Moment matrices and multi-component KP, with applications to random matrix theory

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1 Introduction

Random matrix theory has led to the discovery of novel matrix models and novel statistical distributions, which are defined by means of Fredholm determinants and which, in many cases, satisfy nonlinear ordinary or partial differential equations. A crucial observation is that these matrix integrals, upon appropriate deformation by means of exponentials containing one or several series of time parameters, satisfy (i) integrable equations and (ii) Virasoro constraints with respect to these time parameters. Most of the time, such matrix integrals can be written — by expressing the integrand in "polar coordinates" — as a multiple integral, which then can be expressed in terms of the determinant of

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a moment matrix; this may be a moment matrix with regard to one or several weights. The extra time parameters are added in such a way that each weight has its own exponential time deformation.

The main point is to show that this determinant satisfies (i) and (ii). These features turn out to be extremely robust! The purpose of the present paper is to show point (i) in great generality, which is the determinant of moment matrices associated with one or several weights and defined on various different domains, satisfies the *multi-component KP hierarchy* with regard to the time parameters. This is a very general class of integrable equations.

This determinant will turn out to be the τ-function of this integrable hierarchy; this τ-function with appropriate shifts of the deformation variables will be expressed in terms of the "orthogonal polynomials" defined by the weights and their Cauchy transform. We list below a number of examples having their origin in Hermitian random matrix theory, in random matrices coupled in a chain, in random permutations and in Dyson Brownian motions (non-intersecting Brownian motions) on $\mathbb{R}$ leaving from the origin, where some paths are forced to end up at one point and others at another point, etc... These examples will then be discussed in detail in Section 7.

- *GUE: orthogonal polynomials.*

$$\frac{1}{n!}\int_{E^n}\Delta_n^2(z)\prod_{\ell=1}^n e^{\sum_{k=1}^\infty t_k z_\ell^k}\rho(z_\ell)dz_\ell = \det\Big(\int_{\mathbb{R}} z^{i+j}e^{\sum_{k=1}^\infty t_k z^k}\rho(z)dz\Big)_{0\le i,j\le n-1}$$

- *Coupled random matrices / Dyson Brownian motions: bi-orthogonal polynomials.*

$$\frac{1}{n!}\iint_{E^n}\Delta_n(x)\Delta_n(y)\prod_{\ell=1}^n e^{\sum_{k=1}^\infty (t_k x_\ell^k - s_k y_\ell^k)}\rho(x_\ell,y_\ell)dx_\ell dy_\ell$$
$$= \det\left(\iint_E x^i y^j e^{\sum_{k=1}^\infty (t_k x^k - s_k y^k)}\rho(x,y)dxdy\right)_{0\le i,j\le n-1}$$

- *Longest increasing subsequences in random permutations: orthogonal polynomials on S^1.*

$$\frac{1}{n!}\int_{(S^1)^n}|\Delta_n(z)|^2\prod_{\ell=1}^n\left(e^{\sum_{k=1}^\infty (t_k z_\ell^k - s_k z_\ell^{-k})}\frac{dz_\ell}{2\pi\sqrt{-1}z_\ell}\right)$$
$$= \det\left(\oint_{S^1}\frac{dz}{2\pi\sqrt{-1}z}z^{i-j}e^{\sum_{k=1}^\infty (t_k z^k - s_k z^{-k})}\right)_{0\le i,j\le n-1}$$

- *m_1+m_2 non-intersecting Brownian motions on $\mathbb{R}$ leaving from 0 and $m_{\frac{1}{2}}$ paths forced to end up at $\pm a$: multiple orthogonal polynomials on $\mathbb{R}$.*

$$\frac{1}{m_1!m_2!}\int_{E^{m_1+m_2}}\Delta_{m_1+m_2}(x,y)$$

$$
\begin{aligned}
&\left(\Delta_{m_1}(x)\prod_{\ell=1}^{m_1} e^{-\frac{x_\ell^2}{2}+ax_\ell} e^{\sum_{k=1}^\infty (t_k-s_k)x_\ell^k} dx_\ell\right)\\
&\left(\Delta_{m_2}(y)\prod_{\ell=1}^{m_2} e^{-\frac{y_\ell^2}{2}-ay_\ell} e^{\sum_{k=1}^\infty (t_k-u_k)y_\ell^k} dy_\ell\right)\\
&\qquad = \det\begin{pmatrix} \left(\int_E z^{i+j} e^{-\frac{z^2}{2}+az} e^{\sum_1^\infty (t_k-s_k)z^k} dz\right)_{\substack{0\le i\le m_1-1\\ 0\le j\le m_1+m_2-1}} \\ \left(\int_E z^{i+j} e^{-\frac{z^2}{2}-az} e^{\sum_1^\infty (t_k-u_k)z^k} dz\right)_{\substack{0\le i\le m_2-1\\ 0\le j\le m_1+m_2-1}} \end{pmatrix}
\end{aligned}
$$

- $\sum_{\alpha=1}^q m_\alpha = \sum_{\beta=1}^p n_\beta$ *non-intersecting Brownian motions on* $\mathbb{R}$*, with* m_α *paths starting at* $a_\alpha \in \mathbb{R}$ *and* n_β *paths forced to end up at* b_β*: mixed multiple orthogonal polynomials (mixed mops) on* $\mathbb{R}$.

