TODA VERSUS PFAFF LATTICE AND RELATED POLYNOMIALS

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Abstract

We study the Pfaff lattice, introduced by us in the context of a Lie algebra splitting of gl*(infinity) into* sp*(infinity) and lower-triangular matrices. We establish a set of bilinear identities, which we show to be equivalent to the Pfaff Lattice. In the semi-infinite case, the tau-functions are Pfaffians; interesting examples are the matrix integrals over symmetric matrices (symmetric matrix integrals) and matrix integrals over self-dual quaternionic Hermitian matrices (symplectic matrix integrals).*

There is a striking parallel of the Pfaff lattice with the Toda lattice, and more so, there is a map from one to the other. In particular, we exhibit two maps, dual to each other,

(i) *from the the Hermitian matrix integrals to the symmetric matrix integrals, and*

(ii) *from the Hermitian matrix integrals to the symplectic matrix integrals.*

The map is given by the skew-Borel decomposition of a skew-symmetric operator. We give explicit examples for the classical weights.

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0. Introduction

Consider a weight on $\mathbb{R}$, depending on $t = (t_1, t_2, \dots) \in \mathbb{C}^\infty$,

$$\rho_t(z)\,dz = e^{\sum_1^\infty t_i z^i} \rho(z)\,dz = e^{-V(z)+\sum_1^\infty t_i z^i}\,dz, \quad \text{with} \quad -\frac{\rho'(z)}{\rho(z)} = V'(z) = \frac{g(z)}{f(z)}, \tag{0.1}$$

with rational g and f's, and with $\rho(z)$ decaying fast enough at ∞.

*The Toda lattice, its τ-functions and Hermitian matrix integrals (*revisited*)*
This weight leads to a t-dependent moment matrix

$$m_n(t) = \left(\mu_{k+\ell}(t)\right)_{0\le k,\ell\le n-1} = \left(\int_{\mathbb{R}} z^{k+\ell} \rho_t(z)\,dz\right)_{0\le k,\ell\le n-1},$$

with the semi-infinite moment matrix m_∞, satisfying the commuting equations

$$\frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty = m_\infty \Lambda^k. \tag{0.2}$$

Λ is the customary shift matrix, with zeroes everywhere, except for 1's just above the diagonal, that is, $(\Lambda v)_n = v_{n+1}$. Consider the Borel decomposition into a lower- and an upper-triangular matrix

$$m_\infty = S^{-1} S^{\top -1}, \tag{0.3}$$

and the following t-dependent matrix integrals ($n \ge 0$):

$$\tau_n(t) := \int_{\mathscr{H}_n} e^{\operatorname{Tr}(-V(X)+\sum_1^\infty t_i X^i)}\,dX = \det m_n \qquad \text{and} \qquad \tau_0 = 1, \tag{0.4}$$

where dX is Haar measure on the ensemble $\mathscr{H}_n = \{(n \times n)\text{–Hermitian matrices}\}$. As is well known (e.g., see E. Witten [18] or M. Adler and P. van Moerbeke [6]), integral (0.4) is a solution to the following two systems.

(i) The KP-hierarchy*

$$\left(\mathbf{s}_{k+4}(\tilde{\partial}) - \frac{1}{2}\frac{\partial^2}{\partial t_1 \partial t_{k+3}}\right)\tau_n \circ \tau_n = 0, \quad \text{for } k, n = 0, 1, 2, \ldots. \tag{0.5}$$

(ii) The Toda lattice, that is, the tridiagonal matrix

$$L(t) := S\Lambda S^{-1} = \begin{pmatrix} \frac{\partial}{\partial t_1}\log\frac{\tau_1}{\tau_0} & \left(\frac{\tau_0\tau_2}{\tau_1^2}\right)^{1/2} & 0 & \\ \left(\frac{\tau_0\tau_2}{\tau_1^2}\right)^{1/2} & \frac{\partial}{\partial t_1}\log\frac{\tau_2}{\tau_1} & \left(\frac{\tau_1\tau_3}{\tau_2^2}\right)^{1/2} & \\ 0 & \left(\frac{\tau_1\tau_3}{\tau_2^2}\right)^{1/2} & \frac{\partial}{\partial t_1}\log\frac{\tau_3}{\tau_2} & \ddots \\ & \ddots & \ddots & \end{pmatrix}, \tag{0.6}$$

satisfies the following commuting Toda equations

$$\frac{\partial L}{\partial t_n} = \left[\frac{1}{2}(L^n)_{\mathrm{sk}}, L\right],$$

where $(A)_{\mathrm{sk}}$ denotes the skew-part of the matrix A for the Lie algebra splitting into skew and lower-triangular matrices. Moreover, the following t-dependent polynomials in z, are defined by the S-matrix obtained from the Borel decomposition (0.3); it is also given, on the one hand, in terms of the functions $\tau_n(t)$, and, on the other hand, by a classic determinantal formula (for $a \in \mathbb{C}$, define $[a] := (a, a^2/2, a^3/3, \ldots) \in \mathbb{C}^\infty$)

$$\begin{aligned} p_n(t; z) := \sum_{i=0}^{n} S_{ni}(t) z^i &= z^n \frac{\tau_n(t - [z^{-1}])}{\sqrt{\tau_n \tau_{n+1}}} \\ &= \frac{1}{\sqrt{\tau_n \tau_{n+1}}} \det \left(\begin{array}{ccc|c} & & & 1 \\ & & & z \\ & m_n(t) & & \vdots \\ \hline \mu_{n,0}(t) & \ldots & \mu_{n,n-1}(t) & z^n \end{array}\right). \end{aligned}$$

The $p_n(t; z)$'s are orthonormal with respect to the (symmetric) inner-product $\langle\ ,\ \rangle_{\mathrm{sy}}$, defined by $\langle z^i, z^j\rangle_{\mathrm{sy}} = \mu_{ij}$, which is a restatement of the Borel decomposition (0.3) (see [6]). The vector $p(t; z) = (p_n(t; z))_{n\geq 0}$ is an eigenvector of the matrix $L(t)$ in (0.6):

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$$L(t)p(t; z) = zp(t; z).$$

*The $\mathbf{s}_\ell$'s are the elementary Schur polynomials $e^{\sum_1^\infty t_i z^i} := \sum_{i\geq 0} \mathbf{s}_i(t) z^i$ and $\mathbf{s}_\ell(\tilde{\partial}) := \mathbf{s}_\ell(\partial/\partial t_1, (1/2)(\partial/\partial t_2), \ldots)$. Given a polynomial $p(t_1, t_2, \ldots)$, define the customary Hirota symbol

$$p(\partial_t) f \circ g := p(\partial/\partial y_1, \partial/\partial y_2, \ldots) f(t+y) g(t-y)\Big|_{y=0}.$$

The Pfaff lattice and its τ-functions

For use throughout this paper, define the skew-symmetric matrix

$$J = \begin{pmatrix} \ddots & & & & \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & 0 & \\ & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & \\ & & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \\ & 0 & & & \ddots \end{pmatrix}, \quad \text{with } J^2 = -I, \tag{0.7}$$

and the involution on the space $\mathcal{D} := \mathrm{gl}_\infty$ of infinite matrices,

$$\mathcal{J} : \mathcal{D} \longrightarrow \mathcal{D} : a \longmapsto \mathcal{J}(a) := J a^\top J. \tag{0.8}$$

Also, consider the splitting of $\mathcal{D} = \mathbf{k} + \mathbf{n}$ into two Lie subalgebras $\mathbf{k}$ and $\mathbf{n}$, with the corresponding projections denoted $\pi_{\mathbf{k}}$ and $\pi_{\mathbf{n}}$, where $\mathbf{k}$ is the Lie algebra of lower-triangular matrices with some special feature (see (1.17)) and where

$$\mathbf{n} := \{a \in \mathcal{D} \text{ such that } J a^\top J = a\} = \mathrm{sp}(\infty).$$

Given a skew-symmetric semi-infinite matrix m_∞, consider the commuting differential equations

$$\frac{\partial m_\infty}{\partial t_i} = \Lambda^i m_\infty + m_\infty \Lambda^{\top i}; \tag{0.9}$$

they maintain the skew-symmetry of m_∞. The Borel decomposition of m_∞ into lower-triangular times upper-triangular matrices requires the insertion of the skew-symmetric matrix J:

$$m_\infty(t) = Q^{-1}(t) J Q^{-1\top}(t). \tag{0.10}$$

Dressing up the shift Λ with the lower-triangular matrix $Q(t)$ leads to the commuting equations (0.11) below.

THEOREM 0.1

The Pfaff lattice *equations*

$$\frac{\partial L}{\partial t_i} = [-\pi_{\mathbf{k}} L^i, L] = [\pi_{\mathbf{n}} L^i, L] \tag{0.11}$$

maintain the locus of semi-infinite matrices of the form $(a_i \neq 0)$*:*

$$L = \begin{bmatrix} 0 & 1 & & & & \\ & -d_1 & a_1 & & & O \\ & & d_1 & 1 & & \\ & & & -d_2 & a_2 & \\ & * & & & d_2 & \\ & & & & & \ddots \end{bmatrix}. \tag{0.12}$$

The solutions L *to (0.11) of the form (0.12) are given by*

$$L(t) = Q(t)\Lambda Q^{-1}(t),$$

where Q *is a lower-triangular matrix, whose entries are given by the coefficients of the polynomials, obtained by the finite Taylor expansion in* z^{-1} *of* $\tau_{2n}(t-[z^{-1}])$ *below* $(h_{2n} := \tau_{2n+2}(t)/\tau_{2n}(t))$*:*

$$\begin{aligned} q_{2n}(t;z) &:= \sum_{j=0}^{2n} Q_{2n,j}(t)z^j = z^{2n}h_{2n}^{-1/2}\frac{\tau_{2n}(t-[z^{-1}])}{\tau_{2n}(t)}, \\ q_{2n+1}(t;z) &:= \sum_{j=0}^{2n+1} Q_{2n+1,j}(t)z^j = z^{2n}h_{2n}^{-1/2}\frac{(z+\partial/\partial t_1)\tau_{2n}(t-[z^{-1}])}{\tau_{2n}(t)}, \end{aligned} \tag{0.13}$$

with τ_0, τ_2, τ_4*, a sequence of functions of* $t_1, t_2, \ldots,$ *characterized by the following bilinear identities for all* $n, m \geq 0$*,* $(\tau_0 = 1)$*,*

$$\begin{aligned} &\oint_{z=\infty} \tau_{2n}\big(t-[z^{-1}]\big)\tau_{2m+2}\big(t'+[z^{-1}]\big)e^{\sum_1^\infty (t_i-t_i')z^i} z^{2n-2m-2}\frac{dz}{2\pi i} \\ &\qquad + \oint_{z=0} \tau_{2n+2}\big(t+[z]\big)\tau_{2m}\big(t'-[z]\big)e^{\sum_1^\infty (t_i'-t_i)z^{-i}} z^{2n-2m}\frac{dz}{2\pi i} = 0. \end{aligned} \tag{0.14}$$

Remark
Theorem 0.1 is robust and remains valid for the bi-infinite matrix L. In that case, the summations in the expressions q_{2n} and q_{2n+1} run from $j = -\infty$, instead of running from $j = 0$.

The τ-functions are given by Pfaffians $\operatorname{pf} m_{2n}(t)$ and satisfy, as a consequence of the bilinear relations (0.14), the Pfaffian KP-hierarchy for $k, n = 0, 1, 2, \ldots,$

$$\left(\mathbf{s}_{k+4}(\tilde\partial) - \frac{1}{2}\frac{\partial^2}{\partial t_1 \partial t_{k+3}}\right)\tau_{2n}\circ\tau_{2n} = \mathbf{s}_k(\tilde\partial)\ \tau_{2n+2}\circ\tau_{2n-2}. \tag{0.15}$$

The t-dependent polynomials $q_n(t;z) = (Q(t)\ (1, z, z^2, z^3, \dots)^\top)_n$ in z, obtained in (0.13) are "skew-orthonormal" with respect to the skew inner-product $\langle\ ,\ \rangle_{\mathrm{sk}}$, defined by $\langle y^i, z^j\rangle_{\mathrm{sk}} = \mu_{ij}(t)$, namely,

$$\big(\langle q_i, q_j\rangle_{\mathrm{sk}}\big)_{0\le i,j<\infty} = J,$$

and are eigenvectors for the matrix L:

$$L(t)q(t;z) = zq(t;z). \tag{0.16}$$

Explicit representations of L, in terms of the τ_{2n}'s, are as follows:

$$L = Q\Lambda Q^{-1} = h^{-1/2}\begin{pmatrix} \hat L_{00} & \hat L_{01} & 0 & 0 & \\ \hat L_{10} & \hat L_{11} & \hat L_{12} & 0 & \\ * & \hat L_{21} & \hat L_{22} & \hat L_{23} & \\ * & * & \hat L_{32} & \hat L_{33} & \dots \\ & & & \vdots & \end{pmatrix} h^{1/2},$$

with the entries $\hat L_{ij}$ and the entries of h, being (2×2)-matrices

$$h = \mathrm{diag}(h_0 I_2, h_2 I_2, h_4 I_2, \dots), \qquad h_{2n} = \tau_{2n+2}/\tau_{2n},$$

and $(^{\cdot} = \partial/\partial t_1)$,

$$\hat L_{nn} := \begin{pmatrix} -(\log \tau_{2n})^{\cdot} & 1 \\ -\frac{\mathbf{s}_2(\tilde\partial)\tau_{2n}}{\tau_{2n}} - \frac{\mathbf{s}_2(-\tilde\partial)\tau_{2n+2}}{\tau_{2n+2}} & (\log \tau_{2n+2})^{\cdot} \end{pmatrix}, \qquad \hat L_{n,n+1} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\hat L_{n+1,n} := \begin{pmatrix} * & (\log \tau_{2n+2})^{\cdot\cdot} \\ * & * \end{pmatrix}. \tag{0.17}$$

Also, the t-dependent polynomials $q_n(t,z)$ in z have Pfaffian expressions, in contrast with the determinantal expressions in the Hermitian case, mentioned earlier:

$$q_{2n}(t;z) = \frac{1}{\sqrt{\tau_{2n}\tau_{2n+2}}}\,\mathrm{pf}\left(\begin{array}{ccc|c} & & & 1 \\ & & & z \\ & m_{2n+1}(t) & & \vdots \\ & & & z^{2n} \\ \hline -1 & \dots & -z^{2n} & 0 \end{array}\right),$$

$$\begin{aligned} q_{2n+1}(t;z) \\ = \frac{1}{\sqrt{\tau_{2n}\tau_{2n+2}}}\,\mathrm{pf}\left(\begin{array}{ccc|cc} & & & 1 & \mu_{0,2n+1} \\ & & & z & \mu_{1,2n+1} \\ & m_{2n}(t) & & \vdots & \vdots \\ & & & z^{2n-1} & \mu_{2n-1,2n+1} \\ \hline -1 & -z & \dots & 0 & -z^{2n+1} \\ -\mu_{0,2n+1} & -\mu_{1,2n+1} & \dots & z^{2n+1} & 0 \end{array}\right). \end{aligned} \tag{0.18}$$

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Theorem 0.1 and the subsequent statements are established in Sections 2 and 3. We show how a general skew-symmetric infinite matrix flowing according to (0.11) and its skew-Borel decomposition (0.10), lead to wave vectors Ψ, satisfying bilinear relations and differential equations. Section 3 deals with the existence, in the general setting, of a so-called Pfaffian τ-function, satisfying bilinear equations and the so-called Pfaff-KP hierarchy. In [4], these results were obtained by embedding the system in 2-Toda theory, while in this paper, they are obtained in an intrinsic fashion.

For $k = 0$, the Pfaff-KP equation (0.15) has already appeared in the context of the charged BKP hierarchy, studied by V. Kac and J. van de Leur [13]; the precise relationship between the charged BKP hierarchy of Kac and van de Leur and the Pfaff Lattice, introduced here, deserves further investigation. (See the recent paper by van de Leur [16]).

Examples: Symmetric and "symplectic" matrix integrals
An important example is given by the skew-symmetric matrix $m_\infty = (\mu_{ij})_{i,j\geq 0}$ of moments* defined by ($\alpha = \pm 1$) (see [4])

$$\langle y^i, z^j\rangle_{\mathrm{sk}} := \int\int_{\mathbb{R}^2} y^i z^j e^{\sum t_i(y^i+z^i)} 2D^\alpha \delta(y-z)\rho(y)\rho(z)\,dy\,dz \tag{0.19}$$
$$= \begin{cases} \mu_{ij}^{(1)}(t) = \iint_{\mathbb{R}^2} y^i z^j e^{\sum_1^\infty t_i(y^i+z^i)} \varepsilon(y-z)\rho(y)\rho(z)\,dy\,dz, \\ \qquad\qquad\qquad\qquad \text{for } \alpha = -1, \\ \mu_{ij}^{(2)}(t) = \int_{\mathbb{R}} \{y^i, y^j\}(y) e^{\sum_1^\infty 2t_i y^i} \tilde\rho(y)^2 dy, \quad \text{for } \alpha = +1. \end{cases}$$

The associated moment matrices $m_{2n}^{(1)}$ and $m_{2n}^{(2)}$ satisfy the differential equations (0.9) and lead to "symmetric" matrix integrals

$$\tau_{2n}^{(1)}(t) := \frac{1}{(2n)!}\int_{\mathscr{S}_{2n}} e^{\mathrm{Tr}(-V(X)+\sum_1^\infty t_i X^i)}\,dX = \mathrm{pf}(m_{2n}^{(1)}),$$

and "symplectic" matrix integrals

$$\tau_{2n}^{(1)}(t) := \frac{1}{n!}\int_{\mathscr{T}_{2n}} e^{2\,\mathrm{Tr}(-V(X)+\sum_1^\infty t_i X^i)}\,dX = \mathrm{pf}(m_{2n}^{(2)}),$$

both expressed in terms of the Pfaffian of the upper-left-hand principal minors of the "moment" matrix $m_\infty^{(i)}$, where

(1) for $i = 1$, dX denotes Haar measure on the space $\mathscr{S}_{2n}$ of symmetric matrices, and,

(2) for $i = 2$, dX denotes Haar measure on the $(2n \times 2n)$-matrix realization $\mathscr{T}_{2n}$ of the space of self-dual $(n\times n)$–Hermitian matrices, with quaternionic entries.

A remarkable map from Toda to Pfaff lattice
Remembering the notation (0.1), we act with the z-operator,

$$\mathbf{n}_t := \sqrt{\frac{f}{\rho_t}}\frac{d}{dz}\sqrt{f\rho_t} = e^{-(1/2)\sum t_k z^k}\Big(\frac{d}{dz} f(z) - \frac{f'+g}{2}(z)\Big)e^{1/2\sum t_k z^k} \tag{0.20}$$

on the t-dependent orthonormal polynomials $p_n(t,z)$ in z; in [6], we showed that the matrix $\mathscr{N}$ defined by

$$\mathbf{n}_t\, p(t,z) = \Big(f(L)M - \frac{f'+g}{2}(L)\Big)p(t,z) =: \mathscr{N} p(t,z) \tag{0.21}$$

*We have $\varepsilon(x) = 1$, for $x \geq 0$, $\varepsilon(x) = -1$, for $x < 0$, and $\{f,g\} = f'g - fg'$.

is skew-symmetric. The t-dependent matrix $\mathscr{N}$ is expressed in terms of L and a new matrix M, defined by

$$zp = Lp \qquad \text{and} \qquad e^{-(1/2)\sum t_k z^k} \frac{d}{dz} e^{(1/2)\sum t_k z^k} p = Mp. \tag{0.22}$$

Consider now the skew-Borel decomposition of $\mathscr{N}(2t)$ and its inverse* $\mathscr{N}(2t)^{-1}$, in terms of lower-triangular matrices $O_{(+)}(t)$ and $O_{(-)}(t)$, respectively:†

$$\mathscr{N}(2t)^{\pm 1} = -O_{(\pm)}^{-1}(t) J O_{(\pm)}^{\top -1}(t). \tag{0.23}$$

Then, the lower-triangular matrices $O_{(\pm)}(t)$ map *orthonormal* into *skew-orthonormal* polynomials, and the tridiagonal L-matrix into an $\tilde{L}$-matrix:‡

$$\begin{aligned} p_n(t;z) &\longmapsto q_n^{(\pm)}(t;z) = \big(O_{(\pm)}(t)p(t;z)\big)_n, \\ L(t) &\longmapsto \tilde{L}(t) = O_{(\pm)}(t) L(2t) O_{(\pm)}(t)^{-1} \\ \text{(Toda lattice)} & \qquad \text{(Pfaff lattice).} \end{aligned} \tag{0.24}$$

It also maps the weight into a new weight

$$\rho(z) = e^{-V(z)} \longmapsto \tilde{\rho}_{\pm}(z) = e^{-\tilde{V}(z)} := e^{-(1/2)(V(z) \mp \log f(z))},$$

and the corresponding string of τ-functions into a new string of Pfaffian τ-functions (remember $V_t(z) = V(z) - \sum_1^\infty t_i z^i$):

$$\tau_k(t) = \int_{\mathscr{H}_k} e^{\mathrm{Tr}(-V_t(X))}\, dX \longmapsto \begin{cases} \tau_{2n}^{(+)}(t) := \int_{\mathscr{T}_{2n}} e^{\mathrm{Tr}\, 2(-\tilde{V}_t(X))}\, dX & (\beta = 4), \\ \tau_{2n}^{(-)}(t) := \int_{\mathscr{S}_{2n}} e^{\mathrm{Tr}(-\tilde{V}_t(X))}\, dX & (\beta = 1). \end{cases}$$

For the *classical orthogonal polynomials* $p_n(z)$, we have shown in [6] that $\mathscr{N}(0)$ is not only skew-symmetric but also tridiagonal; that is,

$$L = \begin{bmatrix} b_0 & a_0 & & \\ a_0 & b_1 & a_1 & \\ & a_1 & b_2 & \ddots \\ & & \ddots & \ddots \end{bmatrix}, \qquad -\mathscr{N} = \begin{bmatrix} 0 & c_0 & & \\ -c_0 & 0 & c_1 & \\ & -c_1 & 0 & \ddots \\ & & \ddots & \end{bmatrix}. \tag{0.25}$$

*See Appendix B.
†The upper-signs (resp., lower-signs) correspond to each other throughout this section.
‡We have $p(t;z) := (p_0(t;z), p_1(t;z), \ldots)^\top$.

