inner-product of the type (4.1) for $\alpha = 0$:

$$
(f, g)_t^{sy} = \int f(z)g(z)e^{\sum_\infty u_k z^*_k} \rho(dz),
$$

with moments

$$
\mu_{ij}(t) := \langle z^i, z^j \rangle_t^{sy} = \int \mathbb{R} z^{i+j} e^{\sum_\infty u_k z^*_k} \rho(dz)
$$

satisfying

$$
\frac{\partial \mu_{ij}}{\partial t^j} = \int \mathbb{R} z^{i+j+\ell} e^{\sum_\infty u_k z^*_k} \rho(dz) = \mu_{i+\ell, j}(t).
$$

Therefore the semi-infinite moment matrix $m_\infty(t) = (\mu_{ij}(t))_{i, j \geq 0}$ satisfies

$$
\frac{\partial m_\infty}{\partial t^i} = \Lambda^i m_\infty = m_\infty \Lambda^{T i}.
$$

(4.5)

The point now is that the following integral can be expressed as a determinant of moments, namely,

$$
\int \mathcal{H}_n e^{Tr(-V(X)+\sum_\infty u_i X_i)} dX = \int \mathbb{R}^n \Delta^2(z) \prod_{k=1}^n \rho_t(dz_k)
$$

$$
= \int \mathbb{R}^n \sum_{\sigma \in S_n} \det(z^{\ell-1}_{\sigma(k)} z^{k-1}_{\sigma(k)})_{1 \leq \ell, k \leq n} \prod_{k=1}^n \rho_t(dz_k)
$$

$$
= \int \mathbb{R}^n \sum_{\sigma \in S_n} \det(z^{\ell+k-2}_{\sigma(k)} z_{\sigma(k)})_{1 \leq \ell, k \leq n} \prod_{k=1}^n \rho_t(dz_{\sigma(k)})
$$

$$
= \sum_{\sigma \in S_n} \det \left( \int \mathbb{R} z^{\ell+k-2}_{\sigma(k)} \rho_t(dz_{\sigma(k)}) \right)_{1 \leq \ell, k \leq n}
$$

$$
= n! \det \left( \int \mathbb{R} z^{\ell+k-2} \rho_t(dz) \right)_{1 \leq \ell, k \leq n}
$$

$$
= n! \det(\mu_{ij})_{0 \leq i, j \leq n-1} = n! \tau_n(t)
$$

is a $\tau$-function for the KP-equation; also, in view of (4.5) and the upper-lower Borel decomposition (0.3) of $m_\infty$, the integrals form a vector of $\tau$-functions for the Toda lattice. The polynomials $p_n(t; z)$ defined by the Borel decomposition $m_\infty(t) = S^{-1} S^{T-1}$ and $p(t; z) = S^T(z)$ are orthonormal with regard to the inner-product $\langle z^i, z^j \rangle_t^{sy} = \mu_{ij}(t)$. 

4.2. \( \partial m_\infty / \partial t_k = \Lambda^k m_\infty + m_\infty \Lambda^k \), skew-orthogonal polynomials, and symmetric and symplectic matrix integrals \((\alpha = \pm 1)\)

Consider a skew-symmetric semi-infinite matrix

\[
m_\infty(t) = (\mu_{ij}(t))_{i,j \geq 0}, \quad \text{with } m_n(t) = (\mu_{ij}(t))_{0 \leq i, j \leq n-1},
\]

satisfying

\[
\frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty + m_\infty \Lambda^k.
\]

(4.6)

Then we have shown in Sections 2 and 3 that, upon skew-Borel decomposing \(m_\infty\), these equations ultimately imply the existence of functions \(\tau(t)\) satisfying bilinear equations (3.4). Remember also that

\[
h(t) = \text{diag}(h_0, h_0, h_2, h_2, \ldots) \in \mathcal{D}_0^{-}, \quad \text{with } h_{2n}(t) = \frac{\tau_{2n+2}(t)}{\tau_{2n}(t)}.
\]

Here, we need the Pfaffian \(\text{pf}(A)\) of a skew-symmetric matrix \(A = (a_{ij})_{0 \leq i, j \leq n-1}\) for even \(n\):

\[
\text{pf}(A) dx_0 \wedge \cdots \wedge dx_{n-1} = \frac{1}{(n/2)!} \left( \sum_{0 \leq i < j \leq n-1} a_{ij} dx_i \wedge dx_j \right)^{n/2} / \Gamma \left( \frac{n}{2} \right)
\]

(4.7)

so that \(\text{pf}(A)^2 = \det A\). We now state the following theorem due to Adler, E. Horozov, and van Moerbeke [2], in complete analogy with the discussion of the Hermitian case.

**Theorem 4.1**

Consider a semi-infinite skew-symmetric matrix \(m_\infty\), evolving according to (4.6); setting

\[
\tau_{2n}(t) = \text{pf} \left( m_{2n}(t) \right) \quad \text{and} \quad h_{2n} = \frac{\text{pf}(m_{2n+2}(t))}{\text{pf}(m_{2n}(t))}.
\]

(4.8)

then, modulo the exponential, the wave vector \(\Psi_1(t)\) (defined by (3.3)) is a sequence of polynomials,

\[
\Psi_{1,k}(t, z) = e^{\sum_{l \geq 0} t^l q_k(t, z)}.
\]

(4.9)

where the \(q_k\)'s are skew-orthonormal polynomials of the form (0.13) and (0.18), satisfying

\[
((q_i, q_j)^sk)_{0 \leq i, j < \infty} = J, \quad \text{with} \quad \langle y^i, z^j \rangle^sk := \mu_{ij}.
\]

(4.10)

*In the formula below \((i_0, i_1, \ldots, i_{n-2}, i_{n-1}) = \sigma(0, 1, \ldots, n-1)\), where \(\sigma\) is a permutation and \(\varepsilon(\sigma)\) its parity.*
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The matrix $Q$ defined by $q(z) = Q \chi(z)$ is the unique solution (modulo signs) to the skew-Borel decomposition of $m_\infty$:

$$m_\infty(t) = Q^{-1} J Q^T - 1, \quad \text{with } Q \in \mathbb{k}. \quad (4.11)$$

The matrix $L = Q \Lambda_1 Q^{-1}$, also defined by

$$zq(t, z) = Lq(t, z),$$

and the diagonal matrix $h$ satisfy the equations

$$\frac{\partial L}{\partial t_i} = [-\pi_{\mathbb{k}} L^i, L] \quad \text{and} \quad h^{-1} \frac{\partial h}{\partial t_i} = 2\pi_{\mathbb{k}} (L^i)_0. \quad (4.12)$$

Sketch of proof

At first note that looking for skew-orthogonal polynomials is tantamount to the skew-Borel decomposition of $m_\infty$, so that (4.10) and (4.11) are equivalent. The skew-orthogonality of the polynomials (0.18) follows from expanding the Pfaffians explicitly in terms of $z$-columns, upon using the expression for the Pfaffian in terms of a column

$$\sum_{0 \leq k \leq \ell - 1} (-1)^k a_{ki} \text{pf}(0, \ldots, \hat{k}, \ldots, \ell - 1) = \text{pf}(0, \ldots, \ell - 1, i).$$

For details, see [2]. On the other hand, Theorem 3.2 gives $\Psi(t, z)$ and hence $Q$ in terms of $t_n(t) = \text{pf}_m z_n(t)$ of (4.8). By the uniqueness of decomposition (4.11), the two ways of arriving at $Q$, (0.18), and (3.3) must coincide.

Important remark

The polynomials (0.18) provide an explicit algorithm to perform the skew-Borel decomposition of the skew-symmetric matrix $m_\infty$. Namely, the coefficients of the polynomials $q_i$ provide the entries of the matrix $Q$. This fact is used later in the examples.