A moment matrix for several weights: Define two sets of weights

$$\psi_1(x),\dots,\psi_q(x) \quad \text{and} \quad \varphi_1(y),\dots,\varphi_p(y), \qquad \text{with } x,y\in\mathbb{R},$$

and deformed weights depending on *time* parameters $s_\alpha = (s_{\alpha 1}, s_{\alpha 2},\dots)$ ($1\le \alpha\le q$) and $t_\beta = (t_{\beta 1}, t_{\beta 2},\dots)$ ($1\le\beta\le p$), denoted by

$$\psi_\alpha^{-s}(x) := \psi_\alpha(x) e^{-\sum_{k=1}^\infty s_{\alpha k}x^k} \quad \text{and} \quad \varphi_\beta^t(y) := \varphi_\beta(y) e^{\sum_{k=1}^\infty t_{\beta k}y^k}.$$

That is, each weight goes with its own set of times. For each set of positive integers[4]

$$m=(m_1,\dots,m_q), n=(n_1,\dots,n_p) \text{ with } |m|=|n|,$$

consider the determinant of a moment matrix T_{mn}, composed of blocks and of size $|m|=|n|$, with regard to a (not necessarily symmetric) inner product $\langle\cdot\,|\,\cdot\rangle$

$$
\begin{aligned}
&\tau_{mn}(s_1,\dots,s_q;t_1,\dots,t_p)\\
&\quad := \det T_{mn}\\
&\quad := \det\begin{pmatrix} T_{mn}^{11} & \dots & T_{mn}^{1p}\\ \vdots & & \vdots \\ T_{mn}^{q1} & \dots & T_{mn}^{qp}\end{pmatrix}\\
&\quad := \det\begin{pmatrix} \left(\left\langle x^i\psi_1^{-s}(x)\,\middle|\, y^j\varphi_1^t(y)\right\rangle\right)_{\substack{0\le i<m_1\\0\le j<n_1}} & \dots & \left(\left\langle x^i\psi_1^{-s}(x)\,\middle|\, y^j\varphi_p^t(y)\right\rangle\right)_{\substack{0\le i<m_1\\0\le j<n_p}}\\ \vdots & & \vdots\\ \left(\left\langle x^i\psi_q^{-s}(x)\,\middle|\, y^j\varphi_1^t(y)\right\rangle\right)_{\substack{0\le i<m_q\\0\le j<n_1}} & \dots & \left(\left\langle x^i\psi_q^{-s}(x)\,\middle|\, y^j\varphi_p^t(y)\right\rangle\right)_{\substack{0\le i<m_q\\0\le j<n_p}}\end{pmatrix}.
\end{aligned}
\tag{1}
$$

[4] $|m| = \sum_{\alpha=1}^q m_\alpha$ and $|n| = \sum_{\beta=1}^p n_\beta$.

A typical inner product to keep in mind is

$$\langle f(x) \mid g(y) \rangle = \iint_{\mathbb{R}^2} f(x) g(y)\, d\mu(x,y), \tag{2}$$

where $\mu = \mu(x,y)$ is a fixed measure on $\mathbb{R}^2$, perhaps having support on a line or curve.

From moment matrices to polynomials and their Cauchy transforms:

I. Then, for $1 \le \beta, \beta' \le p$, the following expressions are polynomials (with coefficients depending on s and t)[5]

$$\begin{array}{rcll} z^{n_\beta} \dfrac{\tau_{mn}(t_\beta - [z^{-1}])}{\tau_{mn}} & := & Q_{mn}^{(\beta,\beta)}(z) = z^{n_\beta} + \dots & \\ \varepsilon_{\beta\beta'}(n) z^{n_{\beta'}-1} \dfrac{\tau_{m,n+e_\beta-e_{\beta'}}(t_{\beta'} - [z^{-1}])}{\tau_{mn}} & := & Q_{mn}^{(\beta,\beta')}(z), & \left\{ \begin{array}{l} \text{of degree } < n_{\beta'} \\ \text{for } \beta' \neq \beta, \end{array} \right. \end{array} \tag{3}$$

satisfying, for each β, the following orthogonality conditions

$$\left\langle x^i \psi_\alpha^{-s}(x) \;\middle|\; \sum_{\beta'=1}^{p} Q_{mn}^{(\beta,\beta')}(y) \varphi_{\beta'}^{t}(y) \right\rangle = 0 \quad \text{for} \quad \left\{ \begin{array}{l} 1 \le \alpha \le q \\ 0 \le i \le m_\alpha - 1. \end{array} \right. \tag{4}$$

II. In the same way, the following expressions are polynomials (depending on s and t)

$$\epsilon_{\beta\alpha}(n,m)\, z^{m_\alpha - 1} \frac{\tau_{m-e_\alpha, n-e_\beta}(s_\alpha + [z^{-1}])}{\tau_{mn}} = P_{nm}^{*(\beta,\alpha)}(z) \;\; \text{of degree} < m_\alpha \tag{5}$$

satisfying, for each β, the orthogonality relations

$$\left\{ \begin{array}{l} \left\langle \displaystyle\sum_{\alpha=1}^{q} P_{nm}^{*(\beta,\alpha)}(x) \psi_\alpha^{-s}(x) \;\middle|\; y^j \varphi_{\beta'}^{t}(y) \right\rangle = 0 \;\; \text{for} \left\{ \begin{array}{l} 1 \le \beta' \le p, \;\; 0 \le j \le n_{\beta'} - 1 \\ \text{except } \beta' = \beta, \;\; j = n_\beta - 1 \end{array} \right. \\ \\ \left\langle \displaystyle\sum_{\alpha=1}^{q} P_{nm}^{*(\beta,\alpha)}(x) \psi_\alpha^{-s}(x) \;\middle|\; y^{n_\beta - 1} \varphi_{\beta}^{t}(y) \right\rangle = 1. \end{array} \right. \tag{6}$$

III. The following expressions are Cauchy transforms of the polynomials obtained in II:

[5] Introduce the notation $[\alpha] := (\alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, \dots)$ for $\alpha \in \mathbf{C}$. Only shifted times will be made explicit in the τ-functions; i.e., $\tau_{mn}(t_\ell - [z^{-1}])$ means that τ_{mn} still depends on all time parameters, but the variable t_ℓ only gets shifted. Moreover, here and below all the expressions $\varepsilon_{\alpha\beta}(n)$, $\epsilon_{\alpha\beta}(n,m)$, etc... all equal ± 1 and will be given later. Throughout the paper, we use the standard notation $e_1 = (1,0,0,\dots)$, $e_2 = (0,1,0,\dots)$.