In Sections 6 and 7, we show that the maps $O_{(-)}$ and $O_{(+)}$, as in (0.24), only involve three steps, in the following sense:

$$\begin{aligned} q_{2n}^{(-)}(0;z) &= \sqrt{\frac{c_{2n}}{a_{2n}}}\, p_{2n}(0;z), \\ q_{2n+1}^{(-)}(0;z) &= \sqrt{\frac{a_{2n}}{c_{2n}}} \\ &\quad \times \Big(-c_{2n-1} p_{2n-1}(0,z) + \frac{c_{2n}}{a_{2n}}\Big(\sum_0^{2n} b_i\Big) p_{2n}(0;z) + c_{2n} p_{2n+1}(0;z)\Big) \\ &\quad (\beta = 1), \end{aligned} \tag{0.26}$$

$$\begin{aligned} p_{2n}(0;z) &= -c_{2n-1}\sqrt{\frac{a_{2n-2}}{c_{2n-2}}}\, q_{2n-2}^{(+)}(0;z) + \sqrt{a_{2n}c_{2n}}\, q_{2n}^{(+)}(0;z) \\ p_{2n+1}(0;z) &= -c_{2n}\sqrt{\frac{a_{2n-2}}{c_{2n-2}}}\, q_{2n-2}^{(+)}(0;z), \\ &\quad -\Big(\sum_0^{2n} b_i\Big)\sqrt{\frac{c_{2n}}{a_{2n}}}\, q_{2n}^{(+)}(0;z) + \sqrt{\frac{c_{2n}}{a_{2n}}}\, q_{2n+1}^{(+)}(0;z) \quad (\beta = 4). \end{aligned} \tag{0.27}$$

The abstract map $O_{(-)}$ for $t = 0$ appears already in the work of E. Brézin and H. Neuberger [9]. This has been applied recently by [1] to a problem in the theory of random matrices.

1. Splitting theorems, as applied to the Toda and Pfaff lattices

In this section, we show how each of the equations

$$\frac{\partial m_\infty}{\partial t_i} = \Lambda^i m_\infty \qquad \text{and} \qquad \frac{\partial m_\infty}{\partial t_i} = \Lambda^i m_\infty + m_\infty \Lambda^{\top i} \tag{1.1}$$

lead to commuting Hamiltonian vector fields related to a Lie algebra splitting. First recall the splitting theorem due to Adler, B. Kostant, and W. Symes in [5], and later recall the R-version due to A. Reyman and M. Semenov-Tian-Shansky [15]. The R-version allows for more general initial conditions.

PROPOSITION 1.1
Let $\mathbf{g} = \mathbf{k} + \mathbf{n}$ *be a (vector space) direct sum of a Lie algebra* $\mathbf{g}$ *in terms of Lie subalgebras* $\mathbf{k}$ *and* $\mathbf{n}$*, with* $\mathbf{g}$ *paired with itself via a nondegenerate* ad*-invariant inner product** $\langle\, ,\, \rangle$*; this in turn induces a decomposition* $\mathbf{g} = \mathbf{k}^\perp + \mathbf{n}^\perp$ *and isomorphisms* $\mathbf{g} \simeq \mathbf{g}^*$, $\mathbf{k}^\perp \simeq \mathbf{n}^*$, $\mathbf{n}^\perp \simeq \mathbf{k}^*$. *Let* $\pi_{\mathbf{k}}$ *and* $\pi_{\mathbf{n}}$ *be projections onto* $\mathbf{k}$ *and* $\mathbf{n}$*, respectively.*

*$\langle \mathrm{Ad}_g\, X; Y\rangle = \langle X, \mathrm{Ad}_{g^{-1}}\, Y\rangle$, $g \in G$, and thus $\langle [z,x], y\rangle = \langle x, -[z,y]\rangle$.

Let $\mathscr{G}$, $\mathscr{G}_{\mathbf{k}}$, and $\mathscr{G}_{\mathbf{n}}$ be the groups associated with the Lie algebras $\mathbf{g}$, $\mathbf{k}$, and $\mathbf{n}$. Let $\mathscr{I}(\mathbf{g})$ be the $\mathrm{Ad}^* \simeq \mathrm{Ad}$-invariant functions on $\mathbf{g}^* \simeq \mathbf{g}$.

(i) *Then, given an element*

$$\varepsilon \in \mathbf{g} : [\varepsilon, \mathbf{k}] \subset \mathbf{k}^{\perp} \qquad \textit{and} \qquad [\varepsilon, \mathbf{n}] \subset \mathbf{n}^{\perp},$$

the functions

$$\varphi(\varepsilon + \xi')|_{\mathbf{k}^{\perp}}, \qquad \textit{with } \ \varphi \in \mathscr{I}(\mathbf{g}) \textit{ and } \xi' \in \mathbf{k}^{\perp}, \tag{1.2}$$

*respectively, Poisson commute for the respective Kostant-Kirillov symplectic structures of $n^* \simeq \mathbf{k}^{\perp}$; the associated Hamiltonian flows are expressed in terms of the Lax pairs**

$$\dot{\xi} = [-\pi_{\mathbf{k}} \nabla\varphi(\xi), \xi] = [\pi_{\mathbf{n}} \nabla\varphi(\xi), \xi], \qquad \textit{for } \ \xi \equiv \varepsilon + \xi', \ \ \xi' \in \mathbf{k}^{\perp}. \tag{1.3}$$

(ii) *The splitting also leads to a second Lie algebra $\mathbf{g}_R$, derived from $\mathbf{g}$, such that $\mathbf{g}_R^* \simeq \mathbf{g}_R$, namely,*

$$\mathbf{g}_R : [x, y]_R = \frac{1}{2}[Rx, y] + \frac{1}{2}[x, Ry] = [\pi_{\mathbf{k}} x, \pi_{\mathbf{k}} y] - [\pi_{\mathbf{n}} x, \pi_{\mathbf{n}} y], \tag{1.4}$$

with $R = \pi_{\mathbf{k}} - \pi_{\mathbf{n}}$. The functions

$$\varphi(\xi)|_{\mathbf{g}_R}, \qquad \textit{with } \ \varphi \in \mathscr{I}(\mathbf{g}) \textit{ and } \xi \in \mathbf{g}_R,$$

respectively, Poisson commute for the respective Kostant-Kirillov symplectic structures of $\mathbf{g}_R^* \simeq \mathbf{g}_R$, with the same associated (Hamiltonian) Lax pairs

$$\dot{\xi} = [-\pi_{\mathbf{k}} \nabla\varphi(\xi), \xi] = [\pi_{\mathbf{n}} \nabla\varphi(\xi), \xi], \qquad \textit{for } \ \xi \in \mathbf{g}_R. \tag{1.5}$$

Each of the equations (1.3) and (1.5) has the same solution expressible in two different ways:†

$$\xi(t) = \mathrm{Ad}_{K(t)}\, \xi_0 = \mathrm{Ad}_{S^{-1}(t)}\, \xi_0, \tag{1.6}$$

with‡

$$K(t) = \pi_{\mathscr{G}_{\mathbf{k}}} e^{t\nabla\varphi(\xi_0)} \qquad \textit{and} \qquad S(t) = \pi_{\mathscr{G}_{\mathbf{n}}} e^{t\nabla\varphi(\xi_0)}.$$

Example 1 (The standard Toda lattice and the equations $\partial m/\partial t_i = \Lambda^i m$ for the Hänkel matrix m_∞)

Since, in particular, the matrix m_∞ is symmetric, the Borel decomposition into lower-times upper-triangular matrix must be done with the same lower-triangular matrix S:

$$m_\infty = S^{-1} S^{\top -1}. \tag{1.7}$$

*$\nabla\varphi$ is defined as the element in $\mathbf{g}^*$ such that $d\varphi(\xi) = \langle \nabla\varphi, d\xi \rangle$, $\xi \in \mathbf{g}$.

†We naively write $\mathrm{Ad}_{K(t)}\, \xi_0 = K(t)\xi_0 K(t)^{-1}$, $\mathrm{Ad}_{S^{-1}}\, \xi_0 = S^{-1}(t)\xi_0 S(t)$.

‡Consider the group factorization $A = \pi_{\mathscr{G}_{\mathbf{k}}} A \ \pi_{\mathscr{G}_{\mathbf{n}}} A$.

In turn, the matrix S defines a wave vector Ψ, and operators* L and M, the same as the ones defined in (0.22),

$$\Psi(t,z) := e^{(1/2)\sum_1^\infty t_i z^i} S\chi, \qquad L := S\Lambda S^{-1},$$
$$M := S\Big(\partial + \frac{1}{2}\sum_1^\infty i t_i \Lambda^{i-1}\Big)S^{-1}, \tag{1.8}$$

satisfying the following well-known equations:†

$$\begin{aligned} &L\Psi = z\Psi, && M\Psi = \frac{\partial}{\partial z}\Psi, \quad \text{with } [L,M]=1,\\ &\frac{\partial S}{\partial t_n} = -\frac{1}{2}(L^n)_{\mathrm{bo}} S, && \frac{\partial \Psi}{\partial t_n} = \frac{1}{2}(L^n)_{\mathrm{sk}}\Psi,\\ &\frac{\partial L}{\partial t_n} = \frac{1}{2}\big[(L^n)_{\mathrm{sk}}, L\big] && \frac{\partial M}{\partial t_n} = \frac{1}{2}\big[(L^n)_{\mathrm{sk}}, M\big]. \end{aligned} \tag{1.9}$$

The wave vector Ψ can then be expressed in terms of a sequence of τ-functions $\tau_n(t) = \det m_n(t)$, but it also has a simple expression in terms of orthonormal polynomials, with respect to the moment matrix m_∞:

$$\begin{aligned} \Psi(t,z) &= e^{(1/2)\sum t_i z^i}\bigg(z^n \frac{\tau_n(t-[z^{-1}])}{\sqrt{\tau_n(t)\tau_{n+1}(t)}}\bigg)_{n\geq 0}\\ &= e^{(1/2)\sum t_i z^i}\big(p_n(t,z)\big)_{n\geq 0}. \end{aligned} \tag{1.10}$$

The vector fields (1.9) on L are commuting Hamiltonian vector fields, in view of the Adler-Kostant-Symes (AKS) splitting theorem (version (i)),

$$\frac{\partial L}{\partial t_i} = [-\pi_{\mathbf{k}}\nabla\mathscr{H}_i, L] = [\pi_{\mathbf{n}}\nabla\mathscr{H}_i, L], \quad \mathscr{H}_i = \frac{\operatorname{tr} L^{i+1}}{i+1}, \quad \nabla\mathscr{H}_i = L^i, \tag{1.11}$$

with

$$L = \Lambda^\top a + b + a\Lambda, \qquad a \text{ and } b \text{ diagonal matrices}, \tag{1.12}$$

for the splitting of the Lie algebra of semi-infinite matrices

$$\begin{aligned} \mathscr{D} = \mathrm{gl}_\infty = \mathbf{k} + \mathbf{n} &:= \{\text{skew-symmetric}\} + \{\text{lower-triangular}\}\\ = \mathbf{k}^\perp + \mathbf{n}^\perp &:= \{\text{symmetric}\} + \{\text{strictly upper-triangular}\}, \end{aligned} \tag{1.13}$$

*In the formulas below $\chi(z) = (z^0, z, z^2, \ldots)^\top$, and ∂ is the matrix such that $(d/dz)\chi(z) = \partial\chi(z)$.
†The notation $(\ )_{\mathrm{sk}}$ and $(\ )_{\mathrm{bo}}$ refers to the skew-part and the lower-triangular (Borel) part, respectively, that is, projection onto $\mathbf{k}$ and $\mathbf{n}$, respectively.

with the form (1.12) of L being preserved in time. Note that the solution (1.6) to the differential equation (1.5) in the AKS theorem is nothing but the factorization of m_∞ followed by the dressing up of Λ.

Example 2 (The Pfaff lattice and the equations $\partial m/\partial t_i = \Lambda^i m + m\Lambda^{\top i}$)
Throughout this paper the Lie algebra $\mathscr{D} = \mathrm{gl}_\infty$ of semi-infinite matrices is viewed as composed of (2×2)-blocks. It admits the natural decomposition into subalgebras:

$$\mathscr{D} = \mathscr{D}_- \oplus \mathscr{D}_0 \oplus \mathscr{D}_+ = \mathscr{D}_- \oplus \mathscr{D}_0^- \oplus \mathscr{D}_0^+ \oplus \mathscr{D}_+, \tag{1.14}$$

where $\mathscr{D}_0$ has (2×2)-blocks along the diagonal with zeros everywhere else and where $\mathscr{D}_+$ (resp., $\mathscr{D}_-$) is the subalgebra of upper-triangular (resp., lower-triangular) matrices with (2×2)-zero matrices along $\mathscr{D}_0$ and zero below (resp., above). As pointed out in (1.14), $\mathscr{D}_0$ can further be decomposed into two Lie subalgebras:

$$\begin{aligned} \mathscr{D}_0^- &= \{\text{all } (2 \times 2)\text{-blocks} \in \mathscr{D}_0 \text{ are proportional to Id}\}, \\ \mathscr{D}_0^+ &= \{\text{all } (2 \times 2)\text{-blocks} \in \mathscr{D}_0 \text{ have trace zero}\}. \end{aligned} \tag{1.15}$$

Remember from (0.7) and (0.8) in the introduction, the matrix J and the associated Lie algebra involution $\mathscr{J}$. The splitting into two Lie subalgebras*

$$\mathscr{D} = \mathbf{k} + \mathbf{n}, \tag{1.16}$$

with

$$\begin{aligned} \mathbf{k} &= \mathscr{D}_- + \mathscr{D}_0^- \\ &= \text{algebra of} \begin{pmatrix} \ddots & & & 0 \\ & \boxed{\begin{matrix} Q_{2n,2n} & 0 \\ 0 & Q_{2n,2n} \end{matrix}} & & \\ & & \boxed{\begin{matrix} Q_{2n+2,2n+2} & 0 \\ 0 & Q_{2n+2,2n+2} \end{matrix}} & \\ & * & & \ddots \end{pmatrix}, \\ \mathbf{n} &= \{a \in \mathscr{D}, \text{ such that } \mathscr{J} a = a\} = \{b + \mathscr{J} b,\ b \in \mathscr{D}\} = \mathrm{sp}(\infty), \end{aligned} \tag{1.17}$$

with corresponding Lie groups.† $\mathscr{G}_{\mathbf{k}}$ and $\mathscr{G}_{\mathbf{n}} = \mathrm{Sp}(\infty)$, play a crucial role here. Let $\pi_{\mathbf{k}}$ and $\pi_{\mathbf{n}}$ be the projections onto $\mathbf{k}$ and $\mathbf{n}$. Notice that $\mathbf{n} = \mathrm{sp}(\infty)$ and $\mathscr{G}_{\mathbf{n}} = \mathrm{Sp}(\infty)$

*Note $\mathbf{n}$ is the fixed point set of $\mathscr{J}$.
†$\mathscr{G}_{\mathbf{k}}$ is the group of invertible elements in $\mathbf{k}$, that is, invertible lower-triangular matrices, with nonzero (2×2)-blocks proportional to Id along the diagonal.

stand for the infinite-rank affine symplectic algebra and group $\mathbf{n}$ (e.g., see [12]). Any element $a \in \mathscr{D}$ decomposes uniquely into its projections onto $\mathbf{k}$ and $\mathbf{n}$, as follows:

$$\begin{aligned} a &= \pi_{\mathbf{k}} a + \pi_{\mathbf{n}} a \\ &= \left\{ \left(a_- - \mathscr{J} a_+\right) + \frac{1}{2}\left(a_0 - \mathscr{J} a_0\right) \right\} + \left\{ \left(a_+ + \mathscr{J} a_+\right) + \frac{1}{2}\left(a_0 + \mathscr{J} a_0\right) \right\}. \end{aligned} \tag{1.18}$$

The following splitting, with

$$\mathbf{k}_+ = \mathscr{D}_+ + \mathscr{D}_0^- \qquad \text{and} \qquad \mathbf{n}_+ = \mathbf{n},$$

is also used in Section 2; the projections take on the following form:

$$\begin{aligned} a &= \pi_{\mathbf{k}_+} a + \pi_{\mathbf{n}_+} a \\ &= \left\{ \left(a_+ - \mathscr{J} a_-\right) + \frac{1}{2}\left(a_0 - \mathscr{J} a_0\right) \right\} + \left\{ \left(a_- + \mathscr{J} a_-\right) + \frac{1}{2}\left(a_0 + \mathscr{J} a_0\right) \right\}. \end{aligned} \tag{1.19}$$

Note that $\mathscr{J}$ intertwines $\pi_{\mathbf{k}}$ and $\pi_{\mathbf{k}_+}$:

$$\mathscr{J} \pi_{\mathbf{k}} = \pi_{\mathbf{k}_+} \mathscr{J}. \tag{1.20}$$

For a skew-symmetric semi-infinite matrix m_∞, the skew-Borel decomposition

$$m_\infty := Q^{-1} J Q^{-1\top}, \qquad \text{with } Q \in \mathscr{G}_{\mathbf{k}}, \tag{1.21}$$

is unique, as was shown in [2]. Here we may assume m_∞ to be bi-infinite, as long as factorization (1.21) is unique, upon imposing a suitable normalization. Then we use Q to dress up Λ:

$$L = Q \Lambda Q^{-1}.$$

Then letting m_∞ run according to the equations $\partial m / \partial t_i = \Lambda^i m + m \Lambda^{\top i}$, we show in the next proposition and corollary that L evolves according to a system of commuting equations, which by virtue of the AKS theorem are Hamiltonian vector fields (for details, see [2]).

PROPOSITION 1.2
For the matrices

$$m_\infty := Q^{-1} J Q^{-1\top} \qquad \textit{and} \qquad L := Q \Lambda Q^{-1}, \quad \textit{with } Q \in \mathscr{G}_{\mathbf{k}},$$

the following three statements are equivalent:

(i) $\quad \dfrac{\partial Q}{\partial t_i} Q^{-1} = -\pi_{\mathbf{k}} L^i,$

(ii) $\quad L^i + \dfrac{\partial Q}{\partial t_i} Q^{-1} \in \mathbf{n},$

(iii) $\quad \dfrac{\partial m_\infty}{\partial t_i} = \Lambda^i m_\infty + m_\infty \Lambda^{\top i}.$

Whenever the vector fields on Q or m satisfy (i), (ii), or (iii), then the matrix $L = Q\Lambda Q^{-1}$ is a solution of the AKS-Lax pair

$$\frac{\partial L}{\partial t_i} = [-\pi_{\mathbf{k}} L^i, L] = [\pi_{\mathbf{n}} L^i, L].$$

Proof

Written out and using (1.18), Proposition 1.2 amounts to showing the equivalence of the three formulas:

(I) $\quad \dfrac{\partial Q}{\partial t_i} Q^{-1} + \left((L^i)_- - J(L^i_+)^\top J\right) + \dfrac{1}{2}\left((L^i)_0 - J((L^i)_0)^\top J\right) = 0,$

(II) $\quad \left(L^i + \dfrac{\partial Q}{\partial t_i} Q^{-1}\right) - J\left(L^i + \dfrac{\partial Q}{\partial t_i} Q^{-1}\right)^\top J = 0,$

(III) $\quad \Lambda^i m_\infty + m_\infty \Lambda^{\top i} - \dfrac{\partial m_\infty}{\partial t_i} = 0.$

The point is to show that

$$(\mathrm{I})_+ = 0, \qquad (\mathrm{I})_- = (\mathrm{II})_- = -J\,(\mathrm{II}_+)^\top\,J, \qquad (\mathrm{I})_0 = \frac{1}{2}\,(\mathrm{II})_0\,,$$
$$Q^{-1}(\mathrm{II})JQ^{-1^\top} = (\mathrm{III}). \tag{1.22}$$

The details of this proof are found in [2]. □

2. Wave functions and their bilinear equations for the Pfaff lattice

Consider the commuting vector fields

$$\partial m_\infty / \partial t_i = \Lambda^i m_\infty + m_\infty \Lambda^{\top i} \tag{2.1}$$

on the skew-symmetric matrix $m_\infty(t)$ and the skew-Borel decomposition

$$m_\infty(t) = Q^{-1}(t) J\, Q^{\top -1}(t), \qquad Q(t) \in \mathscr{G}_{\mathbf{k}}; \tag{2.2}$$

remember from (1.17) that $Q(t) \in \mathscr{G}_{\mathbf{k}}$ means that $Q(t)$ is lower-triangular, with along the “diagonal” (2×2)-matrices $c_{2n} I$, with $c_{2n} \neq 0$.