Symmetric matrix integrals ($\alpha = -1$)

Here we focus on integrals over the space $S_{2n}$ of symmetric matrices of the type

$$\int_{S_{2n}} e^{\text{Tr}(-V(X)+\sum_{i=1}^{\infty} t_i X^i)} dX,$$

where $dX$ denotes Haar measure for $X = U \text{diag}(z_1, \ldots, z_n) U^T$, $UU^T = I$,

$$dX := \prod_{1 \leq i \leq j \leq n} dX_{ij} = |\Delta(z)| dz_1 \cdots dz_n dU. \quad (4.14)$$
As appears below, integral (4.13) leads to a skew-inner-product of the type (4.1) with 
\( \alpha = -1 \), with weight 
\( \rho_t(z) := e^{\sum t_i z_i} \rho(z) = e^{-V(z) + \sum t_i z_i} \):

\[
\langle f(x), g(y) \rangle := \int \int_{\mathbb{R}^2} f(x)g(y) \delta(x-y) \rho_t(x) \rho_r(y) \, dx \, dy, \tag{4.15}
\]

leading to skew-symmetric moments

\[
\mu_{ij}(t) = \int \int_{x \geq y} x^i y^j \delta(x-y) \rho_t(x) \rho_t(y) \, dx \, dy \\
= \int \int_{x \geq y} (x^i y^j - x^j y^i) \rho_t(x) \rho_t(y) \, dx \, dy \\
= \int_{\mathbb{R}} (F_j(x)G_i(x) - F_i(x)G_j(x)) \, dx, \tag{4.16}
\]

where \(( = d/dx)\)

\[
F_i(x) := \int_{-\infty}^{x} y^i e^{\sum t_k y^k} \rho(y) \, dy \quad \text{and} \quad G_i(x) := F'_i(x) = x^i e^{\sum t_k x^k} \rho(x).
\]

By simple inspection, the moments \( \mu_{k\ell}(t) \) satisfy

\[
\frac{\partial \mu_{k\ell}}{\partial t_i} = \int \int_{\mathbb{R}^2} (x^{k+i} y^\ell + x^k y^{\ell+i}) \delta(x-y) e^{\sum t_n (x^n + y^n)} \rho(x) \rho(y) \, dx \, dy \\
= \mu_{k+i, \ell} + \mu_{k, \ell+i},
\]

and so \( m_\infty \) satisfies (4.6).

According to M. Mehta [14], the symmetric matrix integral can now be expressed in terms of the Pfaffian, as follows, taking into account a constant \( c_{2n} \), coming from

\*We have \( \delta(x) = 1 \), for \( x \geq 0 \), and \( \delta(x) = -1 \), for \( x < 0 \).
integrating the orthogonal group:

\[
\frac{1}{(2n)!} \int_{\mathcal{F}_{2n}(E)} e^{\operatorname{Tr}(-V(X) + \sum t_i X_i)} dX = \frac{1}{(2n)!} \int_{\mathbb{R}^{2n}} |\Delta_{2n}(z)| \prod_{i=1}^{2n} e^{\sum t_i^2 \rho(z_i)} dz_i
\]

\[
= \int_{-\infty < z_1 < z_2 < \cdots < z_{2n} < \infty} \det (z_{j+1}^i - \rho_i(z_{j+1}))_{0 \leq i, j \leq 2n-1} \prod_{i=1}^{2n} dz_i
\]

\[
= \int_{-\infty < z_1 < z_4 < \cdots < z_{2n} < \infty} \prod_{k=1}^{n} \rho(z_{2k}) dz_{2n} \times \det \left( \int_{-\infty}^{z_{2i}} z_{2i} \rho(z_1) dz_1, z_{2i}^2, \ldots, \int_{-\infty}^{z_{2i}} z_{2n-1} \rho(z_{2n-1}) dz_{2n-1}, z_{2i}^2 \right)_{0 \leq i \leq 2n-1}
\]

\[
= \int_{-\infty < z_1 < z_4 < \cdots < z_{2n} < \infty} \prod_{i=1}^{n} dz_{2i}
\]

\[
\times \det (F_i(z_2), G_i(z_3), \ldots, F_i(z_{2n}), G_i(z_{2n}))_{0 \leq i \leq 2n-1}
\]

\[
= \frac{1}{n!} \int_{\mathbb{R}^n} \prod_{i=1}^{n} dy_i \det (F_i(y_1), G_i(y_1), \ldots, F_i(y_n), G_i(y_n))_{0 \leq i \leq 2n-1}
\]

\[
= \det^{1/2} \left( \int_{\mathbb{R}} (G_i(y) F_j(y) - F_i(y) G_j(y)) dy \right)_{0 \leq i, j \leq 2n-1}
\]

(\text{using de Bruijn’s lemma (see [14, p. 446]))}

\[
= \text{pf} \left( \int_{\mathbb{R}^2} y^k z^\ell e^{(y-z) e^{t_i^2 \rho_2(z)}} dy dz \right)_{0 \leq k, \ell \leq 2n-1}
\]

\[
= \text{pf} (\mu(t))_{0 \leq i, j \leq 2n-1} = \tau_{2n}(t),
\]

(4.17)

which is a Pfaffian \( \tau \)-function.

\textbf{Symplectic matrix integrals: (}\( \alpha = +1 \))

Here we concentrate on integrals of the type

\[
\int_{\mathcal{F}_{2n}} e^{2 \operatorname{Tr}(-V(X) + \sum t_i X_i)} dX,
\]

(4.18)
where \( dX \) denotes Haar measure*
\[
dX = \prod_{k=1}^{N} dX_k \prod_{k \leq \ell} dX^{(0)}_{k\ell} dX^{(1)}_{k\ell} d\tilde{X}^{(0)}_{k\ell} d\tilde{X}^{(1)}_{k\ell}
\]
on the space \( \mathcal{P}_{2N} \) of self-dual \((N \times N)\)-Hermitian matrices, with real quaternionic entries; the latter can be realized as the space of \((2N \times 2N)\)-matrices with entries \( X^{(i)}_{k\ell} \in \mathbb{C} \),
\[
\mathcal{P}_{2N} = \left\{ X = (X_{k\ell})_{1 \leq k \leq N, \ell \leq N}, X_{k\ell} = \begin{pmatrix} X^{(0)}_{k\ell} & X^{(1)}_{k\ell} \\ \bar{X}^{(1)}_{k\ell} & \bar{X}^{(0)}_{k\ell} \end{pmatrix} \right\} \text{ with } X_{k\ell} = \tilde{X}^\top_{k\ell}.
\]

Another skew-symmetric moment matrix \( m_\infty \) satisfying (4.6) is given by inner-product (4.1) for \( \alpha = 1 \), with \( \rho_t(y) = \rho(y)e^{\sum du \gamma^u} = e^{-V(y)+\sum du} \),
\[
\mu_{ij}(t) = \int_{\mathbb{R}} \{ y^i, y^j \} \rho_t(y)^2 I_E(y) \, dy
= \int_{\mathbb{R}} \{ y^i \rho_t(y), y^j \rho_t(y) \} I_E(y) \, dy
= \int_{\mathbb{R}} (G_i(y)F_j(y) - F_i(y)G_j(y)) \, dy,
\]
on setting \( t = d/dx \)
\[
F_j(x) = x^i \rho_t(x) \quad \text{and} \quad G_j(x) := F_j'(x) = (x^i \rho_t(x))'.
\]

That \( m_\infty \) satisfies (4.6) follows at once from the first expression (4.18):
\[
\mu_{k\ell}(t) = \int \{ y^k, y^\ell \} \rho_t(y)^2 \, dy = \int (k - \ell) y^{k+\ell-1} \rho_t(y)^2 \, dy
\]
\[
\frac{\partial \mu_{k\ell}}{\partial t} = 2 \int \{ y^k, y^\ell \} y^i \rho_t(y)^2 \, dy
= \int ((k + i - \ell) y^{k+i+\ell-1} + (k - \ell - i) y^{k+i+\ell-1}) \rho_t(y)^2 \, dy
= \mu_{k+i,\ell} + \mu_{k,\ell+i},
\]
thus leading to (4.6). Using the relation
\[
\prod_{1 \leq i, j \leq n} (x_i - x_j)^4 = \det (x_1^i, x_1^j) \det (x_2^i, x_2^j) \cdots \det (x_n^i, x_n^j)_{0 \leq i \leq 2n-1},
\]
*\( \tilde{X} \) means the usual complex conjugate. The condition on the \((2 \times 2)\)-matrices \( X_{k\ell} \) implies that \( X_{k\ell} = X_{k\ell} I_2 \), with \( X_k \in \mathbb{R} \) and \( I_2 \) the identity.
one computes, using again de Bruijn's lemma,
\[
\frac{1}{(n)!} \int_{\mathbb{R}_n^2} e^{2\text{Tr}(-V(X)+\sum \eta_i X_i)} dX = \frac{1}{n!} \int_{\mathbb{R}_n^2} \prod_{1 \leq i, j \leq n} (x_i - x_j)^4 \prod_{i=1}^n \rho_t(x_i)^2 dx_i
\]
\[
= \frac{1}{n!} \int_{\mathbb{R}_n^2} \prod_{k=1}^n \rho_t(x_k)^2 dx_k
\]
\[
\times \det (x_1^i (x_1^i)' x_2^i (x_2^i)' \cdots x_n^i (x_n^i)')_{0 \leq i \leq 2n-1}
\]
\[
= \frac{1}{n!} \int_{\mathbb{R}_n^2} \prod_{i=1}^n dy_i \det (F_i(y_1) G_i(y_1) \cdots F_i(y_{n}) G_i(y_{n}))_{0 \leq i \leq 2n-1}
\]
\[
= \det^{1/2} (\int_{\mathbb{R}} (G_j(y)F_j(y) - F_j(y)G_j(y)) dy)_{0 \leq i, j \leq 2n-1}
\]
\[
= \text{pf} (\mu_{ij}(t))_{0 \leq i, j \leq 2n-1} = \tau_{2n}(t),
\]
(4.20)
which is a Pfaffian $\tau$-function as well.