$$
\begin{aligned}
z^{-n_\beta}\frac{\tau_{mn}(t_\beta+[z^{-1}])}{\tau_{mn}} &= \left\langle \sum_{\alpha=1}^{q} P^{*(\beta,\alpha)}_{nm}(x)\psi_\alpha^{-s}(x) \,\middle|\, \frac{\varphi^t_\beta(y)}{z-y} \right\rangle \\
\varepsilon_{\beta\beta'}(n)z^{-n_{\beta'}-1}\frac{\tau_{m,n+e_{\beta'}-e_\beta}(t_{\beta'}+[z^{-1}])}{\tau_{mn}} &= \left\langle \sum_{\alpha=1}^{q} P^{*(\beta,\alpha)}_{nm}(x)\psi_\alpha^{-s}(x) \,\middle|\, \frac{\varphi^t_{\beta'}(y)}{z-y} \right\rangle
\end{aligned}
\tag{7}
$$

IV. Similarly, the following expressions are Cauchy transforms of the polynomials obtained in I:

$$
\epsilon_{\alpha\beta}(m,n)\, z^{-m_\alpha-1}\,\frac{\tau_{m+e_\alpha,n+e_\beta}(s_\alpha-[z^{-1}])}{\tau_{mn}} = \left\langle \frac{\psi_\alpha^{-s}(x)}{z-x} \,\middle|\, \sum_{\beta'=1}^{p} Q^{(\beta,\beta')}_{mn}(y)\varphi^t_{\beta'}(y) \right\rangle . \tag{8}
$$

The statements I, II, III and IV summarize sections 1, 2 and 3. As will appear in Section 2, the polynomials appearing in (**I**) are called *Type II* $|_{\varphi^t_\beta}$ mixed multiple orthogonal polynomials, whereas those appearing in (**II**) *Type I* $|_{\psi^t_\alpha}$ mixed multiple orthogonal polynomials. These were introduced by E. Daems and A. Kuijlaars [9], in the context of non-intersecting Brownian motions; they are a generalization of multiple orthogonal polynomials, where instead of one set of weights, there are two sets (the classical orthogonal polynomials correspond to one set with one element). They were introduced and studied by Aptekarev, Bleher, Geronimo, Kuijlaars, Van Assche [6, 14, 15, 8]. Around the same time, they were introduced by Adler-van Moerbeke in the context of band matrices and vertex operator solutions to the KP hierarchy [2]. In [7, 8], they were used in the context of non-intersecting Brownian motions and random matrices with external source.

The $(p+q)$**-KP hierarchy**: Define two matrices $W_{mn}(z)$ and $W^*_{mn}(z)$ of size $p+q$, whose entries are given by ratios of determinants τ_{mn} of moment matrices as above, but with appropriately shifted t and s parameters. They turn out to be the wave and dual wave matrices for the *$(p+q)$-KP hierarchy*. It is remarkable that, upon setting all t and s parameters equal to zero, the matrix $W_{mn}(z)$ below is precisely the *Riemann-Hilbert* matrix characterizing the mixed multiple orthogonal polynomials! Similarly $W^*_{mn}(z)$ at $t=s=0$ satisfies the Riemann-Hilbert problem characterizing alternately the "dual" multiple orthogonal polynomials or the inverse transpose matrix of $W_{mn}(z)$ at $t=s=0$. The Riemann-Hilbert matrix for the multiple-orthogonal polynomials has been defined in Daems-Kuijlaars [9], which is a far generalization of the Riemann-Hilbert matrix of Fokas-Its-Kitaev [11] and Deift-Zhou [10]. Using identities as in I to IV, the two left blocks of W_{mn} and the two right blocks of W^*_{mn} are mixed multiple orthogonal polynomials, and the remaining blocks are Cauchy transforms of such polynomials; for explicit expressions, see Section 5. The matrix $W_{mn}(z)$ is defined by

$$W_{mn}(z)\,\mathrm{diag}\left(e^{-\sum_1^\infty t_{1k}z^k},\dots,e^{-\sum_1^\infty t_{pk}z^k},e^{-\sum_1^\infty s_{1k}z^k},\dots,e^{-\sum_1^\infty s_{qk}z^k}\right):=$$

$$\begin{pmatrix} \left(\varepsilon_{\beta\beta'}(n)\dfrac{\tau_{m,n+e_\beta-e_{\beta'}}(t_{\beta'}-[z^{-1}])}{\tau_{mn}}z^{n_{\beta'}+\delta_{\beta\beta'}-1}\right)_{\substack{1\le\beta\le p\\1\le\beta'\le p}} & \left(\epsilon_{\alpha\beta}(m,n)\dfrac{\tau_{m+e_\alpha,n+e_\beta}(s_\alpha-[z^{-1}])}{\tau_{mn}}z^{-m_\alpha-1}\right)_{\substack{1\le\beta\le p\\1\le\alpha\le q}} \\ \left(\epsilon_{\alpha\beta}(m,n)\dfrac{\tau_{m-e_\alpha,n-e_\beta}(t_\beta-[z^{-1}])}{\tau_{mn}}z^{n_\beta-1}\right)_{\substack{1\le\alpha\le q\\1\le\beta\le p}} & \left(\varepsilon_{\alpha'\alpha}(m)\dfrac{\tau_{m+e_\alpha-e_{\alpha'},n}(s_\alpha-[z^{-1}])}{\tau_{mn}}z^{\delta_{\alpha\alpha'}-1-m_\alpha}\right)_{\substack{1\le\alpha'\le q\\1\le\alpha\le q}} \end{pmatrix}, \tag{9}$$

with inverse transpose matrix given by

$$W^*_{mn}(z)\,\mathrm{diag}\left(e^{\sum_1^\infty t_{1k}z^k},\dots,e^{\sum_1^\infty t_{pk}z^k},e^{\sum_1^\infty s_{1k}z^k},\dots,e^{\sum_1^\infty s_{qk}z^k}\right)=$$

$$\begin{pmatrix} \left(\varepsilon_{\beta'\beta}(n)\dfrac{\tau_{m,n+e_\beta-e_{\beta'}}(t_\beta+[z^{-1}])}{\tau_{mn}}z^{\delta_{\beta'\beta}-1-n_\beta}\right)_{\substack{1\le\beta'\le p\\1\le\beta\le p}} & \left(-\epsilon_{\beta\alpha}(n,m)\dfrac{\tau_{m-e_\alpha,n-e_\beta}(s_\alpha+[z^{-1}])}{\tau_{mn}}z^{m_\alpha-1}\right)_{\substack{1\le\beta\le p\\1\le\alpha\le q}} \\ \left(-\epsilon_{\beta\alpha}(n,m)\dfrac{\tau_{m+e_\alpha,n+e_\beta}(t_\beta+[z^{-1}])}{\tau_{mn}}z^{-n_\beta-1}\right)_{\substack{1\le\alpha\le q\\1\le\beta\le p}} & \left(\varepsilon_{\alpha\alpha'}(m)\dfrac{\tau_{m+e_\alpha-e'_\alpha,n}(s_{\alpha'}+[z^{-1}])}{\tau_{mn}}z^{\delta_{\alpha\alpha'}-1+m_{\alpha'}}\right)_{\substack{1\le\alpha\le q\\1\le\alpha'\le q}} \end{pmatrix}. \tag{10}$$

