Integrals over Classical Groups, Random Permutations, Toda and Toeplitz Lattices

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0 Introduction

In recent times, there has been a considerable interest in matrix Fourier-like integrals over the classical groups $\mathrm{O}_+(\ell)$, $\mathrm{O}_-(\ell)$, $\mathrm{Sp}(\ell)$, and $\mathrm{U}(\ell)$ due to their connection with the distribution of the length of the longest increasing sequence in random permutations and random involutions and also with the spectrum of random matrices. This connection first appeared in I. Gessel's work [12], who showed that some generating function for the distribution of the length of the longest increasing sequence can be represented as a Toeplitz matrix. One of the purposes of this paper is to show that all those expressions are unique solutions to the Painlevé V equation, with certain initial conditions. In this work, we present both new results, concerning $\mathrm{O}(\ell)$, and known ones, concerning $\mathrm{U}(\ell)$; all cases are done in the same unified way.

Communications on Pure and Applied Mathematics, Vol. LIII, 0001–0053 (2000)

Our method consists of appropriately adding one set of time variables $t = (t_1, t_2, \dots)$ to the integrals for the real compact groups and two sets of times $(t, s) = (t_1, t_2, \dots, s_1, s_2, \dots)$ for the unitary group. The point is that these new time-dependent integrals satisfy integrable hierarchies:

(i) $\mathrm{O}_\pm(\ell)$ and $\mathrm{Sp}(\ell)$ correspond to the *standard Toda lattice*; the associated moment matrices are Hänkel, whose determinants provide the Toda τ-functions.

(ii) $\mathrm{U}(\ell)$ corresponds to a very special case of the *discrete sinh-Gordon equation*, leading to a new lattice, the *Toeplitz lattice*. This lattice involves a dual pair of infinite variables x_i and y_i, themselves matrix integrals. Its τ-functions are determinants of moment matrices, which are Toeplitz.

Both systems, the standard Toda lattice and the Toeplitz lattice, are peculiar reductions of the 2-Toda lattice. Each reduction has a natural vertex operator, and so a natural Virasoro algebra, a subalgebra of which annihilates the τ-functions. Combining these equations and, in the end, evaluating the result along appropriate (t, s)-loci all lead, in a unifying and quick way, to different versions of the Painlevé V equation for the integrals. More details about the precise nature of the Painlevé equations will be given in Propositions 3.3, 4.1, and 4.2. After this paper was written, we found out that the Toeplitz lattice coincides with the so-called Ablowitz-Ladik system; see Suris [17]. However, our approach to that system is novel.

Let S_n be the group of $n!$ permutations π_n and S^0_{2n} the subset of $(2n-1)!! = \frac{(2n)!}{2^n n!}$ fixed-point free involutions π^0 (i.e., $(\pi^0)^2 = I$ and $\pi^0(k) \neq k$ for $1 \leq k \leq 2n$). π_n refers to a permutation in S_n and π^0_{2n} to an involution in S^0_{2n}. Also consider $S_{n,k} = \{\text{words of length } n \text{ from an alphabet of } k \text{ letters}\}$.

An *increasing subsequence* of $\pi \in S_n$ or S^0_n is a sequence $1 \leq j_1 < \cdots < j_k \leq n$ such that $\pi(j_1) < \cdots < \pi(j_k)$. Define

$$\sigma(\pi_n) = \text{length of the longest increasing subsequence of } \pi_n.$$

In the case of $S_{n,k}$, the definition of σ is the same except that the subsequences must be increasing, without necessarily being strictly increasing.

Notation. The expectations $E_{\mathrm{O}(\ell)}$, $E_{\mathrm{U}(\ell)}, \dots$, refer to integration with regard to the Haar measure, normalized so that $E_{\mathrm{O}(\ell)}(1) = 1$, $E_{\mathrm{U}(\ell)}(1) = 1, \dots$, as it should. Sometimes it will be more convenient to use integrals $\int_{\mathrm{O}(\ell)}$, $\int_{\mathrm{U}(\ell)}, \dots$, which refer to integration with respect to the Haar measure, normalized as in Proposition 1.1 below. For $\mathrm{U}(\ell)$, the two normalizations happen to agree.

THEOREM 0.1 *For every $\ell \geq 0$, the generating functions below have the following expression in terms of specific solutions of the Painlevé V equation*:

$$\text{(0.1)}\quad \text{(i)}\ 2\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\#\{\pi_{2n}^0 \in S_{2n}^0 : \sigma(\pi_{2n}^0) \leq \ell+1\}$$
$$= E_{\mathrm{O}(\ell+1)_-}e^{x\,\mathrm{tr}\,M} + E_{\mathrm{O}(\ell+1)_+}e^{x\,\mathrm{tr}\,M}$$
$$= \exp\left(\int_0^x \frac{f_\ell^-(u)}{u}\,du\right) + \exp\left(\int_0^x \frac{f_\ell^+(u)}{u}\,du\right)$$

$$\text{(0.2)}\quad \text{(ii)}\ \sum_{n=0}^{\infty}\frac{x^n}{(n!)^2}\#\{\pi_n \in S_n : \sigma(\pi_n) \leq \ell\} = E_{\mathrm{U}(\ell)}e^{\sqrt{x}\,\mathrm{tr}(M+\bar{M})}$$
$$= \exp\int_0^x \log\left(\frac{x}{u}\right) g_\ell(u)du\,,$$

$$\text{(iii)}\ \sum_{n=0}^{\infty}\frac{x^n}{n!}\#\{\pi_n \in S_{n,k} : \sigma(\pi_n) \leq \ell\} = E_{\mathrm{U}(\ell)}\det(I+M)^k e^{-x\,\mathrm{tr}\,\bar{M}}$$
$$\text{(0.3)}\qquad = \exp\left(x\ell + (\ell+k)\int_0^x \frac{h_\ell(u)}{u}\,du\right)$$

where f_ℓ, g_ℓ, and h_ℓ are unique solutions to three different versions of the Painlevé V equation, with the initial condition indicated below; *to be precise,*

$$\text{(i)}\begin{cases} f''' + \dfrac{1}{u}f'' + \dfrac{6}{u}f'^2 - \dfrac{4}{u^2}ff' - \dfrac{16u^2+\ell^2}{u^2}f' + \dfrac{16}{u}f + \dfrac{2(\ell^2-1)}{u} = 0 \\ \textit{with } f_\ell^{\pm}(u) = u^2 \pm \dfrac{u^{\ell+1}}{\ell!} + \mathrm{O}(u^{\ell+2}) \textit{ near } u=0\,, \end{cases}$$

$$\text{(ii)}\begin{cases} g'' - \dfrac{g'^2}{2}\left(\dfrac{1}{g-1}+\dfrac{1}{g}\right) + \dfrac{g'}{u} + \dfrac{2}{u}g(g-1) - \dfrac{\ell^2}{2u^2}\dfrac{g-1}{g} = 0 \\ \textit{with } g_\ell(u) = 1 - \dfrac{u^\ell}{(\ell!)^2} + \mathrm{O}(u^{\ell+1}) \textit{ near } u=0\,, \end{cases}$$

$$\text{(iii)}\begin{cases} h''' - \dfrac{h''^2}{2}\left(\dfrac{1}{h'+1}+\dfrac{1}{h'}\right) + \dfrac{h''}{u} + \dfrac{2(\ell+k)}{u}h'(h'+1) \\ \quad - \dfrac{1}{2u^2h'(h'+1)}\big((u-\ell)h' - h - \ell\big)\big((2h+u+\ell)h' + h + \ell\big) = 0 \\ \quad \textit{with } h_\ell(u) = u\dfrac{k-\ell}{k+\ell} - \dfrac{u^{\ell+1}}{(\ell+1)!}\dbinom{k+\ell-1}{\ell} + \mathrm{O}(u^{\ell+2}) \\ \hfill \textit{near } u=0\,. \end{cases}$$

That the orthogonal matrix integrals (i) satisfy Painlevé V is new. The identity (i) involving orthogonal matrix integrals and random involutions is due to Rains

[16]. That the $\mathrm{U}(\ell)$-integral (ii) satisfies Painlevé was first established by Hisakado [13], using our methods (see [1]), and then reestablished by Tracy and Widom [18], using methods of functional analysis. The identity between random permutations and unitary matrix integrals via Toeplitz determinants goes back to Gessel [12]. Similarly, the $\mathrm{U}(\ell)$-integral (iii) was first established by Tracy and Widom [19], again using methods of functional analysis. The relation of the combinatorics to integrals over the groups was extensively studied by Diaconis and Shahshahani [11], Rains [16], and Baik and Rains [8]; see also Johansson [14], Baik, Deift, and Johansson [7], Aldous and Diaconis [5], and Tracy and Widom [18, 19].

Our methods have the benefit of providing a unifying (and also quick) way of establishing these results as well as new and known ones. The relationship with integrable systems can be summarized by Theorems 0.2 and 0.3:

THEOREM 0.2 *Define the integrals*

$$\text{(i)}\ \ I_\ell^{\pm}(x) = \int_{\mathrm{O}_\pm(\ell)} e^{x\,\mathrm{tr}\,M}\,dM \quad \textit{and} \quad \text{(ii)}\ \ I_\ell(x,y) = \int_{\mathrm{U}(\ell)} e^{\mathrm{tr}(xM - y\bar M)}\,dM\,.$$

The expressions[1]

$$\text{(i)} \qquad q_\ell(x) = \log e_\ell^{\pm}\frac{I_{\ell+2}^{\pm}}{I_\ell^{\pm}} \quad \textit{with } e_\ell^{+} = \frac{2}{[\ell+2]_{\mathrm{even}}} \textit{ and } e_\ell^{-} = \frac{2}{[\ell+1]_{\mathrm{even}}}\,,$$

$$\text{(ii)}\ \ q_\ell(x,y) = \log\frac{I_{\ell+1}}{I_\ell}\,,$$

satisfy, respectively,

$$\text{(i)}\ \ \frac{1}{4}\frac{\partial^2 q_\ell}{\partial x^2} = -e^{q_\ell - q_{\ell-1}} + e^{q_{\ell+1}-q_\ell} \quad (\textit{standard Toda lattice})$$

$$\text{(ii)}\ \ \frac{\partial^2 q_\ell}{\partial x\,\partial y} = e^{q_\ell - q_{\ell-1}} - e^{q_{\ell+1}-q_\ell} \qquad (\textit{discrete sinh-Gordon equation})\,.$$

Remark. Note that if the lattice is 2-periodic, i.e., $q_\ell = q_{\ell+2k}$, then (ii) becomes the sinh-Gordon equation for $r = q_\ell - q_{\ell-1}$:

$$\frac{\partial^2 r}{\partial x\,\partial y} = 4\sinh r\,.$$

Define the following probability measure on the unitary group $\mathrm{U}(n)$,

$$P^{t,s}_{\mathrm{U}(n)}(M \in dM) := \tau_n(t,s)^{-1} e^{\sum_1^\infty \mathrm{Tr}(t_i M^i - s_i \bar M^i)}\,dM\,,$$

and $h = \mathrm{diag}(h_0, h_1, \dots)$, $h_n = \tau_{n+1}/\tau_n$, with

$$\tau_n(t,s) := \int_{\mathrm{U}(n)} e^{\sum_1^\infty \mathrm{Tr}(t_i M^i - s_i \bar M^i)}\,dM\,.$$

[1] In this statement, we use the following notation: $[n]_{\mathrm{even}} := \max\{\text{even } x \text{ such that } x \le n\}$.

Also, let $p_i^{(1)}(t,s;z)$ and $p_i^{(2)}(t,s;z)$ be bi-orthogonal monic polynomials in z, depending on t and s, satisfying $\langle p_i^{(1)}(t,s;z), p_j^{(2)}(t,s;z)\rangle_{t,s} = \delta_{ij}h_i$ with regard to the inner product

$$\langle f(z), g(z)\rangle_{t,s} := \oint_{S^1} \frac{dz}{2\pi iz} f(z)g(z^{-1})e^{\sum_1^\infty (t_iz^i - s_iz^{-i})}\,, \quad t,s\in\mathbb{C}^\infty\,.$$

The statement of Theorem 0.3 contains the elementary Schur polynomial[2] p_n, defined by $e^{\sum_1^\infty t_iz^i} := \sum_{i\geq 0} p_i(t_1,t_2,\ldots)z^i$ and applied to the spectrum $x_k = e^{i\theta_k}$ of the unitary matrix $M\in U(n)$.

THEOREM 0.3 *Consider the following variables, expressed in terms of the expectation for the distribution above or expressed in terms of the bi-orthogonal polynomials evaluated at $z=0$:*

$$\begin{aligned} x_n(t,s) &:= E^{t,s}_{\mathrm{U}(n)} p_n\left(-\operatorname{Tr} M, -\frac{1}{2}\operatorname{Tr} M^2, -\frac{1}{3}\operatorname{Tr} M^3,\ldots\right) \\ &= \frac{p_n(-\tilde{\partial}_t)\tau_n(t,s)}{\tau_n(t,s)} = p_n^{(1)}(t,s;0)\,, \\ y_n(t,s) &:= E^{t,s}_{\mathrm{U}(n)} p_n\left(-\operatorname{Tr}\bar{M}, -\frac{1}{2}\operatorname{Tr}\bar{M}^2, -\frac{1}{3}\operatorname{Tr}\bar{M}^3,\ldots\right) \\ &= \frac{p_n(\tilde{\partial}_s)\tau_n(t,s)}{\tau_n(t,s)} = p_n^{(2)}(t,s;0)\,. \end{aligned}$$

The x_n and y_n satisfy the following integrable Hamiltonian system:

$$\begin{aligned} \frac{\partial x_n}{\partial t_i} &= (1-x_ny_n)\frac{\partial H_i^{(1)}}{\partial y_n}\,, & \frac{\partial y_n}{\partial t_i} &= -(1-x_ny_n)\frac{\partial H_i^{(1)}}{\partial x_n}\,, \\ \frac{\partial x_n}{\partial s_i} &= (1-x_ny_n)\frac{\partial H_i^{(2)}}{\partial y_n}\,, & \frac{\partial y_n}{\partial s_i} &= -(1-x_ny_n)\frac{\partial H_i^{(2)}}{\partial x_n}\,, \end{aligned} \qquad (\textit{Toeplitz lattice})$$

with initial condition $x_n(0,0) = y_n(0,0) = 0$ for $n\geq 1$ and boundary condition $x_0(t,s) = y_0(t,s) = 1$. The traces

$$H_i^{(k)} = -\frac{1}{i}\operatorname{Tr} L_k^i\,, \quad i=1,2,3,\ldots,\ k=1,2\,,$$

of the matrices L_i below are integrals in involution with regard to the symplectic structure

$$\omega := \sum_1^\infty \frac{dx_k\wedge dy_k}{1-x_ky_k}\,,$$

[2]The Schur polynomial should not be confused with the bi-orthogonal polynomials $p_i^{(k)}(t,s;z)$.

where L_1 and L_2 are given by the "rank 2" semi-infinite matrices

$$h^{-1}L_1h := \begin{pmatrix} -x_1y_0 & 1-x_1y_1 & 0 & 0 & \cdots \\ -x_2y_0 & -x_2y_1 & 1-x_2y_2 & 0 & \cdots \\ -x_3y_0 & -x_3y_1 & -x_3y_2 & 1-x_3y_3 & \cdots \\ -x_4y_0 & -x_4y_1 & -x_4y_2 & -x_4y_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$L_2 := \begin{pmatrix} -x_0y_1 & -x_0y_2 & -x_0y_3 & -x_0y_4 & \cdots \\ 1-x_1y_1 & -x_1y_2 & -x_1y_3 & -x_1y_4 & \cdots \\ 0 & 1-x_2y_2 & -x_2y_3 & -x_2y_4 & \cdots \\ 0 & 0 & 1-x_3y_3 & -x_3y_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Moreover, the precise "rank 2" structure of L_1 and L_2 is preserved by the equations

$$\frac{\partial L_i}{\partial t_n} = \left[(L_1^n)_+, L_i\right] \quad \textit{and} \quad \frac{\partial L_i}{\partial s_n} = \left[(L_2^n)_-, L_i\right], \quad i = 1, 2, \ n = 1, 2, \ldots \qquad \text{(\textit{2-Toda lattice})}$$

Remark. The first equation in the hierarchy above, corresponding to the Hamiltonians

$$H_1^{(1)} = -\operatorname{Tr} L_1 = \sum_0^\infty x_{i+1}y_i\,, \quad H_1^{(2)} = -\operatorname{Tr} L_2 = \sum_0^\infty x_i y_{i+1}\,,$$

reads

$$\begin{aligned} \frac{\partial x_n}{\partial t_1} &= x_{n+1}(1 - x_n y_n)\,, & \frac{\partial y_n}{\partial t_1} &= -y_{n-1}(1 - x_n y_n)\,, \\ \frac{\partial x_n}{\partial s_1} &= x_{n-1}(1 - x_n y_n)\,, & \frac{\partial y_n}{\partial s_1} &= -y_{n+1}(1 - x_n y_n)\,. \end{aligned}$$

Here we outline the ideas and the results in the paper. Throughout, consider a weight $\rho(x)dx$ on an interval $F \subset \mathbb{R}$ satisfying[3]

$$\text{(0.4)} \qquad -\frac{\rho'(x)}{\rho(x)} = \frac{\sum_{i\geq 0} b_i x^i}{\sum_{i\geq 0} a_i x^i} = \frac{g(x)}{f(x)} \quad \text{with } \rho(x) \text{ decaying rapidly at } \partial F\,.$$

We now define two time-dependent inner products, one given by a weight $\rho(x)dx$ on the real line $\mathbb{R}$ and another given by a contour integration about the unit circle

[3]Decaying rapidly means $\rho(x)f(x) = 0$ at finite boundary points of F, or $\rho(x)f(x)x^k \to 0$ when $x \to$ {an infinite boundary point} for all $k = 0, 1, 2, \ldots$.

$S^1 \subset \mathbb{C}$,

$$
\text{(0.5)} \quad \begin{cases} \langle f(x), g(x)\rangle_t := \int_{\mathbb{R}} f(x)g(x)e^{\sum_1^\infty t_i x^i}\rho(x)dx\,, & t \in \mathbb{C}^\infty \\ \langle f(z), g(z)\rangle_{t,s} := \oint_{S^1} \frac{dz}{2\pi i z} f(z)g(z^{-1})e^{\sum_1^\infty (t_i z^i - s_i z^{-i})}\,, & t,s \in \mathbb{C}^\infty\,. \end{cases}
$$

These inner products lead to Hänkel and Toeplitz moment matrices, respectively,

$$
\begin{cases} m_n(t) := \left(\langle x^i, x^j\rangle_t\right)_{0\le i,j\le n-1} & \text{(Hänkel)} \\ m_n(t,s) := \left(\langle z^i, z^j\rangle_{t,s}\right)_{0\le i,j\le n-1} & \text{(Toeplitz).} \end{cases}
$$

The determinants τ_n of the m_n's have different representations: On the one hand, as multiple integrals, involving Vandermonde's $\Delta_n(z)$, and on the other hand, as inductive expressions in terms of τ_{n-1} involving a vertex operator,[4]

$$
\text{(0.6)} \qquad \mathbb{X}_{12}(t,s;u,v) = \Lambda^\top e^{\sum_1^\infty (t_i u^i - s_i v^i)} e^{-\sum_1^\infty \left(\frac{u^{-i}}{i}\frac{\partial}{\partial t_i} - \frac{v^{-i}}{i}\frac{\partial}{\partial s_i}\right)} \chi(uv)\,,
$$

to be explained in (0.7). The $\tau_n(t)$ and $\tau_n(t,s)$ are, respectively, solutions to the standard Toda lattice, and the so-called Toeplitz lattice, both reductions of the semi-infinite 2D Toda lattice:[5]

$$
\text{(0.7)} \quad I_n = n!\det \tau_n = n!\det m_n =
$$

$$
\begin{cases} \int_{\mathbb{R}^n} \Delta_n^2 \prod_{k=1}^n e^{\sum_{i=1}^\infty t_i z_k^i}\rho(z_k)dz_k = \int_{\mathbb{R}} du\, \rho(u)\left(\mathbb{X}_{12}\left(\frac{s+t}{2}, \frac{s-t}{2}; u, u\right) I\right)_n \\ \quad \text{(standard Toda } \tau\text{-functions)} \\ \oint_{(S^1)^n} |\Delta_n|^2 \prod_{k=1}^n e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k} = \int_{S^1} \frac{du}{2\pi i u}\left(\mathbb{X}_{12}(t,s;u,u^{-1})I\right)_n \\ \quad \text{(2-Toda } \tau\text{-functions)} \end{cases}
$$

where $\tau_n(t)$ and $\tau_n(t,s)$ satisfy the following differential equations (the second one is new)

$$
\begin{cases} \frac{\partial^4}{\partial t_1^4}\log\tau_n + 6\left(\frac{\partial^2}{\partial t_1^2}\log\tau_n\right)^2 + 3\frac{\partial^2}{\partial t_2^2}\log\tau_n - 4\frac{\partial^2}{\partial t_1 \partial t_3}\log\tau_n = 0 \\ \quad \text{(KP equation)} \\ \frac{\partial^2}{\partial s_2 \partial t_1}\log\tau_n = -2\frac{\partial}{\partial s_1}\log\frac{\tau_n}{\tau_{n-1}} \cdot \frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n - \frac{\partial^3}{\partial s_1^2 \partial t_1}\log\tau_n \\ \quad \text{(2-Toda equation).} \end{cases}
$$

---

[4]For $v = (v_0, v_1, \dots)^\top$, $(\Lambda v)_n = v_{n+1}$, $(\Lambda^\top v)_n = v_{n-1}$ and $\chi(z) := (1, z, z^2, \dots)$.

[5]The expression $\mathbb{X}_{12}(\frac{s+t}{2}, \frac{s-t}{2}; u, u)$ is actually independent of s! The expressions in (0.7) are inductive in n because of the presence of the downward shift $\Lambda^\top$ in $\mathbb{X}_{12}$.