5. A map from the Toda to the Pfaff lattice
Remember from (0.1) the notation $\rho_t(z) = \rho(z) e^{\sum \eta_k z^k}$ and $\rho' / \rho = -g / f$. Assuming, in addition, that $f(z) \rho(z)$ vanishes at the endpoints of the interval under consideration (which could be finite, infinite, or semi-infinite), one checks that the $t$-dependent operator in $z$,
\[
\mathcal{H}_t = \{1, z, z^2, \ldots \}
\]
and is skew-symmetric with respect to the $t$-dependent inner-product $\langle \cdot, \cdot \rangle_t^{\text{Sy}}$, defined by the weight $\rho_t(z) dz$,
\[
\langle n_t, \varphi, \psi \rangle_t^{\text{Sy}} = \int_E \langle n_t, \varphi(z) \rangle_t \psi(z) \rho_t(z) dz = - \int_E \varphi(z) n_t \psi(z) \rho_t(z) dz = -\langle \varphi, n_t \psi \rangle_t^{\text{Sy}}.
\]
The orthonormality of the $t$-dependent polynomials $p_n(t, z)$ in $z$ imply
\[
\langle p_n(t, z), p_m(t, z) \rangle_t^{\text{Sy}} = \delta_{mn}.
\]
The matrices $L$ and $M$ are defined by
\[
z p = L p \quad \text{and} \quad e^{-(1/2) \sum \eta_k z^k} \frac{d}{dz} e^{1/2 \sum \eta_k z^k} p = M p.
\]
The skewness of \( n_t \) implies the skew-symmetry of the matrix

\[
N(t) = f(L)M - \frac{f' + g}{2}(L)
\]

such that \( n_t p(t, z) = N p(t, z); \) (5.2)

so \( N(t) \) can be viewed as the operator \( n_t \), expressed in the polynomial basis \( (p_0(t, z), p_1(t, z), \ldots) \).

In the next theorem, we consider functions \( F \) of two (noncommutative) variables \( z \) and \( n_t \) so that the (pseudo-)differential operator \( u_t := F(z, n_t) \) in \( z \) and the matrix \( U := F(L, N) \) related by

\[
F(z, n_t) p(t, z) = F(L, N) p(t, z),
\]

are skew-symmetric as well. Examples of \( F \)'s are

\[
F(z, n_t) := n_t, \ n_t^{-1}, \text{ or } \{z^\ell, n_t^{2k+1}\}^\dagger,
\]

corresponding to

\[
F(L, N) = N, \ N^{-1}, \text{ or } \{N^{2k+1}, L\}^\dagger.
\]

**THEOREM 5.1**

Any Hankel matrix \( m_\infty \) evolving according to the vector fields

\[
\frac{\partial m_\infty(t)}{\partial t_k} = \Lambda^k m_\infty
\]

leads to matrices \( L \) and \( M \), evolving according to the Toda lattice equations \( \partial L/\partial t_n = (1/2)\left[(L^n)_{sk}, L\right] \) and \( \partial M/\partial t_n = (1/2)\left[(L^n)_{sk}, M\right] \) (see (1.9)). Consider a function \( F \) of two variables such that the operator \( u_t := F(z, n_t) \) is skew-symmetric with respect to \( \langle \cdot, \rangle \) and so the matrix

\[
U(t) = F(L(t), N(t)), \text{ defined by } u_t p(t, z) = U p(t, z),
\]

is skew-symmetric. This induces a natural lower-triangular matrix \( O(t) \), mapping the Toda lattice into the Pfaff lattice (for notation \( \pi_{bo}, \pi_k, \pi_k, \text{ etc.}, \text{ see (1.9), (1.18), (1.17)}):\)

\[
O(t) = \begin{cases} 
p_n(t, z) = (S(t)\chi(z))_n \text{ orthonormal with respect to} \\
m_\infty(t) = (\langle z^i, z^j \rangle t)^{sy}_{0 \leq i, j \leq \infty} = S^{-1}S^{T-1}, \\
L(t) = SAS^{-1} \text{ satisfies} \\
\frac{\partial L}{\partial t_j} = \left[ -\frac{1}{2} \pi_{bo} L^j, L \right], j = 1, 2, \ldots 
\end{cases}
\]

*It is to be understood that \( F(L, N) \) reverses the order of \( z, u \) in \( F(z, u) \).

†We define \( [A, B]^\dagger := AB + BA \).
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map $O(2t)$ such that

\[ -\mathcal{U}(2t) = O^{-1}(2t)J O^T^{-1}(2t), \]

$O(2t)$ is lower-triangular,

$O(2t)S(2t) \in \mathcal{G}_k$

\[
\begin{aligned}
q_n(t, z) &= (O(2t)p(2t, z))_n, \text{ skew-orthonormal with regard to} \\
\tilde{m}_\infty(t) &:= -S^{-1}(2t)\mathcal{U}(2t)S^T(2t) = Q^{-1}(t)\tilde{Q}^T(t) \\
&= \left(\langle z^i, z^j \rangle_{k}^2 \right)_{0 \leq i, j \leq \infty} \\
&= \left(\langle z^i, u_{2t}z^j \rangle_{2t}^2 \right)_{0 \leq i, j \leq \infty}, \\
\tilde{L}(t) &:= O(2t)L(2t)O(2t)^{-1} \text{ satisfies} \\
\frac{\partial \tilde{L}}{\partial t_j} &= [-\pi_k \tilde{L}^j, \tilde{L}], j = 1, \ldots
\end{aligned}
\]

Proof

Since $\mathcal{U}(t)$ is skew-symmetric, it admits a skew-Borel decomposition

\[ -\mathcal{U}(t) = O^{-1}(t)J O^T^{-1}(t), \quad \text{with lower-triangular } O(t). \quad (5.3) \]

But the new matrix, defined by

\[ \tilde{m}_\infty(t) := -S^{-1}(2t)\mathcal{U}(2t)S^T(2t), \quad (5.4) \]

is skew-symmetric and thus admits a unique skew-Borel decomposition

\[ \tilde{m}_\infty(t) = \tilde{Q}^{-1}(t)J \tilde{Q}(t)^T, \quad \text{with } \tilde{Q}(t) \in \mathcal{G}_k. \quad (5.5) \]

Comparing (5.3), (5.4), and (5.5) leads to a unique choice of matrix $O(t)$, skew-Borel decomposing $-\mathcal{U}(2t)$, as in (5.3), such that

\[ O(2t)S(2t) = \tilde{Q}(t) \in \mathcal{G}_k. \quad (5.6) \]

Using, as a consequence of (5.2) and (1.9),

\[ \frac{\partial \mathcal{U}}{\partial t_k}(2t) = [\pi_s L^k(2t), \mathcal{U}(2t)] \]
and
\[ \frac{\partial S}{\partial t_k}(2t) = -(\pi_{bo} L^k(2t)) S(2t), \]

we compute
\[ \frac{\partial \tilde{m}_\infty}{\partial t_k}(t) = S^{-1}(2t) \frac{\partial S}{\partial t_k}(2t) S^{-1}(2t) \mathcal{A} - S^{-1}(2t) \left( \frac{\partial}{\partial t_k} \mathcal{A} \right) S^{-1}(2t) \]
\[ = -S^{-1}(\pi_{bo} L^k(2t)) \mathcal{A} S^{-1}(2t) - S^{-1}(\pi_{sy} L^k, \mathcal{A}) S^{-1}(2t) \]
\[ = -S^{-1}(\pi_{bo} L^k + \pi_{sy} L^k) \mathcal{A} S^{-1}(2t) - S^{-1}(\pi_{bo} L^k - \pi_{sy} L^k) S^{-1}(2t) \]
\[ = -S^{-1} L^k \mathcal{A} S^{-1}(2t) - S^{-1} L^k \mathcal{T} S^{-1}(2t) \text{ (using (5.7))} \]
\[ = -\Lambda^k S^{-1} \mathcal{A} S^{-1}(2t) - S^{-1} \mathcal{A} S^{-1}(2t) \Lambda^k \mathcal{T} S^{-1}(2t) \text{ (using } L^k = S \Lambda^k S^{-1}) \]
\[ = \Lambda^k \tilde{m}_\infty(t) + \tilde{m}_\infty(t) \Lambda^k \text{ (by (5.4))}. \]

For an arbitrary matrix \( A \), we have
\[ A = A^\top \iff A = (A_{bo})^\top - A_{sy}. \quad (5.7) \]

Indeed, remembering that \( A_{bo} = 2A_+ + A_0 \) and \( A_{sy} = A_+ - A_- \), one checks
\[ (A_{bo})^\top - A_{sy} - A = 2(A_+ - A_-) + A_0 = 2(A_+ - A_-) - A_+ + A_0 = -2(A_+ - A_-)^\top, \]
so that the left-hand side vanishes, if the right-hand side does; the latter means \( A \) is symmetric.