The matrices $W_{mn}(z)$ and $W^*_{m^*n^*}(z)$ satisfy the bilinear identities *which characterize the τ-function of the $(p+q)$-KP hierarchy*

$$\oint_\infty W_{mn}(z;s,t)W^*_{m^*n^*}(z;s^*,t^*)^\top dz=0, \tag{11}$$

for all m,n,m^*,n^* such that $|m|=|n|$, $|m^*|=|n^*|$ and all $s,t,s^*,t^*\in\mathbf{C}^\infty$. The integral above is taken along a small circle about $z=\infty$; writing out the identity above componentwise and using the expressions (9) and (10) for W and W^*, the bilinear identity (11) is equivalent to the single identity

$$\sum_{\beta=1}^p\oint_\infty(-1)^{\sigma_\beta(n)}\,\tau_{m,n-e_\beta}(t_\beta-[z^{-1}])\tau_{m^*,n^*+e_\beta}(t^*_\beta+[z^{-1}])e^{\sum_1^\infty(t_{\beta k}-t^*_{\beta k})z^k}\,z^{n_\beta-n^*_\beta-2}\,dz=$$

$$\sum_{\alpha=1}^q\oint_\infty(-1)^{\sigma_\alpha(m)}\,\tau_{m+e_\alpha,n}(s_\alpha-[z^{-1}])\tau_{m^*-e_\alpha,n^*}(s^*_\alpha+[z^{-1}])e^{\sum_1^\infty(s_{\alpha k}-s^*_{\alpha k})z^k}\,z^{m^*_\alpha-m_\alpha-2}\,dz,$$

where $|m^*|=|n^*|+1$ and $|m|=|n|-1$ and

$$\sigma_\alpha(m)=\sum_{\alpha'=1}^{\alpha}(m_{\alpha'}-m^*_{\alpha'})\quad\text{and}\quad\sigma_\beta(n)=\sum_{\beta'=1}^{\beta}(n_{\beta'}-n^*_{\beta'}).$$

It remains an open problem to have a clear understanding of why the $W_{mn}(z;s,t)$-matrix above, evaluated at $t=s=0$, coincides with the Riemann-Hilbert matrix for the mixed multiple orthogonal polynomials.

PDE's for the determinant of moment matrices: Upon actually computing the residues in the contour integrals above, the functions τ_{mn}, with $|m|=|n|$, satisfy the following PDE's expressed in terms of the Hirota symbol[6]:

$$\begin{aligned}
\tau_{mn}^2 \frac{\partial^2}{\partial t_{\beta,\ell+1}\partial t_{\beta',1}} \ln \tau_{mn} &= S_{\ell+2\delta_{\beta\beta'}}(\tilde{\partial}_{t_\beta})\tau_{m,n+e_\beta-e_{\beta'}} \circ \tau_{m,n+e_{\beta'}-e_\beta} \\
\tau_{mn}^2 \frac{\partial^2}{\partial s_{\alpha,\ell+1}\partial s_{\alpha',1}} \ln \tau_{mn} &= S_{\ell+2\delta_{\alpha\alpha'}}(\tilde{\partial}_{s_\alpha})\tau_{m+e_{\alpha'}-e_\alpha,n} \circ \tau_{m+e_\alpha-e_{\alpha'},n} \\
\tau_{mn}^2 \frac{\partial^2}{\partial s_{\alpha,1}\partial t_{\beta,\ell+1}} \ln \tau_{mn} &= -S_\ell(\tilde{\partial}_{t_\beta})\tau_{m+e_\alpha,n+e_\beta} \circ \tau_{m-e_\alpha,n-e_\beta} \\
\tau_{mn}^2 \frac{\partial^2}{\partial t_{\beta,1}\partial s_{\alpha,\ell+1}} \ln \tau_{mn} &= -S_\ell(\tilde{\partial}_{s_\alpha})\tau_{m-e_\alpha,n-e_\beta} \circ \tau_{m+e_\alpha,n+e_\beta}.
\end{aligned} \tag{12}$$

Whereas the formulae above have in their right hand side different τ_{mn}'s, one can combine these relations to yield PDE's in a single τ_{mn}; so, these are PDE's for the determinant of the moment matrix (1). In particular, one finds the following $\binom{p+q}{2}$ PDE's, which play a fundamental role in chains of random matrices and in the transition probabilities for critical infinite-dimensional diffusions:

$$\begin{aligned}
\frac{\partial}{\partial t_{\beta',1}} \left(\frac{\frac{\partial^2}{\partial t_{\beta,2}\partial t_{\beta',1}} \ln \tau_{mn}}{\frac{\partial^2}{\partial t_{\beta,1}\partial t_{\beta',1}} \ln \tau_{mn}} \right) + \frac{\partial}{\partial t_{\beta,1}} \left(\frac{\frac{\partial^2}{\partial t_{\beta',2}\partial t_{\beta,1}} \ln \tau_{mn}}{\frac{\partial^2}{\partial t_{\beta',1}\partial t_{\beta,1}} \ln \tau_{mn}} \right) &= 0, \\
\frac{\partial}{\partial s_{\alpha',1}} \left(\frac{\frac{\partial^2}{\partial s_{\alpha,2}\partial s_{\alpha',1}} \ln \tau_{mn}}{\frac{\partial^2}{\partial s_{\alpha,1}\partial s_{\alpha',1}} \ln \tau_{mn}} \right) + \frac{\partial}{\partial s_{\alpha,1}} \left(\frac{\frac{\partial^2}{\partial s_{\alpha',2}\partial s_{\alpha,1}} \ln \tau_{mn}}{\frac{\partial^2}{\partial s_{\alpha',1}\partial s_{\alpha,1}} \ln \tau_{mn}} \right) &= 0, \\
\frac{\partial}{\partial s_{\alpha,1}} \left(\frac{\frac{\partial^2}{\partial t_{\beta,2}\partial s_{\alpha,1}} \ln \tau_{mn}}{\frac{\partial^2}{\partial t_{\beta,1}\partial s_{\alpha,1}} \ln \tau_{mn}} \right) + \frac{\partial}{\partial t_{\beta,1}} \left(\frac{\frac{\partial^2}{\partial s_{\alpha,2}\partial t_{\beta,1}} \ln \tau_{mn}}{\frac{\partial^2}{\partial s_{\alpha,1}\partial t_{\beta,1}} \ln \tau_{mn}} \right) &= 0.
\end{aligned}$$

2 Tau functions and mixed multiple orthogonal polynomials

Following [9] we introduce the notion of mixed multiple orthogonal polynomials (mixed mops), with regard to two sets of weights $\{\varphi_1,\varphi_2,\dots,\varphi_p\}$ and $\{\psi_1,\psi_2,\dots,\psi_q\}$:

[6]For a given polynomial $p(t_1,t_2,\dots)$, the Hirota symbol between functions $f=f(t_1,t_2,\dots)$ and $g=g(t_1,t_2,\dots)$ is defined by:

$$p(\frac{\partial}{\partial t_1},\frac{\partial}{\partial t_2},\dots)f\circ g := p(\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2},\dots)f(t+y)g(t-y)\Big|_{y=0}.$$

We also need the elementary Schur polynomials S_ℓ, defined by $e^{\sum_1^\infty t_k z^k} := \sum_{k\geq 0} S_k(t)z^k$ for $\ell\geq 0$ and $S_\ell(t)=0$ for $\ell<0$; moreover, set

$$S_\ell(\tilde{\partial}_t) := S_\ell(\frac{\partial}{\partial t_1},\frac{1}{2}\frac{\partial}{\partial t_2},\frac{1}{3}\frac{\partial}{\partial t_3},\dots).$$

Definition 2.1 *Let $A_1,\ A_2, \dots, A_p$ be p polynomials in y and set*

$$Q(y) := A_1(y)\varphi_1(y) + A_2(y)\varphi_2(y) + \cdots + A_p(y)\varphi_p(y).$$

Type I *For $\alpha \in \{1, 2, \dots, q\}$ the polynomials $A_1,\ A_2, \dots, A_p$ are said to be* Type I normalized *with respect to ψ_α, denoted* Type I $_{|\psi_\alpha}$, *if $\deg(A_\beta) < n_\beta$ for $\beta = 1, \dots, p$ and Q satisfies the following orthogonality conditions*

$$\left\langle x^i \psi_{\alpha'}(x) \mid Q(y) \right\rangle = \delta_{\alpha\alpha'}\delta_{i,m_\alpha - 1}, \qquad i = 0, \dots, m_{\alpha'} - 1,\ 1 \le \alpha' \le q. \tag{13}$$

Type II *For $\beta \in \{1, \dots, p\}$ the polynomials $A_1,\ A_2, \dots, A_p$ are said to be* Type II normalized *with respect to φ_β, denoted* Type II $_{|\varphi_\beta}$, *if A_β is monic of degree n_β and $\deg(A_{\beta'}) < n_{\beta'}$ for $1 \le \beta' \le p$, with $\beta' \ne \beta$, and Q satisfies the following orthogonality conditions*

$$\left\langle x^i \psi_\alpha(x) \mid Q(y) \right\rangle = 0, \qquad i = 0, \dots, m_\alpha - 1,\ 1 \le \alpha \le q. \tag{14}$$

In both cases, the polynomials $A_1, \dots, A_p$ are called multiple orthogonal polynomials of mixed type*, or* mixed mops *for brevity.*

Proposition 2.2 *For $\beta = 1, \dots, p$, let*

$$Q_{mn}^{(\beta)}(y) := Q_{mn}^{(\beta,1)}(y)\varphi_1^t(y) + \cdots + Q_{mn}^{(\beta,p)}(y)\varphi_p^t(y), \tag{15}$$

where $Q_{mn}^{(\beta,\beta')}$, with $1 \le \beta, \beta' \le p$ are the polynomials, defined by

$$\begin{aligned} Q_{mn}^{(\beta,\beta)}(z) &:= z^{n_\beta}\frac{\tau_{mn}(t_\beta - [z^{-1}])}{\tau_{mn}} \\ Q_{mn}^{(\beta,\beta')}(z) &:= \varepsilon_{\beta\beta'}(n) z^{n_{\beta'}-1}\frac{\tau_{m,n+e_\beta-e_{\beta'}}(t_{\beta'} - [z^{-1}])}{\tau_{mn}}, \qquad \beta' \ne \beta, \end{aligned} \tag{16}$$

and

$$\varepsilon_{\beta\beta'}(n) = \begin{cases} (-1)^{n_{\beta'+1}+n_{\beta'+2}+\cdots+n_\beta+1} & \text{if} \quad \beta > \beta', \\ (-1)^{n_{\beta+1}+n_{\beta+2}+\cdots+n_{\beta'}} & \text{if} \quad \beta < \beta'. \end{cases} \tag{17}$$

Then $Q_{mn}^{(\beta,1)}(y), \dots, Q_{mn}^{(\beta,p)}(y)$ are Type II $_{|\varphi_\beta^t}$ *mixed mops.*