In this section, we give the properties of the wave vectors and their bilinear relations. In this and the next section, the matrices are assumed to be bi-infinite; the semi-infinite case is dealt with by specialization. Upon setting

$$Q_1 = Q(t) \qquad \text{and} \qquad Q_2 = J Q^{\top -1}(t), \tag{2.3}$$

the matrix $Q(t)$ defines wave operators

$$W_1(t) := Q_1(t)e^{\sum_1^\infty t_i \Lambda^i}, \qquad W_2(t) := Q_2(t)e^{-\sum_1^\infty t_i \Lambda^{\top i}} = JW_1^{-1\top}(t), \tag{2.4}$$

L-matrices

$$L := L_1 := Q_1 \Lambda Q_1^{-1}, \qquad L_2 := -\mathscr{J}(L_1) = Q_2 \Lambda^\top Q_2^{-1}, \tag{2.5}$$

and wave and dual wave vectors

$$\begin{aligned} \Psi_1(t,z) &= W_1(t)\chi(z)\Psi_1^*(t,z) = W_1^{-1}(t)^\top \chi(z^{-1}) = -J\Psi_2(t,z^{-1}), \\ \Psi_2(t,z) &= W_2(t)\chi(z)\Psi_2^*(t,z) = W_2^{-1}(t)^\top \chi(z^{-1}) = J\Psi_1(t,z^{-1}). \end{aligned} \tag{2.6}$$

From the definition, it follows that the wave functions Ψ_1 have the following asymptotics:

$$\left\{ \begin{array}{ll} \Psi_{1,2n}(t,z) := e^{\sum t_k z^k} z^{2n} c_{2n}(t) \psi_{1,2n}(t,z), & \psi_{1,2n} = 1 + O(z^{-1}), \\ \Psi_{1,2n+1}(t,z) = e^{\sum t_k z^k} z^{2n+1} c_{2n}(t) \psi_{1,2n+1}(t,z), & \psi_{1,2n+1} = 1 + O(z^{-2}), \end{array} \right.$$

$$\left\{ \begin{array}{ll} \Psi_{2,2n}(t,z) = e^{-\sum t_k z^{-k}} z^{2n+1} c_{2n}^{-1}(t) \psi_{2,2n}(t,z), & \psi_{2,2n} = 1 + O(z), \\ \Psi_{2,2n+1}(t,z) = e^{-\sum t_k z^{-k}} z^{2n} (-c_{2n}^{-1}(t)) \psi_{2,2n+1}(t,z), & \psi_{2,2n+1} = 1 + O(z^2), \end{array} \right. \tag{2.7}$$

where the c_i are the elements of the diagonal part of Q.

THEOREM 2.1

The following statements are equivalent:

(i) m_∞ *satisfies* $\partial m_\infty / \partial t_i = \Lambda^i m_\infty + m_\infty \Lambda^{\top i}$,

(ii) Q_1 *satisfies the hierarchy of equations (with* L_i *defined in (2.5))*

$$\frac{\partial Q_1}{\partial t_i} = -(\pi_{\mathbf{k}} L_1^i) Q_1, \tag{2.8}$$

(iii) $Q_2 = JQ_1^{\top -1}$ *satisfies*

$$\frac{\partial Q_2}{\partial t_i} = -\big(\mathscr{J}(\pi_{\mathbf{k}} L_1^i)\big) Q_2 = (\pi_{\mathbf{k}_+} L_2^i) Q_2,$$

(iv) $\Psi_1 = e^{\sum_1^\infty t_i z^i} Q_1(t)\chi(z)$ *satisfies*

$$\frac{\partial \Psi_1}{\partial t_i} = (\pi_{\mathbf{n}} L_1^i)\Psi_1, \tag{2.9}$$

(v) $\Psi_2 = e^{-\sum_1^\infty t_i z^{-i}} JQ_1^{\top -1}(t)\chi(z)$ *satisfies*

$$\frac{\partial \Psi_2}{\partial t_i} = -(L_2^i - \pi_{\mathbf{k}_+} L_2^i)\Psi_2 = -(\pi_{\mathbf{n}_+} L_2^i),\, \Psi_2,$$

(vi) Ψ_1 *and* Ψ_2 *satisfy the bilinear identity for all* $n, m \in \mathbb{Z}$,

$$\oint_\infty \Psi_{1,n}(t,z)\Psi_{2,m}(t',z^{-1})\frac{dz}{2\pi iz} + \oint_0 \Psi_{2,n}(t,z)\Psi_{1,m}(t',z^{-1})\frac{dz}{2\pi iz} = 0. \tag{2.10}$$

If any one of these six conditions is satisfied, then

$$\frac{\partial L_1}{\partial t_i} = [-\pi_{\mathbf{k}} L_1^i, L_1], \qquad \frac{\partial L_2}{\partial t_i} = [\pi_{\mathbf{k}_+} L_2^i, L_2], \tag{2.11}$$

and

$$\begin{aligned} L_1\Psi_1 &= z\Psi_1, \qquad & L_2\Psi_2 &= z^{-1}\Psi_2,\\ L_1^\top \Psi_1^\star &= z\Psi_1^\star, \qquad & L_2^\top\Psi_2^\star &= z^{-1}\Psi_2^\star. \end{aligned} \tag{2.12}$$

For later use, we also consider the "monic" wave functions, with the factors $c_{2n}(t)$ removed; that is,

$$\hat\Psi_1(t,z) := Q_0^{-1}\Psi_1 \qquad \text{and} \qquad \hat\Psi_2(t,z) := Q_0\Psi_2 \tag{2.13}$$

and the matrix $\hat L_1$, normalized so as to have 1's above the main diagonal, with $\hat Q := Q_0^{-1}Q$,

$$\begin{aligned} \hat L_1 &= Q_0^{-1}L_1Q_0 = (Q_0^{-1}Q)\Lambda(Q_0^{-1}Q)^{-1} = \hat Q\Lambda\hat Q^{-1},\\ \hat L_2 &= Q_0L_2Q_0^{-1} = -Q_0\mathscr{J}(L_1)Q_0^{-1} = -\mathscr{J}(\hat L_1) \end{aligned} \tag{2.14}$$

Then, in terms of the elements $\hat q_{ij}$ of the matrix $\hat Q := Q_0^{-1}Q$, one easily computes by conjugation that $\hat L_1$ has the following block structure:

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$$\begin{aligned} \hat L_1 &= Q_0^{-1}L_1Q_0 = (Q_0^{-1}Q)\Lambda(Q_0^{-1}Q)^{-1}\\ &= \begin{pmatrix} & \vdots & & & & \\ \dots & \hat L_{00} & \hat L_{01} & 0 & 0 & \\ & \hat L_{10} & \hat L_{11} & \hat L_{12} & 0 & \\ & * & \hat L_{21} & \hat L_{22} & \hat L_{23} & \\ & * & * & \hat L_{32} & \hat L_{33} & \dots \\ & & & & \vdots & \end{pmatrix}, \end{aligned}$$

with

$$\begin{aligned} \hat L_{ii} &:= \begin{pmatrix} \hat q_{2i,2i-1} & 1\\ \hat q_{2i+1,2i-1} - \hat q_{2i+2,2i} & -\hat q_{2i+2,2i+1}\end{pmatrix}, \qquad \hat L_{i,i+1} := \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix},\\ \hat L_{i+1,i} &:= \begin{pmatrix} * & -\hat q^2_{2i+2,2i+1} - \hat q_{2i+3,2i+1} + \hat q_{2i+2,2i}\\ * & * \end{pmatrix}. \end{aligned} \tag{2.15}$$

These definitions lead to a new statement that is equivalent to Theorem 2.1.

THEOREM 2.2

$\hat{L}_i$, $\hat{Q}$, $\hat{\Psi}_1$, $\hat{\Psi}_2$ *satisfy the following equations:*

$$\frac{\partial \hat{Q}}{\partial t_n} = -\big((\hat{L}_1^n)_- - Q_0^{-2} J((\hat{L}_1^n)_+)^\top J Q_0^2\big)\hat{Q}, \tag{2.16}$$

and

$$\hat{L}_1\hat{\Psi}_1 = z\hat{\Psi}_1, \qquad \hat{L}_2\hat{\Psi}_2 = z^{-1}\hat{\Psi}_2, \tag{2.17}$$

with

$$\begin{aligned}
\frac{\partial}{\partial t_n}\hat{\Psi}_1(t,z) &= \big((\hat{L}_1^n)_+ + (\hat{L}_1^n)_0 + Q_0^{-2}\mathscr{J}((\hat{L}_1^n)_+)Q_0^2\big)\hat{\Psi}_1(t,z),\\
\frac{\partial}{\partial t_n}\hat{\Psi}_2(t,z) &= \mathscr{J}\big((\hat{L}_1^n)_+ + (\hat{L}_1^n)_0 + Q_0^{-2}\mathscr{J}((\hat{L}_1^n)_+)Q_0^2\big)\hat{\Psi}_2(t,z)\\
&= -\big((\hat{L}_2^n)_- + (\hat{L}_2^n)_0 + Q_0^2\mathscr{J}((\hat{L}_2^n)_-)Q_0^{-2}\big)\hat{\Psi}_2(t,z).
\end{aligned}$$

The proof of Theorem 2.1 hinges on the following proposition.

PROPOSITION 2.3

The following three statements are equivalent:

(i) $\partial m_\infty/\partial t_i = \Lambda^i m_\infty + m_\infty \Lambda^{\top i}$,

(ii) *the matrices* $W_1(t)$ *and* $W_2(t)$ *(defined in (2.4)) satisfy*

$$W_1(t)W_1(t')^{-1} = W_2(t)W_2(t')^{-1}, \tag{2.18}$$

(iii) *the* $\Psi_i(t,z) = W_i(t)\chi(z)$ *satisfy the bilinear identity*

$$\oint_\infty \Psi_{1,n}(t,z)\Psi_{2,m}(t',z^{-1})\frac{dz}{2\pi i z} + \oint_0 \Psi_{2,n}(t,z)\Psi_{1,m}(t',z^{-1})\frac{dz}{2\pi i z} = 0.$$

Proof

The solution to (2.1) is given by

$$m_\infty(t) = e^{\sum t_k\Lambda^k} m_\infty(0) e^{\sum t_k \Lambda^{\top k}}.$$

Therefore skew-Borel decomposing $m_\infty(t)$ and $m_\infty(0)$, we find

$$Q^{-1}(0)JQ^{\top -1}(0) = e^{-\sum t_i\Lambda^i} Q^{-1}(t)JQ^{\top -1}(t)e^{-\sum t_i\Lambda^{\top i}}, \tag{2.19}$$

and so, from the definition of W_1 and W_2,

$$\begin{aligned}
W_1^{-1}(0)W_2(0) &= Q^{-1}(0)JQ^{\top -1}(0)\\
&= \big(Q(t)e^{\sum_1^\infty t_i\Lambda^i}\big)^{-1}J\big(Q(t)e^{\sum t_i\Lambda^i}\big)^{\top -1} \quad \text{(using (2.19))}\\
&= W_1(t)^{-1}JW_1(t)^{\top -1}\\
&= W_1(t)^{-1}W_2(t),
\end{aligned} \tag{2.20}$$

implying the independence in t of the right-hand side of (2.20). Therefore, we have

$$W_1(t)^{-1}W_2(t) = W_1(t')^{-1}W_2(t'), \quad \text{for all } t, t' \in \mathbb{C}^\infty,$$

and so

$$W_1(t)W_1^{-1}(t') = W_2(t)W_2^{-1}(t'),$$

thus yielding (ii). Reversing the steps yields the differential equation (i).

Finally, the proof of the bilinear identity (iii) proceeds as follows. Using the well-known formula (see [3, Prop. 4.1]),

$$W_{\substack{1\\2}}(t)W_{\substack{1\\2}}(t')^{-1} = \oint_{\substack{\infty\\0}} \Psi_{\substack{1\\2}}(t,z) \otimes \Psi^*_{\substack{1\\2}}(t',z)\frac{dz}{2\pi iz},$$

statement (ii) becomes

$$\oint_\infty \Psi_1(t,z) \otimes \Psi_1^*(t',z)\frac{dz}{2\pi iz} = \oint_0 \Psi_2(t,z) \otimes \Psi_2^*(t',z)\frac{dz}{2\pi iz},$$

whose (m,n)th component is

$$\oint_\infty \Psi_{1,n}(t,z)\Psi^*_{1,m}(t',z)\frac{dz}{2\pi iz} - \oint_0 \Psi_{2,n}(t,z)\Psi^*_{2,m}(t',z)\frac{dz}{2\pi iz} = 0.$$

Next we use the relations $\Psi_1^*(t,z) = -J\Psi_2(t,z^{-1})$ and $\Psi_2^*(t,z) = J\Psi_1(t,z^{-1})$ to yield

$$\oint_\infty \Psi_1(t,z) \otimes J\Psi_2(t',z^{-1})\frac{dz}{2\pi iz} + \oint_0 \Psi_2(t,z) \otimes J\Psi_1(t',z^{-1})\frac{dz}{2\pi iz} = 0,$$

which again leads to (iii). That (iii) implies (ii) is obtained by reversing the arguments. □

Proof of Theorem 2.1

The proof of statement (ii) for Q_1, namely,

$$\frac{\partial Q_1}{\partial t_i} = -(\pi_{\mathbf{k}} L_1^i)Q_1,$$

follows at once from Proposition 1.2.

The proof of (iii) for $Q_2 = JQ_1^{\top -1}$ is based on the identity $\mathscr{J}\pi_{\mathbf{k}} a = \pi_{\mathbf{k}_+}\mathscr{J}a$.

Indeed, we compute

$$
\begin{aligned}
\frac{\partial Q_2}{\partial t_i} Q_2^{-1} &= -J Q_1^{\top -1} \frac{\partial Q_1^{\top}}{\partial t_i} Q_1^{\top -1} Q_2^{-1} \\
&= -J Q_1^{\top -1} Q_1^{\top} (\pi_{\mathbf{k}} L_1^i)^{\top} Q_1^{\top -1} Q_1^{\top} J \\
&= -J (\pi_{\mathbf{k}} L_1^i)^{\top} J \\
&= -\mathscr{J} (\pi_{\mathbf{k}} L_1^i) \\
&= -\pi_{\mathbf{k}_+} \mathscr{J} L_1^i \\
&= -\pi_{\mathbf{k}_+} \mathscr{J} (-\mathscr{J} L_2)^i \quad \text{(using (2.5))} \\
&= -\pi_{\mathbf{k}_+} \mathscr{J} (-1)^i (\mathscr{J} L_2)^i \\
&= -\pi_{\mathbf{k}_+} \mathscr{J} (-1)^i (-1)^{i-1} \mathscr{J} L_2^i \\
&= \pi_{\mathbf{k}_+} L_2^i .
\end{aligned}
$$

Statements (iv) and (v) for $\Psi_1,\ \Psi_2$ are straightforwardly equivalent to (ii) and (iii), respectively. According to Propositions 1.2 and 2.3 combined, the bilinear identity (2.10) in (vi) is equivalent to statement (i), (ii), or (iii). The hierarchy concerning the L_i's follows at once from (ii) and (iii), thus ending the proof of Theorem 2.1. □

Proof of Theorem 2.2
To prove (2.16), remember from Theorem 2.1 that

$$
\frac{\partial Q}{\partial t_n} Q^{-1} = -\pi_{\mathbf{k}} L^n = -\big((L^n)_- - J (L^n_+)^{\top} J\big) - \frac{1}{2}\big((L^n)_0 - J((L^n)_0)^{\top} J\big);
$$

hence, taking the $(\ )_0$-part of this expression yields

$$
\frac{\partial \log Q_0}{\partial t_n} = \Big(\frac{\partial Q}{\partial t_n} Q^{-1}\Big)_0 = -\pi_{\mathbf{k}} (L^n)_0 = -\frac{1}{2} (L^n)_0 + \frac{1}{2} J (L^n)_0^{\top} J.
$$

Using the fact that $Q_0, Q_0^{-1}, \dot{Q}_0 \in G_k \cap \mathscr{D}_0$ commute among themselves and commute with J and the fact that $\mathscr{D}_0 \mathscr{D}_\pm, \mathscr{D}_\pm \mathscr{D}_0 \subset \mathscr{D}_\pm$, we compute for $\hat{Q} = Q_0^{-1} Q,\ \hat{L}_1 = Q_0^{-1} L_1 Q_0$ (see (2.14))

$$
\begin{aligned}
\frac{\partial \hat{Q}}{\partial t_n} \hat{Q}^{-1} &= -Q_0^{-1} \dot{Q}_0 Q_0^{-1} Q Q^{-1} Q_0 + Q_0^{-1} \dot{Q} Q^{-1} Q_0 \\
&= -Q_0^{-1} \dot{Q}_0 + Q_0^{-1} \dot{Q} Q^{-1} Q_0 \\
&= Q_0^{-1} (-\dot{Q}_0 Q_0^{-1} + \dot{Q} Q^{-1}) Q_0 \\
&= Q_0^{-1} \big(-(L_1^n)_- + J (L_{1+}^n)^{\top} J \big) Q_0 \\
&= -(Q_0^{-1} L_1^n Q_0)_- + Q_0^{-1} J \big(Q_0 (Q_0^{-1} L_1^n Q_0)_+ Q_0^{-1} \big)^{\top} J Q_0 \\
&= -(\hat{L}_1^n)_- + Q_0^{-2} J ((\hat{L}_1^n)_+)^{\top} J Q_0^2 .
\end{aligned}
$$

Using this result and $\hat{L}_1\hat{\Psi}_1(t,z) = z\hat{\Psi}_1(t,z)$, we find

$$
\begin{aligned}
\frac{\partial \hat{\Psi}_1(t,z)}{\partial t_n} &= \frac{\partial}{\partial t_n} e^{\sum t_i z^i}\hat{Q}\chi(z) \\
&= z^n e^{\sum_1^\infty t_i z^i}\hat{Q}\chi(z) + e^{\sum_1^\infty t_i z^i}\big(-(\hat{L}_1^n)_- + Q_0^{-2}J((\hat{L}_1^n)_+)^\top JQ_0^2\big)\hat{Q}\chi(z) \\
&= \big(\hat{L}_1^n - (\hat{L}_1^n)_- + Q_0^{-2}J((\hat{L}_1^n)_+)^\top JQ_0^2\big)\hat{\Psi}_1(t,z) \\
&= \big((\hat{L}_1^n)_+ + (\hat{L}_1^n)_0 + Q_0^{-2}(\mathscr{J}(\hat{L}_1^n)_+)Q_0^2\big)\hat{\Psi}_1(t,z).
\end{aligned} \tag{2.21}
$$

But, we also have that $\hat{\Psi}_1 = Q_0^{-1}\Psi_1(t,z)$ and $\hat{\Psi}_2 = Q_0\Psi_2(t,z)$ satisfy, using $W_2 = JW_1^{-1\top}$,

$$
\frac{\partial \hat{\Psi}_1(t,z)}{\partial t_n} = (Q_0^{-1}W_1)^{\cdot}\chi(z) = (Q_0^{-1}W_1)^{\cdot}(Q_0^{-1}W_1)^{-1}\hat{\Psi}_1(t,z), \tag{2.22}
$$

$$
\begin{aligned}
\frac{\partial \hat{\Psi}_2(t,z)}{\partial t_n} = (Q_0W_2)^{\cdot}\chi(z) &= (Q_0W_2)^{\cdot}(Q_0W_2)^{-1}(Q_0\Psi_2) \\
&= (\dot{Q}_0W_2 + Q_0\dot{W}_2)W_2^{-1}Q_0^{-1}(Q_0\Psi_2) \\
&= (\dot{Q}_0Q_0^{-1} + Q_0\dot{W}_2W_2^{-1}Q_0^{-1})Q_0\Psi_2 \\
&= \big(\dot{Q}_0Q_0^{-1} + Q_0\mathscr{J}(\dot{W}_1W_1^{-1})Q_0^{-1}\big)Q_0\Psi_2 \\
&= \big(\dot{Q}_0Q_0^{-1} + Q_0J(\dot{W}_1W_1^{-1})^\top JQ_0^{-1}\big)Q_0\Psi_2 \\
&= \big(-J\dot{Q}_0Q_0^{-1} + J(Q_0^{-1}\dot{W}_1W_1^{-1}Q_0)^\top\big)JQ_0\Psi_2 \\
&= J(-Q_0^{-1}\dot{Q}_0 + Q_0^{-1}\dot{W}_1W_1^{-1}Q_0)^\top JQ_0\Psi_2 \\
&= \mathscr{J}\big((Q_0^{-1}W_1)^{\cdot}(Q_0^{-1}W_1)^{-1}\big)(Q_0\Psi_2).
\end{aligned} \tag{2.23}
$$

Comparing (2.21), (2.22), and (2.23), and invoking (2.14),

$$
-\mathscr{J}(\hat{L}_1^n) = \hat{L}_2^n,
$$

and so, in particular,

$$
-\mathscr{J}((\hat{L}_1^n)_\pm) = (\hat{L}_2^n)_\mp \qquad \text{and} \qquad -\mathscr{J}((\hat{L}_1^n)_0) = (\hat{L}_2^n)_0,
$$

$$
\frac{\partial \hat{\Psi}_2(t,z)}{\partial t_n} = -\big((\hat{L}_2^n)_- + (\hat{L}_2^n)_0 + Q_0^2(\mathscr{J}(L_2^n)_-)Q_0^{-2}\big)\hat{\Psi}_2(t,z),
$$

which establishes Theorem 2.2. □

3. Existence of the Pfaff τ-function

The point of this section is to show that the solution of the Pfaff lattice can be expressed in terms of a sequence of functions τ, which are not τ-functions in the usual sense but which enjoy a different set of bilinear identities and partial differential equations.