The unique factorization of the time-dependent semi-infinite moment matrices m_∞ (defined just under (0.5)) into lower- times upper-triangular matrices

$$\begin{cases} m_\infty(t) = S(t)^{-1} S^{\top -1}(t) \\ m_\infty(t,s) = S_1(t,s)^{-1} S_2(t,s) \end{cases} \tag{0.8}$$

leads to matrices $S(t)$, $S_1(t,s)$, and $S_2(t,s)$ of the form

$$S(t) = \sum_{i\le 0} a_i \Lambda^i\,, \quad S_1(t,s) = \sum_{i\le 0} b_i \Lambda^i\,, \quad S_2(t,s) = \sum_{i\ge 0} b'_i \Lambda^i\,,$$

with $b_0 = I$ and a_i, b_i, b'_i diagonal matrices. By "dressing up" the shift Λ (defined in footnote 4), they evolve according to the following integrable systems:

$$\begin{cases} L(t) = S(t)\Lambda S^{-1}(t) : \text{symmetric and tridiagonal} & \text{(Toda lattice)} \\ \begin{cases} L_1(t,s) = S_1(t,s)\Lambda S_1^{-1}(t,s) \\ L_2(t,s) = S_2(t,s)\Lambda^\top S_2^{-1}(t,s) \end{cases} & \text{(Toeplitz lattice).} \end{cases}$$

As already pointed out in (0.7), that expression involves the following reduction of the 2-Toda vertex operator $\mathbb{X}_{12}$:

$$\begin{cases} \mathbb{X}_{12}\left(\dfrac{s+t}{2}, \dfrac{s-t}{2}; u, u\right) =: \mathbb{X}(t;u) = \Lambda^\top \chi(u^2) e^{\sum_1^\infty t_i u^i} e^{-2\sum_1^\infty \frac{u^{-i}}{i}\frac{\partial}{\partial t_i}} \\ \mathbb{X}_{12}(t,s;u,u^{-1}) = \Lambda^\top e^{\sum_1^\infty (t_i u^i - s_i u^{-i})} e^{-\sum_1^\infty (\frac{u^{-i}}{i}\frac{\partial}{\partial t_i} - \frac{u^i}{i}\frac{\partial}{\partial s_i})}\,. \end{cases} \tag{0.9}$$

Each of these vertex operators leads to Virasoro algebras $\mathcal{J}_m^{(2)}$ and $\mathcal{V}_m^{(2)}$ of central charge $c=1$ and $c=0$, respectively, defined by

$$\begin{cases} \dfrac{\partial}{\partial u} u^{m+1} f(u)\mathbb{X}(t,u)\rho(u) = \left[\mathcal{J}_m^{(2)}(t), \mathbb{X}(t,u)\rho(u)\right] \\ \dfrac{\partial}{\partial u} u^{m+1} \dfrac{\mathbb{X}_{12}(t,s;u,u^{-1})}{u} = \left[\mathcal{V}_m^{(2)}(t,s), \dfrac{\mathbb{X}_{12}(t,s;u,u^{-1})}{u}\right], \end{cases}$$

and having the explicit expressions (see notation (0.4)).

$$\begin{aligned} \mathcal{J}_m^{(2)}(t) &:= \sum_{i\ge 0} \left(a_i\, {}^\beta\mathbb{J}_{i+m}^{(2)}(t) - b_i\, {}^\beta\mathbb{J}_{i+m+1}^{(1)}(t)\right)\Big|_{\beta=2}\,, \\ \mathcal{V}_m^{(2)}(t,s) &:= {}^\beta\mathbb{J}_m^{(2)}(t) - {}^\beta\mathbb{J}_{-m}^{(2)}(-s) \\ &\qquad - m\left(\theta\, {}^\beta\mathbb{J}_m^{(1)}(t) + (1-\theta)\, {}^\beta\mathbb{J}_{-m}^{(1)}(-s)\right)\Big|_{\beta=1}\,, \end{aligned} \tag{0.10}$$

in terms of generators ${}^\beta\mathbb{J}_m^{(2)}$ defined in (A.3) below and arbitrary θ.

The point is that a big subalgebra of $\mathcal{J}_m^{(2)}$'s and a small one of $\mathcal{V}_m^{(2)}$'s annihilate $\tau_n(t)$ and $\tau_n(t,s)$, respectively, for appropriate θ and for all $n\ge 0$,

$$\begin{cases} \mathcal{J}_m^{(2)}\tau_n(t) = 0 & \text{for } m \ge -1 \\ \mathcal{V}_m^{(2)}\tau_n(t,s) = 0 & \text{for } m = -1, 0, 1 \text{ (SL}(2,\mathbb{Z})\text{-algebra).} \end{cases}$$

To summarize, we have that combining these equations and restricting to the three different loci $\mathcal{L}$ below always leads to Painlevé V:

$$\left.\left\{\begin{array}{l}\mathrm{KP}(\tau_n)=0,\\ \mathcal{J}_m^{(2)}\tau_n(t)=0,\\ \text{for } m=-1,0\end{array}\right\}\right|_{\mathcal{L}=\left\{\begin{array}{l}t_1=x,\ \text{all other}\\ t_i=0\end{array}\right\}} \Longrightarrow \begin{cases}\text{Painlevé } V \text{ for}\\ \mathrm{O}_{\pm}(n)\text{-integral}\end{cases}$$

$$\left.\left\{\begin{array}{l}\text{2-Toda PDE}\\ \mathcal{V}_m^{(2)}\tau_n(t,s)=0\,,\\ \text{for } m=-1,0,1\\ \text{Toeplitz relation}\end{array}\right\}\right|_{\substack{\mathcal{L}\ =\left\{\begin{array}{l}t_1,s_1\neq 0\,,\ \text{all other}\\ t_i,s_i=0\end{array}\right\}\text{ or}\\ \mathcal{L}\ =\left\{\begin{array}{l}\text{all } it_i=-k(-1)^i,\\ s_i=0 \text{ except } s_1=x\end{array}\right\}}} \Rightarrow \begin{cases}\text{Painlevé } V \text{ for}\\ \mathrm{U}(n)\text{-integral.}\end{cases}$$

1 Integrals over Classical Groups and Combinatorics

This section contains a number of useful facts about integrals over groups, its relation with combinatorics, and finally the behavior of some of the integrals near $x=0$.

The situation is quite different depending on whether one integrates over the real ($\mathrm{O}_\pm$, $\mathrm{Sp}(\ell)$) or the complex ($\mathrm{U}(\ell)$). The real group integrals involve the Jacobi weight,

$$\rho_{\alpha\beta}(z)dz := (1-z)^{\alpha}(1+z)^{\beta}\,dz \quad \text{for } \alpha,\beta=\pm\frac{1}{2} \tag{1.1}$$

and the Chebyshev polynomials $T_n(z)$, defined by $T_n(\cos\theta) := \cos n\theta$. In particular, we have $T_1(z)=z$. We now have the following theorem (see Johansson [14]):

PROPOSITION 1.1 (Weyl) *Defining*

$$g(z) := 2\sum_1^{\infty} t_i T_i(z)\,,$$

the following holds:

$$\int_{\mathrm{U}(n)} e^{\sum_1^\infty \operatorname{tr}(t_i M^i - s_i \bar M^i)} dM = \frac{1}{n!}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})}\frac{dz_k}{2\pi i z_k}$$

$$\int_{\mathrm{O}(2n+1)_+} e^{\sum_1^\infty t_i \operatorname{tr} M^i}\,dM = e^{\sum_1^\infty t_i}\int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{g(z_k)}\rho_{(\frac{1}{2},-\frac{1}{2})}(z_k)dz_k$$

$$\int_{\mathrm{O}(2n+1)_-} e^{\sum_1^\infty t_i \operatorname{tr} M^i}\,dM = e^{\sum_1^\infty (-1)^i t_i}\int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{g(z_k)}\rho_{(-\frac{1}{2},\frac{1}{2})}(z_k)dz_k$$

$$\int_{\mathrm{O}(2n)_+} e^{\sum_1^\infty t_i \operatorname{tr} M^i}\, dM = \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{g(z_k)} \rho_{(-\frac12,-\frac12)}(z_k) dz_k$$

$$\int_{\mathrm{O}(2n)_-} e^{\sum_1^\infty t_i \operatorname{tr} M^i}\, dM = e^{\sum_1^\infty 2t_{2i}} \int_{[-1,1]^{n-1}} \Delta_{n-1}(z)^2 \prod_{k=1}^{n-1} e^{g(z_k)} \rho_{(\frac12,\frac12)}(z_k) dz_k$$

$$\int_{\mathrm{Sp}(n)} e^{\sum_1^\infty t_i \operatorname{tr} M^i}\, dM = \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{g(z_k)} \rho_{(\frac12,\frac12)}(z_k) dz_k\,. \tag{1.2}$$

With this normalization, we have (*see Appendix* C)

$$\int_{\mathrm{U}(n)} dM = 1\,,$$

$$\int_{\mathrm{O}(2n+1)_\pm} dM = 2^{n^2} \prod_{j=1}^n \frac{j!\,(j-\frac12)\Gamma^2(j-\frac12)}{(n+j-1)!}\,,$$

$$\int_{\mathrm{O}(2n)_+} dM = 2^{n(n-1)} \prod_{j=1}^n \frac{j!\,\Gamma^2(j-\frac12)}{(n+j-2)!}\,,$$

$$\int_{\mathrm{O}(2n)_-} dM = 2^{n(n-1)} \prod_{j=1}^{n-1} \frac{j!\,\Gamma^2(j+\frac12)}{(n+j-1)!}\,.$$

Letting $\iota \in S_n$ denote the permutation $k \to n+1-k$, we also state:

PROPOSITION 1.2 *The combinatorial quantities that follow have an expression in terms of integrals over groups*:

$$\sum_{n\ge 0} \frac{x^{2n}}{(n!)^2} \#\{\pi \in S_n : \sigma_n(\pi) \le \ell\} = E_{\mathrm{U}(\ell)} e^{x \operatorname{tr}(M+\bar M)}\,,$$

$$\sum_{n\ge 0} \frac{x^{2n}}{(2n)!} \# \left\{ \begin{matrix} \pi \in S_{4n} : \pi^2 = 1,\ (\pi\iota)^2 = 1, \\ \pi(y) \ne y, \iota y,\ \sigma_{4n}(\pi) \le 2\ell \end{matrix} \right\} = E_{\mathrm{U}(\ell)} e^{x \operatorname{tr}(M+\bar M)}\,,$$

$$2\sum_{n\ge 0} \frac{x^{2n}}{(2n)!} \# \left\{ \begin{matrix} \pi \in S_{2n} : \pi^2 = 1, \\ \pi(y) \ne y,\ \sigma_{2n}(\pi) \le \ell \end{matrix} \right\} = E_{\mathrm{O}_-(\ell)} e^{x \operatorname{tr} M} + E_{\mathrm{O}_+(\ell)} e^{x \operatorname{tr} M}\,,$$

$$\sum_{n\ge 0} \frac{x^n}{n!} \#\{\pi \in S_n : \pi^2 = 1,\ \sigma_n(\pi) \le \ell\} = e^x E_{\mathrm{O}_-(\ell+1)} e^{x \operatorname{tr} M}\,,$$

$$\sum_{n\ge 0} \frac{x^n}{n!} \#\{\pi \in S_n : (\iota\pi)^2 = 1,\ \sigma_n(\pi) \le \ell\} = e^x E_{\mathrm{O}_-(\ell+1)} e^{x \operatorname{tr} M}\,,$$

$$\sum_{n\geq 0}\frac{x^{2n}}{(2n)!}\#\left\{\begin{matrix}\pi\in S_{2n}:(\pi\iota)^2=1,\\ \pi(y)\neq\iota y,\ \sigma_{2n}(\pi)\leq 2\ell\end{matrix}\right\}=E_{\mathrm{O}_-(2\ell+2)}e^{x\,\mathrm{tr}\,M}\,,$$

$$\sum_{n\geq 0}\frac{x^{2n}}{(n!)^2}\#\{\pi\in S_{2n}:\pi\iota=\iota\pi,\ \sigma_{2n}(\pi)\leq 2\ell\}=E^2_{\mathrm{U}(\ell)}e^{x\,\mathrm{tr}(M+\bar M)}\,,$$

$$\sum_{n\geq 0}\frac{x^{2n}}{(n!)^2}\#\left\{\begin{matrix}\pi\in S_{2n}:\pi\iota=\iota\pi,\\ \sigma_{2n}(\pi)\leq 2\ell+1\end{matrix}\right\}=E_{\mathrm{U}(\ell)}e^{x\,\mathrm{tr}(M+\bar M)}$$

$$\cdot\, E_{\mathrm{U}(\ell+1)}e^{x\,\mathrm{tr}(M+\bar M)}\,,$$

$$\sum_{n\geq 0}\frac{x^n}{n!}\#\{\pi\in S_{n,k}:\sigma_n(\pi)\leq\ell\}=E_{\mathrm{U}(\ell)}\det(I+M)^k e^{x\,\mathrm{tr}\,\bar M}\,.$$

The first identity goes back to Gessel [12] and in this precise form to Rains [16]. The third, seventh, and eight equalities between the combinatorics and the integral expression above are due to Rains [16], the next ones are due to Baik and Rains [8], and the last one is due to Tracy and Widom [19].

We now state the following elementary lemma:

LEMMA 1.3

$$e^{\frac{x^2}{2}}=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\#\big\{\pi\in S_{2n}:\pi^2=1,\ \textit{fixed-point free}\big\}=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\frac{(2n)!}{2^n n!}\,,$$

$$e^{\frac{x^2}{2}+x}=\sum_{n=0}^{\infty}\frac{x^n}{(n)!}\#\big\{\pi\in S_n:\pi^2=1\big\}=\sum_{n=0}^{\infty}\frac{x^n}{(n)!}\sum_{0\leq m\leq[\frac{n}{2}]}\binom{n}{2m}\frac{(2m)!}{2^m m!}\,.$$

We now estimate the following integrals over $\mathrm{O}(\ell+1)$ and $\mathrm{U}(\ell)$ near $x=0$. In all cases, one notices a big gap in the expansion, roughly speaking, of the order ℓ. This would be hard to obtain at the level of the integrals, but easy to obtain via combinatorics.

PROPOSITION 1.4 *The following estimates hold near $x=0$:*

$$E_{\mathrm{O}_\pm(\ell+1)}e^{x\,\mathrm{Tr}\,M}=\exp\left(\frac{x^2}{2}\pm\frac{x^{\ell+1}}{(\ell+1)!}+\mathrm{O}(x^{\ell+2})\right)\,,$$

$$E_{\mathrm{O}_+(\ell+1)}e^{x\,\mathrm{Tr}\,M}+E_{\mathrm{O}_-(\ell+1)}e^{x\,\mathrm{Tr}\,M}=2\exp\left(\frac{x^2}{2}+\mathrm{O}(x^{\ell+2})\right).\tag{1.3}$$

PROOF: From the second relation in Lemma 1.3, it follows that

$$\#\{\pi\in S_n:\pi^2=1,\ \sigma_n(\pi)\leq\ell\}$$
$$=\begin{cases}\#\{\pi\in S_n:\pi^2=1\}=\displaystyle\sum_{0\leq m\leq[n/2]}\binom{n}{2m}\frac{(2m)!}{2^m m!} & \text{for } n\leq\ell\\ \#\{\pi\in S_{\ell+1}:\pi^2=1\}-1 & \text{for } n=\ell+1\,.\end{cases}$$

Hence we have from the fourth identity of Proposition 1.2 and from Lemma 1.3

$$
\begin{aligned}
e^x E_{\mathrm{O}_-(\ell+1)} e^{x \operatorname{tr} M} &= \sum_{n\geq 0} \frac{x^n}{n!} \#\left\{\pi \in S_n : \pi^2 = 1,\ \sigma_n(\pi) \leq \ell\right\} \\
&= \exp\left(\frac{x^2}{2} + x - \frac{x^{\ell+1}}{(\ell+1)!} + \mathrm{O}(x^{\ell+2})\right),
\end{aligned}
$$

and so

$$
E_{\mathrm{O}_-(\ell+1)} e^{x \operatorname{tr} M}\, dM = \exp\left(\frac{x^2}{2} - \frac{x^{\ell+1}}{(\ell+1)!} + \mathrm{O}(x^{\ell+2})\right). \tag{1.4}
$$

But for $2n \leq \ell + 1$,

$$
\begin{aligned}
&\#\left\{\pi \in S_{2n} : \pi^2 = 1,\ \sigma_{2n}(\pi) \leq \ell + 1,\ \text{fixed-point free}\right\} = \\
&\qquad \left\{\pi \in S_{2n} : \pi^2 = 1,\ \text{fixed-point free}\right\} = \frac{(2n)!}{2^n n!},
\end{aligned}
$$

and so, from the third identity of Proposition 1.2,

$$
\begin{aligned}
&E_{\mathrm{O}_-(\ell+1)} e^{x \operatorname{tr} M} + E_{\mathrm{O}_+(\ell+1)} e^{x \operatorname{tr} M} \\
&\quad = 2 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \#\left\{\pi \in S_{2n} : \pi^2 = 1,\ \sigma_{2n}(\pi) \leq \ell + 1,\ \text{fixed-point free}\right\} \\
&\quad = 2 \exp\left(\frac{x^2}{2} + \mathrm{O}(x^{\ell+2})\right),
\end{aligned}
$$

which establishes the second relation (1.3). Combining this formula with (1.4) leads to the following estimate near $x = 0$:

$$
E_{\mathrm{O}_+(\ell+1)} e^{x \operatorname{tr} M} = \exp\left(\frac{x^2}{2} + \frac{x^{\ell+1}}{(\ell+1)!} + \mathrm{O}(x^{\ell+2})\right),
$$

which establishes the first relation (1.3). □

PROPOSITION 1.5 *The following estimates hold near* $x = 0$:

$$
\begin{aligned}
E_{\mathrm{U}(\ell)} e^{\sqrt{x} \operatorname{tr}(M+\bar{M})} dM &= \exp\left(x - \frac{x^{\ell+1}}{((\ell+1)!)^2} + \mathrm{O}(x^{\ell+2})\right), \\
E_{\mathrm{U}(\ell)} \det(I+M)^k e^{-x \operatorname{tr} \bar{M}} dM &= \exp\left(kx - \frac{x^{\ell+1}}{(\ell+1)!} \binom{k+\ell}{\ell+1} + \mathrm{O}(x^{\ell+2})\right).
\end{aligned}
$$

PROOF: Using the first identity of Proposition 1.2, we have

$$
\begin{aligned}
I_\ell &:= E_{\mathrm{U}(\ell)} e^{\sqrt{x}\,\mathrm{tr}(M+\bar{M})}\,dM \\
&= \sum_0^\infty \frac{x^n}{(n!)^2}\#\{\pi \in S_n : \sigma_n(\pi) \le \ell\} \\
&= \sum_0^\ell \frac{x^n}{n!} + \frac{x^{\ell+1}}{(\ell+1)!^2}((\ell+1)! - 1) + \mathrm{O}(x^{\ell+2}) \\
&= \exp\left(x - \frac{x^{\ell+1}}{(\ell+1)!^2} + \mathrm{O}(x^{\ell+2})\right).
\end{aligned}
$$

Since the number of (increasing) sequences

$$
\overbrace{(1, 1, \dots, 2, 2, 2, \dots, k, k, k)}^{\ell+1}
$$

of length $\ell + 1$ and consisting of k symbols is given by $\binom{k+\ell}{\ell+1}$, one computes for $S_{\ell,k}$ ={words of length ℓ from an alphabet of k letters}:

$$
\begin{aligned}
\#\{\pi \in S_{n,k} : \sigma_n(\pi) \le \ell\} &= k^n && \text{for } n \le \ell\,, \\
\#\{\pi \in S_{\ell+1,k} : \sigma_{\ell+1}(\pi) \le \ell\} &= k^{\ell+1} - \binom{k+\ell}{\ell+1} && \text{for } n = \ell + 1\,.
\end{aligned}
$$

Therefore, using the last identity of Proposition 1.2, one finds

$$
\begin{aligned}
I_\ell &:= E_{\mathrm{U}(\ell)} \det(I+M)^k e^{-x\,\mathrm{tr}\,\bar{M}}\,dM_0 \\
&= \sum_0^\infty \frac{x^n}{n!}\#\{\pi \in S_{n,k} : \sigma_n(\pi) \le \ell\} \\
&= \sum_0^\ell \frac{x^n}{n!}k^n + \frac{x^{\ell+1}}{(\ell+1)!}\left(k^{\ell+1} - \binom{k+\ell}{\ell+1}\right) + \mathrm{O}(x^{\ell+2}) \\
&= \exp\left(kx - \frac{x^{\ell+1}}{(\ell+1)!}\binom{k+\ell}{\ell+1} + \mathrm{O}(x^{\ell+2})\right).
\end{aligned}
$$

$\square$

2 2-Toda Lattice and Reductions (Hänkel and Toeplitz)

2.1 2-Toda on Moment Matrices and Identities for τ-Functions

2-Toda τ-functions $\tau_n(t,s)$, $n \in \mathbb{Z}$, depend on two sets of time variables $t, s \in \mathbb{C}^\infty$ and are defined by the following bilinear identities for all $m, n \in \mathbb{Z}$:

$$
\begin{aligned}
(2.1)\quad & \oint_{z=\infty} \tau_n\big(t-[z^{-1}],s\big)\tau_{m+1}\big(t'+[z^{-1}],s'\big)e^{\sum_1^\infty (t_i-t_i')z^i} z^{n-m-1}\,dz = \\
& \qquad \oint_{z=0} \tau_{n+1}\big(t, s-[z]\big)\tau_m\big(t', s'+[z]\big)e^{\sum_1^\infty (s_i-s_i')z^{-i}} z^{n-m-1}\,dz\,,
\end{aligned}
$$

or, specified in terms of the Hirota symbol,[6] by

$$
\begin{aligned}
(2.2)\quad & \sum_{j=0}^\infty p_{m-n+j}(-2a)\,p_j(\tilde\partial_t)e^{\sum_1^\infty \left(a_k \frac{\partial}{\partial t_k}+b_k\frac{\partial}{\partial s_k}\right)}\tau_{m+1}\circ\tau_n = \\
& \qquad \sum_{j=0}^\infty p_{-m+n+j}(-2b)\,p_j(\tilde\partial_s)e^{\sum_1^\infty \left(a_k \frac{\partial}{\partial t_k}+b_k\frac{\partial}{\partial s_k}\right)}\tau_m\circ\tau_{n+1}\,.
\end{aligned}
$$

For the semi-infinite case, the same definitions hold but for $n, m \geq 0$.