Proof For $j = 0, 1, 2, \dots$ and $\beta = 1, 2, \dots, p$ we define a column vector C_j^β of size $|m|$ by

$$C_j^\beta := \begin{pmatrix} \left(\left\langle x^{i_1}\psi_1^{-s}(x) \,\middle|\, y^j \varphi_\beta^t(y) \right\rangle\right)_{0 \le i_1 < m_1} \\ \left(\left\langle x^{i_2}\psi_2^{-s}(x) \,\middle|\, y^j \varphi_\beta^t(y) \right\rangle\right)_{0 \le i_2 < m_2} \\ \vdots \\ \left(\left\langle x^{i_q}\psi_q^{-s}(x) \,\middle|\, y^j \varphi_\beta^t(y) \right\rangle\right)_{0 \le i_q < m_q} \end{pmatrix}. \tag{18}$$

When its size is important (see the proof of Proposition 2.3) we write $C_j^\beta(m)$ for (18). Notice that the moment matrix T_{mn}, defined in (1), can be expressed in terms of the columns C_j^β, and so

$$\tau_{mn} = \det\left(C_0^1,\ C_1^1,\ \ldots,\ C_{n_1-1}^1,\ C_0^2,\ C_1^2,\ \ldots,\ C_{n_p-1}^p\right). \tag{19}$$

For future use, let us point out that the dependence of τ_{mn} on the t variables is as follows:

$$\begin{aligned}
\tau_{mn}(t_1) &= \det(C_0^1(t_1),\ C_1^1(t_1),\ \ldots,\ C_{n_1-1}^1(t_1),\ C_0^2,\ C_1^2,\ \ldots,\ C_{n_p-1}^p),\\
\tau_{mn}(t_2) &= \det(C_0^1,\ C_1^1,\ \ldots,\ C_{n_1-1}^1,\ C_0^2(t_2),\ C_1^2(t_2),\ \ldots,\ C_{n_p-1}^p),\\
&\ \vdots\\
\tau_{mn}(t_{n_p}) &= \det(C_0^1,\ C_1^1,\ \ldots,\ C_{n_1-1}^1,\ C_0^2,\ C_1^2,\ \ldots,\ C_{n_p-1}^p(t_{n_p})).
\end{aligned} \tag{20}$$

Since

$$Q_{mn}^{(\beta)}(y) = \sum_{\beta'=1}^{p} Q_{mn}^{(\beta,\beta')}(y)\varphi_{\beta'}^t(y) = \sum_{\beta'=1}^{p}\left(\delta_{\beta\beta'}y^{n_\beta} + \sum_{j=0}^{n_{\beta'}-1} A_{\beta\beta'}^j y^j\right)\varphi_{\beta'}^t(y), \tag{21}$$

the orthogonality conditions (14) for $Q = Q_{mn}^{(\beta)}$ can be written as the linear system

$$\sum_{\beta'=1}^{p}\sum_{j=0}^{n_{\beta'-1}} A_{\beta\beta'}^j C_j^{\beta'} = -C_{n_\beta}^\beta,$$

of $|m|$ equations, in the $|n|$ $(=|m|)$ unknowns $A_{\beta\beta'}^j$, where $1 \le \beta' \le p$ and $0 \le j < n_{\beta'-1}$. If we order these unknowns as follows: $A_{\beta 1}^0$, $A_{\beta 1}^1, \ldots, A_{\beta 1}^{n_1-1}$, $A_{\beta 2}^0$, $A_{\beta 2}^1, \ldots, A_{\beta p}^{n_p-1}$, then this linear system has precisely τ_{mn} as determinant, in view of (19). Since $\tau_{mn} \neq 0$, generically, we have by Cramer's rule,

$$A_{\beta\beta'}^j = \frac{\det\left(C_0^1,\ C_1^1,\ \ldots, C_{j-1}^{\beta'},\ -C_{n_\beta}^\beta,\ C_{j+1}^{\beta'},\ \ldots,\ C_{n_p-1}^p\right)}{\tau_{mn}}. \tag{22}$$

Substituted in (21) this yields an explicit expression for the *Type II* $|_{\varphi_\beta^t}$ mixed mops $Q_{mn}^{(\beta,1)}(y), \ldots, Q_{mn}^{(\beta,p)}(y)$.

In order to connect these polynomials with the tau functions τ_{mn} we first expand $\tau_{mn}(t_\beta - [z^{-1}])$ using (20). Thus, we need to compute $C_j^\beta(t_\beta - [z^{-1}])$, which we claim to be given by

$$C_j^\beta(t_\beta - [z^{-1}]) = C_j^\beta(t_\beta) - z^{-1}C_{j+1}^\beta(t_\beta) = C_j^\beta - z^{-1}C_{j+1}^\beta, \tag{23}$$

where the last equality is the notational simplification agreed upon. To prove the first equality in (23), which is an equality of formal series in z^{-1}, let us write a typical entry of the column vector $C_j^\beta(t_\beta)$ with its explicit time-dependence on t_β,

$$\left\langle x^i\psi_\alpha^{-s}(x)\ \middle|\ y^j\varphi_\beta^t(y)\right\rangle = \left\langle x^i\psi_\alpha^{-s}(x)\ \middle|\ y^j\varphi_\beta(y)e^{\sum_{k=1}^\infty t_{\beta k}y^k}\right\rangle,$$

where $1 \le \alpha \le q$ and $0 \le i < m_\alpha$. The following trivial identity will be used over and over again in this paper

$$e^{-\sum_1^\infty \frac{x^i}{i}} = 1 - x. \tag{24}$$

In view of the latter, the same entry of $C_j^\beta(t_\beta - [z^{-1}])$ (as above) is given by

$$\left\langle x^i\psi_\alpha^{-s}(x)\ \middle|\ y^j\varphi_\beta^t(y)\left(1-\frac{y}{z}\right)\right\rangle = \left\langle x^i\psi_\alpha^{-s}(x)\ \middle|\ y^j\varphi_\beta^t(y)\right\rangle - \frac{1}{z}\left\langle x^i\psi_\alpha^{-s}(x)\ \middle|\ y^{j+1}\varphi_\beta^t(y)\right\rangle$$

which proves (23). Using the fact that the determinant is a skew-symmetric multilinear function of its columns, which vanishes when two columns are equal, it follows from (20), (23) and (22) that