PROPOSITION 3.1
There exist functions $\tau_{2n}(t)$ such that

$$\psi_{1,2n}(t,z)=\frac{\tau_{2n}(t-[z^{-1}])}{\tau_{2n}(t)} \qquad \textit{and} \qquad \psi_{2,2n}(t,z)=\frac{\tau_{2n+2}(t+[z])}{\tau_{2n+2}(t)}. \tag{3.1}$$

The proof of Proposition 3.1 is postponed until later. For future use, we define the diagonal matrix

$$h=\operatorname{diag}(\dots,h_{-2},h_{-2},h_0,h_0,h_2,h_2,\dots)\in\mathscr{D}_0^-, \quad \text{with } h_{2n}=\frac{\tau_{2n+2}}{\tau_{2n}}. \tag{3.2}$$

THEOREM 3.2
Ψ_1 and Ψ_2 have the following τ-function representation:

$$\begin{aligned}
\Psi_{1,2n}(t,z)&=e^{\sum t_iz^i}z^{2n}h_{2n}^{-1/2}\frac{\tau_{2n}(t-[z^{-1}])}{\tau_{2n}(t)},\\
\Psi_{1,2n+1}(t,z)&=e^{\sum t_iz^i}z^{2n}h_{2n}^{-1/2}\frac{(z+\partial/\partial t_1)\tau_{2n}(t-[z^{-1}])}{\tau_{2n}(t)},\\
\Psi_{2,2n}(t,z)&=e^{-\sum t_iz^{-i}}z^{2n+1}h_{2n}^{-1/2}\frac{\tau_{2n+2}(t+[z])}{\tau_{2n}(t)},\\
\Psi_{2,2n+1}(t,z)&=e^{-\sum t_iz^{-i}}z^{2n+1}h_{2n}^{-1/2}\frac{(z^{-1}-\partial/\partial t_1)\tau_{2n+2}(t+[z])}{\tau_{2n}(t)},
\end{aligned} \tag{3.3}$$

with the $\tau_{2n}(t)$ satisfying the following bilinear identity for all $n,m\in\mathbb{Z}$:

$$\begin{aligned}
&\oint_{z=\infty}\tau_{2n}\big(t-[z^{-1}]\big)\tau_{2m+2}\big(t'+[z^{-1}]\big)e^{\sum(t_i-t_i')z^i}z^{2n-2m-2}\frac{dz}{2\pi i}\\
&\qquad+\oint_{z=0}\tau_{2n+2}\big(t+[z]\big)\tau_{2m}\big(t'-[z]\big)e^{\sum(t_i'-t_i)z^{-i}}z^{2n-2m}\frac{dz}{2\pi i}=0.
\end{aligned} \tag{3.4}$$

Conversely, this bilinear relation characterizes the τ-function for the Pfaff lattice.

Remark
Then L has the following representation in terms of the Pfaffian τ-functions:

$$h^{1/2}Lh^{-1/2}=\begin{pmatrix}
 & \vdots & & & & \\
\dots & \hat{L}_{00} & \hat{L}_{01} & 0 & 0 & \\
 & \hat{L}_{10} & \hat{L}_{11} & \hat{L}_{12} & 0 & \\
 & * & \hat{L}_{21} & \hat{L}_{22} & \hat{L}_{23} & \\
 & * & * & \hat{L}_{32} & \hat{L}_{33} & \dots\\
 & & & & \vdots &
\end{pmatrix},$$

with $(^{\cdot} = \partial/\partial t_1)$,

$$\hat{L}_{nn} := \begin{pmatrix} -(\log \tau_{2n})^{\cdot} & 1 \\ -\frac{\mathbf{s}_2(\tilde{\partial})\tau_{2n}}{\tau_{2n}} - \frac{\mathbf{s}_2(-\tilde{\partial})\tau_{2n+2}}{\tau_{2n+2}} & (\log \tau_{2n+2})^{\cdot} \end{pmatrix}, \qquad \hat{L}_{n,n+1} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$\hat{L}_{n+1,n} := \begin{pmatrix} * & (\log \tau_{2n+2})^{\cdot\cdot} \\ * & * \end{pmatrix}. \tag{3.5}$$

The following bilinear relations follow from (3.4) and are due to [4].

COROLLARY 3.3
The functions $\tau_{2n}(t)$ satisfy the following "differential Fay identity":*

$$\begin{aligned}\{\tau_{2n}(t-[u]), \tau_{2n}(t-[v])\} \\ + (u^{-1}-v^{-1})\big(\tau_{2n}(t-[u])\tau_{2n}(t-[v]) - \tau_{2n}(t)\tau_{2n}(t-[u]-[v])\big) \\ = uv(u-v)\tau_{2n-2}(t-[u]-[v])\tau_{2n+2}(t),\end{aligned} \tag{3.6}$$

and Hirota-type bilinear equations, always involving nearest neighbours:

$$\left(p_{k+4}(\tilde{\partial}) - \frac{1}{2}\frac{\partial^2}{\partial t_1 \partial t_{k+3}}\right)\tau_{2n} \circ \tau_{2n} = p_k(\tilde{\partial})\, \tau_{2n+2} \circ \tau_{2n-2}, \quad k, n = 0, 1, 2, \ldots. \tag{3.7}$$

LEMMA 3.4
Consider an arbitrary function $\varphi(t, z)$ depending on $t \in \mathbb{C}^\infty$, $z \in \mathbb{C}$, having the asymptotics $\varphi(t, z) = 1 + O(1/z)$ for $z \nearrow \infty$ and satisfying the functional relation

$$\frac{\varphi(t-[z_2^{-1}], z_1)}{\varphi(t, z_1)} = \frac{\varphi(t-[z_1^{-1}], z_2)}{\varphi(t, z_2)}, \quad t \in \mathbb{C}^\infty, \ z \in \mathbb{C}.$$

Then there exists a function $\tau(t)$ such that

$$\varphi(t, z) = \frac{\tau(t-[z^{-1}])}{\tau(t)}.$$

Proof
See Appendix C. □

*We define $\{f, g\} := f'g - fg'$, where $' = \partial/\partial t_1$.

LEMMA 3.5
The following holds for the Pfaffian wave functions Ψ_1 and Ψ_2, as in (2.7),

$$\frac{\psi_{1,2n}(t-[z_2^{-1}],z_1)}{\psi_{1,2n}(t,z_1)}=\frac{\psi_{1,2n}(t-[z_1^{-1}],z_2)}{\psi_{1,2n}(t,z_2)} \tag{3.8}$$

and

$$\psi_{2,2n-2}\bigl(t-[z^{-1}],z^{-1}\bigr)\psi_{1,2n}(t,z)=1. \tag{3.9}$$

Proof
Setting (2.7) in the bilinear equation (2.12), with $n\mapsto 2n,\ m\mapsto 2n-2$, yields

$$\frac{c_{2n}(t)}{c_{2n-2}(t)}\oint_\infty e^{\sum(t_i-t_i')z^i}\psi_{1,2n}(t,z)\psi_{2,2n-2}(t',z^{-1})\frac{dz}{2\pi i} \\ +\frac{c_{2n-2}(t)}{c_{2n}(t)}\oint_0 e^{\sum(t_i'-t_i)z^{-i}}\psi_{2,2n}(t,z)\psi_{1,2n-2}(t',z^{-1})\frac{z^2dz}{2\pi i}=0.$$

Setting

$$t-t'=[z_1^{-1}]+[z_2^{-1}]$$

in the above and using $e^{\sum_1^\infty x^i/i}=1/(1-x)$ yields

$$\frac{c_{2n}}{c_{2n-2}}\oint_\infty\frac{\psi_{1,2n}(t,z)\psi_{2,2n-2}(t',z^{-1})}{(1-z/z_1)(1-z/z_2)}\frac{dz}{2\pi i} \\ =-\frac{c_{2n-2}}{c_{2n}}\oint_0 z^2\Bigl(1-\frac{1}{zz_1}\Bigr)\Bigl(1-\frac{1}{zz_2}\Bigr)\psi_{2,2n}(t,z)\psi_{1,2n-2}(t',z^{-1})\frac{dz}{2\pi i}=0,$$

the latter being equal to zero, because the integrand on the right-hand side is holomorphic. The integral on the left-hand side can be viewed as an integral along a contour encompassing ∞ and the points z_1 and z_2, thus leading to

$$\psi_{1,2n}(t,z_1)\psi_{2,2n-2}\bigl(t-[z_1^{-1}]-[z_2^{-1}],z_1^{-1}\bigr) \\ =\psi_{1,2n}(t,z_2)\psi_{2,2n-2}\bigl(t-[z_1^{-1}]-[z_2^{-1}],z_2^{-1}\bigr) \tag{3.10}$$

with

$$\psi_{1,2n}(t,z)=1+O(z^{-1}),\qquad \psi_{2,2n-2}\bigl(t-[z_1^{-1}]-[z_2^{-1}],z^{-1}\bigr)=1+O(z^{-1}).$$

Therefore, letting $z_2\nearrow\infty$, one finds

$$\psi_{1,2n}(t,z_1)\psi_{2,2n-2}\bigl(t-[z_1^{-1}],z_1^{-1}\bigr)=1, \tag{3.11}$$

yielding (3.9), and so, upon shifting $t\mapsto t-[z_2^{-1}]$,

$$\psi_{2,2n-2}\bigl(t-[z_1^{-1}]-[z_2^{-1}],z_1^{-1}\bigr)=\frac{1}{\psi_{1,2n}(t-[z_2^{-1}],z_1)};$$

similarly,

$$\psi_{2,2n-2}\big(t-[z_1^{-1}]-[z_2^{-1}],z_2^{-1}\big)=\frac{1}{\psi_{1,2n}(t-[z_1^{-1}],z_2)}. \tag{3.12}$$

Setting the two expressions (3.12) in (3.10) yields

$$\frac{\psi_{1,2n}(t-[z_2^{-1}],z_1)}{\psi_{1,2n}(t,z_1)}=\frac{\psi_{1,2n}(t-[z_1^{-1}],z_2)}{\psi_{1,2n}(t,z_2)}. \qquad \square$$

Proof of Proposition 3.1
From Lemmas 3.4 and 3.5, there exists, for each $2n$, a function τ_{2n} such that the first relation of (3.1) is satisfied; that is,

$$\psi_{1,2n}(t,z)=\frac{\tau_{2n}(t-[z^{-1}])}{\tau_{2n}(t)},$$

and so from (3.9)

$$\psi_{2,2n-2}\big(t-[z^{-1}],z^{-1}\big)=\frac{1}{\psi_{1,2n}(t,z)}=\frac{\tau_{2n}(t)}{\tau_{2n}(t-[z^{-1}])},$$

thus leading to

$$\psi_{2,2n-2}(t,z)=\frac{\tau_{2n}(t+[z])}{\tau_{2n}(t)},$$

which is the second relation of (3.1). $\square$

Proof of Theorem 3.2
At first, remembering that $\hat{Q}=Q_0^{-1}Q$, observe that

$$\begin{aligned} e^{\sum t_i z^i}\big((\hat{Q})\chi(z)\big)_{2n} &= \big(Q_0^{-1}\Psi_1(t,z)\big)_{2n}\\ &= e^{\sum t_i z^i} z^{2n}\psi_{1,2n}(t,z)\\ &= e^{\sum t_i z^i} z^{2n}\frac{\tau_{2n}(t-[z^{-1}])}{\tau_{2n}(t)}\\ &= e^{\sum t_i z^i} z^{2n}\Big(1+\sum_{n=1}^{\infty}\frac{\mathbf{s}_k(-\tilde{\partial})\tau_{2n}(t)}{\tau_{2n}(t)}\Big), \end{aligned}$$

showing that a few subdiagonals of the matrix $\hat{Q}$ are given by

$$\hat{Q} = \begin{pmatrix} \ddots & & & & \\ & \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & & & \\ & \boxed{\begin{matrix} \hat{q}_{2n,2n-2} & \hat{q}_{2n,2n-1} \\ \hat{q}_{2n+1,2n-2} & \hat{q}_{2n+1,2n-1} \end{matrix}} & \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & & \\ & & & \ddots & \end{pmatrix}$$

with

$$\hat{q}_{2n,2n-1} = -\frac{\partial}{\partial t_1} \log \tau_{2n}, \qquad \hat{q}_{2n,2n-2} = \frac{\mathbf{s}_2(-\tilde{\partial})\tau_{2n}}{\tau_{2n}}. \tag{3.13}$$

Remembering that (2.7), normalized, becomes

$$\begin{cases} \hat{\Psi}_{1,2n}(t,z) = e^{\sum t_k z^k} z^{2n} \psi_{1,2n}(t,z), & \psi_{1,2n} = 1 + O(z^{-1}), \\ \hat{\Psi}_{1,2n+1}(t,z) = e^{\sum t_k z^k} z^{2n+1} \psi_{1,2n+1}(t,z), & \psi_{1,2n+1} = 1 + O(z^{-2}), \end{cases} \tag{3.14}$$

$$\begin{cases} \hat{\Psi}_{2,2n}(t,z) = e^{-\sum t_k z^{-k}} z^{2n+1} \psi_{2,2n}(t,z), & \psi_{2,2n} = 1 + O(z), \\ \hat{\Psi}_{2,2n+1}(t,z) = -e^{-\sum t_k z^{-k}} z^{2n} \psi_{2,2n+1}(t,z), & \psi_{2,2n+1} = 1 + O(z^2), \end{cases} \tag{3.15}$$

we now show (3.3). Compute, using Theorem 2.2,

$$\begin{aligned} e^{\sum t_i z^i} \Big(\frac{\partial}{\partial t_1} + z\Big) z^{2n} \psi_{1,2n}(t,z) &= \Big(\frac{\partial}{\partial t_1} \hat{\Psi}_1(t,z)\Big)_{2n} \\ &= \Big(\big((\hat{L}_1)_+ + (\hat{L}_1)_0 + Q_0^{-2} J (\hat{L}_{1+})^\top J Q_0^2\big) \hat{\Psi}_1(t,z)\Big)_{2n} \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} e^{-\sum t_i z^{-i}} \Big(\frac{\partial}{\partial t_1} - \frac{1}{z}\Big) z^{2n+1} \psi_{2,2n}(t,z) &= \Big(\frac{\partial}{\partial t_1} \hat{\Psi}_2(t,z)\Big)_{2n} \\ &= \Big(\big(\mathscr{J}\big((\hat{L}_1)_+ + (\hat{L}_1)_0 + Q_0^{-2} J (\hat{L}_{1+})^\top J Q_0^2\big)\big) \hat{\Psi}_2(t,z)\Big)_{2n}. \end{aligned} \tag{3.17}$$

In this expression, the matrix equals, according to (2.14),

$$
\begin{aligned}
&(\hat{L}_1)_+ + (\hat{L}_1)_0 + Q_0^{-2} J (\hat{L}_{1+})^\top J Q_0^2 \\
&= \left(\begin{array}{ccc|cc|ccc}
 & \vdots & & & & & & \\
\cdots & \hat{q}_{0,-1} & 1 & 0 & 0 & 0 & 0 & \\
 & \hat{q}_{1,-1} - \hat{q}_{20} & -\hat{q}_{21} & 1 & 0 & 0 & 0 & \\
\hline
 & 0 & 0 & \hat{q}_{21} & 1 & 0 & 0 & \\
 & c_0^2/c_2^2 & 0 & \hat{q}_{31} - \hat{q}_{42} & -\hat{q}_{43} & 1 & 0 & \\
\hline
 & 0 & 0 & 0 & 0 & \hat{q}_{43} & 1 & \\
 & 0 & 0 & c_2^2/c_4^2 & 0 & \hat{q}_{53} - \hat{q}_{64} & -\hat{q}_{65} & \cdots \\
 & & & & & & \vdots &
\end{array}\right),
\end{aligned}
$$

and, acting with $\mathscr{J}$ on this matrix,

$$
\begin{aligned}
&\mathscr{J}\left((\hat{L}_1)_+ + (\hat{L}_1)_0 + Q_0^{-2} J (\hat{L}_{1+})^\top J Q_0^2\right) \\
&= \left(\begin{array}{ccc|cc|ccc}
 & \vdots & & & & & & \\
\cdots & \hat{q}_{21} & 1 & 0 & 0 & 0 & 0 & \\
 & \hat{q}_{1,-1} - \hat{q}_{20} & -\hat{q}_{0,-1} & c_0^2/c_2^2 & 0 & 0 & 0 & \\
\hline
 & 0 & 0 & \hat{q}_{43} & 1 & 0 & 0 & \\
 & 1 & 0 & \hat{q}_{31} - \hat{q}_{42} & -\hat{q}_{21} & c_2^2/c_4^2 & 0 & \\
\hline
 & 0 & 0 & 0 & 0 & \hat{q}_{65} & 1 & \\
 & 0 & 0 & 1 & 0 & \hat{q}_{53} - \hat{q}_{64} & -\hat{q}_{43} & \cdots \\
 & & & & & & \vdots &
\end{array}\right),
\end{aligned}
$$

using the fact that

$$
\mathscr{J}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}.
$$

Therefore the $2n$th rows of both matrices, respectively, have the form

$$
(0, \dots, 0, \underset{\substack{\uparrow \\ 2n}}{\hat{q}_{2n,2n-1}(t)}, 1, 0, 0, \dots),
$$

$$
(0, \dots, 0, \underset{\substack{\uparrow \\ 2n}}{\hat{q}_{2n+2,2n+1}(t)}, 1, 0, 0, \dots),
$$

and thus from (3.16) and (3.17), and expansions (3.14) and (3.15), we have

$$\begin{aligned}\Big(\frac{\partial}{\partial t_1}+z\Big)z^{2n}\psi_{1,2n}(t,z)&=\hat q_{2n,2n-1}(t)z^{2n}\psi_{1,2n}+z^{2n+1}\psi_{1,2n+1},\\ \Big(\frac{\partial}{\partial t_1}-z^{-1}\Big)z^{2n+1}\psi_{2,2n}(t,z)&=\hat q_{2n+2,2n+1}(t)z^{2n+1}\psi_{2,2n}+z^{2n}\psi_{2,2n+1}.\end{aligned}\tag{3.18}$$

So, using the expression (3.13) for $\hat q_{2n,2n-1}$ and the first expression of (3.1),

$$\begin{aligned}&z^{2n+1}\psi_{1,2n+1}(t,z)\\ &=\Big(z+\frac{\partial}{\partial t_1}\Big)z^{2n}\psi_{1,2n}(t,z)-\hat q_{2n,2n-1}(t)z^{2n}\psi_{1,2n}(t,z)\\ &=\Big(z+\frac{\partial}{\partial t_1}\Big)z^{2n}\psi_{1,2n}(t,z)+\Big(\frac{\partial}{\partial t_1}\log\tau_{2n}(t)\Big)z^{2n}\psi_{1,2n}(t,z)\\ &=\Big(z+\frac{\partial}{\partial t_1}\Big)z^{2n}\frac{\tau_{2n}(t-[z^{-1}])}{\tau_{2n}(t)}+\Big(\frac{\partial}{\partial t_1}\tau_{2n}(t)\Big)z^{2n}\frac{\tau_{2n}(t-[z^{-1}])}{\tau_{2n}^2(t)}\\ &=z^{2n}\frac{(z+\partial/\partial t_1)\tau_{2n}(t-[z^{-1}])}{\tau_{2n}(t)},\end{aligned}\tag{3.19}$$

and similarly, using the second relation (3.18),

$$z^{2n}\psi_{2,2n+1}(t,z)=z^{2n+1}\frac{(-z^{-1}+\partial/\partial t_1)\tau_{2n+2}(t+[z])}{\tau_{2n+2}(t)}.\tag{3.20}$$

This establishes (3.3) modulo the denominators. Therefore, we also have

$$\begin{aligned}\psi_{1,2n+1}(t,z)&=\frac{1}{z}\frac{1}{\tau_{2n}(t)}\Big(z+\frac{\partial}{\partial t_1}\Big)\Big(\tau_{2n}(t)-\frac{\partial\tau_{2n}}{\partial t_1}z^{-1}+\mathbf{s}_2(-\tilde\partial)\tau_{2n}z^{-2}+\cdots\Big)\\ &=1+\frac{1}{\tau_{2n}(t)}\Big(-\frac{\partial^2}{\partial t_1^2}+\mathbf{s}_2(-\tilde\partial)\Big)\tau_{2n}z^{-2}+O(z^{-3});\end{aligned}$$

thus, referring to the matrix $\hat Q$ just preceding (3.13),

$$\hat q_{2n+1,2n}=0,\quad \hat q_{2n+1,2n-1}=\frac{1}{\tau_{2n}}\Big(\mathbf{s}_2(-\tilde\partial)-\frac{\partial^2}{\partial t_1^2}\Big)\tau_{2n}=\frac{-\mathbf{s}_2(\tilde\partial)\tau_{2n}}{\tau_{2n}}.\tag{3.21}$$