THEOREM 2.1 2-*Toda* τ-*functions satisfy the following*:

(i) *the* KP-hierarchy *in t and s separately, of which the first equation reads*

$$
\left(\frac{\partial}{\partial t_1}\right)^4 \log\tau + 6\left(\left(\frac{\partial}{\partial t_1}\right)^2\log\tau\right)^2 + 3\left(\frac{\partial}{\partial t_2}\right)^2\log\tau - 4\frac{\partial^2}{\partial t_1\partial t_3}\log\tau = 0\,,
$$

(ii) *an identity involving t and s and nearest neighbors τ_{n-1} and τ_n*,

$$
\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n = -\frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}\,,
$$

(iii) *a* (*new*) *identity involving t and s and nearest neighbors τ_{n-1} and τ_n*,

$$
(2.3)\qquad \frac{\partial^2}{\partial s_2\partial t_1}\log\tau_n = -2\frac{\partial}{\partial s_1}\log\frac{\tau_n}{\tau_{n-1}}\cdot\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n - \frac{\partial^3}{\partial s_1^2\partial t_1}\log\tau_n\,.
$$

The proof of this theorem will be given later in this section.

[6]For the customary Hirota symbol, $p(\partial_t)\, f\circ g := p\big(\frac{\partial}{\partial y}\big)\, f(t+y)g(t-y)\big|_{y=0}$.

The "wave vectors" defined in terms of $\tau_n(t,s)$,[7]

$$\Psi_1(t,s,z) = \left(\frac{\tau_n(t-[z^{-1}],s)}{\tau_n(t,s)} e^{\sum_1^\infty t_i z^i} z^n\right)_{n\in\mathbb{Z}} =: e^{\sum_1^\infty t_i z^i} S_1 \chi(z)\,, \tag{2.4}$$

$$\Psi_2^*(t,s,z) = \left(\frac{\tau_n(t,s+[z])}{\tau_{n+1}(t,s)} e^{-\sum_1^\infty s_i z^{-i}} z^{-n}\right)_{n\in\mathbb{Z}}$$
$$=: e^{-\sum_1^\infty s_i z^{-i}} (S_2^{-1})^\top \chi(z^{-1})\,,$$

specify lower- and upper-triangular wave matrices S_1 and S_2, respectively. They, in turn, define a pair of matrices L_1 and L_2,

$$\begin{aligned} L_1 &:= S_1 \Lambda S_1^{-1} = \sum_{-\infty < i \le 0} a_i^{(1)} \Lambda^i + \Lambda\,, \\ L_2 &:= S_2 \Lambda^\top S_2^{-1} = \sum_{-1 \le i < \infty} a_i^{(2)} \Lambda^i\,, \end{aligned} \tag{2.5}$$

where $\Lambda = (\delta_{j-i,1})_{i,j\in\mathbb{Z}}$ and $a_i^{(1)}$ and $a_i^{(2)}$ are diagonal matrices depending on $t = (t_1, t_2, \dots)$ and $s = (s_1, s_2, \dots)$. Then

$$z\Psi_1 = L_1 \Psi_1 \quad \text{and} \quad z^{-1}\Psi_2^* = L_2^\top \Psi_2^*\,, \tag{2.6}$$

and the matrices L_i satisfy the 2-Toda lattice equations

$$\frac{\partial L_i}{\partial t_n} = \left[\left(L_1^n\right)_+, L_i\right] \quad \text{and} \quad \frac{\partial L_i}{\partial s_n} = \left[\left(L_2^n\right)_{,} L_i\right] \tag{2.7}$$
$$i = 1,2 \text{ and } n = 1,2,\dots,$$

with Ψ_1 and Ψ_2^* satisfying the following differential equations:

$$\begin{aligned} \frac{\partial \Psi_1}{\partial t_n} &= (L_1^n)_+ \Psi_1\,, & \frac{\partial \Psi_1}{\partial s_n} &= (L_2^n)_- \Psi_1\,, \\ \frac{\partial \Psi_2^*}{\partial t_n} &= -((L_1^n)_+)^\top \Psi_2^*\,, & \frac{\partial \Psi_2^*}{\partial s_n} &= -((L_2^n)_-)^\top \Psi_2^*\,. \end{aligned}$$

For future use, define the diagonal matrix as

$$h := (\dots, h_{-1}, h_0, h_1, \dots) \quad \text{where } h_k(t,s) := \frac{\tau_{k+1}(t,s)}{\tau_k(t,s)}\,. \tag{2.8}$$

---

[7] We have

$$\begin{aligned} \chi(z) &= \left(\dots, z^{-1}, 1, z^1, \dots\right) && \text{in the bi-infinite case} \\ &= \left(1, z, z^2, \dots\right) && \text{in the semi-infinite case.} \end{aligned}$$

Also, Λ^{-1} should always be interpreted as $\Lambda^\top$ in the semi-infinite case.

In [4] we have shown that L_1^k has the following expression in terms of τ-functions:[8]

$$
\begin{aligned}
L_1^k &= \sum_{\ell=0}^{\infty} \operatorname{diag}\left(\frac{p_\ell(\tilde{\partial}_t)\tau_{n+k-\ell+1} \circ \tau_n}{\tau_{n+k-\ell+1}\tau_n}\right)_{n\in\mathbb{Z}} \Lambda^{k-\ell}\,, \\
hL_2^{\top k}h^{-1} &= \sum_{\ell=0}^{\infty} \operatorname{diag}\left(\frac{p_\ell(-\tilde{\partial}_s)\tau_{n+k-\ell+1} \circ \tau_n}{\tau_{n+k-\ell+1}\tau_n}\right)_{n\in\mathbb{Z}} \Lambda^{k-\ell}\,.
\end{aligned} \tag{2.9}
$$

There is a general involution in the equation, which we shall frequently use, namely, $t \longleftrightarrow -s$, $L_1 \longleftrightarrow hL_2^{\top}h^{-1}$.

Finally, we define the 2-Toda vertex operator, which is the generating function for the algebra of symmetries acting on τ-functions (it will play a role later!),

$$
\mathbb{X}_{12}(t,s;u,v) = \Lambda^{-1} e^{\sum_1^\infty (t_i u^i - s_i v^i)} e^{-\sum_1^\infty \left(\frac{u^{-i}}{i}\frac{\partial}{\partial t_i} - \frac{v^{-i}}{i}\frac{\partial}{\partial s_i}\right)} \chi(uv)\,, \tag{2.10}
$$

leading to a Virasoro algebra (see Appendix A) with $\beta = 1$ and thus with central charge $c = -2$,

$$
\begin{aligned}
\frac{\partial}{\partial u} u^{k+1}\mathbb{X}_{12}(t,s;u,v) &= \left[{}^{\beta}\mathbb{J}_k^{(2)}(t)\big|_{\beta=1}, \mathbb{X}_{12}(t,s;u,v)\right], \\
u^k \mathbb{X}_{12}(t,s;u,v) &= \left[{}^{\beta}\mathbb{J}_k^{(1)}(t)\big|_{\beta=1}, \mathbb{X}_{12}(t,s;u,v)\right],
\end{aligned} \tag{2.11}
$$

with generators (in t), explicitly given by (A.3). Similarly, the involution $u \leftrightarrow v$, $t \leftrightarrow -s$, leads to the same Virasoro algebra in s with same central charge.

PROPOSITION 2.2

$$
\begin{aligned}
\left(L_1^k\right)_{n,n} &= \frac{p_k(\tilde{\partial}_t)\tau_{n+1}\circ\tau_n}{\tau_{n+1}\tau_n} = \frac{\partial}{\partial t_k}\log\frac{\tau_{n+1}}{\tau_n}\,, \\
\left(hL_2^{\top k}h^{-1}\right)_{n,n} &= \frac{p_k(-\tilde{\partial}_s)\tau_{n+1}\circ\tau_n}{\tau_{n+1}\tau_n} = -\frac{\partial}{\partial s_k}\log\frac{\tau_{n+1}}{\tau_n}\,,
\end{aligned} \tag{2.12}
$$

and

$$
\begin{aligned}
\left(L_1^k\right)_{n,n+1} &= \frac{p_{k-1}(\tilde{\partial}_t)\tau_{n+2}\circ\tau_n}{\tau_{n+2}\tau_n} = \frac{\frac{\partial^2 \log \tau_{n+1}}{\partial s_1 \partial t_k}}{\frac{\partial^2 \log \tau_{n+1}}{\partial s_1 \partial t_1}}\,, \\
\left(hL_2^{\top k}h^{-1}\right)_{n,n+1} &= \frac{p_{k-1}(-\tilde{\partial}_s)\tau_{n+2}\circ\tau_n}{\tau_{n+2}\tau_n} = \frac{\frac{\partial^2 \log \tau_{n+1}}{\partial t_1 \partial s_k}}{\frac{\partial^2 \log \tau_{n+1}}{\partial s_1 \partial t_1}}\,.
\end{aligned} \tag{2.13}
$$

PROOF: Relations (2.12) follow from (2.7) and a standard argument; see [4, theorem 0.1, formula (0.15)].

---

[8] $p_\ell(\tilde{\partial}) f \circ g$ refers to the Hirota operation, defined before. Here the p_ℓ are the elementary Schur polynomials $e^{\sum_1^\infty t_i z^i} := \sum_{i\geq 0} p_i(t) z^i$. Also $p_\ell(\tilde{\partial}_t) := p_\ell\big(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \dots\big)$ and $p_\ell(-\tilde{\partial}_s) = p_\ell\big(-\frac{\partial}{\partial s_1}, -\frac{1}{2}\frac{\partial}{\partial s_2}, -\frac{1}{3}\frac{\partial}{\partial s_3}, \dots\big)$.

To prove (2.13), set $m = n+1$ and all b_k and $a_k = 0$ except for one a_{j+1} in the Hirota bilinear relation (2.2). The first nonzero term in the sum on the left-hand side of that relation, which is also the only one containing a_{j+1} linearly, reads

$$
\begin{aligned}
(2.14)\quad & p_{j+1}(-2a)\,p_j(\tilde{\partial}_t)e^{a_{j+1}\frac{\partial}{\partial t_{j+1}}}\tau_{n+2}\circ\tau_n + \cdots = \\
& \qquad -2a_{j+1}p_j(\tilde{\partial}_t)\tau_{n+2}\circ\tau_n + \mathrm{O}\big(a_{j+1}^2\big)\,,
\end{aligned}
$$

whereas the right-hand side equals

$$
\begin{aligned}
(2.15)\quad & p_0(0)\,p_1(\tilde{\partial}_s)e^{a_{j+1}\frac{\partial}{\partial t_{j+1}}}\tau_{n+1}\circ\tau_{n+1} = \\
& \qquad \frac{\partial}{\partial s_1}\left(1 + a_{j+1}\frac{\partial}{\partial t_{j+1}} + \cdots\right)\tau_{n+1}\circ\tau_{n+1}\,.
\end{aligned}
$$

Comparing the coefficients of a_{j+1} in (2.14) and (2.15) yields

$$
-2p_j(\tilde{\partial}_t)\tau_{n+2}\circ\tau_n = \frac{\partial^2}{\partial s_1\partial t_{j+1}}\tau_{n+1}\circ\tau_{n+1}\,.
$$

In particular, we have

$$
\frac{p_{k-1}(\tilde{\partial}_t)\tau_{n+2}\circ\tau_n}{\tau_{n+1}^2} = -\frac{\partial^2}{\partial s_1\partial t_k}\log\tau_{n+1}\,, \tag{2.16}
$$

and so, for $k = 1$,

$$
\frac{\tau_n\tau_{n+2}}{\tau_{n+1}^2} = -\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_{n+1}\,. \tag{2.17}
$$

Dividing (2.16) and (2.17) leads to the first equality in (2.13), since according to (2.9), the $(n, n+1)$-entry of L_1^k is precisely given by (2.16). The similar result for L_2^k is given by the involution

$$
t \longleftrightarrow -s \quad \text{and} \quad L_1 \longleftrightarrow hL_2^{\top}h^{-1}\,.
$$

□

LEMMA 2.3 *The first upper subdiagonal of L_1^2 and $hL_2^{\top 2}h^{-1}$ reads*

$$
\begin{aligned}
\big(L_1^2\big)_{n,n+1} &= \frac{\partial}{\partial t_1}\log\frac{\tau_{n+2}}{\tau_n} = \frac{\frac{\partial^2}{\partial s_1\partial t_2}\log\tau_{n+1}}{\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_{n+1}} \\
&= \frac{\partial}{\partial t_1}\log\left(\left(\frac{\tau_{n+1}}{\tau_n}\right)^2\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_{n+1}\right), \\
\big(hL_2^{\top 2}h^{-1}\big)_{n,n+1} &= -\frac{\partial}{\partial s_1}\log\frac{\tau_{n+2}}{\tau_n} = \frac{\frac{\partial^2}{\partial t_1\partial s_2}\log\tau_{n+1}}{\frac{\partial^2}{\partial t_1\partial s_1}\log\tau_{n+1}} \\
&= -\frac{\partial}{\partial s_1}\log\left(\left(\frac{\tau_{n+1}}{\tau_n}\right)^2\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_{n+1}\right).
\end{aligned} \tag{2.18}
$$

PROOF: From Proposition 2.2 ($k = 2$), we have the first two identities in (2.18); it also follows from these identities that

$$\begin{aligned}
&\frac{\partial^2 \log \tau_{n+1}}{\partial s_1 \partial t_2} \\
&\quad = \frac{\partial^2 \log \tau_{n+1}}{\partial s_1 \partial t_1} \frac{\partial}{\partial t_1} \log \frac{\tau_{n+2}}{\tau_n} \\
&\quad = \frac{\partial^2 \log \tau_{n+1}}{\partial s_1 \partial t_1} \left(\frac{\partial}{\partial t_1} \log \frac{\tau_{n+2}}{\tau_{n+1}} + \frac{\partial}{\partial t_1} \log \frac{\tau_{n+1}}{\tau_n} \right) \\
&\quad = \frac{\partial^2 \log \tau_{n+1}}{\partial s_1 \partial t_1} \\
&\qquad \cdot \left(\frac{\partial}{\partial t_1} \log \left(-\frac{\tau_{n+1}}{\tau_n} \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_{n+1} \right) + \frac{\partial}{\partial t_1} \log \frac{\tau_{n+1}}{\tau_n} \right) \quad \text{using (3.17)} \\
&\quad = \frac{\partial^2 \log \tau_{n+1}}{\partial s_1 \partial t_1} \left(2 \frac{\partial}{\partial t_1} \log \frac{\tau_{n+1}}{\tau_n} + \frac{\partial}{\partial t_1} \log \left(\frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_{n+1} \right) \right) \\
&\quad = 2 \frac{\partial}{\partial t_1} \log \frac{\tau_{n+1}}{\tau_n} \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_{n+1} + \frac{\partial}{\partial t_1} \left(\frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_{n+1} \right),
\end{aligned}$$

which establishes the first equation (2.18). The second equation (2.18) is simply the dual of the first one by $t_i \mapsto -s_i$. □

PROOF OF THEOREM 2.1: The first statement concerning the KP hierarchy is standard. The proof of the second identity follows immediately from (2.16) for $k = 1$, and the third identity from the last identity in the proof of Lemma 2.3 and the duality. □

A prominent example of the semi-infinite 2-Toda lattice is given by an (arbitrary) (t, s)-dependent semi-infinite matrix

$$m_\infty(t, s) = (\mu_{ij}(t, s))_{0 \le i, j < \infty} \quad \text{with } m_n(t, s) = (\mu_{ij}(t, s))_{0 \le i, j \le n-1}\,, \tag{2.19}$$

evolving according to the equations

$$\frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty \quad \text{and} \quad \frac{\partial m_\infty}{\partial s_k} = -m_\infty (\Lambda^\top)^k\,. \tag{2.20}$$

According to [2], the formal solution to this problem is given by

$$m_\infty(t, s) = e^{\sum_1^\infty t_i \Lambda^i} m_\infty(0, 0) e^{-\sum_1^\infty s_i \Lambda^{\top i}} = S_1^{-1}(t, s) S_2(t, s)\,, \tag{2.21}$$

where the associated unique factorization into lower- times upper-triangular matrices actually lead to the wave matrices S_1 and S_2, as defined in the general Toda theory (2.4). The expression (2.21) contains the matrix of Schur polynomials[9]

$$e^{\sum_1^\infty t_i \Lambda^i} = \sum_0^\infty \Lambda^i p_i(t) = \big(p_{j-i}(t)\big)_{\substack{1\le i<\infty \\ 1\le j<\infty}} ,$$

of which a truncated version is given by the following $n \times \infty$ submatrix $p_i(t)$:

$$\begin{aligned} E_n(t) &= \left(\begin{array}{ccccc|cc} 1 & p_1(t) & p_2(t) & \dots & p_{n-1}(t) & p_n(t) & \dots \\ 0 & 1 & p_1(t) & \dots & p_{n-2}(t) & p_{n-1}(t) & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & \dots & p_1(t) & p_2(t) & \dots \\ 0 & 0 & 0 & \dots & 1 & p_1(t) & \dots \end{array}\right) \\ &= \big(p_{j-i}(t)\big)_{\substack{1\le i\le n \\ 1\le j<\infty}} . \end{aligned} \tag{2.22}$$

So, for a semi-infinite initial condition $m_\infty(0,0)$, the τ-functions of the 2-Toda problem are given by

$$\tau_n(t,s) := \det m_n(t,s) = \det\big(E_n(t)\, m_\infty(0,0)\, E_n^\top(-s)\big) . \tag{2.23}$$

Incidentally, the wave vectors Ψ_1 and Ψ_2^* define monic polynomials $p^{(1)}(x)$ and $p^{(2)}(y)$,

$$\begin{aligned} \Psi_1 &:= e^{\sum t_k z^k} p^{(1)}(z) = e^{\sum t_k z^k} S_1 \chi(z) , \\ \Psi_2^* &:= e^{-\sum s_k z^{-k}} h^{-1} p^{(2)}(z^{-1}) = e^{-\sum s_k z^{-k}} (S_2^{-1})^\top \chi(z^{-1}) , \end{aligned} \tag{2.24}$$

which are *bi-orthogonal* with regard to the original matrix m_∞; that is, for all t, s,

$$\langle p_n^{(1)}, p_m^{(2)} \rangle = \delta_{n,m} h_n \quad \text{for the inner product defined by } \langle x^i, y^j \rangle := \mu_{ij} \tag{2.25}$$

with h_n as in (2.8).

PROPOSITION 2.4 *Given the semi-infinite initial condition $m_\infty(0,0)$, the 2-Toda τ-function has the following expansion in Schur polynomials,*[10]

$$\tau_n(t,s) = \sum_{\substack{\lambda,\nu \\ \hat\lambda_1, \hat\nu_1 \le n}} \det(m^{\lambda,\nu}) s_\lambda(t) s_\nu(-s) \quad \textit{for } n > 0 , \tag{2.26}$$

where the sum is taken over all Young diagrams λ and ν, with first columns $\hat\lambda_1$ and $\hat\nu_1 \le n$ and where

$$m^{\lambda,\nu} := (\mu_{\lambda_i - i + n, \nu_j - j + n})_{1\le i,j\le n} . \tag{2.27}$$

---

[9]The Schur polynomials p_i, defined by $e^{\sum_1^\infty t_i z^i} = \sum_0^\infty p_k(t) z^k$ and $p_k(t) = 0$ for $k < 0$ are not to be confused with the bi-orthogonal polynomials $p_i^{(k)}$, $k = 1, 2$.

[10]For a given Young diagram $\lambda_1 \ge \dots \ge \lambda_n$, define $s_\lambda(t) = \det(p_{\lambda_i - i + j}(t))_{1\le i,j\le n}$.