$$\begin{aligned}
&z^{n_\beta}\tau_{mn}(t_\beta - [z^{-1}])\\
&\quad= \det(C_0^1,\dots,C_{n_{\beta-1}-1}^{\beta-1},\, zC_0^\beta - C_1^\beta,\dots,zC_{n_\beta-1}^\beta - C_{n_\beta}^\beta,\, C_0^{\beta+1},\dots,C_{n_p-1}^p)\\
&\quad\overset{(*)}{=} \sum_{j=0}^{n_\beta} z^j\det(C_0^1,\, C_1^1,\dots,C_{j-1}^\beta, -C_{j+1}^\beta,\dots,-C_{n_\beta}^\beta,\, C_0^{\beta+1},\dots,C_{n_p-1}^p)\\
&\quad= \sum_{j=0}^{n_\beta} z^j\det(C_0^1,\, C_1^1,\dots,C_{j-1}^\beta,\, -C_{n_\beta}^\beta,\, C_{j+1}^\beta,\dots,C_{n_\beta-1}^\beta,\, C_0^{\beta+1},\dots,C_{n_p-1}^p)\\
&\quad= \sum_{j=0}^{n_\beta} z^j A_{\beta\beta}^j\tau_{mn}\\
&\quad= \tau_{mn}\, Q_{mn}^{(\beta,\beta)}(z).
\end{aligned}$$

In $(*)$ it is understood that all the columns between $-C_{j+1}^\beta$ and $-C_{n_\beta}^\beta$ come with negative signs and no others; this notation shall be used freely in the sequel, without further mention.

For $Q_{mn}^{(\beta,\beta')}$ with $\beta \neq \beta'$ we also need to keep track of signs and of shifts in the first index of the tau function, as is seen in the following computation, where we suppose that $\beta < \beta'$:

$$\begin{aligned}
&z^{n_{\beta'}-1}\tau_{m,n+e_\beta-e_{\beta'}}(t_{\beta'} - [z^{-1}])\\
&\quad= \det(C_0^1,\dots,C_{n_\beta-1}^\beta,\, C_{n_\beta}^\beta,\dots,zC_0^{\beta'} - C_1^{\beta'},\dots,zC_{n_{\beta'}-2}^{\beta'} - C_{n_{\beta'}-1}^{\beta'}, C_0^{\beta'+1},\dots,C_{n_p-1}^p)\\
&\quad= \sum_{j=0}^{n_{\beta'}-1} z^j\det(C_0^1,\dots,\, C_{n_\beta}^\beta,\dots,C_{j-1}^{\beta'},\, -C_{j+1}^{\beta'},\dots,-C_{n_{\beta'-1}}^{\beta'},\, C_0^{\beta'+1},\dots,C_{n_p-1}^p)
\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon_{\beta\beta'}(n) \sum_{j=0}^{n_{\beta'}-1} z^j \det(C_0^1,\ldots,\, C_{n_\beta-1}^\beta, C_0^{\beta+1},\ldots, C_{j-1}^{\beta'},\, -C_{n_\beta}^\beta,\, C_{j+1}^{\beta'},\ldots, C_{n_p-1}^p)\\
&= \varepsilon_{\beta\beta'}(n) \sum_{j=0}^{n_{\beta'}-1} z^j\, A_{\beta\beta'}^j\, \tau_{mn}\\
&= \varepsilon_{\beta\beta'}(n)\, \tau_{mn}\, Q_{mn}^{(\beta,\beta')}(z).
\end{aligned}
$$

The sign $\varepsilon_{\beta\beta'}(n)$ which we introduced when moving the column $C_{n_\beta}^\beta$ to the right is given by $(-1)^{n_{\beta+1}+\cdots+n_{\beta'}}$, in agreement with (17). When $\beta > \beta'$ the column $C_{n_\beta}^\beta$ is moved to the left, which yields a sign $\varepsilon_{\beta\beta'}(n) = -(-1)^{n_{\beta'+1}+\cdots+n_\beta}$, as is easily checked. $\square$

The tau functions τ_{mn} also lead to Type I normalized mixed mops, as given in the following proposition.

Proposition 2.3 *For $\alpha = 1,\ldots,q$, let*

$$
P_{mn}^{(\alpha)}(y) := P_{mn}^{(\alpha,1)}(y)\varphi_1^t(y) + \cdots + P_{mn}^{(\alpha,p)}(y)\varphi_p^t(y), \tag{25}
$$

where $P_{mn}^{(\alpha,\beta)}$ are the polynomials, defined by

$$
P_{mn}^{(\alpha,\beta)}(z) := \epsilon_{\alpha\beta}(m,n) z^{n_\beta - 1}\, \frac{\tau_{m-e_\alpha,n-e_\beta}(t_\beta - [z^{-1}])}{\tau_{mn}}, \tag{26}
$$

with leading sign

$$
\epsilon_{\alpha\beta}(m,n) = (-1)^{m_1+\cdots+m_\alpha}(-1)^{n_1+\cdots+n_\beta}. \tag{27}
$$

Then $P_{mn}^{(\alpha,1)}(y),\ldots,P_{mn}^{(\alpha,p)}(y)$ are Type I $|_{\psi_\alpha^{-s}}$ *mixed mops.*

Proof Letting

$$
P_{mn}^{(\alpha)}(y) = \sum_{\beta=1}^{p} P_{mn}^{(\alpha,\beta)}(y)\varphi_\beta^t(y) = \sum_{\beta=1}^{p}\sum_{j=0}^{n_\beta-1} B_{\alpha\beta}^j\, y^j\, \varphi_\beta^t(y), \tag{28}
$$

the orthogonality conditions (13) for $Q = P_{mn}^{(\alpha)}$ can be written as the linear system

$$
\sum_{\beta=1}^{p}\sum_{j=0}^{m_\beta-1} B_{\alpha\beta}^j C_j^\beta = E_{m_\alpha}^\alpha,
$$

where $E_{m_\alpha}^\alpha$ denotes the column vector of size $|m|$ with a 1 at position m_α of the α-th block (so at position $m_1 + m_2 + \cdots + m_\alpha$), and zeros elsewhere. Cramer's rule now yields