To show (3.4), setting $n\mapsto 2n$ and $m\mapsto 2n$ in bilinear relation (2.10) and substituting, using (2.7) and the expressions for $\psi_{1,2n}(t,z)$ and $\psi_{2,2n}(t,z)$ in the proof of Proposition 3.1,

$$\Psi_{1,2n}(t,z)=e^{\sum t_kz^k}z^{2n}c_{2n}(t)\frac{\tau_{2n}(t-[z^{-1}])}{\tau_{2n}(t)}$$

and

$$\Psi_{2,2n}(t',z)=e^{-\sum t'_kz^{-k}}z^{2n+1}c_{2n}^{-1}(t')\frac{\tau_{2n+2}(t'+[z])}{\tau_{2n+2}(t')}$$

into

$$\oint_\infty \Psi_{1,2n}(t,z)\Psi_{2,2n}(t',z^{-1})\frac{dz}{2\pi i z} + \oint_0 \Psi_{2,2n}(t,z)\Psi_{1,2n}(t',z^{-1})\frac{dz}{2\pi i z} = 0$$

yields

$$\frac{c_{2n}(t)}{c_{2n}(t')}\oint_\infty e^{\sum(t_k-t'_k)z^k}\frac{\tau_{2n}(t-[z^{-1}])\tau_{2n+2}(t'+[z^{-1}])}{\tau_{2n}(t)\tau_{2n+2}(t')}\frac{dz}{2\pi i z^2} + \frac{c_{2n}(t')}{c_{2n}(t)}\oint_0 e^{\sum(t'_k-t_k)z^{-k}}\frac{\tau_{2n+2}(t+[z])\tau_{2n}(t'-[z])}{\tau_{2n+2}(t)\tau_{2n}(t')}\frac{dz}{2\pi i}.$$

Setting $t' = t + [\alpha]$ amounts to replacing the exponential:

$$e^{\sum(t_k-t'_k)z^k} = 1-\alpha z, \qquad e^{\sum(t'_k-t_k)z^{-k}} = \frac{1}{1-\alpha/z},$$

so that the first integral has a simple pole at $z=\infty$ and the second integral has one at $z=\alpha$. Evaluating the integrals yields

$$-\alpha\frac{c_{2n}^2(t)(\tau_{2n+2}(t)/\tau_{2n}(t))}{c_{2n}^2(t')\tau_{2n+2}(t')/\tau_{2n}(t')} + \alpha = 0;$$

that is,

$$(e^{\sum(\alpha^i/i)(\partial/\partial t_i)}-1)c_n^2(t)\frac{\tau_{2n+2}(t)}{\tau_{2n}(t)} = 0$$

yields the following relation, which involves a constant c_n, independent of time,

$$c_{2n}^2(t) = c_n\frac{\tau_{2n}(t)}{\tau_{2n+2}(t)} = c_n\cdot h_{2n}^{-1}(t). \tag{3.22}$$

Rescaling $\tau_{2n} \mapsto \tau_{2n}/(c_1c_2\cdots c_{n-1})$, in effect, sets $c_n = 1$, and then (3.22), (2.7), (3.1), (3.19), and (3.20) yield (3.3); substituting (3.3) into (2.10) yields (3.4).

Finally, identity (3.22) actually says $Q_0 = h^{-1/2}$. To derive the form (3.5) of the matrix L, set (3.13) and (3.21) in the elements just below the main diagonal of matrix (2.15), to yield ($\dot{} = \partial/\partial t_1$)

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$$\begin{aligned}
-\hat q_{2n,2n-1}^2 - \hat q_{2n+1,2n-1} + \hat q_{2n,2n-2} &= -\left(\frac{\dot\tau_{2n}}{\tau_{2n}}\right)^2 - \frac{(\mathbf{s}_2(-\tilde\partial)-(\partial^2/\partial t_1^2))\tau_{2n}}{\tau_{2n}} \\
&\quad + \frac{\mathbf{s}_2(-\tilde\partial)\tau_{2n}}{\tau_{2n}} \\
&= \frac{\ddot\tau_{2n}}{\tau_{2n}} - \left(\frac{\dot\tau_{2n}}{\tau_{2n}}\right)^2 \\
&= (\log\tau_{2n})^{\cdot\cdot}
\end{aligned}$$

and

$$\hat{q}_{2n+1,2n-1} - \hat{q}_{2n+2,2n} = \frac{(\mathbf{s}_2(-\tilde{\partial}) - \partial^2/\partial t_1^2)\tau_{2n}}{\tau_{2n}} - \frac{\mathbf{s}_2(-\tilde{\partial})\tau_{2n+2}}{\tau_{2n+2}}$$
$$= -\frac{\mathbf{s}_2(\tilde{\partial})\tau_{2n}}{\tau_{2n}} - \frac{\mathbf{s}_2(-\tilde{\partial})\tau_{2n+2}}{\tau_{2n+2}},$$

concluding the proof of Theorem 3.2, upon substituting the two relations (3.13) and (3.14) and also $Q_0 = h^{-1/2}$ into (2.15). □

4. Semi-infinite matrices m_∞, (skew-)orthogonal polynomials, and matrix integrals

In this section, consider the following inner-product* for $\alpha = 0, \mp 1$:

$$\langle f, g\rangle_t = \int\int_{\mathbb{R}^2} f(y)g(z)e^{\sum t_i(y^i+z^i)} 2D^\alpha \delta(y-z)\tilde{\rho}(y)\tilde{\rho}(z)\,dy\,dz$$
$$= \begin{cases} \displaystyle\int_{\mathbb{R}} f(y)g(y)e^{\sum 2t_i y^i} 2\tilde{\rho}(y)^2 dy, & \text{for } \alpha = 0, \\ \displaystyle\iint_{\mathbb{R}^2} f(y)g(z)e^{\sum_1^\infty t_i(y^i+z^i)}\varepsilon(y-z)\tilde{\rho}(y)\tilde{\rho}(z)\,dy\,dz, & \text{for } \alpha = -1, \\ \displaystyle\int_{\mathbb{R}} \{f,g\}(y)e^{\sum_1^\infty 2t_i y^i}\tilde{\rho}(y)^2\,dy, & \text{for } \alpha = +1. \end{cases} \tag{4.1}$$

Each type of inner-product is discussed in Sections 4.1 and 4.2.

4.1. $\partial m_\infty/\partial t_k = \Lambda^k m_\infty$, orthogonal polynomials, and Hermitian matrix integrals ($\alpha = 0$)

The inner-product above, with $\alpha = 0$, corresponds to Hermitian matrix integrals; this theory is sketched here for the sake of completeness and analogy; it mainly summarizes [6]. Consider a t-dependent weight

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$$\rho_t(dz) := e^{\sum t_i z^i}, \qquad \rho(dz) = e^{-V(z)+\sum t_i z^i} dz$$

on $\mathbb{R}$, as in (0.1) and the induced t-dependent measure

$$e^{\operatorname{Tr}(-V(X)+\sum t_i X^i)}\,dX, \tag{4.2}$$

on the ensemble $\mathscr{H}_n$ of Hermitian matrices, with Haar measure dX; the latter can be decomposed into a spectral part (radial part) and an angular part:

$$dX := \prod_1^n dX_{ii} \prod_{1\le i<j\le n} (d\Re X_{ij}\, d\Im X_{ij}) = \Delta^2(z)\,dz_1\cdots dz_n\,dU, \tag{4.3}$$

*We have $\varepsilon(x) = \operatorname{sign} x$, having the property $\varepsilon' = 2\delta(x)$. Also, consider the Wronskian $\{f,g\} := (\partial f/\partial y)g - f(\partial g/\partial y)$.

where $\Delta(z) = \prod_{1\le i<j\le n}(z_i - z_j)$ is the Vandermonde determinant. Here we form the following matrix integral

$$\int_{\mathscr{H}_n} e^{\mathrm{Tr}(-V(X)+\sum t_i X^i)}\, dX = c_n \int_{\mathbb{R}^n} \Delta^2(z) \prod_{k=1}^{n} e^{\sum_1^\infty t_i z_k^i} \rho(dz_k). \tag{4.4}$$

The weight $\rho_t(dz)$ defines a (symmetric) t-dependent inner-product of the type (4.1) for $\alpha = 0$:

$$\langle f, g\rangle_t^{\mathrm{sy}} = \int f(z)g(z)e^{\sum_1^\infty t_i z^i} \rho(dz),$$

with moments

$$\mu_{ij}(t) := \langle z^i, z^j\rangle_t^{\mathrm{sy}} = \int_{\mathbb{R}} z^{i+j} e^{\sum t_k z^k} \rho(dz)$$

satisfying

$$\frac{\partial \mu_{ij}}{\partial t_\ell} = \int_{\mathbb{R}} z^{i+j+\ell}\, e^{\sum t_k z^k} \rho(dz) = \mu_{i+\ell,j}(t).$$

Therefore the semi-infinite moment matrix $m_\infty(t) = (\mu_{ij}(t))_{i,j\ge 0}$ satisfies

$$\frac{\partial m_\infty}{\partial t_i} = \Lambda^i m_\infty = m_\infty \Lambda^{\top i}. \tag{4.5}$$

The point now is that the following integral can be expressed as a determinant of moments, namely,

$$\begin{aligned}
\int_{\mathscr{H}_n} e^{\mathrm{Tr}(-V(X)+\sum_1^\infty t_i X^i)}\, dX &= \int_{\mathbb{R}^n} \Delta^2(z) \prod_{k=1}^{n} \rho_t(dz_k) \\
&= \int_{\mathbb{R}^n} \sum_{\sigma\in S_n} \det(z_{\sigma(k)}^{\ell-1} z_{\sigma(k)}^{k-1})_{1\le \ell,k\le n} \prod_{k=1}^{n} \rho_t(dz_k) \\
&= \int_{\mathbb{R}^n} \sum_{\sigma\in S_n} \det(z_{\sigma(k)}^{\ell+k-2})_{1\le \ell,k\le n} \prod_{k=1}^{n} \rho_t(dz_{\sigma(k)}) \\
&= \sum_{\sigma\in S_n} \det\Big(\int_{\mathbb{R}} z_{\sigma(k)}^{\ell+k-2} \rho_t(dz_{\sigma(k)})\Big)_{1\le \ell,k\le n} \\
&= n!\det\Big(\int_{\mathbb{R}} z^{\ell+k-2}\rho_t(dz)\Big)_{1\le \ell,k\le n} \\
&= n!\det(\mu_{ij})_{0\le i,j\le n-1} = n!\tau_n(t)
\end{aligned}$$

is a τ-function for the KP-equation; also, in view of (4.5) and the upper-lower Borel decomposition (0.3) of m_∞, the integrals form a vector of τ-functions for the Toda lattice. The polynomials $p_n(t;z)$ defined by the Borel decomposition $m_\infty(t) = S^{-1}S^{\top -1}$ and $p(t;z) = S\chi(z)$ are orthonormal with regard to the inner-product $\langle z^i, z^j\rangle_t^{\mathrm{sy}} = \mu_{ij}(t)$.

4.2. $\partial m_\infty/\partial t_k = \Lambda^k m_\infty + m_\infty \Lambda^{\top k}$, *skew-orthogonal polynomials, and symmetric and symplectic matrix integrals* ($\alpha = \pm 1$)

Consider a skew-symmetric semi-infinite matrix

$$m_\infty(t) = \big(\mu_{ij}(t)\big)_{i,j\geq 0}, \quad \text{with } m_n(t) = \big(\mu_{ij}(t)\big)_{0\leq i,j\leq n-1},$$

satisfying

$$\partial m_\infty/\partial t_k = \Lambda^k m_\infty + m_\infty \Lambda^{\top n}. \tag{4.6}$$

Then we have shown in Sections 2 and 3 that, upon skew-Borel decomposing m_∞, these equations ultimately imply the existence of functions $\tau(t)$ satisfying bilinear equations (3.4). Remember also that

$$h(t) = \operatorname{diag}(h_0, h_0, h_2, h_2, \ldots) \in \mathscr{D}_0^-, \quad \text{with } h_{2n}(t) = \frac{\tau_{2n+2}(t)}{\tau_{2n}(t)}.$$

Here, we need the Pfaffian $\operatorname{pf}(A)$ of a skew-symmetric matrix $A = (a_{ij})_{0\leq i,j\leq n-1}$ for* even n:

$$\begin{aligned}\operatorname{pf}(A)\, dx_0 \wedge \cdots \wedge dx_{n-1} &= \frac{1}{(n/2)!}\Big(\sum_{0\leq i<j\leq n-1} a_{ij}\, dx_i \wedge dx_j\Big)^{n/2} r \\ &= \frac{1}{2^{n/2}(n/2)!}\Big(\sum_\sigma \varepsilon(\sigma) a_{i_0,i_1} a_{i_2,i_3} \cdots a_{i_{n-2},i_{n-1}}\Big)\, dx_0 \wedge \cdots \wedge dx_{n-1},\end{aligned} \tag{4.7}$$

so that $\operatorname{pf}(A)^2 = \det A$. We now state the following theorem due to Adler, E. Horozov, and van Moerbeke [2], in complete analogy with the discussion of the Hermitian case.

THEOREM 4.1

Consider a semi-infinite skew-symmetric matrix m_∞, evolving according to (4.6); setting

$$\tau_{2n}(t) = \operatorname{pf}\big(m_{2n}(t)\big) \qquad \textit{and} \qquad h_{2n} = \frac{\operatorname{pf}(m_{2n+2}(t))}{\operatorname{pf}(m_{2n}(t))}; \tag{4.8}$$

then, modulo the exponential, the wave vector Ψ_1 (defined by (3.3)) is a sequence of polynomials,

$$\Psi_{1,k}(t,z) = e^{\sum t_i z^i} q_k(t,z), \tag{4.9}$$

where the q_k's are skew-orthonormal polynomials of the form (0.13) and (0.18), satisfying

$$(\langle q_i, q_j\rangle^{\mathrm{sk}})_{0\leq i,j<\infty} = J, \quad \textit{with } \langle y^i, z^j\rangle^{\mathrm{sk}} := \mu_{ij}. \tag{4.10}$$

*In the formula below $(i_0, i_1, \ldots, i_{n-2}, i_{n-1}) = \sigma(0, 1, \ldots, n-1)$, where σ is a permutation and $\varepsilon(\sigma)$ its parity.

The matrix Q defined by $q(z) = Q\chi(z)$ is the unique solution (modulo signs) to the skew-Borel decomposition of m_∞:

$$m_\infty(t) = Q^{-1} J Q^{\top -1}, \quad \textit{with } Q \in \mathbf{k}. \tag{4.11}$$

The matrix $L = Q\Lambda Q^{-1}$, also defined by

$$zq(t,z) = Lq(t,z),$$

and the diagonal matrix h satisfy the equations

$$\frac{\partial L}{\partial t_i} = [-\pi_{\mathbf{k}} L^i, L] \qquad \textit{and} \qquad h^{-1}\frac{\partial h}{\partial t_i} = 2\pi_{\mathbf{k}}(L^i)_0. \tag{4.12}$$

Sketch of proof
At first note that looking for skew-orthogonal polynomials is tantamount to the skew-Borel decomposition of m_∞, so that (4.10) and (4.11) are equivalent. The skew-orthogonality of the polynomials (0.18) follows from expanding the Pfaffians explicitly in terms of z-columns, upon using the expression for the Pfaffian in terms of a column

$$\sum_{0\le k\le \ell-1} (-1)^k a_{ki}\, \mathrm{pf}(0,\dots,\hat{k},\dots,\ell-1) = \mathrm{pf}(0,\dots,\ell-1,i).$$

For details, see [2]. On the other hand, Theorem 3.2 gives $\Psi(t,z)$ and hence Q in terms of $\tau_n(t) = \mathrm{pf}\, m_{2n}(t)$ of (4.8). By the uniqueness of decomposition (4.11), the two ways of arriving at Q, (0.18), and (3.3) must coincide. $\square$

Important remark
The polynomials (0.18) provide an explicit algorithm to perform the skew-Borel decomposition of the skew-symmetric matrix m_∞. Namely, the coefficients of the polynomials q_i provide the entries of the matrix Q. This fact is used later in the examples.

Symmetric matrix integrals ($\alpha = -1$)
Here we focus on integrals over the space S_{2n} of symmetric matrices of the type

$$\int_{\mathscr{S}_{2n}} e^{\mathrm{Tr}(-V(X)+\sum_1^\infty t_i X^i)}\, dX, \tag{4.13}$$

where dX denotes Haar measure for $X = U\,\mathrm{diag}(z_1,\dots,z_n)U^\top,\ UU^\top = I$,

$$dX := \prod_{1\le i\le j\le n} dX_{ij} = |\Delta(z)|\, dz_1\cdots dz_n\, dU. \tag{4.14}$$

As appears below, integral (4.13) leads to a skew-inner-product of the type (4.1) with $\alpha = -1$, with weight $\rho_t(z) := e^{\sum t_i z^i} \rho(z) = e^{-V(z)+\sum t_i z^i}$:

$$\langle f(x), g(y) \rangle := \int\int_{\mathbb{R}^2} f(x) g(y) \varepsilon(x-y) \rho_t(x) \rho_t(y)\, dx\, dy, \tag{4.15}$$

leading to skew-symmetric moments*

$$\begin{aligned} \mu_{ij}(t) &= \int\int_{\mathbb{R}^2} x^i y^j \varepsilon(x-y) \rho_t(x) \rho_t(y)\, dx\, dy \\ &= \int\int_{x \geq y} (x^i y^j - x^j y^i) \rho_t(x) \rho_t(y)\, dx\, dy \\ &= \int_{\mathbb{R}} \big(F_j(x) G_i(x) - F_i(x) G_j(x) \big)\, dx, \end{aligned} \tag{4.16}$$

where $({}' = d/dx)$

$$F_i(x) := \int_{-\infty}^{x} y^i e^{\sum t_k y^k} \rho(y)\, dy \qquad \text{and} \qquad G_i(x) := F_i'(x) = x^i e^{\sum t_k x^k} \rho(x).$$

By simple inspection, the moments $\mu_{k\ell}(t)$ satisfy

$$\begin{aligned} \frac{\partial \mu_{k\ell}}{\partial t_i} &= \int\int_{\mathbb{R}^2} (x^{k+i} y^{\ell} + x^k y^{\ell+i}) \varepsilon(x-y) e^{\sum t_n (x^n + y^n)} \rho(x) \rho(y)\, dx\, dy \\ &= \mu_{k+i,\ell} + \mu_{k,\ell+i}, \end{aligned}$$

and so m_∞ satisfies (4.6).

According to M. Mehta [14], the symmetric matrix integral can now be expressed in terms of the Pfaffian, as follows, taking into account a constant c_{2n}, coming from

*We have $\varepsilon(x) = 1$, for $x \geq 0$, and $\varepsilon(x) = -1$, for $x < 0$.

integrating the orthogonal group:

$$
\begin{aligned}
\frac{1}{(2n)!}\int_{\mathscr{S}_{2n}(E)} e^{\operatorname{Tr}(-V(X)+\sum t_i X^i)\,dX} &= \frac{1}{(2n)!}\int_{\mathbb{R}^{2n}} |\Delta_{2n}(z)| \prod_{i=1}^{2n} e^{\sum t_k z_i^k}\rho(z_i)\,dz_i \\
&= \int_{-\infty<z_1<z_2<\cdots<z_{2n}<\infty} \det\left(z_{j+1}^i \rho_t(z_{j+1})\right)_{0\le i,j\le 2n-1} \prod_{i=1}^{2n} dz_i \\
&= \int_{-\infty<z_2<z_4<\cdots<z_{2n}<\infty} \prod_{k=1}^{n} \rho_t(z_{2k})\,dz_{2k} \times \det\Big(\int_{-\infty}^{z_2} z_1^i\rho_t(z_1)\,dz_1,\, z_2^i, \\
&\qquad \ldots, \int_{z_{2n-2}}^{z_{2n}} z_{2n-1}^i \rho_t(z_{2n-1})\,dz_{2n-1},\ z_{2n}^i\Big)_{0\le i\le 2n-1} \\
&= \int_{-\infty<z_2<z_4<\cdots<z_{2n}<\infty} \prod_{k=1}^{n} \rho_t(z_{2k})\,dz_{2k} \\
&\quad \times \det\big(F_i(z_2),\ z_2^i,\ F_i(z_4)-F_i(z_2), z_4^i, \\
&\qquad\qquad \ldots, F_i(z_{2n})-F_i(z_{2n-2}),\ z_{2n}^i\big)_{0\le i\le 2n-1} \\
&= \int_{-\infty<z_2<z_4<\cdots<z_{2n}<\infty} \prod_{1}^{n} dz_{2i} \\
&\quad \times \det\big(F_i(z_2),\ G_i(z_2), \ldots, F_i(z_{2n}), G_i(z_{2n})\big)_{0\le i\le 2n-1} \\
&= \frac{1}{n!}\int_{\mathbb{R}^n} \prod_{1}^{n} dy_i \det\big(F_i(y_1), G_i(y_1), \ldots, F_i(y_n), G_i(y_n)\big)_{0\le i\le 2n-1} \\
&= \det{}^{1/2}\Big(\int_{\mathbb{R}} \big(G_i(y)F_j(y)-F_i(y)G_j(y)\big)\,dy\Big)_{0\le i,j\le 2n-1} \\
&\qquad\qquad \text{(using de Bruijn's lemma (see [14, p. 446]))} \\
&= \operatorname{pf}\Big(\iint_{\mathbb{R}^2} y^k z^\ell \varepsilon(y-z) e^{\sum_1^\infty t_i(y^i+z^i)}\rho(y)\rho(z)\,dy\,dz\Big)_{0\le k,\ell\le 2n-1} \\
&= \operatorname{pf}\left(\mu_{ij}(t)\right)_{0\le i,j\le 2n-1} = \tau_{2n}(t),
\end{aligned} \tag{4.17}
$$

which is a Pfaffian τ-function.