PROOF: Note that every increasing sequence $1 \leq k_1 < \cdots < k_n < \infty$ can be mapped into a Young diagram $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ by setting $k_j = j + \lambda_{n+1-j}$. Relabeling the indices i, j with $1 \leq i, j \leq n$, by setting $j' := n - j + 1$, $i' := n - i + 1$, we have $1 \leq i', j' \leq n$, $k_j - i = \lambda_{j'} - j' + i'$, and $k_i - 1 = \lambda_{i'} - i' + n$. The same can be done for the sequence $1 \leq \ell_1 < \cdots < \ell_n < \infty$, leading to the Young diagram ν, using the same relabeling. Applying the Cauchy-Binet formula twice, expression (2.23) leads to

$$\begin{aligned}
&\tau_n(t,s)\\
&\quad = \det\left(E_n(t) m_\infty(0,0) E_n^\top(-s)\right)\\
&\quad = \sum_{1\leq k_1<\cdots<k_n<\infty} \det\left(p_{k_j-i}(t)\right)_{1\leq i,j\leq n} \det\left((m_\infty(0,0)E_n^\top(-s))_{k_i,\ell}\right)_{1\leq i,\ell\leq n}\\
&\quad = \sum_{1\leq k_1<\cdots<k_n<\infty} \det\left(p_{k_j-i}(t)\right)_{1\leq i,j\leq n} \det\left((\mu_{k_i-1,j-1})_{\substack{1\leq i\leq n\\ 1\leq j<\infty}} (p_{i-j}(-s))_{\substack{1\leq i<\infty\\ 1\leq j\leq n}}\right)\\
&\quad = \sum_{1\leq k_1<\cdots<k_n<\infty} \det\left(p_{k_j-i}(t)\right)_{1\leq i,j\leq n}\\
&\qquad\quad \sum_{1\leq \ell_1<\cdots<\ell_n<\infty} \det\left(\mu_{k_i-1,\ell_j-1}\right)_{1\leq i,j\leq n} \det\left(p_{\ell_i-j}(-s)\right)_{1\leq i,j\leq n}\\
&\quad = \sum_{\substack{\lambda\\ \hat\lambda_1\leq n}} \det\left(p_{\lambda_{j'}-j'+i'}(t)\right)_{1\leq i',j'\leq n}\\
&\qquad\quad \sum_{\substack{\nu\\ \hat\nu_1\leq n}} \det\left(\mu_{\lambda_{i'}-i'+n,\nu_{j'}-j'+n}\right)_{1\leq i',j'\leq n} \det\left(p_{\nu_{i'}-i'+j'}(-s)\right)_{1\leq i',j'\leq n}\\
&\quad = \sum_{\substack{\lambda,\nu\\ \hat\lambda_1,\hat\nu_1\leq n}} \det(\mu^{\lambda,\nu}) s_\lambda(t) s_\nu(-s)\,.
\end{aligned}$$

$\square$

2.2 Reduction to Hänkel Matrices: The Standard Toda Lattice and a Virasoro Algebra of Constraints

In the notation of (2.7), consider the locus of (L_1, L_2)'s such that $L_1 = L_2$. This means the matrix $L_1 = L_2$ is tridiagonal. From equations (2.7), it follows that along that locus,

$$\frac{\partial(L_1 - L_2)}{\partial t_n} = 0\,, \qquad \frac{\partial(L_1 - L_2)}{\partial s_n} = 0\,.$$

We now define new variables t_n' and s_n' by

$$t_n' = t_n - s_n \quad \text{and} \quad s_n' = t_n + s_n\,, \tag{2.28}$$

and thus

$$\frac{\partial}{\partial t'_n}=\frac{1}{2}\left(\frac{\partial}{\partial t_n}-\frac{\partial}{\partial s_n}\right),\quad \frac{\partial}{\partial s'_n}=\frac{1}{2}\left(\frac{\partial}{\partial t_n}+\frac{\partial}{\partial s_n}\right). \tag{2.29}$$

Then

$$\frac{\partial L_i}{\partial s'_n}=\frac{1}{2}\left(\frac{\partial}{\partial t_n}+\frac{\partial}{\partial s_n}\right)L_i=[(L_1^n)_++(L_2^n)_-,L_i]=[L_i^n,L_i]=0 \tag{2.30}$$

and

$$\begin{aligned}\frac{\partial L_i}{\partial t'_n}=\frac{1}{2}\left(\frac{\partial}{\partial t_n}-\frac{\partial}{\partial s_n}\right)L_i&=\frac{1}{2}[(L_1^n)_+-(L_2^n)_-+L_i^n,L_i]\\ &=[(L_1^n)_+,L_i]\quad\text{using } L_1=L_2\,.\end{aligned} \tag{2.31}$$

So, equation (2.30) implies that $L_1=L_2$ is independent of s'. Since $\tau(t,s)$ is a function of $t-s$ only, we may set $\tau(t'):=\tau(t-s)$. After noting (see (2.12))

$$\frac{\partial}{\partial t'_k}\log h_n=\left(L_1^k\right)_{n,n}\ ,$$

this situation leads to the standard Toda lattice equations on symmetric and tridiagonal matrices $L=h^{-1/2}L_1h^{1/2}$ and wave vectors (expressed in terms of the 2-Toda wave vectors (2.4))

$$\begin{aligned}\Psi(t',z):=h^{-1/2}\Psi_1(t,s;z)e^{-\frac{1}{2}\sum_1^\infty(t_i+s_i)z^i}&=h^{1/2}\Psi_2^*(t,s;z^{-1})e^{\frac{1}{2}\sum_1^\infty(t_i+s_i)z^i}\\ &=e^{\frac{1}{2}\Sigma t'_iz^i}\left(z^n\frac{\tau_n(t'-[z^{-1}])}{\sqrt{\tau_n(t')\tau_{n+1}(t')}}\right)_{n\geq 0}\ ,\end{aligned} \tag{2.32}$$

namely,

$$L\Psi=z\Psi\ ,\quad \frac{\partial\Psi}{\partial t'_n}=\frac{1}{2}(L^n)_{sk}\Psi\ ,\quad\text{and}\quad \frac{\partial L}{\partial t'_n}=\frac{1}{2}\big[(L^n)_{sk},L\big]\,, \tag{2.33}$$

(standard Toda lattice)

where $()_{\rm sk}$ refers to the skew part in the skew and lower-triangular Lie decomposition.

We now define the standard Toda vertex operator as the reduction of the 2-Toda vertex operator $\mathbb{X}_{12}$, defined in (2.10), using (2.28) and (2.29):

$$\mathbb{X}(t';z):=\mathbb{X}_{12}(t,s;z,z)\Big|_{\substack{t=(s'+t')/2\\ s=(s'-t')/2}}=\Lambda^\top\chi(z^2)e^{\sum_1^\infty t'_iz^i}e^{-2\sum_1^\infty\frac{z^{-i}}{i}\frac{\partial}{\partial t'_i}}\,. \tag{2.34}$$

In the rest of this section, we shall omit $'$ in t' and s'. This vertex operator $\mathbb{X}(t;z)$ generates a Virasoro algebra with $\beta=2$ and thus central charge $c=1$ (see Appendix A and (A.5))

$$\frac{d}{du}u^{k+1}\mathbb{X}(t,u)=\left[{}^\beta\mathbb{J}_k^{(2)}(t)\Big|_{\beta=2},\mathbb{X}(t;u)\right],\quad k\in\mathbb{Z}\,. \tag{2.35}$$

An interesting *semi-infinite example* of the standard Toda lattice is obtained by considering a weight $\rho(z)dz = e^{-V(z)}\,dz$ defined on an interval $F \subset \mathbb{R}$, satisfying (0.4) and an inner product

$$\langle f, g\rangle_t := \int_{\mathbb{R}} f(z)g(z)e^{\sum_1^\infty t_i z^i}\rho(z)dz\,, \tag{2.36}$$

leading to a t-dependent moment matrix

$$m_\infty(t) = \big(\mu_{ij}(t)\big)_{0\le i,j\le\infty} = \big(\langle y^i, y^j\rangle_t\big)_{0\le i,j<\infty} \quad \text{(Hänkel matrix)}\,. \tag{2.37}$$

THEOREM 2.5 (Adler-van Moerbeke [1]) *The vector $\tau(t) = (\tau_0 = 1, \tau_1(t), \dots)$ of integrals*

$$\begin{aligned}\tau_n(t) := \frac{1}{n!}\int_{\mathcal{H}_n} e^{\operatorname{tr}(-V(M)+\sum_1^\infty t_i M^i)}\,dM &= \frac{1}{n!}\int_{F^n}\Delta_n(z)^2\prod_{k=1}^n e^{\sum_{i=1}^\infty t_i z_k^i}\rho(z_k)dz_k \\ &= \det\big(\mu_{ij}(t)\big)_{0\le i,j\le n-1}\end{aligned} \tag{2.38}$$

is a set of τ-functions for the standard Toda lattice. Also, each $\tau_n(t)$ satisfies the KP hierarchy, of which the first equation is given in Theorem 2.1. *It also satisfies* (i) *of Theorem* 0.2 *with $t_1 = x$. Moreover, the τ_n's satisfy the following Virasoro constraints for $\beta = 2$ (see* (2.35)*):*

$$\begin{aligned} 0 &= \mathcal{J}_m^{(2)}\tau(t)\,, \quad m \ge -1\,, \\ &:= \sum_{k\ge 0}\Bigg(-a_k\sum_{i+j=k+m} :{}^\beta\mathbb{J}_i^{(1)}\,{}^\beta\mathbb{J}_j^{(1)}: + b_k\,{}^\beta\mathbb{J}_{k+m+1}^{(1)}\Bigg)\Bigg|_{\beta=2}\tau(t) \\ &= \Bigg(\sum_{k\ge 0}\Big(-a_k\big({}^\beta J_{k+m}^{(2)} + 2n\,{}^\beta J_{k+m}^{(1)} + n^2 J_{k+m}^{(0)}\big) \\ &\qquad + b_k\big({}^\beta J_{k+m+1}^{(1)} + n\delta_{k+m+1,0}\big)\Big)\Big|_{\beta=2}\tau_n(t)\Bigg)_{n\ge 0}\end{aligned} \tag{2.39}$$

where the a_k and b_k are the coefficients (0.4) *of the rational function ρ'/ρ, and where the ${}^\beta\mathbb{J}_k^{(i)}$ and ${}^\beta J_k^{(i)}$ for $\beta = 2$ are given by* (A.1), (A.3), *and* (A.5)*. The relation between the vertex operator $\mathbb{X}(t,u)$ and $\mathcal{J}_m^{(2)}$ is given by*

$$\frac{\partial}{\partial u}u^{m+1}f(u)\mathbb{X}(t,u)\rho(u) = \big[-\mathcal{J}_m^{(2)}, \mathbb{X}(t,u)\rho(u)\big]\,. \tag{2.40}$$

SKETCH OF PROOF: Formula (2.40) is a direct consequence of (2.35), while (2.39) hinges on the fact that the vector $I = (\tau_0, \tau_1, \dots, n!\tau_n, \dots)$ of Toda lattice τ-functions is a fixed point for a certain integrated vertex operator in the following sense:

$$(\mathcal{Y}I)_n := I_n \quad \text{for } n \ge 1 \text{ where } \mathcal{Y} := \int_F du\,\rho(u)\mathbb{X}(t,u)\,,$$

which is just an iterated integral formula. Upon integrating (2.40) on the full range F, deduce $[\mathcal{J}_m^{(2)}, \mathcal{Y}] = 0$ from the boundary conditions (0.4). Acting on I with this relation, one deduces (2.39) by induction on n and the fact that $\tau_0 = 1$. □

2.3 Reduction to Toeplitz Matrices: 2-Toda Lattice and an SL(2, $\mathbb{Z}$)-Algebra of Constraints

Consider the following inner product, depending on (t, s):

$$\langle f(z), g(z)\rangle_{t,s} := \oint_{S^1} \frac{\rho(dz)}{2\pi i z} f(z) g(z^{-1}) e^{\sum_1^\infty (t_i z^i - s_i z^{-i})}, \tag{2.41}$$

where the integral is taken over the unit circle S^1 around the origin in the complex plane $\mathbb{C}$. Instead of having $z^{k\top} = z^k$ in the Hänkel inner product, we have $z^{k\top} = z^{-k}$ in this inner product:

$$\langle z^k f(z), g(z)\rangle_{t,s} = \langle f(z), z^{-k} g(z)\rangle_{t,s} . \tag{2.42}$$

Thus the moment matrix m_∞, with entries

$$\mu_{k\ell}(t,s) = \langle z^k, z^\ell\rangle_{t,s} = \oint_{S^1} \frac{\rho(dz)}{2\pi i z} z^{k-\ell} e^{\sum_1^\infty (t_i z^i - s_i z^{-i})}, \tag{2.43}$$

is a Toeplitz matrix for all (t, s), satisfying the differential equations (2.20) of the 2-Toda lattice, i.e.,

$$\frac{\partial \mu_{k\ell}}{\partial t_i} = \mu_{k+i,\ell} \quad \text{and} \quad \frac{\partial \mu_{k\ell}}{\partial s_i} = -\mu_{k,\ell+i} .$$

THEOREM 2.6 *For $\rho(dz) = dz$, the vector $\tau(t,s) = (\tau_0 = 1, \tau_1(t,s), \dots)$ with*

$$\begin{aligned} \tau_n(t,s) &= \int_{\mathrm{U}(n)} e^{\sum_1^\infty \mathrm{Tr}(t_i M^i - s_i \bar{M}^i)}\, dM \\ &= \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k} \right) \\ &= \det\left(\mu_{k\ell}(t,s)\right)_{0\le k,\ell\le n-1} \\ &= \sum_{\{\text{Young diagrams } \lambda \text{ with first column } \le n\}} s_\lambda(t) s_\lambda(-s) \end{aligned} \tag{2.44}$$

is a vector of τ-functions for the 2*-Toda lattice. Hence, they satisfy the identity* (2.3) *and* (ii) *of Theorem* 0.2 *with $t_1 = x$, $s_1 = y$. Moreover, they are annihilated by the following algebra of three Virasoro partial differential operators, which form a*

SL$(2,\mathbb{Z})$-*algebra* (*note that the expression below is a semi-infinite vector*):

$$0=\mathcal{V}_k^{(2)}\tau(t,s)\quad \textit{for}\ \left\{\begin{array}{l} k=-1\,,\ \theta=0\,,\\ k=0\,,\ \theta\ \textit{arbitrary}\,,\\ k=1\,,\ \theta=1\,,\end{array}\right\}\ \textit{only}$$

(2.45)
$$=\Big({}^{\beta}\mathbb{J}_k^{(2)}(t)-{}^{\beta}\mathbb{J}_{-k}^{(2)}(-s)-k\big(\theta\,{}^{\beta}\mathbb{J}_k^{(1)}(t)+(1-\theta)\,{}^{\beta}\mathbb{J}_{-k}^{(1)}(-s)\big)\Big)\Big|_{\beta=1}\tau(t,s)\,,$$

where θ *is an arbitrary parameter and* ${}^{\beta}\mathbb{J}_k^{(2)}$ *for* $\beta=1$ *is given in* (A.4)*. This is a subalgebra of a Virasoro algebra* (*generated by the 2-Toda vertex operator* (2.10)):

$$\frac{d}{du}\,u^{k+1}\frac{\mathbb{X}_{12}(t,s;u,u^{-1})}{u}=\left[\mathcal{V}_k^{(2)}(t,s),\,\frac{\mathbb{X}_{12}(t,s;u,u^{-1})}{u}\right],\quad k\in\mathbb{Z}\,,\tag{2.46}$$

of central charge $c=0$:

$$\left[\mathcal{V}_k^{(2)},\,\mathcal{V}_\ell^{(2)}\right]=(k-\ell)\mathcal{V}_{k+\ell}^{(2)}\,.\tag{2.47}$$

The statement hinges on the following statement about the vertex operator:

PROPOSITION 2.7 *For general weight* $\rho(dz)=e^{V(z)}\,dz$*, the column vector of* 2-*Toda* τ*-functions* (*slightly rescaled*), $\tau(t,s)=(\tau_0,\tau_1,\dots)$*, with*

$$I_n(t,s)=n!\,\tau_n=n!\int_{\mathrm{U}(n)}e^{\sum_1^\infty \mathrm{Tr}(t_iM^i-s_i\bar M^i)}e^{\mathrm{Tr}\,V(M)}\,dM\tag{2.48}$$

is a fixed point in the sense that

$$(\mathcal{Y}(t,s;\rho)I)_n=I_n\quad \textit{for } n\ge 1\tag{2.49}$$

for the operator

$$\mathcal{Y}(t,s;\rho)=\int_{S^1}\frac{\rho(du)}{2\pi iu}\mathbb{X}_{12}(t,s;u,u^{-1})\,.\tag{2.50}$$

PROOF OF PROPOSITION 2.7 AND THEOREM 2.6: On the one hand, using the fact that $\bar z=1/z$ on the circle S^1 and a property of Vandermonde determinants[11] and Theorem 1.1, we have, for a general weight $\rho(dz)=e^{V(z)}\,dz$,

$$\begin{aligned} n!\,\tau_n(t,s)&=n!\int_{\mathrm{U}(n)}e^{\sum_1^\infty \mathrm{Tr}(t_iM^i-s_i\bar M^i)}e^{\mathrm{Tr}\,V(M)}\,dM\\ &=\int_{(S^1)^n}|\Delta_n(z)|^2\prod_{k=1}^n\left(e^{\sum_1^\infty(t_iz_k^i-s_iz_k^{-i})}\frac{\rho(dz_k)}{2\pi iz_k}\right)\end{aligned}$$

[11] The following holds:

$$\det(u_k^{\ell-1})_{1\le\ell,k\le N}\det(v_k^{\ell-1})_{1\le\ell,k\le N}=\sum_{\sigma\in S_N}\det\big(u_{\sigma(k)}^{\ell-1}v_{\sigma(k)}^{k-1}\big)_{1\le\ell,k\le N}\,.$$

$$
\begin{aligned}
&= \int_{(S^1)^n} \Delta_n(z)\Delta_n(\bar{z}) \prod_{k=1}^n \left(e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{\rho(dz_k)}{2\pi i z_k} \right) \\
&= \int_{(S^1)^n} \sum_{\sigma \in S_n} \det \left(z_{\sigma(m)}^{\ell-1} \bar{z}_{\sigma(m)}^{m-1} \right)_{1 \le \ell, m \le n} \prod_{k=1}^n \left(e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{\rho(dz_k)}{2\pi i z_k} \right) \\
&= \sum_{\sigma \in S_n} \det \left(\int_{S^1} z^{\ell-1} \bar{z}^{m-1} e^{\sum_1^\infty (t_i z^i - s_i z^{-i})} \frac{\rho(dz)}{2\pi i z} \right)_{1 \le \ell, m \le n} \\
&= n! \det \left(\int_{S^1} z^{\ell-m} e^{\sum_1^\infty (t_i z^i - s_i z^{-i})} \frac{\rho(dz)}{2\pi i z} \right)_{1 \le \ell, m \le n} \\
&= n! \det \left(\mu_{\ell m}(t,s) \right)_{0 \le \ell, m \le n-1}, \qquad (2.51)
\end{aligned}
$$

yielding the third equality of (2.44). On the other hand, for $n \geq 1$, we have

$$
\begin{aligned}
I_n(t,s) &= n!\, \tau_n(t,s) \\
&= \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{\rho(dz_k)}{2\pi i z_k} \right) \\
&= \int_{S^1} \frac{\rho(du)}{2\pi i u} e^{\sum_1^\infty (t_i u^i - s_i u^{-i})} u^{n-1} u^{-n+1} \\
&\qquad \int_{(S^1)^{n-1}} \Delta_{n-1}(z) \bar{\Delta}_{n-1}(z) \prod_{k=1}^{n-1} \left(1 - \frac{z_k}{u}\right)\left(1 - \frac{u}{z_k}\right) e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{\rho(dz_k)}{2\pi i z_k} \\
&= \int_{S^1} \frac{\rho(du)}{2\pi i u} e^{\sum_1^\infty (t_i u^i - s_i u^{-i})} e^{-\sum_1^\infty \left(\frac{u^{-i}}{i} \frac{\partial}{\partial t_i} - \frac{u^i}{i} \frac{\partial}{\partial s_i} \right)} \\
&\qquad \int_{(S^1)^{n-1}} \Delta_{n-1}(z) \bar{\Delta}_{n-1}(z) \prod_{k=1}^{n-1} e^{\sum_1^\infty (t_i z_k^i - s_i z_k^{-i})} \frac{\rho(dz_k)}{2\pi i z_k} \\
&= \int_{S^1} \frac{\rho(du)}{2\pi i u} e^{\sum_1^\infty (t_i u^i - s_i u^{-i})} e^{-\sum_1^\infty \left(\frac{u^{-i}}{i} \frac{\partial}{\partial t_i} - \frac{u^i}{i} \frac{\partial}{\partial s_i} \right)} I_{n-1}(t,s) \\
&= \left(\mathcal{Y}(t,s;\rho) I(t,s) \right)_n, \qquad (2.52)
\end{aligned}
$$

from which (2.46) follows.

Given the vertex operator $\mathbb{X}_{12}(t,s;u,u^{-1})$, we now compute the corresponding Virasoro algebra using (2.11):

$$
\begin{aligned}
& u\frac{d}{du}u^k\mathbb{X}_{12}\big(t,s;u,u^{-1}\big) \\
&\quad = \left(u^{k+1}\frac{d}{du}+ku^k\right)\mathbb{X}_{12}\big(t,s;u,u^{-1}\big) \\
&\quad = \left(u^{k+1}\frac{\partial}{\partial u}-v^{1-k}\frac{\partial}{\partial v}+ku^k\right)\mathbb{X}_{12}(t,s;u,v)\Big|_{v=u^{-1}} \\
&\quad = \left(\frac{\partial}{\partial u}u^{k+1}-\frac{\partial}{\partial v}v^{1-k}-ku^k\right)\mathbb{X}_{12}(t,s;u,v)\Big|_{v=u^{-1}} \\
&\quad = \left(\frac{\partial}{\partial u}u^{k+1}-\frac{\partial}{\partial v}v^{1-k}-k\theta u^k-k(1-\theta)v^{-k}\right)\mathbb{X}_{12}(t,s;u,v)\Big|_{v=u^{-1}} \\
&\quad = \Big[{}^{\beta}\mathbb{J}_k^{(2)}(t)-{}^{\beta}\mathbb{J}_{-k}^{(2)}(-s) \\
&\qquad\quad -k\Big(\theta\,{}^{\beta}\mathbb{J}_k^{(1)}(t)+(1-\theta)\,{}^{\beta}\mathbb{J}_{-k}^{(1)}(-s)\Big)\mathbb{X}_{12}\big(t,s;u,u^{-1}\big)\Big]\Big|_{\beta=1} \\
&\quad = \Big[\mathcal{V}_k^{(2)},\mathbb{X}_{12}\big(t,s;u,u^{-1}\big)\Big],
\end{aligned}
\tag{2.53}
$$

from which (2.46) follows. Verifying (2.47) goes by explicit computation using (A.2).