We now define \( \tilde{L}(t) \) by conjugation of \( L(2t) \) by \( O(2t) \):
\[ \tilde{L}(t) := O(2t) L(2t) O(2t)^{-1} = O(2t) S(2t) \Lambda S^{-1}(2t) O(2t)^{-1} = \tilde{Q}(t) \Lambda \tilde{Q}^{-1}(t); \]
thus, by Proposition 1.2, \( \tilde{L}(t) \) satisfies the Pfaff Lax equation. Therefore the sequence of polynomials
\[ q(t, z) := O(2t) p(2t, z) = O(2t) S(2t) \chi(z) = \tilde{Q}(t) \chi(z) \]
is skew-orthonormal
\[ (q_i(t, z), q_j(t, z))^k = J_{ij}. \]

*\( A_\pm \) means the usual strictly upper-(lower-)triangular part, and \( A_0 \) means the diagonal part in the common sense.
with regard to the skew inner-product specified by the matrix $\tilde{m}_\infty$:

$$\langle z^i, z^j \rangle^s = \tilde{m}_{ij}(t).$$

In the last step, we show that $\langle \varphi, \psi \rangle^s = \langle \varphi, u \psi \rangle^s_{2t}$. Since

$$U(2t) = -O^{-1}(2t)JO^{-1}(2t) = -\mathcal{H}^T(2t), \quad (5.8)$$

we compute

$$\langle q^i(t, z), (u_2q^j(t, z))_{2t}^s \rangle = \langle (Op)_i(2t), (uOp)_j(2t) \rangle_{2t}^s$$

$$= \langle (Op)_i(2t), (Opu)_j(2t) \rangle_{2t}^s$$

$$= \langle (Op)_i(2t), (Op\mathcal{H})_j(2t) \rangle_{2t}^s$$

$$= \langle (Op)_i(2t), p_{k}(2t)_{k, t \geq 0}(O\mathcal{H})^T(2t) \rangle_{ij}$$

$$= \langle (O2t)(Op\mathcal{H})^T(2t) \rangle_{ij}$$

$$= \langle O2t\mathcal{H}^T(2t)O^T(2t) \rangle_{ij}$$

$$= -\langle O2t\mathcal{H}(2t)O^T(2t) \rangle_{ij}$$

$$= J_{ij} \quad (\text{using (5.8)}). \quad (5.9)$$

Therefore, defining a new skew-inner-product $\langle \cdot, \cdot \rangle^s$:

$$\langle \varphi, \psi \rangle^s := \langle \varphi, u \psi \rangle^s_{2t},$$

we have shown

$$\langle q^i, q^j \rangle_{t}^s = \langle q^i, q^j \rangle_{t}^s = J_{ij},$$

and so by completeness of the basis $q^i$, we have

$$\langle \cdot, \cdot \rangle_{t}^s = \langle \cdot, \cdot \rangle_{t}^s,$$

thus ending the proof of Theorem 4.1.

6. Example 1: From Hermitian to symmetric matrix integrals

Striking examples are given by using the map $O(t)$ obtained from skew-Borel decomposing $N^{-1}(t)$ and $N(t)$ (see (5.2)). This section deals with $N^{-1}(t)$, whereas Section 7 deals with $N(t)$.

**Proposition 6.1**

The special transformation

$$\mathcal{H}(t) = N^{-1}(t) = \left( f(L)M - \frac{f'(L) + g}{2}(L) \right)^{-1}(t)$$
maps the Toda lattice $\tau$-functions with initial weight $\rho = e^{-V}$, $V' = -g/f$ (Hermitean matrix integral) to the Pfaff lattice $\tau$-functions (symmetric matrix integral), with initial weight

$$\tilde{\rho}_t(z) := \left(\frac{\rho_{2t}(z)}{f(z)}\right)^{1/2} = e^{-(1/2)(V(z)+\log f(z)-2\sum \mu c')}$$

$$= e^{\tilde{V}(z)+\sum \mu c'} = \tilde{\rho}(z)e^{\sum \mu c'} .$$

To be precise:

$$\left\{ \begin{array}{l}
\mu_{ij}(t) = \langle \zeta^i, \zeta^j \rangle, \text{ and } m_n = (\mu_{ij})_{0 \leq i,j \leq n-1}, \\
\tau_t(z) = \det m_n = \frac{1}{n!} \int_{\mathcal{O}_n} e^{\text{Tr}(-V(X)+\sum \mu_i X_i)} dX
\end{array} \right.$$ 

map $O(2t)$ such that

$$\left\{ \begin{array}{l}
-A^{-1}(2t) = O^{-1}(2t)J O^T-1(2t), \\
O(2t) \text{ is lower-triangular,} \\
O(2t)S(2t) \in \mathfrak{g}_k
\end{array} \right.$$ 

$$\left\{ \begin{array}{l}
q_n(t, z) = (O(2t) p(2t, z))_n \text{ skew-orthonormal polynomials} \\
in z for the skew-inner-product (weight}\tilde{\rho}, \\
\langle \varphi, \psi \rangle_{\tilde{\rho}} := \langle \varphi, n_{2t}^{-1}\psi \rangle_{2t} \\
= v \frac{1}{2} \int_{\mathbb{R}^2} \varphi(x) \psi(y) e^{(x-y)\sum \mu (x'+y')} \\
\times \sqrt{\frac{\rho}{f}}(x) \sqrt{\frac{\rho}{f}}(y) dx dy, \\
\tilde{\mu}_{ij}(t) = \langle x^i, y^j \rangle, \text{ and } \tilde{m}_n = (\tilde{\mu}_{ij})_{0 \leq i,j \leq n-1}, \\
\tilde{\tau}_{2n}(t) = \text{pf}(\tilde{m}_{2n}) = \frac{1}{2^{n(2n)!}} \int_{\mathcal{O}_{2n}} e^{\text{Tr}(-\tilde{V}(X)+\sum \mu_i X_i)} dX,
\end{array} \right.$$ 

with $\tilde{V}(z) = \frac{1}{2} (V(z) + \log f(z))$. 

$$\left\{ \begin{array}{l}
\end{array} \right.$$
In the first integral defining $\tau_n(t)$, $dX$ denotes Haar measure on Hermitian matrices (see Section 4.1), whereas the second integral $\tilde{\tau}_{2n}(t)$ involves Haar measure on symmetric matrices (see Section 4.2).

**Proof**

At first, check that

$$\left( \frac{d}{dx} \right)^{-1} \varphi(x) = \frac{1}{2} \int \delta(x - y)\varphi(y) \, dy. \quad (6.1)$$

Indeed,

$$\frac{d}{dx} \left( \frac{d}{dx} \right)^{-1} \varphi(x) = \int \frac{1}{2} \frac{\partial}{\partial x} \delta(x - y)\varphi(y) \, dy = \int \delta(x - y)\varphi(y) \, dy \quad \text{(using } \frac{\partial}{\partial x} \delta(x) = 2\delta(x))$$

$$= \varphi(x).$$

Consider now the operator

$$u_t = \mathbf{n}^{-1}_t = \left( \frac{f_t}{\rho_t} \frac{d}{dx} \sqrt{f_t} \right)^{-1}, \quad \text{so that } u_t p = \mathbf{n}^{-1}_t p = \mathcal{N}^{-1} p,$$

according to (5.2). Let it act on a function $\varphi(x)$:

$$\mathbf{n}^{-1}_t \varphi(x) = \left( \frac{1}{\sqrt{f_t(x)\rho_t(x)}} \left( \frac{d}{dx} \right)^{-1} \frac{\rho_t(x)}{f_t(x)} \right) \varphi(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{f_t(x)\rho_t(x)}} \frac{\varepsilon(x - y)}{2} \sqrt{\frac{\rho_t(y)}{f_t(y)}} \varphi(y) \, dy \quad \text{(using (6.1)).}$$