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Symplectic matrix integrals: $(\alpha=+1)$
Here we concentrate on integrals of the type

$$
\int_{\mathscr{T}_{2n}} e^{2\operatorname{Tr}(-V(X)+\sum_1^\infty t_i X^i)}\,dX, \tag{4.18}
$$

where dX denotes Haar measure*

$$dX = \prod_1^N dX_k \prod_{k<\ell} dX_{k\ell}^{(0)} d\bar{X}_{k\ell}^{(0)} dX_{k\ell}^{(1)} d\bar{X}_{k\ell}^{(1)}$$

on the space $\mathscr{T}_{2N}$ of self-dual $(N \times N)$–Hermitian matrices, with real quaternionic entries; the latter can be realized as the space of $(2N \times 2N)$-matrices with entries $X_{\ell k}^{(i)} \in \mathbb{C}$,

$$\mathscr{T}_{2N} = \left\{ X = (X_{k\ell})_{1\le k,\ell \le N}, X_{k\ell} = \begin{pmatrix} X_{k\ell}^{(0)} & X_{k\ell}^{(1)} \\ -\bar{X}_{k\ell}^{(1)} & \bar{X}_{k\ell}^{(0)} \end{pmatrix} \text{ with } X_{\ell k} = \bar{X}_{k\ell}^{\top} \right\}.$$

Another skew-symmetric moment matrix m_∞ satisfying (4.6) is given by inner-product (4.1) for $\alpha = 1$, with $\rho_t(y) = \rho(y) e^{\sum t_\alpha y^\alpha} = e^{-V(y)+\sum t_\alpha y^\alpha}$,

$$\begin{aligned} \mu_{ij}(t) &= \int_{\mathbb{R}} \{y^i, y^j\} \rho_t(y)^2 I_E(y)\, dy \\ &= \int_{\mathbb{R}} \left\{ y^i \rho_t(y), y^j \rho_t(y) \right\} I_E(y)\, dy \\ &= \int_{\mathbb{R}} \big(G_i(y) F_j(y) - F_i(y) G_j(y) \big)\, dy, \end{aligned} \tag{4.19}$$

upon setting $(' = d/dx)$

$$F_j(x) = x^j \rho_t(x) \qquad \text{and} \qquad G_j(x) := F_j'(x) = \big(x^j \rho_t(x) \big)'.$$

That m_∞ satisfies (4.6) follows at once from the first expression (4.18):

$$\begin{aligned} \mu_{k\ell}(t) &= \int \{y^k, y^\ell\} \rho_t(y)^2\, dy = \int (k-\ell) y^{k+\ell-1} \rho_t(y)^2\, dy \\ \frac{\partial \mu_{k\ell}}{\partial t_i} &= 2 \int \{y^k, y^\ell\} y^i \rho_t(y)^2\, dy \\ &= \int \big((k+i-\ell) y^{k+i+\ell-1} + (k-\ell-i) y^{k+i+\ell-1} \big) \rho_t(y)^2\, dy \\ &= \mu_{k+i,\ell} + \mu_{k,\ell+i}, \end{aligned}$$

thus leading to (4.6). Using the relation

$$\prod_{1\le i,j\le n} (x_i - x_j)^4 = \det \big(x_1^i \quad (x_1^i)' \quad x_2^i \quad (x_2^i)' \cdots x_n^i \quad (x_n^i)' \big)_{0\le i\le 2n-1},$$

*$\bar{X}$ means the usual complex conjugate. The condition on the (2×2)-matrices $X_{k\ell}$ implies that $X_{kk} = X_k I$, with $X_k \in \mathbb{R}$ and I the identity.

one computes, using again de Bruijn's lemma,

$$\begin{aligned}
\frac{1}{(n)!}\int_{\mathscr{T}_{2n}} e^{2\,\mathrm{Tr}(-V(X)+\sum t_i X^i)}\,dX &= \frac{1}{n!}\int_{\mathbb{R}^n}\prod_{1\le i,j\le n}(x_i-x_j)^4\prod_{i=1}^n \rho_t(x_i)^2\,dx_i\\
&= \frac{1}{n!}\int_{\mathbb{R}^n}\prod_{k=1}^n \rho_t(x_k)^2\,dx_k\\
&\quad\times \det\left(x_1^i \quad (x_1^i)' \quad x_2^i \quad (x_2^i)'\cdots x_n^i \quad (x_n^i)'\right)_{0\le i\le 2n-1}\\
&= \frac{1}{n!}\int_{\mathbb{R}^n}\prod_1^n dy_i\,\det\big(F_i(y_1)\quad G_i(y_1)\ldots F_i(y_n)\quad G_i(y_n)\big)_{0\le i\le 2n-1}\\
&= \det{}^{1/2}\left(\int_{\mathbb{R}}\big(G_i(y)F_j(y)-F_i(y)G_j(y)\big)\,dy\right)_{0\le i,j\le 2n-1}\\
&= \mathrm{pf}\,\big(\mu_{ij}(t)\big)_{0\le i,j\le 2n-1} = \tau_{2n}(t),
\end{aligned}\tag{4.20}$$

which is a Pfaffian τ-function as well.

5. A map from the Toda to the Pfaff lattice

Remember from (0.1) the notation $\rho_t(z)=\rho(z)e^{\sum t_k z^k}$ and $\rho'/\rho=-g/f$. Assuming, in addition, that $f(z)\rho(z)$ vanishes at the endpoints of the interval under consideration (which could be finite, infinite, or semi-infinite), one checks that the t-dependent operator in z,

$$\begin{aligned}
\mathbf{n}_t &:= \sqrt{\frac{f}{\rho_t}}\frac{d}{dz}\sqrt{f\rho_t}\\
&= e^{(1/2)\sum t_k z^k}\left(\frac{d}{dz}f(z)-\frac{f'+g}{2}(z)\right)e^{1/2\sum t_k z^k}\\
&= \frac{d}{dz}f(z)-\frac{f'+g_t}{2}(z), \quad \text{with } g_t(z)=g(z)-f(z)\sum_1^\infty k t_k z^{k-1},
\end{aligned}\tag{5.1}$$

maintains $\mathscr{H}_+=\{1,z,z^2,\ldots\}$ and is skew-symmetric with respect to the t-dependent inner-product $\langle\ ,\ \rangle_t^{\mathrm{sy}}$, defined by the weight $\rho_t(z)\,dz$,

$$\langle \mathbf{n}_t\varphi,\psi\rangle_t^{\mathrm{sy}} = \int_E(\mathbf{n}_t\varphi)(z)\psi(z)\rho_t(z)\,dz = -\int_E \varphi(\mathbf{n}_t\psi)\rho_t\,dz = -\langle\varphi,\mathbf{n}_t\psi\rangle_t^{\mathrm{sy}}.$$

The orthonormality of the t-dependent polynomials $p_n(t,z)$ in z imply

$$\big\langle p_n(t,z),\,p_m(t,z)\big\rangle_t^{\mathrm{sy}}=\delta_{mn}.$$

The matrices L and M are defined by

$$zp=Lp \qquad \text{and} \qquad e^{-(1/2)\sum t_k z^k}\frac{d}{dz}e^{1/2\sum t_k z^k}p=Mp.$$

The skewness of $\mathbf{n}_t$ implies the skew-symmetry of the matrix

$$\mathscr{N}(t) = f(L)M - \frac{f' + g}{2}(L) \quad \text{such that } \mathbf{n}_t\, p(t,z) = \mathscr{N} p(t,z); \tag{5.2}$$

so $\mathscr{N}(t)$ can be viewed as the operator $\mathbf{n}_t$, expressed in the polynomial basis $(p_0(t,z), p_1(t,z), \dots)$.

In the next theorem, we consider functions F of two (noncommutative) variables z and $\mathbf{n}_t$ so that the (pseudo-)differential operator $\mathbf{u}_t := F(z, \mathbf{n}_t)$ in z and the matrix $\mathscr{U} := F(L, \mathscr{N})$ related by*

$$F(z, \mathbf{n}_t)\, p(t,z) = F(L, \mathscr{N})\, p(t,z),$$

are skew-symmetric as well. Examples of F's are†

$$F(z, \mathbf{n}_t) := \mathbf{n}_t,\ \mathbf{n}_t^{-1},\ \text{or } \{z^\ell, \mathbf{n}_t^{2k+1}\}^\dagger,$$

corresponding to

$$F(L, \mathscr{N}) = \mathscr{N}, \mathscr{N}^{-1}, \text{or } \{\mathscr{N}^{2k+1}, L^\ell\}^\dagger.$$

THEOREM 5.1
Any Hänkel matrix m_∞ evolving according to the vector fields

$$\frac{\partial m_\infty(t)}{\partial t_k} = \Lambda^k m_\infty$$

leads to matrices L and M, evolving according to the Toda lattice equations $\partial L/\partial t_n = (1/2)[(L^n)_{\mathrm{sk}}, L]$ and $\partial M/\partial t_n = (1/2)[(L^n)_{\mathrm{sk}}, M]$ (see (1.9)). Consider a function F of two variables such that the operator $\mathbf{u}_t := F(z, \mathbf{n}_t)$ is skew-symmetric with respect to $\langle\ ,\ \rangle_t^{\mathrm{sy}}$ and so the matrix

$$\mathscr{U}(t) = F\big(L(t), \mathscr{N}(t)\big), \quad \text{defined by } \mathbf{u}_t\, p(t,z) = \mathscr{U} p(t,z),$$

is skew-symmetric. This induces a natural lower-triangular matrix $O(t)$, mapping the Toda lattice into the Pfaff lattice (for notation π_{bo}, $\pi_{\mathbf{k}}$, $\mathscr{G}_{\mathbf{k}}$, etc., see (1.9), (1.18), (1.17)):

$$\text{Toda lattice} \begin{cases} p_n(t,z) = (S(t)\chi(z))_n \ \text{ orthonormal with respect to} \\ \qquad\qquad m_\infty(t) = \big(\langle z^i, z^j\rangle_t^{\mathrm{sy}}\big)_{0\le i,j\le\infty} = S^{-1}S^{\top -1}, \\ L(t) = S\Lambda S^{-1} \ \text{ satisfies } \dfrac{\partial L}{\partial t_j} = \Big[-\dfrac{1}{2}\pi_{\mathrm{bo}} L^j, L\Big],\ j = 1, 2, \dots \end{cases}$$

*It is to be understood that $F(L, \mathscr{N})$ reverses the order of z, $\mathbf{u}$ in $F(z, \mathbf{u})$.
†We define $\{A, B\}^\dagger := AB + BA$.

$$
\Bigg\downarrow \quad \textit{map } O(2t) \textit{ such that } \begin{cases} -\mathscr{U}(2t) = O^{-1}(2t) J O^{\top -1}(2t), \\ O(2t) \text{ is lower-triangular,} \\ O(2t) S(2t) \in \mathscr{G}_{\mathbf{k}} \end{cases}
$$

$$
\textit{Pfaff lattice} \begin{cases} q_n(t,z) = \big(O(2t)p(2t,z)\big)_n, \text{ skew-orthonormal with regard to} \\ \qquad \tilde{m}_\infty(t) := -S^{-1}(2t)\mathscr{U}(2t)S^{\top -1}(2t) = Q^{-1}(t) J Q^{\top -1}(t) \\ \qquad\qquad = \big(\langle z^i, z^j\rangle_t^{\mathrm{sk}}\big)_{0\le i,j\le\infty} \\ \qquad\qquad = \big(\langle z^i, \mathbf{u}_{2t} z^j\rangle_{2t}^{\mathrm{sy}}\big)_{0\le i,j\le\infty}, \\ \tilde{L}(t) := O(2t)L(2t)O(2t)^{-1} \text{ satisfies} \\ \qquad \dfrac{\partial \tilde{L}}{\partial t_j} = [-\pi_{\mathbf{k}} \tilde{L}^j, \tilde{L}],\ j = 1, \dots . \end{cases}
$$

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Proof

Since $\mathscr{U}(t)$ is skew-symmetric, it admits a skew-Borel decomposition

$$
-\mathscr{U}(t) = O^{-1}(t) J O^{\top -1}(t), \quad \text{with lower-triangular } O(t). \tag{5.3}
$$

But the new matrix, defined by

$$
\tilde{m}_\infty(t) := -S^{-1}(2t)\mathscr{U}(2t)S^{\top -1}(2t), \tag{5.4}
$$

is skew-symmetric and thus admits a unique skew-Borel decomposition

$$
\tilde{m}_\infty(t) = \tilde{Q}^{-1}(t) J \tilde{Q}(t)^{\top -1}, \quad \text{with } \tilde{Q}(t) \in \mathscr{G}_{\mathbf{k}}. \tag{5.5}
$$

Comparing (5.3), (5.4), and (5.5) leads to a unique choice of matrix $O(t)$, skew-Borel decomposing $-\mathscr{U}(2t)$, as in (5.3), such that

$$
O(2t)S(2t) = \tilde{Q}(t) \in \mathscr{G}_{\mathbf{k}}. \tag{5.6}
$$

Using, as a consequence of (5.2) and (1.9),

$$
\frac{\partial \mathscr{U}}{\partial t_k}(2t) = \big[\pi_{\mathrm{sy}} L^k(2t), \mathscr{U}(2t)\big]
$$

and

$$\frac{\partial S}{\partial t_k}(2t) = -\big(\pi_{\mathrm{bo}} L^k(2t)\big) S(2t),$$

we compute

$$\begin{aligned}
\frac{\partial \tilde m_\infty}{\partial t_k}(t) &= S^{-1}\frac{\partial S}{\partial t_k}(2t) S^{-1}\mathscr{U}(2t)S^{\top -1}(2t) - S^{-1}(2t)\Big(\frac{\partial}{\partial t_k}\mathscr{U}(2t)\Big)S^{\top -1}(2t)\\
&\quad + S^{-1}(2t)\mathscr{U}(2t)S^{\top -1}\frac{\partial S^\top}{\partial t_k}(2t)S^{\top -1}\\
&= -S^{-1}\big(\pi_{\mathrm{bo}}L^k(2t)\big)\mathscr{U}S^{\top -1} - S^{-1}[\pi_{\mathrm{sy}}L^k, \mathscr{U}]S^{\top -1}\\
&\quad - S^{-1}\mathscr{U}(\pi_{\mathrm{bo}}L^k)^\top S^{\top -1}\\
&= -S^{-1}(\pi_{\mathrm{bo}}L^k + \pi_{\mathrm{sy}}L^k)\mathscr{U}S^{\top -1} - S^{-1}\mathscr{U}\big((\pi_{\mathrm{bo}}L^k)^\top - \pi_{\mathrm{sy}}L^k\big)S^{\top -1}\\
&= -S^{-1}L^k\mathscr{U}S^{\top -1} - S^{-1}\mathscr{U}L^{\top k}S^{\top -1} \text{ (using (5.7))}\\
&= -\Lambda^k S^{-1}\mathscr{U}S^{\top -1} - S^{-1}\mathscr{U}S^{\top -1}\Lambda^{\top k}S^\top S^{\top -1} \quad \text{(using } L^k = S\Lambda^k S^{-1}\text{)}\\
&= \Lambda^k \tilde m_\infty(t) + \tilde m_\infty(t)\Lambda^{\top k} \quad \text{(by (5.4))}.
\end{aligned}$$

For an arbitrary matrix A, we have

$$A = A^\top \iff A = (A_{\mathrm{bo}})^\top - A_{\mathrm{sy}}. \tag{5.7}$$

Indeed, remembering that* $A_{\mathrm{bo}} = 2A_- + A_0$ and $A_{\mathrm{sy}} = A_+ - A_-$, one checks

$$(A_{\mathrm{bo}})^\top - A_{\mathrm{sy}} - A = 2(A_-)^\top + A_0 - (A_+ - A_-) - A_- - A_+ - A_0 = -2(A_+ - (A_-)^\top),$$

so that the left-hand side vanishes, if the right-hand side does; the latter means A is symmetric.

We now define $\tilde L(t)$ by conjugation of $L(2t)$ by $O(2t)$:

$$\tilde L(t) := O(2t)L(2t)O(2t)^{-1} = O(2t)S(2t)\Lambda S^{-1}(2t)O(2t)^{-1} = \tilde Q(t)\Lambda\tilde Q^{-1}(t);$$

thus, by Proposition 1.2, $\tilde L(t)$ satisfies the Pfaff Lax equation. Therefore the sequence of polynomials

$$q(t,z) := O(2t)p(2t,z) = O(2t)S(2t)\chi(z) = \tilde Q(t)\chi(z)$$

is skew-orthonormal

$$\big\langle q_i(t,z), q_j(t,z)\big\rangle^{\mathrm{sk}} = J_{ij}$$

*$A_\pm$ means the usual strictly upper-(lower-)triangular part, and A_0 means the diagonal part in the common sense.

with regard to the skew inner-product specified by the matrix $\tilde{m}_\infty$:

$$\langle z^i, z^j\rangle_t^{\mathrm{sk}} = \tilde{\mu}_{ij}(t).$$

In the last step, we show that $\langle\varphi,\psi\rangle_t^{\mathrm{sk}} = \langle\varphi,\mathbf{u}\psi\rangle_{2t}^{\mathrm{sy}}$. Since

$$\mathscr{U}(2t) = -O^{-1}(2t)JO^{\top-1}(2t) = -\mathscr{U}^\top(2t), \tag{5.8}$$

we compute

$$\begin{aligned}
\big\langle q_i(t,z), (\mathbf{u}_{2t}q)_j(t,z)\big\rangle_{2t}^{\mathrm{sy}} &= \big\langle (Op)_i(2t), (\mathbf{u}Op)_j(2t)\big\rangle_{2t}^{\mathrm{sy}}\\
&= \big\langle (Op)_i(2t), (O\mathbf{u}p)_j(2t)\big\rangle_{2t}^{\mathrm{sy}}\\
&= \big\langle (Op)_i(2t), (O\mathscr{U}p)_j(2t)\big\rangle_{2t}^{\mathrm{sy}}\\
&= \Big(O(2t)\big\langle p_k(2t), p_\ell(2t)\big\rangle_{k,\ell\ge 0}^{\mathrm{sy}}(O\mathscr{U})^\top(2t)\Big)_{ij}\\
&= \Big(O(2t)I(O\mathscr{U})^\top(2t)\Big)_{ij}\\
&= \Big(O(2t)\mathscr{U}^\top(2t)O^\top(2t)\Big)_{ij}\\
&= -\Big(O(2t)\mathscr{U}(2t)O^\top(2t)\Big)_{ij}\\
&= J_{ij}\quad \text{(using (5.8))}.
\end{aligned}\tag{5.9}$$

Therefore, defining a new skew-inner-product $\langle\ ,\ \rangle^{\mathrm{sk}'}$

$$\langle\varphi,\psi\rangle^{\mathrm{sk}'} := \langle\varphi,\mathbf{u}\psi\rangle_{2t}^{\mathrm{sy}},$$

we have shown

$$\langle q_i, q_j\rangle_t^{\mathrm{sk}'} = \langle q_i, q_j\rangle_t^{\mathrm{sk}} = J_{ij},$$

and so by completeness of the basis q_i, we have

$$\langle\ ,\ \rangle_t^{\mathrm{sk}'} = \langle\ ,\ \rangle_t^{\mathrm{sk}},$$

thus ending the proof of Theorem 4.1. □

6. Example 1: From Hermitian to symmetric matrix integrals

Striking examples are given by using the map $O(t)$ obtained from skew-Borel decomposing $\mathscr{N}^{-1}(t)$ and $\mathscr{N}(t)$ (see (5.2)). This section deals with $\mathscr{N}^{-1}(t)$, whereas Section 7 deals with $\mathscr{N}(t)$.