Since (by virtue of (2.46) for Lebesgue measure on S^1,

$$
\begin{aligned}
\Big[\mathcal{V}_k^{(2)},\mathcal{Y}(t,s,\rho=1)\Big] &= \left[\mathcal{V}_k^{(2)},\int_{S^1}\mathbb{X}_{12}\big(t,s;u,u^{-1}\big)\frac{du}{2\pi iu}\right] \\
&= \int_{S^1}\left[\mathcal{V}_k^{(2)},\mathbb{X}_{12}\big(t,s;u,u^{-1}\big)\frac{du}{2\pi iu}\right] \\
&= \int_{S^1}\frac{du}{2\pi i}\,\frac{d}{du}u^{k+1}\frac{\mathbb{X}_{12}(t,s;u,u^{-1})}{u} \\
&= 0\,,
\end{aligned}
$$

we have, using the notation (2.50) and the fact that, for $n\geq 0$ and $\rho(dz)=dz$, the integrals $I_n=n!\tau_n(t,s)$ are fixed points for $\mathcal{Y}(t,s,dz)$; hence

$$
\begin{aligned}
0 &= \Big([\mathcal{V}_k^{(2)},(\mathcal{Y}(t,s;dz))^n]I\Big)_n \\
&= \Big(\mathcal{V}_k^{(2)}\mathcal{Y}(t,s;dz)^nI-\mathcal{Y}(t,s;dz)^n\mathcal{V}_k^{(2)}I\Big)_n \\
&= \Big(\mathcal{V}_k^{(2)}I-\mathcal{Y}(t,s;dz)^n\mathcal{V}_k^{(2)}I\Big)_n\ .
\end{aligned}
$$

Taking the n^{th} component and taking into account the presence of Λ^{-1} in $\mathbb{X}_{12}(t,s;u,u^{-1})$, we find

$$\begin{aligned} 0 &= \left(\mathcal{V}_k^{(2)} I - \mathcal{Y}(t,s;dz)^n \mathcal{V}_k^{(2)} I\right)_n \\ &= \mathcal{V}_k^{(2)} I_n - \int_{S_1} \frac{du}{2\pi i u}\, e^{\sum_1^\infty (t_i u^i - s_i u^{-i})} e^{-\sum_1^\infty \left(\frac{u^{-i}}{i}\frac{\partial}{\partial t_i} - \frac{u^i}{i}\frac{\partial}{\partial s_i}\right)} \\ &\qquad \cdots \int_{S_1} \frac{du}{2\pi i u}\, e^{\sum_1^\infty (t_i u^i - s_i u^{-i})} e^{-\sum_1^\infty \left(\frac{u^{-i}}{i}\frac{\partial}{\partial t_i} - \frac{u^i}{i}\frac{\partial}{\partial s_i}\right)} \mathcal{V}_k^{(2)} I_0 \,. \end{aligned}$$

In the notation of (2.45) and (A.4), $\mathcal{V}_k^{(2)}(t,s)$ has the following form:

$$\begin{aligned} &\mathcal{V}_k^{(2)}(t,s) \\ &\quad = \frac{1}{2}\left(J_k^{(2)}(t) - J_{-k}^{(2)}(-s) + (2n+k+1) J_k^{(1)}(t) - (2n-k+1) J_{-k}^{(1)}(-s)\right) \\ &\qquad - k\left(\theta J_k^{(1)}(t) + (1-\theta) J_{-k}^{(1)}(-s)\right). \end{aligned}$$

Working out the expression above leads to the expression written out in (4.7), and one checks immediately that, given $\tau_0 = 1$,

$$\mathcal{V}_k^{(2)}(t,s)\tau_0 = 0 \quad \text{only for} \left\{ \begin{array}{l} k=-1\,, \theta = 0 \\ k = 0\,, \theta \text{ arbitrary} \\ k = 1\,, \theta = 1 \end{array} \right\},$$

ending the proof of Theorem 2.6 except for the last equality of (2.44), which follows easily from Proposition 2.4. Indeed,

$$m_\infty(0,0) = \left(\oint_{S^1} z^{k-\ell} \frac{dz}{2\pi i z}\right)_{1 \le k,\ell < \infty} = I_\infty\,,$$

where I_∞ denotes the semi-infinite identity matrix. Finally, one uses identity (2.27) and the fact that all the determinants of submatrices $(I_\infty)^{\lambda\mu}$ (in the notation of (2.27)) are zero except when the Young diagrams λ and μ are equal. $\square$

Remark. According to the strong Szegö theorem, we have

$$\tau_n = \exp\left(-\sum_1^\infty k t_k s_k\right) \quad \text{for } n \to \infty$$

provided $\sum_1^\infty k(|t_k|^2 + |s_k|^2) < \infty$. Therefore the Toeplitz case yields boundary conditions for τ_n at both extremities, namely $n = 0$ and $n = \infty$.

2.4 Toeplitz Matrices and the Structure of L_1 and L_2

The associated 2-Toda matrices L_1 and L_2 have a very peculiar structure when the initial m_∞ matrix is Toeplitz, as we shall see in the main theorem of this section. Throughout we shall be using the multiplication operator identity $z^\top = z^{-1}$ with regard to the inner product (2.41). This characterizes the Toeplitz case. Remember from Section 2.1 that the polynomials (combining (2.4) and (2.24))

$$p_n^{(1)}(z) = z^n \frac{\tau_n(t-[z^{-1}],s)}{\tau_n(t,s)} \quad \text{and} \quad p_n^{(2)}(z) = z^n \frac{\tau_n(t,s+[z^{-1}])}{\tau_n(t,s)}$$

are bi-orthogonal for the special inner product (2.41); also consider the vector notation

$$p^{(i)} = \left(p_0^{(i)}, p_1^{(i)}, \dots\right), \quad p_\Lambda^{(i)} = \left(p_1^{(i)}, p_2^{(i)}, \dots\right), \quad \text{and}$$
$$h = \operatorname{diag}\left(\frac{\tau_1}{\tau_0}, \frac{\tau_2}{\tau_1}, \dots\right).$$

THEOREM 2.8 *The lower-triangular parts*[12] *of the matrices L_1 and $hL_2^\top h^{-1}$, arising in the context of a Toeplitz matrix m_∞, are the projection of a rank 2 matrix*:

$$L_1 = -\left(hp_\Lambda^{(1)}(0) \otimes h^{-1}p^{(2)}(0)\right)_{-0} + \Lambda\,,$$
$$hL_2^\top h^{-1} = -\left(hp_\Lambda^{(2)}(0) \otimes h^{-1}p^{(1)}(0)\right)_{-0} + \Lambda\,.$$

COROLLARY 2.9 (Unsymmetric Identities) *In particular,*[13]

$$p_{n+1}^{(1)}(0)p_{n+1}^{(2)}(0) = 1 - \frac{h_{n+1}}{h_n}\,, \qquad p_{n+1}^{(2)}(0)p_{n+1}^{(1)}(0) = 1 - \frac{h_{n+1}}{h_n}\,,$$

$$p_{n+1}^{(1)}(0)p_n^{(2)}(0) = -\frac{\partial}{\partial t_1}\log h_n\,, \qquad p_{n+1}^{(2)}(0)p_n^{(1)}(0) = \frac{\partial}{\partial s_1}\log h_n\,,$$

$$p_{n+1}^{(1)}(0)p_{n-1}^{(2)}(0) = -\frac{h_{n-1}}{h_n}\left(\frac{\partial}{\partial t_1}\right)^2 \log\tau_n\,,$$
$$p_{n+1}^{(2)}(0)p_{n-1}^{(1)}(0) = -\frac{h_{n-1}}{h_n}\left(\frac{\partial}{\partial s_1}\right)^2 \log\tau_n\,,$$
$$p_{n+1}^{(1)}(0)p_{n-k}^{(2)}(0) = -\frac{h_{n-k}}{h_n}\frac{p_{k+1}(\tilde\partial_t)\tau_{n-k+1}\circ\tau_n}{\tau_{n-k+1}\tau_n}$$
$$p_{n+1}^{(2)}(0)p_{n-k}^{(1)}(0) = -\frac{h_{n-k}}{h_n}\frac{p_{k+1}(-\tilde\partial_s)\tau_{n-k+1}\circ\tau_n}{\tau_{n-k+1}\tau_n}\,, \quad k \geq 0\,. \tag{2.54}$$

[12] In the formulae below, A_{-0} denotes the lower-triangular part of A, including the diagonal.

[13] See footnote 8 for notation $p_k(\tilde\partial_t)$ and $p_k(-\tilde\partial_s)$. The bi-orthogonal polynomials $p_k^{(i)}$ should not be confused with the Schur polynomials p_k.

COROLLARY 2.10 (Symmetrized Identities) *We also have for $n > m$,*

$$\left(\frac{h_n}{h_{m+1}}\right)^2\left(1-\frac{h_{n+1}}{h_n}\right)\left(1-\frac{h_{m+1}}{h_m}\right)= \frac{1}{\tau_{m+2}^2\tau_n^2}\left(p_{n-m}(\tilde{\partial}_t)\tau_{m+2}\circ\tau_n\right)\cdot\left(p_{n-m}(-\tilde{\partial}_s)\tau_{m+2}\circ\tau_n\right).$$

In particular, for $m = n-1$,

$$\left(1-\frac{h_{n+1}}{h_n}\right)\left(1-\frac{h_n}{h_{n-1}}\right) = -\frac{\partial}{\partial t_1}\log h_n \frac{\partial}{\partial s_1}\log h_n\,. \tag{2.55}$$

Identity (2.55) was already observed by Hisakado in [13]. We first need a lemma that explains the peculiar structure of the bi-orthogonal polynomials $p_n^{(1)}(y)$ and $p_n^{(2)}(z)$ associated with the inner product (2.41).

LEMMA 2.11 (Hisakado [13]) *The following holds*:

$$\begin{aligned} p_{n+1}^{(1)}(z) - zp_n^{(1)}(z) &= p_{n+1}^{(1)}(0)z^n p_n^{(2)}\left(z^{-1}\right),\\ p_{n+1}^{(2)}(z) - zp_n^{(2)}(z) &= p_{n+1}^{(2)}(0)z^n p_n^{(1)}\left(z^{-1}\right). \end{aligned} \tag{2.56}$$

PROOF: The following orthogonality relations hold for $1 \le i \le n$:

$$\langle p_{n+1}^{(1)}(z) - zp_n^{(1)}(z), z^i\rangle = \langle p_{n+1}^{(1)}(z), z^i\rangle - \langle p_n^{(1)}(z), z^{i-1}\rangle = 0$$

and

$$\langle z^n p_n^{(2)}(z^{-1}), z^i\rangle = \langle z^{n-i}, p_n^{(2)}(z)\rangle = 0\,.$$

Therefore the two n^{th} degree polynomials $p_{n+1}^{(1)}(z) - zp_n^{(1)}(z)$ and $z^n p_n^{(2)}(z^{-1})$ must be proportional, and since

$$p_{n+1}^{(1)}(z) - zp_n^{(1)}(z)\Big|_{z=0} = p_{n+1}^{(1)}(0) \quad\text{and}\quad z^n p_n^{(2)}(z^{-1})\Big|_{z=0} = 1\,,$$

the first identity (2.56) follows. The second one follows by duality. □

PROOF OF THEOREM 2.8 AND COROLLARIES 2.9 AND 2.10: On one hand,[14]

$$\begin{aligned} &\left\langle p_{n+1}^{(1)}(z) - zp_n^{(1)}(z),\, p_{m+1}^{(2)}(z) - zp_m^{(2)}(z)\right\rangle, \quad n > m \ge -1\,,\\ &= -\left\langle zp_n^{(1)}(z),\, p_{m+1}^{(2)}(z)\right\rangle\\ &= -\left\langle p_{n+1}^{(1)}(z) + \cdots + (L_1)_{n,m+1}p_{m+1}^{(1)}(z) + \cdots,\; p_{m+1}^{(2)}(z)\right\rangle\\ &= -(L_1)_{n,m+1}\left\langle p_{m+1}^{(1)}(z),\, p_{m+1}^{(2)}(z)\right\rangle\\ &= -(L_1)_{n,m+1}h_{m+1}\,, \end{aligned} \tag{2.57}$$

and on the other hand,

$$\begin{aligned} &\left\langle p_{n+1}^{(1)}(z) - zp_n^{(1)}(z),\, p_{m+1}^{(2)}(z) - zp_m^{(2)}(z)\right\rangle, \quad n \ge m \ge -1\,,\\ &= \left\langle p_{n+1}^{(1)}(0)z^n p_n^{(2)}(z^{-1}),\, p_{m+1}^{(2)}(0)z^m p_m^{(1)}(z^{-1})\right\rangle \end{aligned}$$

[14] Define $p_{-1}^{(2)}(z) = 0$.

$$
\begin{aligned}
&= p^{(1)}_{n+1}(0)\,p^{(2)}_{m+1}(0)\big\langle z^{n-m}p^{(1)}_m(z),\, p^{(2)}_n(z)\big\rangle \\
&= p^{(1)}_{n+1}(0)\,p^{(2)}_{m+1}(0)\big\langle p^{(1)}_n(z)+\cdots,\, p^{(2)}_n(z)\big\rangle \\
&= p^{(1)}_{n+1}(0)\,p^{(2)}_{m+1}(0)h_n\,. \qquad (2.58)
\end{aligned}
$$

Comparing (2.57) and (2.58) yields

$$
(L_1)_{n,m+1} = -h_n p^{(1)}_{n+1}(0)h^{-1}_{m+1}p^{(2)}_{m+1}(0)\,, \quad n>m\geq -1\,, \qquad (2.59)
$$

proving the first expression of Theorem 2.8. The second one is obtained by the usual duality $L_1 \mapsto hL_2^{\top}h^{-1}$, $t\leftrightarrow -s$, and so $p^{(1)}\leftrightarrow p^{(2)}$ (see formulae in the beginning of this section). For $n=m$, we compute

$$
\begin{aligned}
&\Big\langle p^{(1)}_{n+1}(z)-zp^{(1)}_n(z),\, p^{(2)}_{n+1}(z)-zp^{(2)}_n(z)\Big\rangle \\
&\quad= \Big\langle p^{(1)}_{n+1}(z),\, p^{(2)}_{n+1}(z)\Big\rangle + \Big\langle zp^{(1)}_n(z), zp^{(2)}_n(z)\Big\rangle \\
&\qquad- \Big\langle zp^{(1)}_n(z),\, p^{(2)}_{n+1}(z)\Big\rangle - \Big\langle p^{(1)}_{n+1}(z), zp^{(2)}_n(z)\Big\rangle \\
&\quad= h_{n+1}+h_n-h_{n+1}-h_{n+1} = h_n-h_{n+1}\,,
\end{aligned}
$$

which upon comparison with (2.58) for $n=m$ yields the first line of Corollary 2.9:

$$
p^{(1)}_{n+1}(0)\,p^{(2)}_{n+1}(0)h_n = h_n - h_{n+1}\,. \qquad (2.60)
$$

Remember from (2.9) and (2.12), we have

$$
\begin{aligned}
L_1 &= \sum_{k=-\infty}^{-2}\operatorname{diag}\left(\frac{p_{1-k}(\tilde\partial_t)\tau_{n+k+1}\circ\tau_n}{\tau_{n+k+1}\tau_n}\right)_{n\in\mathbb{Z}}\Lambda^k \\
&\quad+\left(\frac{\partial}{\partial t_1}\right)^2\log\tau_n\Lambda^{-1}+\frac{\partial}{\partial t_1}\log h_n\Lambda^0+\Lambda \\
hL_2^{\top}h^{-1} &= \sum_{k=-\infty}^{-2}\operatorname{diag}\left(\frac{p_{1-k}(-\tilde\partial_s)\tau_{n+k+1}\circ\tau_n}{\tau_{n+k+1}\tau_n}\right)\Lambda^k \\
&\quad+\left(\frac{\partial}{\partial s_1}\right)^2\log\tau_n\Lambda^{-1}-\frac{\partial}{\partial s_1}\log h_n\Lambda^0+\Lambda\,.
\end{aligned}
$$

Together with the theorem, this yields Corollary 2.9.

Finally, upon multiplying relations (2.54), setting $m+1=n-k$, $n>m$, and using the relation above, one obtains, using Corollary 2.9,

$$
\begin{aligned}
&\left(\frac{h_{m+1}}{h_n}\right)^2\frac{\big(p_{n-m}(\tilde\partial_t)\tau_{m+2}\circ\tau_n\big)\big(p_{n-m}(-\tilde\partial_s)\tau_{m+2}\circ\tau_n\big)}{\tau^2_{m+2}\tau^2_n} \\
&\quad= p^{(1)}_{n+1}(0)\,p^{(2)}_{m+1}(0)\,p^{(2)}_{n+1}(0)\,p^{(1)}_{m+1}(0)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(1 - \frac{h_{n+1}}{h_n}\right)\left(1 - \frac{h_{m+1}}{h_m}\right) \\
&= \frac{1}{\tau_{n+1}^2 \tau_{m+1}^2}\left(\tau_{n+1}^2 - \tau_n \tau_{n+2}\right)\left(\tau_{m+1}^2 - \tau_m \tau_{m+2}\right), \qquad (2.61)
\end{aligned}
$$

which is precisely Corollary 2.10. Relation (2.55) is a special case of (2.61) by setting $m = n - 1$. □

PROOF OF THEOREM 0.3: The structure of L_1 and L_2 follows from Theorem 2.8. The statement about the mathematical expectation follows from

$$
\begin{aligned}
&p_n^{(1)}(t, s; z) \\
&= z^n \frac{\tau_n(t - [z^{-1}], s)}{\tau_n(t, s)} \\
&= \sum_{k=0}^{n} z^k \, \frac{p_{n-k}(-\tilde{\partial}_t)\tau_n(t, s)}{\tau_n(t, s)} \\
&= \frac{1}{\tau_n} \sum_{k=0}^{n} z^k \\
&\quad \int_{\mathrm{U}(n)} p_{n-k}\left(- \operatorname{Tr} M, -\frac{1}{2} \operatorname{Tr} M^2, -\frac{1}{3} \operatorname{Tr} M^3, \dots \right) e^{\sum_1^\infty \operatorname{tr}(t_i M^i - s_i \bar{M}^i)} \, dM ,
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&p_n^{(2)}(t, s; z) \\
&= z^n \frac{\tau_n(t, s + [z^{-1}])}{\tau_n(t, s)} \\
&= \sum_{k=0}^{n} z^k \, \frac{p_{n-k}(\tilde{\partial}_s)\tau_n(t, s)}{\tau_n(t, s)} \\
&= \frac{1}{\tau_n} \sum_{k=0}^{n} z^k \\
&\quad \int_{\mathrm{U}(n)} p_{n-k}\left(- \operatorname{Tr} \bar{M}, -\frac{1}{2} \operatorname{Tr} \bar{M}^2, -\frac{1}{3} \operatorname{Tr} \bar{M}^3, \dots \right) e^{\sum_1^\infty \operatorname{tr}(t_i M^i - s_i \bar{M}^i)} \, dM .
\end{aligned}
$$

Finally, we check the Hamiltonian flow statement for the first flow. Indeed, from the equations for Ψ (after (2.7)), (2.23), Theorem 2.8, and the first relation of

Corollary 2.9, it follows that ($h_{-1} = 0$)

$$\begin{aligned}
\frac{\partial x_n}{\partial t_1} &= \left.\frac{\partial p_n^{(1)}(t,s;z)}{\partial t_1}\right|_{z=0} \\
&= -\left((L_1)_- p^{(1)}\right)_n\Big|_{z=0} \\
&= h_n p_{n+1}^{(1)}(t,s;0) \sum_{i=0}^{n-1} \frac{p_i^{(1)}(t,s;0) p_i^{(2)}(t,s;0)}{h_i} \\
&= h_n x_{n+1} \sum_{i=0}^{n-1} \frac{x_i y_i}{h_i} \\
&= h_n x_{n+1} \sum_{i=0}^{n-1} \left(\frac{1}{h_i} - \frac{1}{h_{i-1}}\right) = x_{n+1} \frac{h_n}{h_{n-1}} = x_{n+1}(1 - x_n y_n)\,,
\end{aligned}$$

and similarly for the other coordinates; this ends the proof of Theorem 0.3. $\square$

3 Painlevé Equations for O(n) and Sp(n) Integrals

3.1 Painlevé Equations Associated with the Jacobi Weight

THEOREM 3.1 (Painlevé Equation and the Jacobi Weight) *The function* $H_n(x) = x \frac{d}{dx} \log \tau_n(x)$, *with*

$$\tau_n(x) := c_n \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^{n} e^{x z_k} (1 - z_k)^{\alpha} (1 + z_k)^{\beta}\, dz_k\,, \tag{3.1}$$

satisfies the Painlevé V equation ($a := \alpha + \beta,\ b := \alpha - \beta$, *and* $\alpha, \beta > -1$)

$$\begin{aligned}
x^2 H''' + x H'' + 6x H'^2 - \left(4H + 4x^2 - 4bx + (2n + a)^2\right) H' \\
+ (4x - 2b) H + 2n(n + a)x - bn(2n + a) = 0\,,
\end{aligned}$$

with initial condition

$$H(0) = 0 \quad \textit{and} \quad H'(0) = \frac{-nb}{a + 2n}\,. \tag{3.2}$$

COROLLARY 3.2

$$\tilde{H}_n(x) = x \frac{d}{dx} \log e^{-cx} \tau_n(2x) \tag{3.3}$$

satisfies the Painlevé V equation

$$\begin{aligned}
&\frac{1}{2} x^2 \tilde{H}''' + \frac{1}{2} x \tilde{H}'' + 3x\left(\tilde{H}'\right)^2 \\
&\quad - \frac{1}{2}\left(4\tilde{H} + 16x^2 - 8(b + c)x + (2n + a)^2\right)\tilde{H}'
\end{aligned}$$

$$+\big(8x-2(b+c)\big)\tilde H+\big(4n(n+a)+c(2b+c)\big)x$$
$$-\frac12(2n+a)\big(2n(b+c)+ac\big)=0\,. \tag{3.4}$$

with

$$\tilde H(0)=0 \quad \textit{and} \quad \tilde H'(0)=-\frac{2n(b+c)+ac}{2n+a}\,. \tag{3.5}$$