One computes

$$\langle \varphi, \psi \rangle_{t}^{sk} = \langle \varphi, \mathbf{u}_t \psi \rangle_{2t}^{sy} = \langle \varphi, \mathbf{n}^{-1}_t \psi \rangle_{2t}^{sy} = \frac{1}{2} \int \int_{\mathbb{R}^2} \sqrt{\frac{\rho_t(x)}{f_t(x)}} \varepsilon(x - y) \sqrt{\frac{\rho_t(y)}{f_t(y)}} \varphi(x) \psi(y) \, dx \, dy$$

$$= \frac{1}{2} \int \int_{\mathbb{R}^2} \tilde{\rho}(x) \tilde{\rho}(y) e^{\sum_{i=1}^{\infty} n_i (x^i + y^i)} e^{\varepsilon(x - y) \varphi(x) \psi(y)} \, dx \, dy.$$ 

So, finally setting $\tilde{V}(x) = (1/2)(V(x) + \log f(x))$ yields by (4.17) that

$$\tilde{\tau}_{2n}(t) = \text{pf}(\tilde{m}_{2n}) = \frac{1}{(2n)!} \int_{\mathcal{S}_{2n}} e^{\text{Tr}(-\tilde{V}(X) + \sum_{i=1}^{\infty} \tilde{n}_i X^i)} \, dX.$$
The map $O$ for the classical orthogonal polynomials at $t = 0$

Then, the matrix $O$ mapping orthonormal $p_k$ into skew-orthonormal polynomials $q_k$, is given by a lower-triangular three-step relation:

$$q_{2n}(0,z) = \sqrt{\frac{c_{2n}}{a_{2n}}} p_{2n}(0,z),$$

$$q_{2n+1}(0,z) = \sqrt{\frac{a_{2n}}{c_{2n}}} \left( -c_{2n-1} p_{2n-1}(0,z) + \frac{c_{2n}}{a_{2n}} \left( \sum_{i=0}^{2n} b_i \right) p_{2n}(0,z) + c_{2n} p_{2n+1}(0,z) \right),$$

(6.2)

where the $a_i$ and $b_i$ are the entries in the tridiagonal matrix defining the orthonormal polynomials, and the $c_i$'s are the entries of the skew-symmetric matrix $\mathcal{N}$.

In [6], we showed that, in the classical cases below, $\mathcal{N}$ is tridiagonal, at the same time as $L$ (see Appendix B):

$$L = \begin{bmatrix} b_0 & a_0 & & & \\ a_0 & b_1 & a_1 & & \\ & a_1 & b_2 & & \\ & & & \ddots & \\ & & & & \end{bmatrix}, \quad -\mathcal{N} = \begin{bmatrix} 0 & c_0 & & & \\ -c_0 & 0 & c_1 & & \\ & -c_1 & 0 & & \\ & & & \ddots & \\ & & & & \end{bmatrix},$$

(6.3)

with the following precise entries:

**Hermite:** $\rho(z) = e^{-z^2}$, $a_{n-1} = \sqrt{n/2}$, $b_n = 0$, $c_n = a_n$;

**Laguerre:** $\rho(z) = e^{-z} z^\alpha I_{(0,\infty)}(z)$, $a_{n-1} = \sqrt{n(1+\alpha)}$, $b_n = 2n + \alpha + 1$, $c_n = a_n/2$;

**Jacobi:** $\rho(z) = (1-z)^\alpha (1+z)^\beta I_{[-1,1]}(z)$;

$$a_{n-1} = \left( \frac{4n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)(2n+\alpha+\beta+1)} \right)^{1/2}$$

$$b_n = \left( \frac{\alpha^2 - \beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} \right),$$

$$c_n = a_n \left( \frac{\alpha + \beta}{2} + n + 1 \right).$$

If the skew-symmetric matrix $\mathcal{N}$ has the tridiagonal form above, then one checks that
its inverse has the following form:

\[
-N^{-1} = \begin{pmatrix}
0 & -\frac{1}{c_0} & 0 & -\frac{c_1}{c_0c_2} & 0 & -\frac{c_1c_3}{c_0c_2c_4} & 0 & -\frac{c_1c_3c_5}{c_0c_2c_4c_6} \\
\frac{1}{c_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{c_2} & 0 & -\frac{c_3}{c_2c_4} & 0 & -\frac{c_3c_5}{c_2c_4c_6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{c_1}{c_0c_2} & 0 & \frac{1}{c_2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{c_1}{c_0c_2c_4} & 0 & \frac{1}{c_2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{c_6} & 0 \\
\end{pmatrix}
\]

(6.4)

In order to find the matrix \(O\), we must perform the skew-Borel decomposition of the matrix \(-\mathcal{U}\):

\[-\mathcal{U} = -N^{-1} = O^{-1}JO^{-1}.

The recipe for doing so is given in Theorem 4.1 (see also the important remark following that theorem). It suffices to form the Pfaffians (0.18) by appropriately bordering the matrix \(-N^{-1}\), as in (0.18), with rows and columns of powers of \(z\), yielding skew-orthonormal polynomials; we choose to call them \(r_k\)'s, instead of the \(q\)'s of Theorem 4.1, with \(O^k(z) = r(z)\). They turn out to be the following simple polynomials, with \(1/r_{2n} = c_0c_2c_4 \cdots c_{2n-2}z^{n-1}\):

\[
r_{2n}(z) = \frac{1}{\sqrt{\frac{z}{r_{2n}}}} \frac{c_{2n}z^{2n}}{c_0c_2 \cdots c_{2n}} = \frac{1}{\sqrt{c_{2n}}} c_{2n}z^{2n},
\]

\[
r_{2n+1}(z) = \frac{1}{\sqrt{\frac{z}{r_{2n}}}} \frac{c_{2n}z^{2n+1} - c_{2n-1}z^{2n-1}}{c_0c_2 \cdots c_{2n}} = \frac{1}{\sqrt{c_{2n}}} (c_{2n}z^{2n+1} - c_{2n-1}z^{2n-1}).
\]

Then, also from Appendix A, in order to get \(O \rightarrow \hat{O}\) in the correct form, we compute the skew-orthonormal polynomials \(\hat{r}_k\), with \(O^\hat{k}(z) = \hat{r}(z)\):

\[
\hat{r}_{2n}(z) = \frac{1}{\sqrt{a_{2n}}} r_{2n}(z) = \frac{c_{2n}}{a_{2n}} z^{2n},
\]

\[
\hat{r}_{2n+1}(z) = \frac{\sum_{b} b_{i} \sqrt{a_{2n}}}{\sqrt{a_{2n}}} r_{2n}(z) + \sqrt{a_{2n}} \hat{r}_{2n+1}(z) = \sqrt{\frac{a_{2n}}{c_{2n}}} \left( -c_{2n-1}z^{2n-1} + \frac{c_{2n}}{a_{2n}} \left( \sum_{b} b_{i} \right) z^{2n} + c_{2n}z^{2n+1} \right). \quad (6.5)
\]

From the coefficients of the polynomial \(\hat{r}_k\), one reads off the transformation matrix from orthonormal to skew-orthonormal polynomials; it is given by the matrix
\(\hat{O}\) such that \(\hat{O} X(z) = \hat{r}(z)\). Therefore \(q(t, z) = \hat{O}(2t)p(2t, z)\) yields, after setting \(t = 0\),

\[
q_{2n}(0, z) = \sqrt{\frac{c_{2n}}{a_{2n}}} p_{2n}(0, z),
\]

\[
q_{2n+1}(0, z) = \sqrt{\frac{d_{2n}}{c_{2n}}} 
\times \left( - c_{2n-1} p_{2n-1}(0, z) + \frac{c_{2n}}{d_{2n}} \sum_{0}^{2n} b_i \right) p_{2n}(0, z) + c_{2n} p_{2n+1}(0, z),
\]

(6.6)

confirming (6.2).