PROPOSITION 6.1
The special transformation

$$\mathscr{U}(t) = \mathscr{N}^{-1}(t) = \Big(f(L)M - \frac{f'+g}{2}(L)\Big)^{-1}(t)$$

maps the Toda lattice τ-functions with initial weight $\rho = e^{-V}$, $V' = -g/f$ (Hermitian matrix integral) to the Pfaff lattice τ-functions (symmetric matrix integral), with initial weight

$$\begin{aligned}\tilde{\rho}_t(z) &:= \left(\frac{\rho_{2t}(z)}{f(z)}\right)^{1/2} = e^{-(1/2)(V(z)+\log f(z)-2\sum_1^\infty t_i z^i)} \\ &=: e^{-\tilde{V}(z)+\sum_1^\infty t_i z^i} = \tilde{\rho}(z)e^{\sum t_i z^i}.\end{aligned}$$

To be precise:

$$\text{Toda lattice}\begin{cases} p_n(t,z) \ \textit{orthonormal polynomials in } z \textit{ for the inner-product} \\ \qquad\qquad\qquad\qquad \langle\varphi,\psi\rangle_t^{\mathrm{sy}} = \displaystyle\int \varphi(z)\psi(z)e^{\sum t_i z^i}\rho(z)\,dz, \\ \mu_{ij}(t) = \langle z^i, z^j\rangle_t^{\mathrm{sy}} \ \textit{and } m_n = (\mu_{ij})_{0\le i,j\le n-1}, \\ \tau_n(t) = \det m_n = \dfrac{1}{n!}\displaystyle\int_{\mathscr{H}_n} e^{\mathrm{Tr}(-V(X)+\sum_1^\infty t_i X^i)}\,dX \end{cases}$$

$$\Bigg\downarrow \ \textit{map } O(2t) \textit{ such that } \begin{cases} -\mathscr{N}^{-1}(2t) = O^{-1}(2t)JO^{\top-1}(2t), \\ O(2t) \textit{ is lower-triangular,} \\ O(2t)S(2t) \in \mathscr{G}_{\mathbf{k}} \end{cases}$$

$$\text{Pfaff lattice}\begin{cases} q_n(t,z) = \big(O(2t)p(2t,z)\big)_n \ \textit{skew-orthonormal polynomials} \\ \qquad\qquad \textit{in } z \textit{ for the skew-inner-product (weight} \tilde{\rho}), \\ \langle\varphi,\psi\rangle_t^{\mathrm{sk}} := \langle\varphi, \mathbf{n}_{2t}^{-1}\psi\rangle_{2t}^{\mathrm{sy}} \\ \qquad = v\dfrac{1}{2}\displaystyle\int\!\!\int_{\mathbb{R}^2}\varphi(x)\psi(y)\varepsilon(x-y)e^{\sum t_i(x^i+y^i)} \\ \qquad\quad \times\sqrt{\dfrac{\rho}{f}}(x)\sqrt{\dfrac{\rho}{f}}(y)\,dx\,dy, \\ \tilde{\mu}_{ij}(t) = \langle x^i, y^j\rangle_t^{\mathrm{sk}} \ \textit{and } \tilde{m}_n = (\tilde{\mu}_{ij})_{0\le i,j\le n-1}, \\ \tilde{\tau}_{2n}(t) = \mathrm{pf}(\tilde{m}_{2n}) = \dfrac{1}{2^n(2n)!}\displaystyle\int_{\mathscr{S}_{2n}} e^{\mathrm{Tr}(-\tilde{V}(X)+\sum_1^\infty t_i X^i)}\,dX, \\ \qquad \textit{with } \tilde{V}(z) = \dfrac{1}{2}\big(V(z)+\log f(z)\big). \end{cases}$$

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In the first integral defining $\tau_n(t)$, dX denotes Haar measure on Hermitian matrices (see Section 4.1), whereas the second integral $\tilde{\tau}_{2n}(t)$ involves Haar measure on symmetric matrices (see Section 4.2).

Proof
At first, check that

$$\left(\frac{d}{dx}\right)^{-1}\varphi(x) = \frac{1}{2}\int \varepsilon(x-y)\varphi(y)\,dy. \tag{6.1}$$

Indeed,

$$\begin{aligned}\frac{d}{dx}\left(\frac{d}{dx}\right)^{-1}\varphi(x) &= \int \frac{1}{2}\frac{\partial}{\partial x}\varepsilon(x-y)\varphi(y)\,dy\\ &= \int \delta(x-y)\varphi(y)\,dy \quad \left(\text{using } \frac{\partial}{\partial x}\varepsilon(x) = 2\delta(x)\right)\\ &= \varphi(x).\end{aligned}$$

Consider now the operator

$$\mathbf{u}_t = \mathbf{n}_t^{-1} = \left(\sqrt{\frac{f}{\rho_t}}\frac{d}{dz}\sqrt{f\rho_t}\right)^{-1}, \quad \text{so that } \mathbf{u}_t p = \mathbf{n}_t^{-1} p = \mathscr{N}^{-1} p,$$

according to (5.2). Let it act on a function $\varphi(x)$:

$$\begin{aligned}\mathbf{n}_t^{-1}\varphi(x) &= \left(\frac{1}{\sqrt{f(x)\rho_t(x)}}\left(\frac{d}{dx}\right)^{-1}\sqrt{\frac{\rho_t(x)}{f(x)}}\right)\varphi(x)\\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{f(x)\rho_t(x)}}\frac{\varepsilon(x-y)}{2}\sqrt{\frac{\rho_t(y)}{f(y)}}\varphi(y)\,dy \quad \text{(using (6.1))}.\end{aligned}$$

One computes

$$\begin{aligned}\langle\varphi,\psi\rangle_t^{\mathrm{sk}} &= \langle\varphi,\mathbf{u}_{2t}\psi\rangle_{2t}^{\mathrm{sy}}\\ &= \langle\varphi,\mathbf{n}_{2t}^{-1}\psi\rangle_{2t}^{\mathrm{sy}}\\ &= \frac{1}{2}\int\int_{\mathbb{R}^2}\sqrt{\frac{\rho_{2t}(x)}{f(x)}}\varepsilon(x-y)\sqrt{\frac{\rho_{2t}(y)}{f(y)}}\varphi(x)\psi(y)\,dx\,dy\\ &= \frac{1}{2}\int\int_{\mathbb{R}^2}\tilde{\rho}(x)\tilde{\rho}(y)e^{\sum_1^\infty t_k(x^k+y^k)}\varepsilon(x-y)\varphi(x)\psi(y)\,dx\,dy.\end{aligned}$$

So, finally setting $\tilde{V}(x) = (1/2)(V(x)+\log f(x))$ yields by (4.17) that

$$\tilde{\tau}_{2n}(t) = \mathrm{pf}(\tilde{m}_{2n}) = \frac{1}{(2n)!}\int_{\mathscr{S}_{2n}} e^{\mathrm{Tr}(-\tilde{V}(X)+\sum_1^\infty t_i X^i)}\,dX.$$

The map O for the classical orthogonal polynomials at $t=0$
Then, the matrix O, mapping orthonormal p_k into skew-orthonormal polynomials q_k, is given by a lower-triangular three-step relation:

$$\begin{aligned} q_{2n}(0,z) &= \sqrt{\frac{c_{2n}}{a_{2n}}}\, p_{2n}(0,z), \\ q_{2n+1}(0,z) &= \sqrt{\frac{a_{2n}}{c_{2n}}} \\ &\qquad \Big(-c_{2n-1}p_{2n-1}(0,z) + \frac{c_{2n}}{a_{2n}}\Big(\sum_0^{2n} b_i\Big) p_{2n}(0,z) + c_{2n}p_{2n+1}(0,z)\Big), \end{aligned} \tag{6.2}$$

where the a_i and b_i are the entries in the tridiagonal matrix defining the orthonormal polynomials, and the c_i's are the entries of the skew-symmetric matrix $\mathscr{N}$.

In [6], we showed that, in the classical cases below, $\mathscr{N}$ is tridiagonal, at the same time as L (see Appendix B):

$$L = \begin{bmatrix} b_0 & a_0 & & \\ a_0 & b_1 & a_1 & \\ & a_1 & b_2 & \ddots \\ & & \ddots & \ddots \end{bmatrix}, \qquad -\mathscr{N} = \begin{bmatrix} 0 & c_0 & & \\ -c_0 & 0 & c_1 & \\ & -c_1 & 0 & \ddots \\ & & \ddots & \end{bmatrix}, \tag{6.3}$$

with the following precise entries:

$$\begin{aligned} &\textit{Hermite: } \rho(z) = e^{-z^2},\ a_{n-1} = \sqrt{n/2},\ b_n = 0,\ c_n = a_n; \\ &\textit{Laguerre: } \rho(z) = e^{-z}z^{\alpha}I_{[0,\infty)}(z),\ a_{n-1} = \sqrt{n(n+\alpha)}, \\ &\qquad\qquad b_n = 2n+\alpha+1,\ c_n = a_n/2; \\ &\textit{Jacobi: } \rho(z) = (1-z)^{\alpha}(1+z)^{\rho}I_{[-1,1]}(z); \\ &\qquad a_{n-1} = \Big(\frac{4n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)(2n+\alpha+\beta-1)}\Big)^{1/2}, \\ &\qquad b_n = \frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \\ &\qquad c_n = a_n\Big(\frac{\alpha+\beta}{2}+n+1\Big). \end{aligned}$$

If the skew-symmetric matrix $\mathscr{N}$ has the tridiagonal form above, then one checks that its inverse has the following form:

$$
-\mathcal{N}^{-1} = \begin{pmatrix}
0 & -\frac{1}{c_0} & 0 & \frac{-c_1}{c_0c_2} & 0 & \frac{-c_1c_3}{c_0c_2c_4} & 0 & \frac{-c_1c_3c_5}{c_0c_2c_4c_6} & \\
\frac{1}{c_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & -\frac{1}{c_2} & 0 & \frac{-c_3}{c_2c_4} & 0 & \frac{-c_3c_5}{c_2c_4c_6} & \\
\frac{c_1}{c_0c_2} & 0 & \frac{1}{c_2} & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{c_4} & 0 & \frac{-c_5}{c_4c_6} & \\
\frac{c_1c_3}{c_0c_2c_4} & 0 & \frac{c_3}{c_2c_4} & 0 & \frac{1}{c_4} & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{c_6} & \\
\frac{c_1c_3c_5}{c_0c_2c_4c_6} & 0 & \frac{c_3c_5}{c_2c_4c_6} & 0 & \frac{c_5}{c_4c_6} & 0 & \frac{1}{c_6} & 0 & \\
 & & & & & & & & \ddots
\end{pmatrix} \tag{6.4}
$$

In order to find the matrix O, we must perform the skew-Borel decomposition of the matrix $-\mathcal{U}$:

$$
-\mathcal{U} = -\mathcal{N}^{-1} = O^{-1} J O^{\top -1}.
$$

The recipe for doing so is given in Theorem 4.1 (see also the important remark following that theorem). It suffices to form the Pfaffians (0.18) by appropriately bordering the matrix $-\mathcal{N}^{-1}$, as in (0.18), with rows and columns of powers of z, yielding skew-orthonormal polynomials; we choose to call them r's, instead of the q's of Theorem 4.1, with $O\chi(z) = r(z)$. They turn out to be the following simple polynomials, with $1/\tilde{\tilde{\tau}}_{2n} = c_0c_2c_4 \cdots c_{2n-2}$:

$$
\begin{aligned}
r_{2n}(z) &= \frac{1}{\sqrt{\tilde{\tilde{\tau}}_{2n}\tilde{\tilde{\tau}}_{2n+2}}} \frac{c_{2n}z^{2n}}{c_0c_2\cdots c_{2n}} = \frac{1}{\sqrt{c_{2n}}} c_{2n}z^{2n}, \\
r_{2n+1}(z) &= \frac{1}{\sqrt{\tilde{\tilde{\tau}}_{2n}\tilde{\tilde{\tau}}_{2n+2}}} \frac{c_{2n}z^{2n+1} - c_{2n-1}z^{2n-1}}{c_0c_2\cdots c_{2n}} = \frac{1}{\sqrt{c_{2n}}} (c_{2n}z^{2n+1} - c_{2n-1}z^{2n-1}).
\end{aligned}
$$

Then, also from Appendix A, in order to get $O \to \hat{O}$ in the correct form, we compute the skew-orthonormal polynomials $\hat{r}_k$, with $\hat{O}\chi(z) = \hat{r}(z)$:

$$
\begin{aligned}
\hat{r}_{2n}(z) &= \frac{1}{\sqrt{a_{2n}}}\, r_{2n}(z) = \sqrt{\frac{c_{2n}}{a_{2n}}} z^{2n}, \\
\hat{r}_{2n+1}(z) &= \frac{\sum_0^{2n} b_i}{\sqrt{a_{2n}}}\, r_{2n}(z) + \sqrt{a_{2n}}\, r_{2n+1}(z) \\
&= \sqrt{\frac{a_{2n}}{c_{2n}}} \left(-c_{2n-1}z^{2n-1} + \frac{c_{2n}}{a_{2n}} \left(\sum_0^{2n} b_i \right) z^{2n} + c_{2n}z^{2n+1} \right).
\end{aligned} \tag{6.5}
$$

From the coefficients of the polynomial $\hat{r}_k$, one reads off the transformation matrix from orthonormal to skew-orthonormal polynomials; it is given by the matrix

$\hat{O}$ such that $\hat{O}\chi(z) = \hat{r}(z)$. Therefore $q(t,z) = \hat{O}(2t)p(2t,z)$ yields, after setting $t = 0$,

We added times symbol to (6.6). OK?

$$\begin{aligned} q_{2n}(0,z) &= \sqrt{\frac{c_{2n}}{a_{2n}}}\, p_{2n}(0,z), \\ q_{2n+1}(0,z) &= \sqrt{\frac{a_{2n}}{c_{2n}}} \\ &\quad \times \left(-c_{2n-1}p_{2n-1}(0,z) + \frac{c_{2n}}{a_{2n}}\Big(\sum_0^{2n} b_i\Big) p_{2n}(0,z) + c_{2n}p_{2n+1}(0,z)\right), \end{aligned} \tag{6.6}$$

confirming (6.2). □

7. Example 2: From Hermitian to symplectic matrix integrals

PROPOSITION 7.1
The matrix transformation

$$\mathscr{N} = f(L)M - \frac{f'+g}{2}(L)$$

maps the Toda lattice τ-functions with t-dependent weight

$$\rho_t(z) = e^{-V(z)+\sum_1^\infty t_i z^i}, \quad V' = g/f$$

(Hermitian matrix integral) to the Pfaff lattice τ-functions (symplectic matrix integral), with t-dependent weight

Is this break OK?

$$\begin{aligned} \tilde{\rho}_t(z) &:= \big(\rho_{2t}(z) f(z)\big)^{1/2} = e^{-(1/2)(V(z)-\log f(z)-2\sum_1^\infty t_i z^i)} \\ &=: e^{-\tilde{V}(z)+\sum t_i z^i} = \tilde{\rho}(z) e^{\sum t_i z^i}. \end{aligned}$$

To be precise:

$$\text{Toda lattice} \begin{cases} p_n(t,z) \text{ orthonormal polynomials in } z \text{ for the inner-product} \\ \qquad\qquad \langle \varphi, \psi\rangle_t^{\mathrm{sy}} = \int \varphi(z)\psi(z) e^{\sum t_i z^i}\rho(z)\,dz, \\ \mu_{ij}(t) = \langle z^i, z^j\rangle_t^{\mathrm{sy}} \text{ and } m_n = (\mu_{ij})_{0\le i,j\le n-1}, \\ \tau_n(t) = \det m_n(t) = \dfrac{1}{n!}\displaystyle\int_{\mathscr{H}_n} e^{\mathrm{Tr}(-V(X)+\sum t_i X^i)}\,dX \end{cases}$$

$$
\Bigg\downarrow \ \textit{map } O(2t) \textit{ such that } \begin{cases} -\mathscr{N}(2t) = O^{-1}(2t) J O^{\top -1}(2t), \\ O(2t) \textit{ is lower-triangular,} \\ O(2t) S(2t) \in \mathscr{G}_{\mathbf{k}} \end{cases}
$$

$$
\textit{Pfaff lattice} \begin{cases} q_n(t,z) = \big(O(2t)p(2t,z)\big)_n \ \textit{skew-orthonormal polynomials} \\ \qquad \textit{in } z \textit{ for the skew-inner-product (weight } \tilde{\rho}_t\textit{),} \\ \langle \varphi, \psi \rangle_t^{\mathrm{sk}} := \langle \varphi, \mathbf{n}_{2t} \psi \rangle_{2t}^{\mathrm{sy}} \\ \qquad = -\dfrac{1}{2} \displaystyle\int\!\!\int_{\mathbb{R}^2} \{\varphi(z), \psi(z)\} e^{2\sum t_i z^i} \rho(z) f(z)\, dz, \\ \tilde{\mu}_{ij}(t) = \langle z^i, z^j \rangle_t^{\mathrm{sk}} \textit{ and } \tilde{m}_n = \det(\tilde{\mu}_{ij})_{0 \le i,j \le n-1}, \\ \tilde{\tau}_{2n}(t) = \mathrm{pf}(\tilde{m}_{2n}(t)) = \dfrac{1}{(-2)^n n!} \displaystyle\int_{\mathscr{T}_{2n}} e^{2\,\mathrm{Tr}(-\tilde{V}(X) + \sum t_i X^i)}\, dX, \\ \qquad \textit{with } \tilde{V}(z) = \dfrac{1}{2}\big(V(z) - \log f(z)\big). \end{cases}
$$

Proof

Representing d/dx as an integral operator

$$
\frac{d}{dx}\varphi(x) = \int_{\mathbb{R}} \delta(x-y)\varphi'(y)\, dy = -\int_{\mathbb{R}} \frac{\partial}{\partial y}\delta(x-y)\varphi(y)\, dy = \int_{\mathbb{R}} \delta'(x-y)\varphi(y)\, dy,
$$

compute

$$
\mathbf{u}_t = \mathbf{n}_t = \sqrt{\frac{f}{\rho_t}}\frac{d}{dz}\sqrt{f\rho_t}, \quad \text{so that } \ \mathbf{n}_t\, p(t,z) = \mathscr{N} p(t,z);
$$

remember $\mathscr{N}$ from (5.2). Let it act on a function $\varphi(x)$:

$$
\begin{aligned}
\mathbf{u}_t \varphi(x) &= \left(\sqrt{\frac{f}{\rho_t}}\frac{d}{dx}\sqrt{f\rho_t}\right)\varphi(x) \\
&= \int_{\mathbb{R}} \sqrt{\frac{f(x)}{\rho_t(x)}}\,\delta'(x-y)\sqrt{f(y)\rho_t(y)}\,\varphi(y)\, dy.
\end{aligned}
$$

Then

$$
\begin{aligned}
\langle\varphi,\psi\rangle_t^{\mathrm{sk}} &= \langle\varphi,\mathbf{u}_{2t}\psi\rangle_{2t}^{\mathrm{sy}} = \langle\varphi,\mathbf{n}_{2t}\psi\rangle_{2t}^{\mathrm{sy}}\\
&= \int\int_{\mathbb{R}^2} \rho_{2t}(x)\varphi(x)\sqrt{\frac{f(x)}{\rho_{2t}(x)}}\delta'(x-y)\sqrt{f(y)\rho_{2t}(y)}\psi(y)\,dx\,dy\\
&= \int\int_{\mathbb{R}^2} \sqrt{f(x)\rho_{2t}(x)}\varphi(x)\delta'(x-y)\sqrt{f(y)\rho_{2t}(y)}\psi(y)\,dx\,dy\\
&= -\int\int_{\mathbb{R}^2} \Big(\frac{\partial}{\partial x}\sqrt{f(x)\rho_{2t}(x)}\varphi(x)\Big)\delta(x-y)\sqrt{f(y)\rho_{2t}(y)}\psi(y)\,dx\,dy\\
&= -\int_{\mathbb{R}} \Big(\frac{\partial}{\partial x}\sqrt{f(x)\rho_{2t}(x)}\varphi(x)\Big)\sqrt{f(x)\rho_{2t}(x)}\psi(x)\,dx\\
&= -\frac{1}{2}\int_{\mathbb{R}} \Big(\frac{\partial}{\partial x}\sqrt{f(x)\rho_{2t}(x)}\varphi(x)\Big)\sqrt{f(x)\rho_{2t}(x)}\psi(x)\,dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}} \sqrt{f(x)\rho_{2t}(x)}\varphi(x)\Big(\frac{\partial}{\partial x}\sqrt{f(x)\rho_{2t}(x)}\psi(x)\Big)\,dx\\
&= -\frac{1}{2}\int_{\mathbb{R}} \Big\{\sqrt{f(x)\rho_{2t}(x)}\varphi(x),\sqrt{f(x)\rho_{2t}(x)}\psi(x)\Big\}\,dx\\
&= -\frac{1}{2}\int_{\mathbb{R}} \big\{\varphi(x),\psi(x)\big\}\tilde\rho_0^2(x)e^{2\sum_1^\infty t_i x^i}\,dx,
\end{aligned}
$$

using the notation in the statement of this proposition. Setting $\tilde\rho(x)=e^{-\tilde V(x)}$, with $\tilde V(x)=(1/2)(V(x)-\log f)$,

$$
\begin{aligned}
\langle x^i,x^j\rangle^{\mathrm{sk}} &= -\frac{1}{2}\int_{\mathbb{R}}\{x^i,x^j\}\tilde\rho^2(x)e^{2\sum_1^\infty t_i x^i}\,dx\\
&= -\frac{1}{2}\int_{\mathbb{R}}\{x^i,x^j\}e^{-2(\tilde V(x)-\sum t_i x^i)}\,dx,
\end{aligned}
$$

and so

$$
\tau_{2n}(t)=\mathrm{pf}\big(\tilde m_{2n}(t)\big)=\frac{1}{(-2)^n n!}\int_{\mathscr{T}_{2n}} e^{2\,\mathrm{Tr}(-\tilde V(x)+\sum t_i x^i)}\,dx.
$$

The map O^{-1} for the classical orthogonal polynomials at $t = 0$
Then, the matrix O, mapping orthonormal p_k into skew-orthonormal polynomials q_k, is given by a lower-triangular three-step relation:

$$\begin{aligned} p_{2n}(0,z) &= -c_{2n-1}\sqrt{\frac{a_{2n-2}}{c_{2n-2}}}q_{2n-2}(0,z) + \sqrt{a_{2n}c_{2n}}\, q_{2n}(0,z), \\ p_{2n+1}(0,z) &= -c_{2n}\sqrt{\frac{a_{2n-2}}{c_{2n-2}}}q_{2n-2}(0,z) \\ &\qquad - \Big(\sum_0^{2n} b_i\Big)\sqrt{\frac{c_{2n}}{a_{2n}}}q_{2n}(0,z) + \sqrt{\frac{c_{2n}}{a_{2n}}}q_{2n+1}(0,z), \end{aligned} \tag{7.1}$$

where the a_i and b_i are the entries in the tridiagonal matrix defining the orthonormal polynomials, and the c_i are the entries in the skew-symmetric matrix.