PROOF OF THEOREM 3.1: Since $\alpha,\beta>-1$, the boundary condition (0.4) on $\rho(z)$ is fulfilled, so we may apply Theorem 2.5. We set

$$a_0=1\,,\quad a_1=0\,,\quad a_2=-1\,,\quad b_0=\alpha-\beta=:b\,,\quad b_1=\alpha+\beta=:a\,,$$

and all other $a_i=b_j=0$ in (2.39), implying that

$$\tau_n(t_1,t_2,\dots):=\int_{[-1,1]^n}\Delta_n(z)^2\prod_{k=1}^n e^{\sum_1^\infty t_i z_k^i}(1-z_k)^\alpha(1+z_k)^\beta\,dz_k$$

satisfies the equations ($m=1,2,3,\dots$):[15]

$$\begin{aligned}0&=\mathcal{J}^{(2)}_{m-2}\tau_n\\&=\sum_{k\ge0}\left(-a_k\sum_{i+j=k+m-2}:{}^{\beta}\mathbb{J}^{(1)}_i\,{}^{\beta}\mathbb{J}^{(1)}_j:+b_k\,{}^{\beta}\mathbb{J}^{(1)}_{k+m-1}\right)\Bigg|_{\beta=2}\tau_n\\&=\big(J^{(2)}_m-J^{(2)}_{m-2}-2nJ^{(1)}_{m-2}+(2n+a)J^{(1)}_m+bJ^{(1)}_{m-1}-n^2\delta_{m,2}+nb\delta_{m,1}\big)\tau_n\,.\end{aligned}$$

Then introducing the function $F_n:=\log\tau_n(t)$, the two first Virasoro constraints for $m=1,2$ divided by τ_n are given by

$$\begin{aligned}\frac{\mathcal{J}^{(2)}_{-1}\tau_n}{\tau_n}&=\left(\sum_{i\ge1}it_i\frac{\partial}{\partial t_{i+1}}-\sum_{i\ge2}it_i\frac{\partial}{\partial t_{i-1}}+(2n+a)\frac{\partial}{\partial t_1}\right)F_n+n(b-t_1)=0\,,\\ \frac{\mathcal{J}^{(2)}_{0}\tau_n}{\tau_n}&=\left(\sum_{i\ge1}it_i\frac{\partial}{\partial t_{i+2}}-\sum_{i\ge1}it_i\frac{\partial}{\partial t_i}+b\frac{\partial}{\partial t_1}+\frac{\partial^2}{\partial t_1^2}+(2n+a)\frac{\partial}{\partial t_2}\right)F_n\\&\quad+\left(\frac{\partial F_n}{\partial t_1}\right)^2-n^2=0\,.\end{aligned} \tag{3.6}$$

These expressions and their first t_1- and t_2- derivatives, evaluated along the locus

$$\mathcal{L}:=\{t_1=x,\ \text{all other } t_i=0\}$$

[15]The $J^{(i)}_m$ below are the ones of (A.3) for $\beta=2$.

read as follows:

$$
\begin{aligned}
0 &= \frac{\mathcal{J}_{-1}^{(2)}\tau_n}{\tau_n}\bigg|_{\mathcal{L}} = \left(t_1\frac{\partial}{\partial t_2} + (2n+a)\frac{\partial}{\partial t_1}\right)F_n + n(b-t_1)\bigg|_{\mathcal{L}}, \\
0 &= \frac{\mathcal{J}_{0}^{(2)}\tau_n}{\tau_n}\bigg|_{\mathcal{L}} = \left(t_1\frac{\partial}{\partial t_3} + (b-t_1)\frac{\partial}{\partial t_1} + (2n+a)\frac{\partial}{\partial t_2} + \frac{\partial^2}{\partial t_1^2}\right)F_n \\
&\qquad + \left(\frac{\partial F_n}{\partial t_1}\right)^2 - n^2\bigg|_{\mathcal{L}}, \\
0 &= \frac{\partial}{\partial t_1}\frac{\mathcal{J}_{-1}^{(2)}\tau_n}{\tau_n}\bigg|_{\mathcal{L}} = \left(\sum_{i\geq 1} i t_i \frac{\partial^2}{\partial t_{i+1}\partial t_1} + \frac{\partial}{\partial t_2} - \sum_{i\geq 2} i t_i \frac{\partial^2}{\partial t_{i-1}\partial t_1}\right. \\
&\qquad \left. + (2n+a)\frac{\partial^2}{\partial t_1^2}\right)F_n\bigg|_{\mathcal{L}} - n \\
&= \left(t_1\frac{\partial^2}{\partial t_2\partial t_1} + \frac{\partial}{\partial t_2} + (2n+a)\frac{\partial^2}{\partial t_1^2}\right)F_n\bigg|_{\mathcal{L}} - n\,, \\
0 &= \frac{\partial}{\partial t_1}\frac{\mathcal{J}_{0}^{(2)}\tau_n}{\tau_n}\bigg|_{\mathcal{L}} = \left(\sum_{i\geq 1} i t_i \frac{\partial^2}{\partial t_{i+2}\partial t_1} + \frac{\partial}{\partial t_3} - \sum_{i\geq 1} i t_i \frac{\partial^2}{\partial t_i\partial t_1} - \frac{\partial}{\partial t_1} + b\frac{\partial^2}{\partial t_1^2}\right. \\
&\qquad \left. + \frac{\partial^3}{\partial t_1^3} + (2n+a)\frac{\partial^2}{\partial t_2\partial t_1}\right)F_n + 2\frac{\partial F_n}{\partial t_1}\frac{\partial^2 F_n}{\partial t_1^2}\bigg|_{\mathcal{L}}, \\
&= \left(t_1\frac{\partial^2}{\partial t_3\partial t_1} + \frac{\partial}{\partial t_3} + (b-t_1)\frac{\partial^2}{\partial t_1^2} - \frac{\partial}{\partial t_1} + \frac{\partial^3}{\partial t_1^3}\right. \\
&\qquad \left. + (2n+a)\frac{\partial^2}{\partial t_2\partial t_1}\right)F_n + 2\frac{\partial F_n}{\partial t_1}\frac{\partial^2 F_n}{\partial t_1^2}\bigg|_{\mathcal{L}}, \\
0 &= \frac{\partial}{\partial t_2}\frac{\mathcal{J}_{-1}^{(2)}\tau_n}{\tau_n}\bigg|_{\mathcal{L}} = \left(\sum_{i\geq 1} i t_i \frac{\partial^2}{\partial t_{i+1}\partial t_2} + 2\frac{\partial}{\partial t_3} - \sum_{i\geq 2} i t_i \frac{\partial^2}{\partial t_{i-1}\partial t_2} - 2\frac{\partial}{\partial t_1}\right. \\
&\qquad \left. + (2n+a)\frac{\partial^2}{\partial t_1\partial t_2}\right)F_n\bigg|_{\mathcal{L}} \\
&= \left(t_1\frac{\partial^2}{\partial t_2^2} + 2\frac{\partial}{\partial t_3} - 2\frac{\partial}{\partial t_1} + (2n+a)\frac{\partial^2}{\partial t_1\partial t_2}\right)F_n\,.
\end{aligned}
$$

The five equations above form a (triangular) linear system in five unknowns

$$
\frac{\partial F_n}{\partial t_2}\bigg|_{\mathcal{L}}, \quad \frac{\partial F_n}{\partial t_3}\bigg|_{\mathcal{L}}, \quad \frac{\partial^2 F_n}{\partial t_1\partial t_2}\bigg|_{\mathcal{L}}, \quad \frac{\partial^2 F_n}{\partial t_1\partial t_3}\bigg|_{\mathcal{L}}, \quad \frac{\partial^2 F_n}{\partial t_2^2}\bigg|_{\mathcal{L}}.
$$

Setting $t_1 = x$ and $F_n' = \partial F_n/\partial x$, the solution is given by the following expressions:

$$
\frac{\partial F_n}{\partial t_2}\bigg|_{\mathcal{L}} = -\frac{1}{x}\big((2n+a)F_n' + n(b-x)\big),
$$

$$\begin{aligned}
\left.\frac{\partial F_n}{\partial t_3}\right|_{\mathcal{L}} &= -\frac{1}{x^2}\Big(x\big(F_n'' + F_n'^2 + (b-x)F_n' + n(n+a)\big) \\
&\qquad - (2n+a)\big((2n+a)F_n' + bn\big)\Big), \\
\left.\frac{\partial^2 F_n}{\partial t_1 \partial t_2}\right|_{\mathcal{L}} &= -\frac{1}{x^2}\big((2n+a)(xF_n'' - F_n') - bn\big), \\
\left.\frac{\partial^2 F_n}{\partial t_1 \partial t_3}\right|_{\mathcal{L}} &= -\frac{1}{x^3}\Big(x^2\big(F_n''' + 2F_n'F_n''\big) \\
&\qquad - x\big((x^2 - bx + 1)F_n'' + F_n'^2 + bF_n' + (2n+a)^2F_n'' + n(n+a)\big) \\
&\qquad + 2(2n+a)^2F_n' + 2bn(2n+a)\Big), \\
\left.\frac{\partial^2 F_n}{\partial t_2^2}\right|_{\mathcal{L}} &= \frac{1}{x^3}\Big(x\big(2F_n'^2 + 2bF_n' + ((2n+a)^2+2)F_n'' + 2n(n+a)\big) \\
&\qquad - 3(2n+a)^2F_n' - 3bn(2n+a)\Big).
\end{aligned}$$

Putting these expressions into the KP equation (Theorem 2.1) and setting

$$G(x) := F_n'(x) = \frac{d}{dx}\log\tau_n(x)\,,$$

we find

$$\begin{aligned}
(3.7)\quad & x^3G''' + 4x^2G'' + x\big(-4x^2 + 4bx + 2 - (2n+a)^2\big)G' \\
&\qquad + 8x^2GG' + 6x^3G'^2 + 2xG^2 + \big(2bx - (2n+a)^2\big)G \\
&\qquad\qquad + n(2x-b)(n+a) - bn^2 = 0\,.
\end{aligned}$$

Finally, the function

$$H(x) := xG(x) = x\frac{d}{dx}\log\tau_n(x)$$

satisfies

$$\begin{aligned}
(3.8)\quad & x^2H''' + xH'' + 6xH'^2 - \big(4H + 4x^2 - 4bx + (2n+a)^2\big)H' \\
&\qquad + (4x-2b)H + 2n(n+a)x - bn(2n+a) = 0\,.
\end{aligned}$$

According to Cosgrove [9], this third-order equation can be transformed into a master Painlevé equation, which one recognizes to be Painlevé V; see Appendix B.

From Appendix D, identity (D.5), it now follows that

$$H_n'(0) = \frac{\tau_n'(0)}{\tau_n(0)} = \frac{\sum_{i=1}^n \int_{[-1,1]^n} \Delta_n(z)^2 z_i \prod_{k=1}^n \rho_{(\alpha,\beta)}(z_k)dz_k}{\int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n \rho_{(\alpha,\beta)}(z_k)dz_k} = n\langle y_1\rangle = \frac{-nb}{a+2n}\,.$$

This ends the proof of Theorem 3.1. □

PROOF OF COROLLARY 3.2: The differential equation for

$$\tilde{H}(x) = x \frac{d}{dx} \log e^{-cx} \tau_n(2x) = H(2x) - cx$$

is obtained by first setting $x \mapsto 2x$ in the differential equation (3.2) and then setting $H(2x) = \tilde{H}(x) + cx$. This leads to the differential equation (3.5), which is, of course, also Painlevé V. Relation (3.6) follows at once from (3.4). □

3.2 Proof of Theorem 0.1 (O(n) and Sp(n))

We give here a more detailed version of Theorem 0.1(i).

PROPOSITION 3.3 *Given the integral*[16]

$$I_\ell^{\pm}(x) = \int_{\mathrm{O}_\pm(\ell)} e^{x \operatorname{tr} M} \, dM \,,$$

the expressions[17]

$$q_\ell(x) = \log e_\ell^{\pm} \frac{I_{\ell+2}^{\pm}}{I_\ell^{\pm}} \quad \textit{with } e_\ell^{+} = \frac{2}{[\ell+2]_{\text{even}}} \;\; \textit{and } e_\ell^{-} = \frac{2}{[\ell+1]_{\text{even}}}$$

satisfy the standard Toda lattice equations

$$\frac{1}{4} \frac{\partial^2 q_\ell}{\partial x^2} = -e^{q_\ell - q_{\ell-1}} + e^{q_{\ell+1} - q_\ell} \,.$$

PROPOSITION 3.4 *The function*

$$f_\ell^{\pm}(x) = x \frac{d}{dx} \log \int_{\mathrm{O}(\ell+1)_\pm \ \textit{or} \ \mathrm{Sp}(\frac{\ell-1}{2})} e^{x \operatorname{tr} M} \, dM \tag{3.9}$$

is the unique *solution to the third-order equation* (i) *in Theorem* 0.1:

(3.10)

$$\begin{cases} f''' + \dfrac{1}{x} f'' + \dfrac{6}{x} f'^2 - \dfrac{4}{x^2} f f' - \dfrac{16x^2 + \ell^2}{x^2} f' + \dfrac{16}{x} f + \dfrac{2(\ell^2 - 1)}{x} = 0 \\ \quad \textit{with } f_\ell^{\pm}(x) = x^2 \pm \dfrac{x^{\ell+1}}{\ell!} + \mathrm{O}(x^{\ell+2}) \quad \textit{near } x = 0 \,. \end{cases}$$

This third-order equation can be transformed into the following second-order equation in f, quadratic in f'':

$$\frac{x^2}{4} f''^2 = -\left(x f'^2 - \left(4x^2 + \frac{\ell^2}{4} \right) f' + x(\ell^2 - 1) \right) f' \\ + \left(f'^2 - 8x f' + \ell^2 - 1 \right) f + 4 f^2 \,,$$

[16] The integral over the symplectic group Sp($n-1$) can be identified with O($2n$)$_-$.

[17] In this statement, we use the following notation:

$$[n]_{\text{even}} := \max\{\text{even } x \text{ such that } x \leq n\}.$$

which in turn leads to the standard Painlevé equation (B.3) *for*

$$\alpha = -\beta = \frac{(\ell+1)^2}{8}\,, \quad \gamma = 0\,, \quad \delta = -8\,.$$

PROOF OF PROPOSITION 3.3: Proposition 1.1 and identity (2.38) imply

- $I^+_{2n+1}(x) = n!\, e^x \tau_n(2x, 0, \dots)$,
- $I^+_{2n}(x) = n!\, \tau_n(2x, 0, \dots)$,
- $I^-_{2n+1}(x) = n!\, e^{-x} \tau_n(2x, 0, \dots)$,
- $I^-_{2n}(x) = (n-1)!\, \tau_{n-1}(2x, 0, \dots)$.

Note that, since the functions $\tau_n(t-s) = \tau_n(t,s)$ satisfy differential equation (ii) of Theorem 2.1, we obtain for the function $\tau_n(t)$, by subtracting two consecutive equations,

$$\frac{\partial^2}{\partial t_1^2} \log \frac{\tau_{n+1}}{\tau_n} = \frac{\tau_n \tau_{n+2}}{\tau_{n+1}^2} - \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}\,,$$

from which the standard Toda lattice equations follow. □

PROOF OF PROPOSITION 3.4: From Corollary 3.2 it follows that

$$\tilde{H}_n(x) = x \frac{d}{dx} \log \left(e^{-cx} \int\limits_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{2xz_k} (1-z_k)^\alpha (1+z_k)^\beta \, dz_k \right) \tag{3.11}$$

satisfies the Painlevé V equation (3.4). Then in view of Theorem 1.1, $\tilde{H}_n(x)$ corresponds to $f_\ell(x)$ in (3.9) when the parameters n, $a = \alpha+\beta$, $b = \alpha - \beta$, and c take on the following values:

$$\begin{array}{ll}
\mathrm{O}(\ell+1)_- \text{ with } \ell \text{ even:} & n = \frac{\ell}{2},\ a = 0,\ b = -1,\ c = 1\,, \\[6pt]
\mathrm{O}(\ell+1)_- \text{ with } \ell \text{ odd:} & n = \frac{\ell-1}{2},\ a = 1,\ b = 0,\ c = 0\,, \\[6pt]
\mathrm{O}(\ell+1)_+ \text{ with } \ell \text{ even:} & n = \frac{\ell}{2},\ a = 0,\ b = 1,\ c = -1\,, \\[6pt]
\mathrm{O}(\ell+1)_+ \text{ with } \ell \text{ odd:} & n = \frac{\ell+1}{2},\ a = -1,\ b = 0,\ c = 0\,, \\[6pt]
\mathrm{Sp}\left(\frac{\ell-1}{2}\right) \text{ with } \ell \text{ odd :} & n = \frac{\ell-1}{2},\ a = 1,\ b = 0,\ c = 0\,.
\end{array}$$

Setting these values into equation (3.4) leads at once to equation (i) of Theorem 0.1, namely

$$f''' + \frac{1}{x} f'' + \frac{6}{x} f'^2 - \frac{4}{x^2} f f' - \frac{16x^2 + \ell^2}{x^2} f' + \frac{16}{x} f + \frac{2(\ell^2-1)}{x} = 0\,.$$

Moreover, for these values, we have that $b + c = ac = 0$, and so from (3.5), it follows that

$$f_\ell(0) = \tilde{H}_\ell(0) = 0 \quad \text{and} \quad f'_\ell(0) = \tilde{H}'_\ell(0) = -\frac{2n(b+c)+ac}{2n+a} = 0\,. \tag{3.12}$$

According to Appendix B (see Cosgrove [9]), this third-order equation has a first integral, which is second order in f and quadratic in f'', thus introducing a constant c:

$$\frac{x^2}{4} f''^2 = -\left(x f'^2 - \left(4x^2 + \frac{\ell^2}{4} \right) f' + x(\ell^2 - 1) \right) f' \\ + \left(f'^2 - 8x f' + \ell^2 - 1 \right) f + 4 f^2 - \frac{c}{4}\,.$$

Evaluating this differential equation at $x = 0$ leads to, since $f(0) = 0$,

$$c = \ell^2 f'(0)^2 = 0 \quad \text{using (3.12).}$$

Setting $f = \bar{f} - \ell^2/4$ in order to get the equation in Cosgrove's form [10],

$$\frac{x^2}{4} \bar{f}''^2 = -\left(x \bar{f}'^2 - 4x^2 \bar{f}' - x(\ell^2 + 1) \right) \bar{f}' \\ + \left(\bar{f}'^2 - 8x \bar{f}' - (\ell^2 + 1) \right) \bar{f} + 4 \bar{f}^2 + \frac{\ell^2}{4}\,.$$

In the notation (B.2), we have

$$a_1 = 16\,, \quad a_2 = 4(\ell^2 + 1)\,, \quad a_3 = 0\,, \quad c = -\frac{\ell^2}{4}\,.$$

Solving (B.3) for α, β, γ, and δ leads to the canonical form for Painlevé V, with

$$\alpha = -\beta = \frac{(1+\ell)^2}{8}\,, \quad \gamma = 0\,, \quad \delta = -8\,,$$

and according to Appendix D, $f''_\ell(0) = 2$, ending the proof of the first half of Theorem 0.1.

Of course, from combinatorics (Proposition 1.4), we have a much stronger statement:

$$E_{\mathrm{O}_\pm(\ell+1)} e^{x \operatorname{Tr} M} = \exp\left(\frac{x^2}{2} \pm \frac{x^{\ell+1}}{(\ell+1)!} + \mathrm{O}\big(x^{\ell+2}\big) \right),$$

and thus

$$f_\ell^\pm(x) = x \frac{d}{dx} \log E_{\mathrm{O}(\ell+1)_\pm} e^{x \operatorname{tr} M}\, dM = x^2 \pm \frac{x^{\ell+1}}{\ell!} + \mathrm{O}\big(x^{\ell+2}\big) \quad \text{near } x = 0\,.$$

It remains to show the uniqueness of the solution to the initial value problem (3.10). Indeed, substituting $f(x) = x^2 + \sum_{i \geq 3} a_i x^i$ into the third-order differential equation (3.10) for f yields the recursive formula for the coefficients:

$$3(4-\ell^2)a_3 = 0\,,$$

$$(i+1)(i^2-\ell^2)a_{i+1} - 16(i-2)a_{i-1} + \sum_{\substack{2\le m,n\le i-1\\ n+m=i+1}} n a_n(6m-4)a_m = 0 \quad \text{for } i \ge 3\,. \tag{3.13}$$

Therefore, if $\ell \ge 3$, we have inductively $a_3 = \cdots = a_\ell = 0$ from (3.13). Setting $i = \ell$ in the equation above shows that the coefficient $a_{\ell+1}$ is free and can therefore be specified; it is specified by the combinatorics, namely, $a_{\ell+1} = \pm((\ell+1)!)^{-1}$. Once $a_{\ell+1}$ is fixed, all the subsequent a_i's are determined by (3.13). □

4 Painlevé Equations for U(n) Integrals

In this section we show items (ii) and (iii) of Theorem 0.1.

PROPOSITION 4.1

$$g_n(x) = \frac{d}{dx}\, x\, \frac{d}{dx} \log \int_{\mathrm{U}(n)} e^{\sqrt{x}\,\mathrm{tr}(M+\bar M)}\, dM \tag{4.1}$$

is the unique solution to the initial value problem (*Painlevé V equation*):

$$\begin{cases} g_n'' - \dfrac{g_n'^2}{2}\left(\dfrac{1}{g_n-1}+\dfrac{1}{g_n}\right) + \dfrac{g_n'}{x} - \dfrac{n^2}{2x^2}\dfrac{(g_n-1)}{g_n} + \dfrac{2}{x} g_n(g_n-1) = 0 \\ \quad \text{with } g_n(x) = 1 - \dfrac{x^n}{(n!)^2} + \mathrm{O}(x^{n+1}) \text{ near } x = 0\,. \end{cases}$$

PROPOSITION 4.2

$$\begin{aligned} h_n(x) &= \frac{E_{\mathrm{U}(n)}\, \mathrm{tr}\, M \det(I+M)^k e^{-x\,\mathrm{tr}\,\bar M}}{E_{\mathrm{U}(n)} \det(I+M)^k e^{-x\,\mathrm{tr}\,\bar M}} \\ &= \frac{1}{n+k} x \frac{d}{dx} \log E_{\mathrm{U}(n)} \det(I+M)^k e^{-x\,\mathrm{tr}(I+\bar M)}\, dM \end{aligned} \tag{4.2}$$

is the unique solution to the initial value Painlevé V equation as well:

$$\begin{cases} h''' - \dfrac{1}{2}\left(\dfrac{1}{h'}+\dfrac{1}{h'+1}\right) h''^2 + \dfrac{h''}{x} + \dfrac{2(n+k)}{x} h'(h'+1) \\ \quad - \dfrac{1}{2x^2 h'(h'+1)}\big((x-n)h' - h - n\big)\big((2h+x+n)h' + h + n\big) = 0 \\ \quad \text{with } h := h_n(x) = x\dfrac{k-n}{k+n} - \dfrac{x^{n+1}}{(n+1)!}\dbinom{k+n-1}{n} + \mathrm{O}(x^{n+2}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{near } x = 0\,. \end{cases}$$

PROOF OF PROPOSITION 4.1: The proofs of Propositions 4.1 and 4.2 are almost identical except in the end one specializes to a different locus.