7. Example 2: From Hermitian to symplectic matrix integrals

**Proposition 7.1**

The matrix transformation

\[ N = f(L)M - \frac{f'}{2} + g(L) \]

maps the Toda lattice \(\tau\)-functions with \(t\)-dependent weight

\[ \rho_t(z) = e^{-V(z)+\sum \infty i z^i}, \quad V' = g/f \]

(Hermitian matrix integral) to the Pfaff lattice \(\tau\)-functions (symplectic matrix integral), with \(t\)-dependent weight

\[ \tilde{\rho}_t(z) := \left( \rho_{2t}(z) f(z) \right)^{1/2} = e^{-\left((1/2)(V(z)-\log f(z)) - \sum \infty i z^i\right)} \]

\[ = e^{-V(z)+\sum \infty i z^i} \tilde{\rho}(z)e^{\sum \infty i z^i}. \]

To be precise:

- \(p_n(t, z)\) orthonormal polynomials in \(z\) for the inner-product
  \[ (\varphi, \psi)_t^S = \int \varphi(z) \psi(z) e^{\sum \infty i z^i} \rho(z) \rho(z) \] with \(\rho_t(z) = (\rho_{2t}(z) f(z))^{1/2} = e^{-\left((1/2)(V(z)-\log f(z)) - \sum \infty i z^i\right)} \]

- \(\mu_{ij}(t) = (z^i, z^j)_t^S\) and \(m_n = \mu_{ij} \sum_{i, j} \leq n-1\)

- \(\tau_n(t) = \det m_n(t) = \frac{1}{n!} \int e^{Tr(-V(X)+\sum \infty i x^i)} dX \)
map $O(2t)$ such that

\[
\begin{align*}
-\mathcal{N}(2t) &= O^{-1}(2t)JO^T(2t), \\
O(2t) &= \text{lower-triangular}, \\
O(2t)S(2t) &= \mathbb{G}_k
\end{align*}
\]

\[
q_n(t, z) = \left( O(2t)p(2t, z) \right)_n \quad \text{skew-orthonormal polynomials in } z \text{ for the skew-inner-product (weight } \tilde{\rho}_t),
\]

\[
\langle \varphi, \psi \rangle_{sk} := \langle \varphi, n_{2t}\psi \rangle_{2t}
\]

\[
\tilde{\mu}_{ij}(t) = \langle z^i, z^j \rangle_{sk} \quad \text{and } \tilde{m}_n = \det(\tilde{\mu}_{ij})_{0 \leq i, j \leq n-1},
\]

\[
\tilde{r}_{2n}(t) = \text{pf}(\tilde{m}_n(t)) = \frac{1}{(-2)^n n!} \int_{\mathbb{R}^n} e^{2Tr(-\tilde{V}(X)+\sum_i x_i)} dX,
\]

with $\tilde{V}(z) = \frac{1}{2}(V(z) - \log f(z))$.

**Proof**

Representing $d/dx$ as an integral operator

\[
\frac{d}{dx}\varphi(x) = \int_{\mathbb{R}} \delta(x-y)\varphi'(y) dy = -\int_{\mathbb{R}} \frac{\partial}{\partial y} \delta(x-y)\varphi(y) dy = \int_{\mathbb{R}} \delta'(x-y)\varphi(y) dy,
\]

compute

\[
u_t = n_t = \sqrt{\frac{f}{\rho_t}} \frac{d}{dz} \sqrt{f \rho_t}, \quad \text{so that } n_t p(t, z) = \mathcal{N} p(t, z);
\]

remember $\mathcal{N}$ from (5.2). Let it act on a function $\varphi(x)$:

\[
\text{u}_t \varphi(x) = \left( \sqrt{\frac{f}{\rho_t} \frac{d}{dx} \sqrt{f \rho_t}} \right) \varphi(x)
\]

\[
= \int_{\mathbb{R}} \frac{f(x)}{\rho_t(x)} \delta'(x-y)\sqrt{f(y)\rho_t(y)}\varphi(y) dy.
\]
Then
\[ \langle \varphi, \psi \rangle_{sk}^t = \langle \varphi, \mathbf{u}_t \psi \rangle_{sk}^t = \langle \varphi, \mathbf{n}_t \psi \rangle_{sk}^t \]
\[ = \int \int_{\mathbb{R}^2} \rho_{2t}(x)\varphi(x) \sqrt{\frac{f(x)}{\rho_{2t}(x)}} \delta'(x-y) \sqrt{f(y)}\rho_{2t}(y)\psi(y) \, dx \, dy \]
\[ = \int \int_{\mathbb{R}^2} \sqrt{f(x)\rho_{2t}(x)}\varphi(x) \delta'(x-y) \sqrt{f(y)}\rho_{2t}(y)\psi(y) \, dx \, dy \]
\[ = -\int \int_{\mathbb{R}^2} \left( \frac{\partial}{\partial x} \sqrt{f(x)\rho_{2t}(x)}\varphi(x) \right) \delta(x-y) \sqrt{f(y)}\rho_{2t}(y)\psi(y) \, dx \, dy \]
\[ = -\int_{\mathbb{R}} \left( \frac{\partial}{\partial x} \sqrt{f(x)\rho_{2t}(x)}\varphi(x) \right) \sqrt{f(x)\rho_{2t}(x)} \psi(x) \, dx \]
\[ = -\frac{1}{2} \int_{\mathbb{R}} \left( \frac{\partial}{\partial x} \sqrt{f(x)\rho_{2t}(x)}\varphi(x) \right) \sqrt{f(x)\rho_{2t}(x)} \psi(x) \, dx \]
\[ + \frac{1}{2} \int_{\mathbb{R}} \sqrt{f(x)\rho_{2t}(x)}\varphi(x) \left( \frac{\partial}{\partial x} \sqrt{f(x)\rho_{2t}(x)} \psi(x) \right) \, dx \]
\[ = -\frac{1}{2} \int_{\mathbb{R}} \left\{ \sqrt{f(x)\rho_{2t}(x)}\varphi(x), \sqrt{f(x)\rho_{2t}(x)} \psi(x) \right\} \, dx \]
\[ = -\frac{1}{2} \int_{\mathbb{R}} \left\{ \varphi(x), \psi(x) \right\} \tilde{\rho}_0^2(x) e^{2 \sum \limits_{i=1}^{\infty} t_i x_i} \, dx, \]

using the notation in the statement of this proposition. Setting \( \tilde{\rho}(x) = e^{-\tilde{V}(x)} \), with \( \tilde{V}(x) = (1/2)(V(x) - \log f) \),
\[ \langle x^i, x^j \rangle_{sk} = -\frac{1}{2} \int_{\mathbb{R}} \{ x^i, x^j \} \tilde{\rho}^2(x) e^{2 \sum \limits_{i=1}^{\infty} t_i x_i} \, dx \]
\[ = -\frac{1}{2} \int_{\mathbb{R}} \{ x^i, x^j \} e^{-2(\tilde{V}(x) - \sum \limits_{i=1}^{\infty} t_i x_i)} \, dx, \]

and so
\[ \tau_{2n}(t) = \text{pf} \left( \bar{m}_{2n}(t) \right) = \frac{1}{(-2)^n n!} \int_{\mathbb{R}^2n} e^{2 \text{Tr}(\tilde{V}(x) + \sum \limits_{i=1}^{\infty} t_i x_i)} \, dx. \]
The map $O^{-1}$ for the classical orthogonal polynomials at $t = 0$

Then, the matrix $O$, mapping orthonormal $p_k$ into skew-orthonormal polynomials $q_k$, is given by a lower-triangular three-step relation:

\[
p_{2n}(0, z) = -c_{2n-1} \sqrt{\frac{a_{2n-2}}{c_{2n-2}}} q_{2n-2}(0, z) + \sqrt{a_{2n}c_{2n}} q_{2n}(0, z),
\]

\[
p_{2n+1}(0, z) = -c_{2n} \sqrt{\frac{a_{2n-2}}{c_{2n-2}}} q_{2n-2}(0, z)
- \left( \sum_{i=0}^{2n} b_i \right) \sqrt{\frac{c_{2n}}{a_{2n}}} q_{2n}(0, z) + \sqrt{\frac{c_{2n}}{a_{2n}}} q_{2n+1}(0, z),
\]

(7.1)

where the $a_i$ and $b_i$ are the entries in the tridiagonal matrix defining the orthonormal polynomials, and the $c_i$ are the entries in the skew-symmetric matrix.