In this case, we need to perform the following skew-Borel decomposition at $t = 0$:

$$-\mathscr{U} = -\mathscr{N} = O^{-1} J O^{\top -1},$$

where $\mathscr{N}$ is the matrix (6.3). Here again, in order to find O, we use the recipe given in Theorem 4.1, namely, writing down the corresponding skew-orthogonal polynomials (0.18), but where the μ_{ij} are the entries of $-\mathscr{U} = -\mathscr{N}$: consider the Pfaffians of the bordered matrices (0.18); they have the leading term

$$\tilde{\tilde{\tau}}_{2n} = \prod_0^{n-1} c_{2j}.$$

Then one computes

$$\begin{aligned} r_{2n} &= \frac{1}{\sqrt{\tilde{\tilde{\tau}}_{2n}\tilde{\tilde{\tau}}_{2n+2}}} \sum_{i=0}^{n} z^{2n-2i}\Big(\prod_0^{n-i-1} c_{2j}\Big)\Big(\prod_0^{i-1} c_{2n-2j-1}\Big), \\ r_{2n+1} &= \frac{1}{\sqrt{\tilde{\tilde{\tau}}_{2n}\tilde{\tilde{\tau}}_{2n+2}}}\Bigg(z^{2n+1}\prod_0^{n-1} c_{2j} + \sum_{i=1}^{n} z^{2n-2i}\Big(\prod_0^{n-i-1} c_{2j}\Big)\Big(\prod_0^{i-1} c_{2n-2j-1}\Big)\Bigg), \end{aligned} \tag{7.2}$$

with

$$\sqrt{\tilde{\tilde{\tau}}_{2n}\tilde{\tilde{\tau}}_{2n+2}} = c_0 c_2 \cdots c_{2n-2}\sqrt{c_{2n}}, \qquad \sqrt{\tilde{\tilde{\tau}}_0\tilde{\tilde{\tau}}_2} = \sqrt{c_0}.$$

Setting

$$D := \operatorname{diag}\big(\sqrt{\tilde{\tilde{\tau}}_0\tilde{\tilde{\tau}}_2}, \sqrt{\tilde{\tilde{\tau}}_0\tilde{\tilde{\tau}}_2}, \sqrt{\tilde{\tilde{\tau}}_2\tilde{\tilde{\tau}}_4}, \sqrt{\tilde{\tilde{\tau}}_2\tilde{\tilde{\tau}}_4}, \ldots\big),$$

the matrix O is the set of coefficients of the polynomials above, that is,

$$
\begin{aligned}
O = D^{-1} & \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\mathrm{c}_1 & 0 & \mathrm{c}_0 & 0 & 0 & 0 & 0 & 0 & \\
\mathrm{c}_2 & 0 & 0 & \mathrm{c}_0 & 0 & 0 & 0 & 0 & \\
\mathrm{c}_1\,\mathrm{c}_3 & 0 & \mathrm{c}_0\,\mathrm{c}_3 & 0 & \mathrm{c}_0\,\mathrm{c}_2 & 0 & 0 & 0 & \\
\mathrm{c}_1\,\mathrm{c}_4 & 0 & \mathrm{c}_0\,\mathrm{c}_4 & 0 & 0 & \mathrm{c}_0\,\mathrm{c}_2 & 0 & 0 & \\
\mathrm{c}_1\,\mathrm{c}_3\,\mathrm{c}_5 & 0 & \mathrm{c}_0\,\mathrm{c}_3\,\mathrm{c}_5 & 0 & \mathrm{c}_0\,\mathrm{c}_2\,\mathrm{c}_5 & 0 & \mathrm{c}_0\,\mathrm{c}_2\,\mathrm{c}_4 & 0 & \\
\mathrm{c}_1\,\mathrm{c}_3\,\mathrm{c}_6 & 0 & \mathrm{c}_0\,\mathrm{c}_3\,\mathrm{c}_6 & 0 & \mathrm{c}_0\,\mathrm{c}_2\,\mathrm{c}_6 & 0 & 0 & \mathrm{c}_0\,\mathrm{c}_2\,\mathrm{c}_4 & \\
 & & & & & & & & \ddots
\end{pmatrix} \\
=:{} & D^{-1}R.
\end{aligned}
\tag{7.3}
$$

As before, in order to get the skew-symmetric polynomials in the right form, from the orthogonal ones, one needs to multiply to the left with the matrix E, defined in (A.2):

$$
\hat{O} = EO = ED^{-1}R, \tag{7.4}
$$

and so,

$$
\hat{O}^{-1} = R^{-1}DE^{-1}; \tag{7.5}
$$

it turns out the matrix $\hat{O}$ is complicated, but its inverse is simple. Namely, compute

We made all c's roman here and in (7.3). OK?

$$
R^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
-\frac{\mathrm{c}_1}{\mathrm{c}_0} & 0 & \frac{1}{\mathrm{c}_0} & 0 & 0 & 0 & 0 & 0 & \\
-\frac{\mathrm{c}_2}{\mathrm{c}_0} & 0 & 0 & \frac{1}{\mathrm{c}_0} & 0 & 0 & 0 & 0 & \\
0 & 0 & -\frac{\mathrm{c}_3}{\mathrm{c}_0\,\mathrm{c}_2} & 0 & \frac{1}{\mathrm{c}_0\,\mathrm{c}_2} & 0 & 0 & 0 & \\
0 & 0 & -\frac{\mathrm{c}_4}{\mathrm{c}_0\,\mathrm{c}_2} & 0 & 0 & \frac{1}{\mathrm{c}_0\,\mathrm{c}_2} & 0 & 0 & \\
0 & 0 & 0 & 0 & -\frac{\mathrm{c}_5}{\mathrm{c}_0\,\mathrm{c}_2\,\mathrm{c}_4} & 0 & \frac{1}{\mathrm{c}_0\,\mathrm{c}_2\,\mathrm{c}_4} & 0 & \\
0 & 0 & 0 & 0 & -\frac{\mathrm{c}_6}{\mathrm{c}_0\,\mathrm{c}_2\,\mathrm{c}_4} & 0 & 0 & \frac{1}{\mathrm{c}_0\,\mathrm{c}_2\,\mathrm{c}_4} & \\
 & & & & & & & & \ddots
\end{pmatrix},
$$

and

$$E^{-1} = \begin{pmatrix} \boxed{\begin{matrix} \alpha_0 & 0 \\ -\beta_0 & \frac{1}{\alpha_0} \end{matrix}} & & & 0 \\ & \boxed{\begin{matrix} \alpha_2 & 0 \\ -\beta_2 & \frac{1}{\alpha_2} \end{matrix}} & & \\ & & \boxed{\begin{matrix} \alpha_4 & 0 \\ -\beta_4 & \frac{1}{\alpha_4} \end{matrix}} & \\ 0 & & & \ddots \end{pmatrix}, \tag{7.6}$$

with α_{2n} and β_{2n} as in (A.5). Carrying out the multiplication (7.5) leads to the matrix $\hat{O}^{-1}$, with a few nonzero bands, yielding the map (7.1), by the recipe of Proposition 6.1 inverted. □

Appendix A. Free parameter in the skew-Borel decomposition

If the Borel decomposition of $-H = O^{-1} J O^{\top -1}$ is given by a matrix $O \in \mathcal{G}_{\mathbf{k}}$, with the diagonal part of O being

$$(O)_0 = \begin{pmatrix} \boxed{\begin{matrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{matrix}} & & & 0 \\ & \boxed{\begin{matrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{matrix}} & & \\ & & \boxed{\begin{matrix} \sigma_4 & 0 \\ 0 & \sigma_4 \end{matrix}} & \\ 0 & & & \ddots \end{pmatrix}, \tag{A.1}$$

then the new matrix

Is "O=:EO," a separate equation?

$$\hat{O} := \begin{pmatrix} \boxed{\begin{matrix} 1/\alpha_0 & 0 \\ \beta_0 & \alpha_0 \end{matrix}} & & & 0 \\ & \boxed{\begin{matrix} 1/\alpha_2 & 0 \\ \beta_2 & \alpha_2 \end{matrix}} & & \\ & & \boxed{\begin{matrix} 1/\alpha_4 & 0 \\ \beta_4 & \alpha_4 \end{matrix}} & \\ 0 & & & \ddots \end{pmatrix} O =: EO, \tag{A.2}$$

with free parameters α_{2n}, β_{2n}, is a solution of the Borel decomposition $-H = \hat{O}^{-1} J \hat{O}^{\top -1}$, as well. The diagonal part of $\hat{O}$ consists of (2×2)-blocks

$$\begin{pmatrix} 1/\alpha_{2n} & 0 \\ \beta_{2n} & \alpha_{2n} \end{pmatrix} \begin{pmatrix} \sigma_{2n} & 0 \\ 0 & \sigma_{2n} \end{pmatrix} = \begin{pmatrix} \sigma_{2n}/\alpha_{2n} & 0 \\ \beta_{2n}\sigma_{2n} & \alpha_{2n}\sigma_{2n} \end{pmatrix}.$$

Imposing the condition that

$$q_i(z) = \sum_{0 \le j \le i} \hat{O}_{ij} \, p_j(z), \quad \text{with } p_k(z) = \sum_{i=0}^{k} p_{ki} z^i,$$

has the required form, that is, the same leading term for q_{2n} and q_{2n+1} and no z^{2n}-term in q_{2n+1},

$$\begin{aligned} q_{2n}(z) &= q_{2n,2n} z^{2n} + \cdots , \\ q_{2n+1}(z) &= q_{2n,2n} z^{2n+1} + q_{2n,2n-1} z^{2n-1} + \cdots \end{aligned} \tag{A.3}$$

implies

$$\begin{aligned} \frac{\sigma_{2n}}{\alpha_{2n}} p_{2n,2n} &= \sigma_{2n}\alpha_{2n} p_{2n+1,2n+1}, \\ \sigma_{2n}\beta_{2n} p_{2n,2n} + \sigma_{2n}\alpha_{2n} p_{2n+1,2n} &= 0 \end{aligned}$$

yielding, upon using the explicit form of the coefficients $p_{k\ell}$ of the polynomials p_k, associated with three-step relations (see Lemma A.1),

$$\begin{aligned} \alpha_{2n}^2 &= \frac{p_{2n,2n}}{p_{2n+1,2n+1}} = a_{2n}, \\ \frac{\beta_{2n}}{\alpha_{2n}} &= -\frac{p_{2n+1,2n}}{p_{2n,2n}} = \frac{\sum_0^{2n} b_i}{a_{2n}}. \end{aligned} \tag{A.4}$$

Hence

$$\alpha_{2n} = \sqrt{a_{2n}} \qquad \text{and} \qquad \beta_{2n} = \frac{1}{\sqrt{a_{2n}}} \sum_0^{2n} b_i. \tag{A.5}$$

So, if

$$r(z) = O\chi(z),$$

then (A.2) yields

$$\hat{r}(z) := \hat{O}\chi(z) = \begin{pmatrix} \boxed{\begin{matrix} 1/\alpha_0 & 0 \\ \beta_0 & \alpha_0 \end{matrix}} & & & 0 \\ & \boxed{\begin{matrix} 1/\alpha_2 & 0 \\ \beta_2 & \alpha_2 \end{matrix}} & & \\ & & \boxed{\begin{matrix} 1/\alpha_4 & 0 \\ \beta_4 & \alpha_4 \end{matrix}} & \\ 0 & & & \ddots \end{pmatrix} r(z) = Er(z),$$

and thus

$$\hat{r}_{2n}(z) = \frac{1}{\sqrt{a_{2n}}} r_{2n}(z), \tag{A.6}$$

Is "r(z) = Er(z)," a separate equation?

$$\hat{r}_{2n+1}(z) = \frac{\sum_0^{2n} b_i}{\sqrt{a_{2n}}} r_{2n} + \sqrt{a_{2n}}\, r_{2n+1}(z). \tag{A.7}$$

LEMMA A.1

A sequence of polynomials $p_n(z) = \sum_{i=0}^{n} p_{ni} z^i$ *of degree* n *satisfying three-step recursion relation**

$$zp_n = a_{n-1}p_{n-1} + b_n p_n + a_n p_{n+1}, \quad n = 0, 1, \ldots, \tag{A.8}$$

has the form

$$p_{n+1}(z) = \frac{p_{n,n}}{a_n}\left(z^{n+1} - \left(\sum_0^n b_i\right) z^n + \cdots\right).$$

Proof

Equating the z^{n+1} and z^n coefficients of (A.8) divided by $p_{n,n}$ yields

$$\frac{p_{n+1,n+1}}{p_{n,n}} = \frac{1}{a_n}$$

and

$$\frac{p_{n,n-1}}{p_{n,n}} = a_n \frac{p_{n+1,n}}{p_{n,n}} + b_n.$$

Combining both equations leads to

$$a_n \frac{p_{n+1,n}}{p_{n,n}} - a_{n-1}\frac{p_{n,n-1}}{p_{n-1,n-1}} = -b_n,$$

*We set $a_{-1} = 0$.

yielding

$$a_n \frac{p_{n+1,n}}{p_{n,n}} = -\sum_0^n b_i \quad (\text{using } a_{-1} = 0). \qquad \square$$

Appendix B. Simultaneous (skew-)symmetrization of L and $\mathcal{N}$

CLAIM

For the classical polynomials, the matrices L and $\mathcal{N}$ can be simultaneously symmetrized and skew-symmetrized.

Sketch of proof

This statement has been established by us in [6]. Given the monic orthogonal polynomials $\tilde{p}_n$ with respect to the weight ρ, with $\rho'/\rho = -g/f$, we have that the operators z and

$$\mathbf{n} = \sqrt{\frac{f}{\rho}}\frac{d}{dz}\sqrt{f\rho} = f\frac{d}{dz} + \frac{f'-g}{2}$$

acting on the polynomials $\tilde{p}_n$'s have the following form:

$$\begin{aligned} z\tilde{p}_n &= a_{n-1}^2 \tilde{p}_{n-1} + b_n \tilde{p}_n + \tilde{p}_{n+1}, \\ \mathbf{n}\tilde{p}_n &= \ldots - \gamma_n \tilde{p}_{n+1}, \end{aligned} \tag{B.1}$$

in view of the fact that for the classical orthogonal polynomials,*

$$\begin{cases} \textit{Hermite:} & \mathbf{n} = \frac{d}{dz} - z, \\ \textit{Laguerre:} & \mathbf{n} = z\frac{d}{dz} - \frac{1}{2}(z-\alpha-1), \\ \textit{Jacobi:} & \mathbf{n} = (1-z^2)\frac{d}{dz} - \frac{1}{2}((\alpha+\beta+2)z + (\alpha-\beta)). \end{cases}$$

For the orthonormal polynomials, the matrices L and $-\mathcal{N}$ are symmetric and skew-symmetric, respectively. Therefore the right-hand side of these expressions must have the form:

$$\begin{aligned} z\tilde{p}_n &= a_{n-1}^2 \tilde{p}_{n-1} + b_n \tilde{p}_n + \tilde{p}_{n+1}, \\ \mathbf{n}\tilde{p}_n &= a_{n-1}^2 \gamma_{n-1} \tilde{p}_{n-1} - \gamma_n \tilde{p}_{n+1}. \end{aligned}$$

Therefore, upon rescaling the $\tilde{p}_n$'s, to make them orthonormal, we have

$$\begin{aligned} zp_n &= (Lp)_n = a_{n-1} p_{n-1} + b_n p_n + a_n p_{n+1}, \\ \mathbf{n}p_n &= (\mathcal{N} p)_n = a_{n-1}\gamma_{n-1} \tilde{p}_{n-1} - a_n \gamma_n \tilde{p}_{n+1}, \end{aligned}$$

*They have the respective weights $\rho = e^{-z^2}$, $\rho = e^{-z} z^{\alpha}$, $\rho = (1-z)^{\alpha}(1+z)^{\beta}$.

from which it follows that

$$-\mathcal{N} = \begin{bmatrix} 0 & c_0 & & & \\ -c_0 & 0 & c_1 & & \\ & -c_1 & 0 & \ddots & \\ & & \ddots & & \end{bmatrix}, \quad \text{with } c_n = a_n \gamma_n \ ,$$

where $-\gamma_n$ is the leading term in expression (B.1). □

Appendix C. Proof of Lemma 3.4

For future use, consider the first-order differential operators

$$\eta(t,z) = \sum_{j=1}^{\infty} \frac{z^{-j}}{j} \frac{\partial}{\partial t_j} \qquad \text{and} \qquad B(z) = -\frac{\partial}{\partial z} + \sum_{j=1}^{\infty} z^{-j-1} \frac{\partial}{\partial t_j} \tag{C.1}$$

having the property

$$B(z) e^{-\eta(z)} f(t) = B(z) f\big(t - [z^{-1}]\big) = 0. \tag{C.2}$$

LEMMA C.1

Consider an arbitrary function $\varphi(t,z)$ depending on $t \in \mathbb{C}^\infty$, $z \in \mathbb{C}$, having the asymptotics $\varphi(t,z) = 1 + O(1/z)$ for $z \nearrow \infty$ and satisfying the functional relation

$$\frac{\varphi(t - [z_2^{-1}], z_1)}{\varphi(t, z_1)} = \frac{\varphi(t - [z_1^{-1}], z_2)}{\varphi(t, z_2)}, \quad t \in \mathbb{C}^\infty, \ z \in \mathbb{C}. \tag{C.3}$$

Then there exists a function $\tau(t)$ such that

$$\varphi(t,z) = \frac{\tau(t - [z^{-1}])}{\tau(t)}. \tag{C.4}$$

Proof

Applying $B_1 := B(z_1)$ to the logarithm of (C.3) and using (C.1) and (C.2) yields

$$\begin{aligned}(e^{-\eta(z_2)} - 1) B_1 \log \varphi(t; z_1) &= -B_1 \log \varphi(t, z_2) \\ &= -\sum_{j=1}^{\infty} z_1^{-j-1} \frac{\partial}{\partial t_j} \log \varphi(t, z_2),\end{aligned}$$

which, upon setting

$$f_j(t) = \operatorname{Res}_{z_1=\infty} z_1^j B_1 \log \varphi(t, z_1),$$

yields termwise in z_1,

$$(e^{-\eta(z_2)}-1)\,f_j(t)=-\frac{\partial}{\partial t_j}\log\varphi(t,z_2). \tag{C.5}$$

Acting with $\partial/\partial t_i$ on the latter expression and with $\partial/\partial t_j$ on the same expression with j replaced by i, and subtracting,* one finds

$$(e^{-\eta(z_2)}-1)\Big(\frac{\partial f_i}{\partial t_j}-\frac{\partial f_j}{\partial t_i}\Big)=0,$$

yielding

$$\frac{\partial f_i}{\partial t_j}-\frac{\partial f_j}{\partial t_i}=0;$$

the constant vanishes because $\partial f_i/\partial t_j$ never contains constant terms.

Therefore there exists a function $\log\tau(t_1,t_2,\dots)$ such that

$$-\frac{\partial}{\partial t_j}\log\tau\;=\;f_j(t)\;=\;\operatorname{Res}_{z=\infty}z^jB\log\varphi,$$

and hence, using (C.5),

$$\frac{\partial}{\partial t_j}\log\varphi(t,z)=(e^{-\eta(z)}-1)\frac{\partial}{\partial t_j}\log\tau,$$

or, what is the same,

$$\frac{\partial}{\partial t_j}\big(\log\varphi-(e^{-\eta}-1)\log\tau\big)=0,$$

from which it follows that

$$\log\varphi-(e^{-\eta}-1)\log\tau=-\sum_1^\infty\frac{b_i}{i}z^{-i}$$

is, at worst, a holomorphic series in z^{-1} with constant coefficients, which we call $-b_i/i$. Hence

$$\begin{aligned}\varphi(t,z)&=\frac{\tau(t-[z^{-1}]e^{-\sum_1^\infty(b_i/i)z^{-i}}}{\tau(t)}\\&=\frac{\tau(t-[z^{-1}])e^{\sum_1^\infty b_i(t_i-z^{-i}/i)}}{\tau(t)e^{\sum_1^\infty b_it_i}};\end{aligned}$$

*It is obvious that $[\partial/\partial t_i,e^{-\eta(z)}]=0$.

that is,

$$\varphi(t,z)=\frac{\tilde{\tau}(t-[z^{-1}])}{\tilde{\tau}(t)},$$

where

$$\tilde{\tau}=\tau(t)e^{\sum_1^\infty b_i t_i}. \qquad \square$$

Thus Lemma 3.4 is proved. $\square$

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