Throughout we shall be using the diagonal elements (2.12) of L_1 and $hL_2^\top h^{-1}$:

$$
\begin{aligned}
b_n &= \frac{\partial}{\partial t_1}\log\frac{\tau_n}{\tau_{n-1}} = (L_1)_{n-1,n-1}\,,\\
b_n^* &= -\frac{\partial}{\partial s_1}\log\frac{\tau_n}{\tau_{n-1}} = (hL_2^\top h^{-1})_{n-1,n-1}\,.
\end{aligned}
\tag{4.3}
$$

From (2.55), (2.17), (2.3), Theorem 2.6, and (A.4), the integral below, which is also the determinant of a Toeplitz matrix,

$$
\begin{aligned}
\tau_n(t,s) &= \int_{\mathrm{U}(n)} e^{\sum_1^\infty \operatorname{tr}(t_i M^i - s_i \bar M^i)}\,dM\\
&= \det\left(\int_{S^1} z^{k-\ell} e^{\sum_1^\infty (t_i z^i - s_i z^{-i})}\frac{dz}{2\pi i z}\right)_{0\le k,\ell\le n-1}
\end{aligned}
\tag{4.4}
$$

satisfies the following three relations:

(1) Toeplitz

$$
\begin{aligned}
\mathcal{T}(\tau)_n &= \frac{\partial}{\partial t_1}\log\frac{\tau_n}{\tau_{n-1}}\frac{\partial}{\partial s_1}\log\frac{\tau_n}{\tau_{n-1}}\\
&\quad + \left(1+\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n\right)\left(1+\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n - \frac{\partial}{\partial s_1}\left(\frac{\partial}{\partial t_1}\log\frac{\tau_n}{\tau_{n-1}}\right)\right)\\
&= -b_n b_n^* + \left(1+\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n\right)\left(1+\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n - \frac{\partial}{\partial s_1}b_n\right) = 0\,,
\end{aligned}
\tag{4.5}
$$

(2) 2-Toda

$$
\begin{aligned}
\frac{\partial^2\log\tau_n}{\partial s_2\partial t_1} &= -2\frac{\partial}{\partial s_1}\log\frac{\tau_n}{\tau_{n-1}}\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n - \frac{\partial^3}{\partial s_1^2\partial t_1}\log\tau_n\\
&= 2b_n^*\,\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n - \frac{\partial^3}{\partial s_1^2\partial t_1}\log\tau_n\,,
\end{aligned}
\tag{4.6}
$$

(3) Virasoro

$$
\begin{aligned}
\mathcal{V}_{-1}\tau_n &= \Bigg(\sum_{i\ge1}(i+1)t_{i+1}\frac{\partial}{\partial t_i} - \sum_{i\ge2}(i-1)s_{i-1}\frac{\partial}{\partial s_i}\\
&\qquad + n\left(t_1+\frac{\partial}{\partial s_1}\right)\Bigg)\tau_n = 0\,,\\
\mathcal{V}_0\tau_n &= \sum_{i\ge1}\left(it_i\frac{\partial}{\partial t_i} - is_i\frac{\partial}{\partial s_i}\right)\tau_n = 0\,,\\
\mathcal{V}_1\tau_n &= \Bigg(-\sum_{i\ge1}(i+1)s_{i+1}\frac{\partial}{\partial s_i} + \sum_{i\ge2}(i-1)t_{i-1}\frac{\partial}{\partial t_i}\\
&\qquad + n\left(s_1+\frac{\partial}{\partial t_1}\right)\Bigg)\tau_n = 0\,.
\end{aligned}
\tag{4.7}
$$

Therefore we have

$$
\begin{aligned}
0 &= \frac{1}{\tau_n}(\mathcal{V}_{-1}+\mathcal{V}_0)\tau_n \\
&= \Bigg(\sum_{i\geq 1}\big((i+1)t_{i+1}+it_i\big)\frac{\partial}{\partial t_i} \\
&\qquad -\sum_{i\geq 2}\big((i-1)s_{i-1}+is_i\big)\frac{\partial}{\partial s_i}+(n-s_1)\frac{\partial}{\partial s_1}\Bigg)\log\tau_n+nt_1\,, \\
0 &= \frac{1}{\tau_n}(\mathcal{V}_0+\mathcal{V}_1)\tau_n \\
&= \Bigg(\sum_{i\geq 2}\big((i-1)t_{i-1}+it_i\big)\frac{\partial}{\partial t_i} \\
&\qquad -\sum_{i\geq 1}\big((i+1)s_{i+1}+is_i\big)\frac{\partial}{\partial s_i}+(n+t_1)\frac{\partial}{\partial t_1}\Bigg)\log\tau_n+ns_1\,, \\
0 &= \frac{\partial}{\partial t_1}\left(\frac{\mathcal{V}_{-1}\tau_n}{\tau_n}\right) \\
&= \Bigg(\sum_{i\geq 1}(i+1)t_{i+1}\frac{\partial^2}{\partial t_1\partial t_i}-\sum_{i\geq 2}(i-1)s_{i-1}\frac{\partial^2}{\partial t_1\partial s_i} \\
&\qquad +n\frac{\partial^2}{\partial t_1\partial s_1}\Bigg)\log\tau_n+n\,, \\
0 &= \frac{\partial}{\partial t_1}\left(\frac{\mathcal{V}_0\tau_n}{\tau_n}\right)=\left(\sum_{i\geq 1}\left(it_i\frac{\partial^2}{\partial t_1\partial t_i}-is_i\frac{\partial^2}{\partial t_1\partial s_i}\right)+\frac{\partial}{\partial t_1}\right)\log\tau_n\,, \\
0 &= \frac{\partial}{\partial s_1}\left(\frac{\mathcal{V}_1\tau_n}{\tau_n}\right) \\
&= \Bigg(-\sum_{i\geq 1}(i+1)s_{i+1}\frac{\partial^2}{\partial s_1\partial s_i}+\sum_{i\geq 2}(i-1)t_{i-1}\frac{\partial^2}{\partial s_1\partial t_i} \\
&\qquad +n\frac{\partial^2}{\partial s_1\partial t_1}\Bigg)\log\tau_n+n
\end{aligned}
$$

$$
0=\frac{\partial}{\partial s_1}\left(\frac{\mathcal{V}_0\tau_n}{\tau_n}\right)=\left(-\sum_{i\geq 1}\left(is_i\frac{\partial^2}{\partial s_1\partial s_i}-it_i\frac{\partial^2}{\partial s_1\partial t_i}\right)-\frac{\partial}{\partial s_1}\right)\log\tau_n\,. \tag{4.8}
$$

For the sake of this proof, consider the

$$\text{locus } \mathcal{L}=\{\text{all } t_i=s_i=0 \text{ except } t_1,s_1\neq 0\}\,.$$

From (4.7), we have on $\mathcal{L}$,

$$\frac{\mathcal{V}_0\tau_n}{\tau_n}\bigg|_{\mathcal{L}}=\left(t_1\frac{\partial}{\partial t_1}-s_1\frac{\partial}{\partial s_1}\right)\log\tau_n\bigg|_{\mathcal{L}}=0\,,$$

implying $\tau_n(t,s)\big|_{\mathcal{L}}$ is a function of $x := -t_1 s_1$ only. Therefore we may write $\tau_n\big|_{\mathcal{L}} = \tau_n(x)$, and so, along $\mathcal{L}$, we have

$$\frac{\partial}{\partial t_1} = -s_1 \frac{\partial}{\partial x}\,, \qquad \frac{\partial}{\partial s_1} = -t_1 \frac{\partial}{\partial x}\,, \qquad \frac{\partial^2}{\partial t_1 \partial s_1} = -\frac{\partial}{\partial x} x \frac{\partial}{\partial x}\,.$$

Setting

$$f_n(x) = \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \log \tau_n(x) = -\frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_n(t,s)\Big|_{\mathcal{L}}$$

and using $x = -t_1 s_1$, the 2-Toda relation (4.6) takes on the form

$$\begin{aligned} s_1 \frac{\partial^2 \log \tau_n}{\partial s_2 \partial t_1}\Big|_{\mathcal{L}} &= s_1 \left(2 b_n^* \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n - \frac{\partial}{\partial s_1} \left(\frac{\partial^2 \log \tau_n}{\partial s_1 \partial t_1} \right) \right) \\ &= x \left(2 \frac{b_n^*}{t_1} f_n + f_n' \right). \end{aligned} \tag{4.9}$$

Setting this relation (4.9) into the Virasoro relations (4.7) and (4.8), we have

$$0 = \frac{\mathcal{V}_0 \tau_n}{\tau_n} - \frac{\mathcal{V}_0 \tau_{n-1}}{\tau_{n-1}}\Big|_{\mathcal{L}} = \left(t_1 \frac{\partial}{\partial t_1} - s_1 \frac{\partial}{\partial s_1} \right) \log \frac{\tau_n}{\tau_{n-1}}\Big|_{\mathcal{L}} = t_1 b_n + s_1 b_n^*\,, \tag{4.10}$$

$$\begin{aligned} 0 = \frac{\partial}{\partial t_1} \frac{\mathcal{V}_{-1} \tau_n}{\tau_n}\Big|_{\mathcal{L}} &= \left(-s_1 \frac{\partial^2}{\partial s_2 \partial t_1} + n \frac{\partial^2}{\partial t_1 \partial s_1} \right) \log \tau_n \Big|_{\mathcal{L}} + n \\ &= -x \left(2 \frac{b_n^*}{t_1} f_n(x) + f_n'(x) \right) + n(-f_n(x) + 1)\,. \end{aligned} \tag{4.11}$$

This is a system of two linear relations (4.10) and (4.11) in b_n and b_n^*, whose solution, together with its derivatives, is given by

$$\begin{aligned} \frac{b_n^*}{t_1} &= -\frac{b_n}{s_1} = -\frac{n(f_n - 1) + x f_n'}{2 x f_n}\,, \\ \frac{\partial b_n}{\partial s_1} &= \frac{\partial}{\partial x} x \frac{b_n}{s_1} = \frac{x(f_n f_n'' - f_n'^2) + (f_n + n) f_n'}{2 f_n^2}\,. \end{aligned}$$

Setting $\partial^2 \log \tau_n / \partial s_1 \partial t_1 = -f_n$ into the Toeplitz relation (4.5) yields

$$b_n b_n^* = (1 - f_n) \left(1 - f_n - \frac{\partial}{\partial s_1} b_n \right),$$

which, using the expressions above for b_n, b_n^*, and $\partial b_n / \partial s_1$, yields the differential equation

$$f_n'' - \frac{1}{2} f_n'^2 \left(\frac{1}{f_n - 1} + \frac{1}{f_n} \right) + \frac{1}{x} f_n' + \frac{n^2(-f_n + 1)}{2x^2 f_n} - \frac{2}{x} f_n(-f_n + 1) = 0\,. \tag{4.12}$$

Note, along the locus $\mathcal{L}$, we may set $t_1 = \sqrt{x}$ and $s_1 = -\sqrt{x}$, since it respects $t_1 s_1 = -x$. Thus,

$$f_n(x) = \frac{d}{dx} x \frac{d}{dx} \log \tau_n(x)\,,$$

with

$$\tau_n(t,s)\big|_{\mathcal{L}} = \int_{\mathrm{U}(n)} e^{\operatorname{tr}(t_1 M - s_1 \bar{M})}\, dM\Big|_{\mathcal{L}} = \int_{\mathrm{U}(n)} e^{\sqrt{x}\operatorname{tr}(M+\bar{M})}\, dM\,,$$

satisfies (4.12). The behavior of $f_n(x)$ near $x = 0$ is given by Proposition 1.5 and the above formula, with the uniqueness established as in the orthogonal case, thus proving Proposition 4.1. □

Remark. Setting

$$f_n(x) = \frac{w(x)}{w(x)-1}$$

leads to standard Painlevé V, with $\alpha = \delta = 0,\ \beta = -n^2/2,\ \gamma = -2$.

PROOF OF PROPOSITION 4.2: For fixed $k \in \mathbb{R}$, $k \neq 0$, consider the locus

$$\mathcal{L} = \{\text{all } it_i = -k(-1)^i \text{ and } s_i = 0 \text{ except } s_1 = x\}\,.$$

Then setting

$$f_n(x) = \frac{\partial}{\partial t_1}\log\tau_n\Big|_{\mathcal{L}}\,, \tag{4.13}$$

the Toda relations (4.6) become

$$\begin{aligned} -x\frac{\partial^2}{\partial s_2\partial t_1}\log\tau_n\Big|_{\mathcal{L}} &= -2xb_n^*\,\frac{\partial^2}{\partial s_1\partial t_1}\log\tau_n + x\frac{\partial^3}{\partial s_1^2\partial t_1}\log\tau_n \\ &= -2xb_n^*\, f_n' + x f_n''\,. \end{aligned} \tag{4.14}$$

The Virasoro relations (4.8) become, by using (4.3) and the locus,

$$\begin{aligned} 0 &= \left(\frac{(\mathcal{V}_0+\mathcal{V}_1)\tau_n}{\tau_n} - \frac{(\mathcal{V}_0+\mathcal{V}_1)\tau_{n-1}}{\tau_{n-1}}\right)\Bigg|_{\mathcal{L}} \\ &= \left(-x\frac{\partial}{\partial s_1} + (n+k)\frac{\partial}{\partial t_1}\right)\log\tau_n + nx \\ &\quad - \left(-x\frac{\partial}{\partial s_1} + (n-1+k)\frac{\partial}{\partial t_1}\right)\log\tau_{n-1} - (n-1)x \\ &= \left(-x\frac{\partial}{\partial s_1} + (k+n-1)\frac{\partial}{\partial t_1}\right)\log\frac{\tau_n}{\tau_{n-1}} + \frac{\partial}{\partial t_1}\log\tau_n + x \\ &= -x\frac{\partial}{\partial s_1}\log\frac{\tau_n}{\tau_{n-1}} + (k+n-1)\frac{\partial}{\partial t_1}\log\frac{\tau_n}{\tau_{n-1}} + f_n + x \\ &= xb_n^* + (k+n-1)b_n + f_n + x\,, \end{aligned} \tag{4.15}$$

and, by using (4.8) and (4.14),

$$
\begin{aligned}
0 &= \frac{\partial}{\partial t_1} \frac{(\mathcal{V}_{-1}+\mathcal{V}_0)\tau_n}{\tau_n}\Big|_{\mathcal{L}} \\
&= \left(\frac{\partial}{\partial t_1} - x\frac{\partial^2}{\partial t_1 \partial s_2} + (n-x)\frac{\partial^2}{\partial s_1 \partial t_1}\right)\log \tau_n + n = 0 \\
&= f_n + (n-x) f_n' + n - 2x b_n^* f_n' + x f_n'' . \qquad (4.16)
\end{aligned}
$$

So, as before, we have a linear system in b_n and b_n^* whose solution is

$$
\begin{aligned}
b_n &= -\frac{x f_n'' + f_n'(2 f_n + x + n) + f_n + n}{2 f_n'(n+k-1)} , \\
b_n^* &= \frac{x f_n'' - f_n'(x-n) + f_n + n}{2x f_n'} .
\end{aligned}
$$

Substituting this solution into the Toeplitz relation (4.5)

$$
b_n b_n^* = (1+f_n')\left(1 + f_n' - \frac{\partial}{\partial x} b_n\right)
$$

yields

$$
\begin{aligned}
& f_n''' - \frac{1}{2}\left(\frac{1}{f_n'} + \frac{1}{f_n'+1}\right) f_n''^2 + \frac{f_n''}{x} + \frac{2(n+k)}{x} f_n'(f_n'+1) \\
&\quad - \frac{1}{2x^2 f_n'(f_n'+1)}\big((x-n) f_n' - f_n - n\big)\big((2 f_n + x + n) f_n' + f_n + n\big) = 0 .
\end{aligned}
$$

It remains to compute $f_n(x)$ as in (4.13). Note that

$$
\begin{aligned}
\tau_n(x) := \tau_n(t,s)\big|_{\mathcal{L}} &= \int_{\mathrm{U}(n)} e^{\operatorname{tr}\sum_1^\infty (t_i M^i - s_i \bar{M}^i)}\, dM \Big|_{\mathcal{L}} \\
&= \int_{\mathrm{U}(n)} \left(e^{-\operatorname{tr}\sum_1^\infty \frac{(-M)^i}{i}}\right)^k e^{-x \operatorname{tr} \bar{M}}\, dM \\
&= \int_{\mathrm{U}(n)} \det(I+M)^k e^{-x\operatorname{tr}\bar{M}}\, dM .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f_n(x) = \frac{\partial}{\partial t_1}\log \tau_n \Big|_{\mathcal{L}} &= \frac{\int \operatorname{tr} M\, e^{\operatorname{tr}\sum_1^\infty (t_i M^i - s_i \bar{M}^i)}\, dM}{\int e^{\operatorname{tr}\sum_1^\infty (t_i M^i - s_i \bar{M}^i)}\, dM}\Bigg|_{\mathcal{L}} \\
&= \frac{\int \operatorname{tr} M \det(I+M)^k e^{-x\operatorname{tr}\bar{M}}\, dM}{\int \det(I+M)^k e^{-x\operatorname{tr}\bar{M}}\, dM} \\
&\overset{*}{=} \frac{1}{n+k} x \frac{d}{dx} \log \tau_n e^{-nx} . \qquad (4.17)
\end{aligned}
$$

This last equality $\stackrel{*}{=}$ will be shown later in Lemma 4.3. To conclude the proof of Proposition 4.2, observe from (4.17) and Proposition 1.5 that

$$\begin{aligned} f_n(x) &= \frac{1}{n+k}\left(x\frac{d}{dx}\log\int_{\mathrm{U}(n)} \det(I+M)^k e^{-x\,\mathrm{tr}\,\bar{M}}\,dM - nx\right) \\ &= x\frac{k-n}{k+n} - \frac{x^{n+1}}{(n+1)!}\binom{k+n-1}{n} + \mathrm{O}(x^{n+2})\,; \end{aligned}$$

this concludes the proof of Proposition 4.2. □

PROOF OF THEOREM 0.1: Upon integrating expressions (4.1) and (4.2) and exponentiating, one finds expressions (ii) and (iii) of Theorem 0.1 after using, respectively, the initial conditions (0.2) and the first identity of Lemma 4.3. □

Recall that equality (4.17) ($\stackrel{*}{=}$) still needs proof.