In this case, we need to perform the following skew-Borel decomposition at $t = 0$:

\[-\mathcal{U} = -\mathcal{N} = O^{-1} J O^\top - 1,\]

where $\mathcal{N}$ is the matrix (6.3). Here again, in order to find $O$, we use the recipe given in Theorem 4.1, namely, writing down the corresponding skew-orthogonal polynomials (0.18), but where the $\mu_{ij}$ are the entries of $-\mathcal{U} = -\mathcal{N}$; consider the Pfaffians of the bordered matrices (0.18); they have the leading term

\[
\tilde{\tau}_{2n} = \prod_{i=0}^{n-1} c_{2j}.
\]

Then one computes

\[
r_{2n} = \frac{1}{\sqrt{\tau_{2n} \tau_{2n+2}}} \sum_{i=0}^{n} z^{2n-2i} \left( \prod_{j=0}^{n-i-1} c_{2j} \right) \left( \prod_{j=0}^{i-1} c_{2n-2j-1} \right),
\]

\[
r_{2n+1} = \frac{1}{\sqrt{\tau_{2n+1} \tau_{2n+2}}} \left( z^{2n+1} - \prod_{j=0}^{i} c_{2j} + \sum_{i=1}^{n} z^{2n-2i} \left( \prod_{j=0}^{n-i-1} c_{2j} \right) \left( \prod_{j=0}^{i-1} c_{2n-2j-1} \right) \right),
\]

(7.2)

with

\[
\sqrt{\tau_{2n} \tau_{2n+2}} = c_0 c_2 \cdots c_{2n-2} \sqrt{c_{2n}}, \quad \sqrt{\tau_0 \tau_2} = \sqrt{c_0}.
\]

Setting

\[
D := \text{diag}(\sqrt{\tau_0 \tau_2}, \sqrt{\tau_0 \tau_4}, \sqrt{\tau_2 \tau_4}, \sqrt{\tau_2 \tau_6}, \ldots),
\]
the matrix $O$ is the set of coefficients of the polynomials above, that is,

$$O = D^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_1 & 0 & c_0 & 0 & 0 & 0 \\
0 & 0 & c_2 & 0 & 0 & c_0 & 0 & 0 \\
0 & 0 & c_1 c_3 & 0 & c_0 c_3 & 0 & c_0 c_2 & 0 & 0 \\
0 & 0 & c_1 c_4 & 0 & c_0 c_4 & 0 & 0 & c_0 c_2 & 0 \\
0 & 0 & c_1 c_5 & 0 & c_0 c_5 & 0 & c_0 c_2 c_5 & 0 & c_0 c_2 c_4 \\
0 & 0 & c_1 c_6 & 0 & c_0 c_6 & 0 & c_0 c_2 c_6 & 0 & c_0 c_2 c_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix} \quad \vphantom{D}$$

\[=: D^{-1} R. \quad (7.3)\]

As before, in order to get the skew-symmetric polynomials in the right form, from the orthogonal ones, one needs to multiply to the left with the matrix $E$, defined in (A.2):

$$\hat{O} = EO = ED^{-1} R, \quad (7.4)$$

and so,

$$\hat{O}^{-1} = R^{-1} D^{-1} E^{-1}; \quad (7.5)$$

it turns out the matrix $\hat{O}$ is complicated, but its inverse is simple. Namely, compute

$$R^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vphantom{1}c_1 & 0 & \frac{1}{c_0} & 0 & 0 & 0 & 0 & 0 \\
\vphantom{1}c_2 & 0 & 0 & \frac{1}{c_0} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{c_3}{c_0 c_2} & 0 & \frac{1}{c_0 c_2} & 0 & 0 & 0 \\
0 & 0 & -\frac{c_4}{c_0 c_2} & 0 & 0 & \frac{1}{c_0 c_2} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{c_5}{c_0 c_2 c_4} & 0 & \frac{1}{c_0 c_2 c_4} & 0 \\
0 & 0 & 0 & 0 & -\frac{c_6}{c_0 c_2 c_4} & 0 & 0 & \frac{1}{c_0 c_2 c_4} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}.$$

We made all $c$'s roman here and in (7.3). OK?
and

$$E^{-1} = \begin{pmatrix}
\alpha_0 & 0 & 0 & 0 \\
-\beta_0 & \frac{1}{\alpha_0} & 0 & 0 \\
\alpha_2 & 0 & 0 & 0 \\
-\beta_2 & \frac{1}{\alpha_2} & 0 & 0 \\
\alpha_4 & 0 & 0 & 0 \\
-\beta_4 & \frac{1}{\alpha_4} & 0 & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}, \quad (7.6)$$

with $\alpha_{2n}$ and $\beta_{2n}$ as in (A.5). Carrying out the multiplication (7.5) leads to the matrix $\hat{O}^{-1}$, with a few nonzero bands, yielding the map (7.1), by the recipe of Proposition 6.1 inverted.

---

**Appendix A. Free parameter in the skew-Borel decomposition**

If the Borel decomposition of $-H = O^{-1} J O^{-1}$ is given by a matrix $O \in \mathcal{B}_k$, with the diagonal part of $O$ being

$$(O)_0 = \begin{pmatrix}
\sigma_0 & 0 & 0 & 0 \\
0 & \sigma_0 & 0 & 0 \\
\sigma_2 & 0 & 0 & 0 \\
0 & \sigma_2 & 0 & 0 \\
\sigma_4 & 0 & 0 & 0 \\
0 & \sigma_4 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}, \quad (A.1)$$

then the new matrix

$$\hat{O} := \begin{pmatrix}
\frac{1}{\alpha_0} & 0 & 0 & 0 \\
\beta_0 & \alpha_0 & 0 & 0 \\
\frac{1}{\alpha_2} & 0 & 0 & 0 \\
\beta_2 & \alpha_2 & 0 & 0 \\
\frac{1}{\alpha_4} & 0 & 0 & 0 \\
\beta_4 & \alpha_4 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix} O =: EO, \quad (A.2)$$
with free parameters \( \alpha_{2n}, \beta_{2n} \), is a solution of the Borel decomposition \( -H = \hat{\mathbf{O}}^{-1}J\hat{\mathbf{O}}^{\top}^{-1} \), as well. The diagonal part of \( \hat{\mathbf{O}} \) consists of \((2 \times 2)\)-blocks.

\[
\begin{pmatrix}
\frac{1}{\alpha_{2n}} & 0 \\
\beta_{2n} & \alpha_{2n}
\end{pmatrix}
\begin{pmatrix}
\sigma_{2n} \\
0
\end{pmatrix}
= \begin{pmatrix}
\sigma_{2n}/\alpha_{2n} & 0 \\
\beta_{2n}\sigma_{2n} & \alpha_{2n}\sigma_{2n}
\end{pmatrix}.
\]

Imposing the condition that

\[
q_i(z) = \sum_{0 \leq j \leq i} \hat{\mathbf{O}}_{ij} p_j(z), \quad \text{with} \quad p_k(z) = \sum_{i=0}^{k} p_{ki}z^i,
\]

has the required form, that is, the same leading term for \( q_{2n} \) and \( q_{2n+1} \) and no \( z^{2n} \)-term in \( q_{2n+1} \),

\[
q_{2n}(z) = q_{2n,2n}z^{2n} + \cdots,
q_{2n+1}(z) = q_{2n,2n}z^{2n+1} + q_{2n,2n-1}z^{2n-1} + \cdots \quad (A.3)
\]

implies

\[
\frac{\sigma_{2n}}{\alpha_{2n}}p_{2n,2n} = \sigma_{2n}\alpha_{2n}p_{2n+1,2n+1},
\sigma_{2n}\beta_{2n}p_{2n,2n} + \sigma_{2n}\alpha_{2n}p_{2n+1,2n} = 0
\]

yielding, upon using the explicit form of the coefficients \( p_{k\ell} \) of the polynomials \( p_k \), associated with three-step relations (see Lemma A.1),

\[
\alpha_{2n}^2 = \frac{p_{2n,2n}}{p_{2n+1,2n+1}} = a_{2n},
\beta_{2n} = -\frac{p_{2n+1,2n}}{p_{2n,2n}} = \sum_{i=0}^{2n} b_i \quad (A.4)
\]

Hence

\[
\alpha_{2n} = \sqrt{a_{2n}} \quad \text{and} \quad \beta_{2n} = \frac{1}{\sqrt{a_{2n}}} \sum_{i=0}^{2n} b_i. \quad (A.5)
\]

So, if

\[
r(z) = O \chi(z),
\]
then (A.2) yields
\[
\hat{r}(z) := \hat{O}\chi(z) = \begin{pmatrix}
\frac{1}{\alpha_0} & 0 \\
\beta_0 & \alpha_0 \\
\frac{1}{\alpha_2} & 0 \\
\beta_2 & \alpha_2 \\
\frac{1}{\alpha_4} & 0 \\
\beta_4 & \alpha_4 \\
0 & \ddots
\end{pmatrix}
\]
and thus
\[
\hat{r}_2n(z) = \frac{1}{\sqrt{a_{2n}}}r_{2n}(z), \quad (A.6)
\]
\[
\hat{r}_{2n+1}(z) = \frac{\sum_{i=0}^{2n} b_i}{\sqrt{a_{2n}}}r_{2n} + \sqrt{a_{2n}}r_{2n+1}(z). \quad (A.7)
\]

**Lemma A.1**

A sequence of polynomials \( p_n(z) = \sum_{i=0}^{n} p_{ni}z^i \) of degree \( n \) satisfying three-step recursion relation
\[
zp_n = a_{n-1}p_{n-1} + b_np_n + a_np_{n+1}, \quad n = 0, 1, \ldots, \quad (A.8)
\]
has the form
\[
p_{n+1}(z) = \frac{p_{n,n}}{a_n} \left( z^{n+1} - \left( \sum_{i=0}^{n} b_i \right) z^n + \ldots \right).
\]

**Proof**

Equating the \( z^{n+1} \) and \( z^n \) coefficients of (A.8) divided by \( p_{n,n} \) yields
\[
\frac{p_{n+1,n+1}}{p_{n,n}} = \frac{1}{a_n}
\]
and
\[
\frac{p_{n,n-1}}{p_{n,n}} = a_n \frac{p_{n+1,n}}{p_{n,n}} + b_n.
\]
Combining both equations leads to
\[
\frac{a_n p_{n+1,n}}{p_{n,n}} - a_{n-1} \frac{p_{n,n-1}}{p_{n-1,n-1}} = -b_n,
\]

*We set \( a_{-1} = 0. \)
yielding
\[ a_n \frac{p_{n+1,n}}{p_{n,n}} = - \sum_{0}^{n} b_i \quad \text{(using } a_{-1} = 0). \]

Appendix B. Simultaneous (skew-)symmetrization of \( L \) and \( N \)

**Claim**

For the classical polynomials, the matrices \( L \) and \( N \) can be simultaneously symmetrized and skew-symmetrized.

**Sketch of proof**

This statement has been established by us in [6]. Given the monic orthogonal polynomials \( \tilde{p}_n \) with respect to the weight \( \rho \), with \( \rho'/\rho = -g/f \), we have that the operators \( z \) and
\[ n = \sqrt{f} \frac{d}{\rho d z} \sqrt{f \rho} = f \frac{d}{d z} + \frac{f' - g}{2} \]
acting on the polynomials \( \tilde{p}_n \)'s have the following form:
\[ z \tilde{p}_n = a_{n-1}^2 \tilde{p}_{n-1} + b_n \tilde{p}_n + \tilde{p}_{n+1}, \]
\[ n \tilde{p}_n = \ldots - \gamma_n \tilde{p}_{n+1}, \quad (B.1) \]
in view of the fact that for the classical orthogonal polynomials,\(^*\)

\[
\begin{align*}
\text{Hermite:} & \quad n = \frac{d}{dz} - z, \\
\text{Laguerre:} & \quad n = z \frac{d}{dz} - \frac{1}{2} (z - \alpha - 1), \\
\text{Jacobi:} & \quad n = (1 - z^2) \frac{d}{dz} - \frac{1}{2} (\alpha + \beta + 2) z + (\alpha - \beta)).
\end{align*}
\]

For the orthonormal polynomials, the matrices \( L \) and \( -N \) are symmetric and skew-symmetric, respectively. Therefore the right-hand side of these expressions must have the form:
\[ z \tilde{p}_n = a_{n-1}^2 \tilde{p}_{n-1} + b_n \tilde{p}_n + \tilde{p}_{n+1}, \]
\[ n \tilde{p}_n = a_{n-1}^2 \gamma_{n-1} \tilde{p}_{n-1} - \gamma_n \tilde{p}_{n+1}. \]

Therefore, upon rescaling the \( \tilde{p}_n \)'s, to make them orthonormal, we have
\[ zp_n = (Lp)_n = a_{n-1} p_{n-1} + b_n p_n + a_n p_{n+1}, \]
\[ np_n = (Np)_n = a_{n-1} \gamma_{n-1} \tilde{p}_{n-1} - a_n \gamma_n \tilde{p}_{n+1}. \]

\(^*\)They have the respective weights \( \rho = e^{-z^2}, \rho = e^{-z} z^\alpha, \rho = (1-z)\alpha(1+z)\beta. \)
from which it follows that
\[ -\mathcal{N} = \begin{bmatrix} 0 & c_0 \\ -c_0 & 0 & c_1 \\ -c_1 & 0 & \ddots \end{bmatrix}, \quad \text{with } c_n = a_n\gamma_n, \]

where \(-\gamma_n\) is the leading term in expression (B.1). \(\square\)

**Appendix C. Proof of Lemma 3.4**

For future use, consider the first-order differential operators
\[ \eta(t, z) = \sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial t_j} \quad \text{and} \quad \mathcal{B}(z) = -\frac{\partial}{\partial z} + \sum_{j=1}^{\infty} z^{-j-1} \frac{\partial}{\partial t_j} \quad (C.1) \]

having the property
\[ \mathcal{B}(z)e^{-\eta(z)}f(t) = \mathcal{B}(z)f(t - [z^{-1}]) = 0. \quad (C.2) \]

**LEMMA C.1**

Consider an arbitrary function \(\phi(t, z)\) depending on \(t \in \mathbb{C}^\infty, z \in \mathbb{C}\), having the asymptotics \(\phi(t, z) = 1 + O(1/z)\) for \(z \not\to \infty\) and satisfying the functional relation
\[ \frac{\phi(t - [z^{-1}], z_1)}{\phi(t, z_1)} = \frac{\phi(t - [z_1^{1}], z_2)}{\phi(t, z_2)}, \quad t \in \mathbb{C}^\infty, \ z \in \mathbb{C}. \quad (C.3) \]

Then there exists a function \(\tau(t)\) such that
\[ \phi(t, z) = \frac{\tau(t - [z^{-1}])}{\tau(t)}. \quad (C.4) \]

**Proof**

Applying \(\mathcal{B}_1 := \mathcal{B}(z_1)\) to the logarithm of (C.3) and using (C.1) and (C.2) yields
\[ (e^{-\eta(z)} - 1)\mathcal{B}_1 \log\phi(t; z_1) = -\mathcal{B}_1 \log\phi(t, z_2) \]
\[ = -\sum_{j=1}^{\infty} z_1^{-j-1} \frac{\partial}{\partial t_j} \log\phi(t, z_2), \]

which, upon setting
\[ f_j(t) = \text{Res}_{z_1=\infty} z_1^{j} B_1 \log\phi(t, z_1), \]

...
yields termwise in $z_1$,

$$(e^{-\eta(z)} - 1) f_j(t) = -\frac{\partial}{\partial t_j} \log \varphi(t, z_2). \tag{C.5}$$

Acting with $\partial / \partial t_i$ on the latter expression and with $\partial / \partial t_j$ on the same expression with $j$ replaced by $i$, and subtracting, one finds

$$(e^{-\eta(z)} - 1) \left( \frac{\partial f_i}{\partial t_j} - \frac{\partial f_j}{\partial t_i} \right) = 0,$$

yielding

$$\frac{\partial f_i}{\partial t_j} - \frac{\partial f_j}{\partial t_i} = 0;$$

the constant vanishes because $\partial f_i / \partial t_j$ never contains constant terms.

Therefore there exists a function $\log \tau(t_1, t_2, \ldots)$ such that

$$-\frac{\partial}{\partial t_j} \log \tau(t_1, t_2, \ldots) = f_j(t) = \text{Res}_{z=\infty} z^j B \log \varphi,$$

and hence, using (C.5),

$$\frac{\partial}{\partial t_j} \log \varphi(t, z) = (e^{-\eta(z)} - 1) \frac{\partial}{\partial t_j} \log \tau,$$

or, what is the same,

$$\frac{\partial}{\partial t_j} \left( \log \varphi - (e^{-\eta} - 1) \log \tau \right) = 0,$$

from which it follows that

$$\log \varphi - (e^{-\eta} - 1) \log \tau = -\sum_{i=1}^{\infty} \frac{b_i}{i} z^{-i}$$

is, at worst, a holomorphic series in $z^{-1}$ with constant coefficients, which we call $-b_i / i$. Hence

$$\varphi(t, z) = \frac{\tau(t) - [z^{-1}] e^{-\sum_{i=1}^{\infty} (b_i / i) z^{-i}}}{\tau(t)}$$

$$= \frac{\tau(t) - [z^{-1}] e^{\sum_{i=1}^{\infty} b_i (z^{-1} / i)}}{\tau(t) e^{\sum_{i=1}^{\infty} b_i / i}};$$

*It is obvious that $[\partial / \partial t_i, e^{-\eta(z)}] = 0.$
that is,

\[
\varphi(t, z) = \frac{\bar{\tau}(t - [z^{-1}])}{\bar{\tau}(t)},
\]

where

\[
\bar{\tau} = \tau(t) e^{\sum_{i=1}^{\infty} b_i h}.\]

Thus Lemma 3.4 is proved. □

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