LEMMA 4.3

$$\tau_n(0) = \int_{\mathrm{U}(n)} \det(I+M)^k\,dM = 1\,, \quad \frac{\partial\tau_n}{\partial t_1}(0) = \int_{\mathrm{U}(n)} \mathrm{tr}\,M\det(I+M)^k\,dM = 0\,,$$

and

$$\begin{aligned} &\frac{\int \mathrm{tr}\,M\det(I+M)^k e^{-x\,\mathrm{tr}\,\bar{M}}\,dM}{\int \det(I+M)^k e^{-x\,\mathrm{tr}\,\bar{M}}\,dM} \\ &\quad = \frac{1}{n+k}x\frac{d}{dx}\log e^{-nx}\int_{\mathrm{U}(n)} \det(I+M)^k e^{-x\,\mathrm{tr}\,\bar{M}}\,dM \\ &\quad = \frac{-x}{n+k}\left(\frac{\int \mathrm{tr}\,\bar{M}\det(I+M)^k e^{-xtr\;\bar{M}}\,dM}{\int \det(I+M)^k e^{-x\,\mathrm{tr}\,\bar{M}}\,dM} + n\right). \end{aligned} \tag{4.18}$$

PROOF: Recall from (4.17) that $f(x)$ is the left-hand side of (4.18). At first we show, using the Toeplitz matrix representation (2.51) in the third identity, that $f(0) = 0$; indeed,

$$\begin{aligned} \tau_n(0)\;f(0) &= \int_{\mathrm{U}(n)} \mathrm{tr}\,M\det(I+M)^k\,dM \\ &= \frac{d}{d\varepsilon}\int \det(I+M)^k\det(I+\varepsilon M)dM\Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}\det\left(\int_{S^1} z^{\ell-m}(1+z)^k(1+\varepsilon z)\frac{dz}{2\pi\,iz}\right)_{0\le\ell,m\le n-1}\Bigg|_{\varepsilon=0} \end{aligned}$$

$$\stackrel{*}{=} \frac{d}{d\varepsilon} \det \left. \begin{pmatrix} 1 & & & \\ & 1 & * & \\ & & \ddots & \\ O & & & 1 \end{pmatrix} \right|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}(1) = 0\,. \tag{4.19}$$

The equality $\stackrel{*}{=}$ is due to the fact that

$$\int_{S^1} z^{\ell-m}(1+z)^k(1+\varepsilon z)\frac{dz}{2\pi\, iz} = \begin{cases} 0 & \text{for } \ell - m \geq 1 \\ 1 & \text{for } \ell = m\,. \end{cases}$$

The same but even simpler argument shows $\tau_n(0) = 1$ by replacing $1 + \varepsilon z$ by 1 in (4.19). From (4.8), it also follows that

$$\begin{aligned} 0 = \frac{\partial}{\partial s_1} \left. \frac{(\mathcal{V}_0 + \mathcal{V}_1)\tau_n}{\tau_n} \right|_{\mathcal{L}} &= \left.\left((n+k)\frac{\partial^2}{\partial s_1 \partial t_1} - s_1 \frac{\partial^2}{\partial s_1^2} - \frac{\partial}{\partial s_1} \right) \log \tau_n \right|_{\mathcal{L}} + n \\ &= (n+k)\, \frac{\partial f}{\partial x} - \left. \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \log \tau_n \right|_{\mathcal{L}} + n \\ &= (n+k)\, \frac{\partial f}{\partial x} - \left. \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \log \tau_n e^{-nx} \right|_{\mathcal{L}} . \end{aligned}$$

Integrating this expression from 0 to x yields

$$(n+k)(f(x) - f(0)) = x\frac{\partial}{\partial x} \log \tau_n e^{-nx}\,;$$

the fact that $f(0) = 0$ establishes the first identity of (4.18). The second identity of (4.18) follows from

$$\begin{aligned} f(x) &= \frac{1}{n+k} x \frac{d}{dx} \log e^{-nx} \int_{\mathrm{U}(n)} \det(I+M)^k e^{-x \operatorname{tr} \bar{M}}\, dM \\ &= \frac{-x}{n+k} \left(\frac{\int \operatorname{tr} \bar{M} \det(I+M)^k e^{-x \operatorname{tr} \bar{M}}\, dM}{\int \det(I+M)^k e^{-x \operatorname{tr} \bar{M}}\, dM} + n \right), \end{aligned}$$

ending the proof of Lemma 4.3. □

Appendix A: Virasoro Algebras

In [3], we defined a Heisenberg and Virasoro algebra of vector operators ${}^{\beta}\mathbb{J}_k^{(i)}$, depending on a parameter $\beta > 0$:

$$\left({}^{\beta}\mathbb{J}_k^{(1)}\right)_n = {}^{\beta}J_k^{(1)} + nJ_k^{(0)} \quad \text{and} \quad \left(\mathbb{J}_k^{(0)}\right)_n = nJ_k^{(0)} = n\delta_{0k}\,,$$

and

$$
\begin{aligned}
{}^{\beta}\mathbb{J}_k^{(2)}(\beta) &= \frac{\beta}{2}\sum_{i+j=k} : {}^{\beta}\mathbb{J}_i^{(1)}{}^{\beta}\mathbb{J}_j^{(1)} : + \left(1-\frac{\beta}{2}\right)\left((k+1)\,{}^{\beta}\mathbb{J}_k^{(1)} - k\mathbb{J}_k^{(0)}\right) \\
&= \left(\frac{\beta}{2}\cdot {}^{\beta}J_k^{(2)} + \left(n\beta + (k+1)\left(1-\frac{\beta}{2}\right)\right) . {}^{\beta}J_k^{(1)}\right. \\
&\qquad \left. + \frac{n((n-1)\beta+2)}{2} J_k^{(0)}\right)_{n\in\mathbb{Z}} . \qquad \text{(A.1)}
\end{aligned}
$$

The ${}^{\beta}\mathbb{J}_k^{(2)}$'s satisfy the commutation relations (see [3]):

$$
\begin{aligned}
\left[{}^{\beta}\mathbb{J}_k^{(1)}, {}^{\beta}\mathbb{J}_\ell^{(1)}\right] &= \frac{k}{\beta}\delta_{k,-\ell} , \\
\left[{}^{\beta}\mathbb{J}_k^{(2)}, \ {}^{\beta}\mathbb{J}_\ell^{(1)}\right] &= -\ell\,{}^{\beta}\mathbb{J}_{k+\ell}^{(1)} + k(k+1)\left(\frac{1}{\beta}-\frac{1}{2}\right)\delta_{k,-\ell} , \\
\left[{}^{\beta}\mathbb{J}_k^{(2)}, \ {}^{\beta}\mathbb{J}_\ell^{(2)}\right] &= (k-\ell)\,{}^{\beta}\mathbb{J}_{k+\ell}^{(2)} + c\left(\frac{k^3-k}{12}\right)\delta_{k,-\ell} , \qquad \text{(A.2)}
\end{aligned}
$$

with central charge

$$
c = 1 - 6\left(\left(\frac{\beta}{2}\right)^{1/2} - \left(\frac{\beta}{2}\right)^{-1/2}\right)^2 .
$$

In the expressions above,

$$
{}^{\beta}J_k^{(1)} = \begin{cases} \frac{\partial}{\partial t_k} & \text{for } k>0 \\ \frac{1}{\beta}(-k)t_{-k} & \text{for } k<0 \\ 0 & \text{for } k=0 , \end{cases}
$$

$$
{}^{\beta}J_k^{(2)} = \sum_{i+j=k}\frac{\partial^2}{\partial t_i\,\partial t_j} + \frac{2}{\beta}\sum_{-i+j=k} i t_i \frac{\partial}{\partial t_j} + \frac{1}{\beta^2}\sum_{-i-j=k} i t_i\, j t_j . \qquad \text{(A.3)}
$$

In particular, for $\beta = 1$ and 2, the ${}^{\beta}\mathbb{J}_k^{(2)}$ take on the form

$$
{}^{\beta}\mathbb{J}_k^{(2)}(t)\big|_{\beta=1} = \frac{1}{2}\left({}^{\beta}J_k^{(2)} + (2n+k+1)\,{}^{\beta}J_k^{(1)} + n(n+1)J_k^{(0)}\right)_{n\in\mathbb{Z}}\Big|_{\beta=1} , \qquad \text{(A.4)}
$$

$$
{}^{\beta}\mathbb{J}_k^{(2)}(t)\big|_{\beta=2} = \left({}^{\beta}J_k^{(2)} + 2n\,{}^{\beta}J_k^{(1)} + n^2 J_k^{(0)}\right)_{n\in\mathbb{Z}}\Big|_{\beta=2} . \qquad \text{(A.5)}
$$

Appendix B: Chazy Classes

Given arbitrary polynomials $P(z)$, $Q(z)$, and $R(z)$ of degree 3, 2, and 1, respectively, Cosgrove [9, (A.3)], shows that the third-order equation

$$
f''' + \frac{P'}{P}f'' + \frac{6}{P}f'^2 - \frac{4P'}{P^2}ff' + \frac{P''}{P^2}f^2 + \frac{4Q}{P^2}f' - \frac{2Q'}{P^2}f + \frac{2R}{P^2} = 0 \qquad \text{(B.1)}
$$

has a first integral, which is second order in f and quadratic in f'',

$$
\text{(B.2)} \quad f''^2 + \frac{4}{P^2}\Big(\big(Pf'^2 + Qf' + R\big) f' - \big(P' f'^2 + Q' f' + R'\big) f + \frac{1}{2}(P'' f' + Q'') f^2 - \frac{1}{6} P''' f^3 + c \Big) = 0 ;
$$

c is the integration constant. This is a master Painlevé equation, containing the six Painlevé equations. When the polynomials P, Q, and R have the forms

$$
P = x , \quad Q = -\frac{a_1}{4} x^2 , \quad R = -\frac{1}{4}(a_2 x + a_3) ,
$$

then equation (B.2) can be reduced to the Painlevé V equation [10, p. 70]:

$$
\text{(B.3)} \quad w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right) w'^2 - \frac{1}{x} w' + \frac{(w-1)^2}{x^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{x} + \frac{\delta w (w+1)}{w-1} ,
$$

with

$$
a_1 = -2\delta , \quad a_2 = \frac{1}{4}\gamma^2 + 2\beta\delta - \delta(1 - \sqrt{2\alpha})^2 , \quad a_3 = \beta\gamma + \frac{1}{2}\gamma(1 - \sqrt{2\alpha})^2 ,
$$

$$
c = -\frac{1}{32}\gamma^2\big((1 - \sqrt{2\alpha})^2 - 2\beta\big) + \frac{1}{32}\delta\big((1 - \sqrt{2\alpha})^2 + 2\beta\big)^2 .
$$

Appendix C: The Volume of the Orthogonal and Symplectic Groups

Selberg's integral (see Mehta [15, p. 340]), renormalized over $[-1, 1]$,

$$
\begin{aligned}
\int_{[-1,1]^n} & \Delta_n(x)^{2\gamma} \prod_{j=1}^{n} (1 - x_j)^{\alpha} (1 + x_j)^{\beta} \, dx_j \\
&= 2^{n(\alpha+\beta+\gamma(n-1)+1)} \prod_{j=0}^{n-1} \frac{\Gamma(\alpha + j\gamma + 1)\Gamma(\beta + j\gamma + 1)\Gamma(\gamma + j\gamma + 1)}{\Gamma(\gamma + 1)\Gamma(\alpha + \beta - \gamma + \gamma(n + j) + 2)} \\
&= 2^{n(n+\alpha+\beta)} \prod_{j=1}^{n} \frac{j!\Gamma(j + \alpha)\Gamma(j + \beta)}{\Gamma(n + j + \alpha + \beta)} \quad \text{upon setting } \gamma = 1 ,
\end{aligned}
$$

leads to the value of $c^{\pm}_{2n}$ and $c^{\pm}_{2n-1}$ in Theorem 1.1:

$$
\alpha = -\beta = \pm\frac{1}{2} \longrightarrow \int_{\mathrm{O}(2n+1)_{\pm}} dM = 2^{n^2} \prod_{j=1}^{n} \frac{j!(j - \frac{1}{2})\Gamma^2(j - \frac{1}{2})}{(n + j - 1)!}
$$

$$
\alpha = \beta = -\frac{1}{2} \longrightarrow \int_{\mathrm{O}(2n)_{+}} dM = 2^{n(n-1)} \prod_{j=1}^{n} \frac{j!\Gamma^2(j - \frac{1}{2})}{(n + j - 2)!}
$$

$$\alpha = \beta = \frac{1}{2},\ n \mapsto n-1 \longrightarrow \int_{\mathrm{O}(2n)_-} dM = 2^{n(n-1)} \prod_{j=1}^{n-1} \frac{j!\Gamma^2(j+\frac{1}{2})}{(n+j-1)!}.$$

Appendix D: Direct Evaluation of Integrals over the Orthogonal Group and Their Derivatives at $x = 0$

Referring to Theorem 3.1, formulae (3.3) and (3.4), we evaluate $d/dx \log \tau_n(x)$ and $d^2/dx^2 \log \tau_n(x)$ directly from the integral representation, not using the combinatorial interpretation of the integrals. To do this, we need the Aomoto extension [6] (see Mehta [15, p. 340]) of Selberg's integral:[18]

$$\langle x_1 \dots x_m \rangle := \frac{\int_0^1 \cdots \int_0^1 x_1 \cdots x_m \, |\Delta(x)|^{2\gamma} \prod_{j=1}^n x_j^\alpha (1-x_j)^\beta \, dx_1 \cdots dx_n}{\int_0^1 \cdots \int_0^1 |\Delta(x)|^{2\gamma} \prod_{j=1}^n x_j^\alpha (1-x_j)^\beta \, dx_1 \cdots dx_n}$$

$$= \prod_{j=1}^{m} \frac{\alpha + 1 + (n-j)\gamma}{\alpha + \beta + 2 + (2n-j-1)\gamma}. \tag{D.1}$$

In particular, by setting $\gamma = 1$, formula (D.1) implies

$$\langle x_1 \rangle = \frac{n+\alpha}{2\,n+\beta+\alpha} \quad \text{and} \quad \langle x_1 x_2 \rangle = \frac{(n+\alpha-1)\,(n+\alpha)}{(2\,n+\beta+\alpha-1)\,(2\,n+\beta+\alpha)}, \tag{D.2}$$

and from the identity (see [15, p. 349])

$$(2n+\beta+\alpha+1)\,\langle x_1^2 \rangle = (2n+\alpha)\,\langle x_1 \rangle - (n-1)\langle x_1 x_2 \rangle$$

we derive

$$\left\langle x_1^2 \right\rangle = \frac{(n+\alpha)\left(3n^2 + 2\beta n + 3\alpha n + \alpha\beta + \alpha^2 - 1\right)}{(2n+\beta+\alpha-1)\,(2n+\beta+\alpha)\,(2n+\beta+\alpha+1)}. \tag{D.3}$$

We now consider the following ratio of integrals (remember $\rho_{\alpha\beta}(z) := (1-z)^\alpha (1+z)^\beta$)

$$\langle y_1 \cdots y_m \rangle_{[-1,1]} := \frac{\int_{[-1,1]^n} y_1 \cdots y_m \, \Delta_n(y)^2 \prod_{k=1}^n \rho_{(\alpha,\beta)}(y_k) dy_k}{\int_{[-1,1]^n} \Delta_n(y)^2 \prod_{k=1}^n \rho_{(\alpha,\beta)}(y_k) dy_k}. \tag{D.4}$$

The relationship between the two integrals (D.1) and (D.4) is obtained by setting $x_j = (1-y_j)/2$; so we have

$$\langle x_1 \rangle = \frac{1}{2}(1 - \langle y_1 \rangle), \quad \langle x_1 x_2 \rangle = \frac{1}{4}(1 - 2\langle y_1 \rangle + \langle y_1 y_2 \rangle),$$
$$\langle x_1^2 \rangle = \frac{1}{4}(1 - 2\langle y_1 \rangle + \langle y_1^2 \rangle).$$

---

[18] $\operatorname{Re}\alpha, \operatorname{Re}\beta > -1$, $\operatorname{Re}\gamma > -\min\left(\frac{1}{n}, \frac{\operatorname{Re}\alpha+1}{n-1}, \frac{\operatorname{Re}\beta+1}{n-1}\right)$.

Thus, these relations, upon using (D.2) and (D.3) and upon setting $\alpha = (a+b)/2$ and $\beta = (a-b)/2$, yield

$$\langle y_1 \rangle = \frac{-b}{a+2n}, \qquad \langle y_1 y_2 \rangle = \frac{b^2 - a - 2n}{(a+2n-1)(a+2n)},$$
$$\langle y_1^2 \rangle = \frac{b^2(a+n) + n(a+2n)^2 - (a+2n)}{(a+2n-1)(a+2n)(a+2n+1)}. \tag{D.5}$$

Hence, setting

$$I_n^{(\alpha,\beta)}(x) := \int_{[-1,1]^n} \Delta_n(z)^2 \prod_1^n e^{2xz_k} \rho_{\alpha,\beta}(z_k) dz_k\ ,$$

we compute for future use:

$$\begin{aligned}
\gamma(n) := 2\, \frac{I_n''}{I_n}\bigg|_{x=0} = 8 \left\langle \left(\sum_1^n y_i \right)^2 \right\rangle &= 8\ (n\langle y_1^2 \rangle + n(n-1)\langle y_1 y_2 \rangle) \\
&= 8n\ (\langle y_1^2 \rangle + (n-1)\langle y_1 y_2 \rangle) \\
&= 8n\ \frac{(a+2n)(b^2 n + a + n) - b^2}{(a+2n-1)(a+2n)(a+2n+1)}.
\end{aligned} \tag{D.6}$$

Note this formula applies to a general $I_n^{(\alpha,\beta)}(x)$, where a combinatorial interpretation is absent. These considerations will now be applied to the orthogonal case. Indeed, considering the special values of α and β and thus for a and b, we evaluate:

- $a = -1,\ b = 0 : \gamma(n) = 2,$
- $a = 1,\ b = 0 : \gamma(n) = 2,$
- $a = 0,\ b = 1 : \gamma(n) = 4,$
- $a = 0,\ b = -1 : \gamma(n) = 4\,.$

It is easily seen that

$$\left(x \frac{d}{dx} \log \int e^{x \operatorname{tr} M}\, dM \right)'' = 2 \frac{\left(\int e^{x \operatorname{tr} M}\, dM \right)''}{\int e^{x \operatorname{tr} M}\, dM} - 2 \left(\frac{\left(\int e^{x \operatorname{tr} M}\, dM \right)'}{\int e^{x \operatorname{tr} M}\, dM} \right)^2 + \mathrm{O}(x)$$

and so, using (3.12) and the fact that the volume $\int dM$ does not vanish,

$$\left(x \frac{d}{dx} \log \int e^{x \operatorname{tr} M}\, dM \right)'' \bigg|_{x=0} = 2\ \frac{\left(\int e^{x \operatorname{tr} M}\, dM \right)''}{\int e^{x \operatorname{tr} M}\, dM} \bigg|_{x=0}.$$

Using $f'_\ell(0) = 0$ to evaluate $I'_n(0)$ below and using (1.2) and (D.6), we now verify in each of the cases:

$$f''_{2n-1}(0) = \left(x \frac{d}{dx} \log \int_{\mathrm{O}(2n)_+} e^{x \operatorname{Tr} M}\, dM \right)'' \Bigg|_{x=0}$$

$$= 2\, \frac{I_n''^{(-\frac{1}{2},-\frac{1}{2})}}{I_n^{(-\frac{1}{2},-\frac{1}{2})}} \Bigg|_{x=0} = \gamma(n)\big|_{a=-1,b=0} = 2\,,$$

$$f''_{2n-1}(0) = \left(x \frac{d}{dx} \log \int_{\mathrm{O}(2n)_-} e^{x \operatorname{Tr} M}\, dM \right)'' \Bigg|_{x=0}$$

$$= 2 \frac{I_{n-1}''^{(\frac{1}{2},\frac{1}{2})}}{I_{n-1}^{(\frac{1}{2},\frac{1}{2})}} = \gamma(n-1)\big|_{a=1,b=0} = 2\,,$$

$$f''_{2n}(0) = \left(x \frac{d}{dx} \log \int_{\mathrm{O}(2n+1)_+} e^{x \operatorname{Tr} M}\, dM \right)'' \Bigg|_{x=0} = 2\, \frac{(e^x I_n^{(\frac{1}{2},-\frac{1}{2})})''}{e^x I_n^{(\frac{1}{2},-\frac{1}{2})}} \Bigg|_{x=0}$$

$$= 2 \left(\frac{I_n'' + 2I_n' + I_n}{I_n} \right) \Bigg|_{x=0}$$

$$= 2 \left(\frac{I_n''}{I_n} - 1 \right) \quad \text{from } \left(e^x I_n(x)\right)' \Big|_{x=0} = I_n'(0) + I_n(0) = 0$$

$$= -2 + \gamma(n)\big|_{a=0,b=+1} = 2\,,$$

$$f''_{2n}(0) = \left(x \frac{d}{dx} \log \int_{\mathrm{O}(2n+1)_-} e^{x \operatorname{Tr} M} \right)'' \Bigg|_{x=0} = 2\, \frac{(e^{-x} I_n^{(-\frac{1}{2},\frac{1}{2})})''}{e^{-x} I_n^{(-\frac{1}{2},\frac{1}{2})}} \Bigg|_{x=0}$$

$$= 2\, \frac{I_n'' - 2I_n' + I_n}{I_n} \Bigg|_{x=0} \qquad \text{from } \left(e^{-x} I_n(x)\right)' \Big|_{x=0} = 0$$

$$= 2\left(\frac{I_n''}{I_n} - 1\right) = -2 + \gamma(n)\big|_{a=0,b=-1} = 2\,.$$

Acknowledgments. We wish to thank Jinho Baik, Craig Tracy, and Harold Widom for several informative discussions.

Bibliography

[1] Adler, M.; van Moerbeke, P. Matrix integrals, Toda symmetries, Virasoro constraints and orthogonal polynomials. *Duke Math. J.* **80** (1995), no. 3, 863–911.

[2] Adler, M.; van Moerbeke, P. String-orthogonal polynomials, string equations, and 2-Toda symmetries. *Comm. Pure and Appl. Math.* **50** (1997), no. 3, 241–290.

[3] Adler, M.; van Moerbeke, P. The spectrum of coupled random matrices. *Ann. of Math. (2)* **149** (1999), no. 3, 921–976.

[4] Adler, M.; van Moerbeke, P. Vertex operator solutions to the discrete KP-hierarchy. *Comm. Math. Phys.* **203** (1999), no. 1, 185–210.

[5] Aldous, D.; Diaconis, P. Longest increasing subsequences: from patience sorting to the Baik-Deift-Johansson theorem. *Bull. Amer. Math. Soc. (N.S.)* **36** (1999), no. 4, 413–432.

[6] Aomoto, K. Jacobi polynomials associated with Selberg integrals. *SIAM J. Math. Anal.* **18** (1987), no. 2, 545–549.

[7] Baik, J.; Deift, P.; Johansson, K. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.* **12** (1999), no. 4, 1119–1178.

[8] Baik, J.; Rains, E. Algebraic aspects of increasing subsequences. Preprint Math. CO/9905083 B, 1999.

[9] Cosgrove, C. M. Chazy classes IX-XI of third-order differential equations. *Stud. Appl. Math.* **104** (2000), no. 3, 171–228.

[10] Cosgrove, C. M.; Scoufis, G. Painlevé classification of a class of differential equations of the second order and second degree. *Stud. Appl. Math.* **88** (1993), no. 1, 25–87.

[11] Diaconis, P.; Shahshahani, M. On the eigenvalues of random matrices. Studies in applied probability. *J. Appl. Probab.* **31A** (1994), 49–62.

[12] Gessel, I. M. Symmetric functions and P-recursiveness. *J. Combin. Theory Ser. A* **53** (1990), no. 2, 257–285.

[13] Hisakado, M. Unitary matrix models and Painlevé III. *Modern Phys. Lett. A* **11** (1996), no. 38, 3001–3010.

[14] Johansson, K. On random matrices from the compact classical groups. *Ann. of Math. (2)* **145** (1997), no. 3, 519–545.

[15] Mehta, M. L. *Random matrices.* Second edition. Academic Press, Boston, 1991.

[16] Rains, E. M. Increasing subsequences and the classical groups. *Electron. J. Combin.* **5** (1998), no. 1, Research Paper 12, 9 pp.

[17] Suris, Y. B. A note on an integrable discretization of the nonlinear Schrödinger equation. *Inverse Problems* **13** (1997), no. 4, 1121–1136.

[18] Tracy, C. A.; Widom, H. Random unitary matrices, permutations and Painlevé. *Comm. Math. Phys.* **207** (1999), no. 3, 665–685.

[19] Tracy, C. A.; Widom, H. On the distribution of the lengths of the longest monotone subsequences in random words. Preprint Math. CO/9904042, 1999.

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Received December 1999.