$$
\begin{aligned}
B_{\alpha\beta}^j &= \frac{\det(C_0^1,\, C_1^1,\ldots, C_{j-1}^\beta,\, E_{m_\alpha}^\alpha,\, C_{j+1}^\beta,\ldots, C_{n_p-1}^p)}{\tau_{mn}},\\
&= \epsilon_{\alpha\beta}(m,n)(-1)^{j+1-n_\beta}\, \frac{\det(D_0^1,\, D_1^1,\ldots, D_{j-1}^\beta,\, \widehat{D_j^\beta},\, D_{j+1}^\beta,\ldots, D_{n_p-1}^p)}{\tau_{mn}},
\end{aligned}
$$

where the last line was obtained by expanding the determinants along the $E^{\alpha}_{m_\alpha}$ column, $\epsilon_{\alpha\beta}(m,n)$ is given by (27) and $\widehat{D^{\gamma}_k}$ is the column vector D^{γ}_k with its $(m_1+\cdots+m_\alpha)$-th entry removed, i.e., $\widehat{D^{\gamma}_k} := C^{\gamma}_k(m-e_\alpha)$. This yields explicit expressions for the *Type I* $|_{\psi^{-s}_\alpha}$ mixed mops. To connect them with tau functions, we notice on the one hand that the columns $\widehat{D^{\gamma}_k}$ appear in the matrices which define the tau functions $\tau_{m-e_\alpha,\star}$, and on the other hand that these columns behave in the same way (23) as C^{γ}_k under shifts. Therefore we can compute, as before

$$
\begin{aligned}
&z^{n_\beta-1}\tau_{m-e_\alpha,n-e_\beta}(t_\beta-\left[z^{-1}\right])\\
&= \det(\widehat{D^1_0},\ldots,\widehat{D^{\beta-1}_{n_{\beta-1}-1}},\, z\widehat{D^{\beta}_0}-\widehat{D^{\beta}_1},\ldots,z\widehat{D^{\beta}_{n_\beta-2}}-\widehat{D^{\beta}_{n_\beta-1}},\widehat{D^{\beta+1}_0},\ldots,\widehat{D^{p}_{n_p-1}})\\
&= \sum_{j=0}^{n_\beta-1} z^j \det(\widehat{D^1_0},\ldots,\widehat{D^{\beta}_{j-1}},\,-\widehat{D^{\beta}_{j+1}},\ldots,-\widehat{D^{\beta}_{n_\beta-1}},\,\widehat{D^{\beta+1}_0},\ldots,\widehat{D^{p}_{n_p-1}})\\
&= \sum_{j=0}^{n_\beta-1} (-1)^{n_\beta-j-1} z^j \det(\widehat{D^1_0},\ldots,\widehat{D^{\beta}_{j-1}},\,\widehat{D^{\beta}_{j}},\,\widehat{D^{\beta}_{j+1}},\ldots,\widehat{D^{p}_{n_p-1}})\\
&= \epsilon_{\alpha\beta}(m,n)\sum_{j=0}^{n_\beta-1} z^j\, B^j_{\alpha\beta}\,\tau_{mn}\\
&= \epsilon_{\alpha\beta}(m,n)\, P^{(\alpha,\beta)}_{mn}(z)\,\tau_{mn}.
\end{aligned}
$$

□

3 Cauchy transforms

We now show that certain shifts of the tau function, appearing in the Riemann-Hilbert matrix of [9], are (formal) Cauchy transforms. For a function F and a weight ψ, define its Cauchy transform as

$$\mathcal{C}_\psi G(z) := \left\langle \frac{\psi(x)}{z-x} \mid G(y) \right\rangle = \sum_{i=0}^{\infty} \frac{1}{z^{i+1}} \left\langle x^i\psi(x) \mid G(y) \right\rangle, \tag{29}$$

i.e., our Cauchy transforms will be formal in the sense that we always think of z as being large, and this is precisely how it will be used. The first type of Cauchy transforms which we are interested in are given in the following proposition.

Proposition 3.1 *For $\alpha=1,\ldots,q$ and $\beta=1,\ldots,p$, the Cauchy transforms of $Q^{(\beta)}_{mn}(y)=Q^{(\beta,1)}_{mn}(y)\varphi^t_1(y)+\cdots+Q^{(\beta,p)}_{mn}(y)\varphi^t_p(y)$, with respect to ψ^{-s}_α can be expressed in terms of tau functions as follows.*

$$\mathcal{C}_{\psi^{-s}_\alpha} Q^{(\beta)}_{mn}(z) = \epsilon_{\alpha\beta}(m,n)\, z^{-m_\alpha-1}\,\frac{\tau_{m+e_\alpha,n+e_\beta}(s_\alpha-\left[z^{-1}\right])}{\tau_{mn}}. \tag{30}$$

Proof The proof is based on an investigation of the moment matrix by row. Therefore we define, for $\alpha = 1, \ldots, q$ and for $i = 0, 1, 2, \ldots$ the row R^i_α of size $|n|$ by

$$R^i_\alpha := \left(\left(\left\langle x^i \psi^{-s}_\alpha(x) \,\middle|\, y^{j_1} \varphi^t_1(y) \right\rangle\right)_{0 \le j_1 < n_1} \cdots \left(\left\langle x^i \psi^{-s}_\alpha(x) \,\middle|\, y^{j_p} \varphi^t_p(y) \right\rangle\right)_{0 \le j_p < n_p} \right).$$

When its size is important we write $R^i_\alpha(n)$ for $R^i_\alpha(n)$. The moment matrix T_{mn} can now be expressed in terms of the rows R^i_α, and so

$$\tau_{mn} = \det \begin{pmatrix} R^0_1 \\ R^1_1 \\ \vdots \\ R^{m_1-1}_1 \\ R^0_2 \\ \vdots \\ R^{m_q-1}_q \end{pmatrix}. \tag{31}$$

The tau function which we need to compute is $\tau_{m+e_\alpha, n+e_\beta}(s_\alpha - [z^{-1}])$; so throughout the proof, $R^i_{\alpha'}$ stands for $R^i_{\alpha'}(n + e_\beta)$ for all $1 \le \alpha' \le q$ and $i = 0, 1, 2, \ldots$. Notice that the only rows which depend on the time variables $s_\alpha = (s_{\alpha 1}, s_{\alpha 2}, \ldots)$ are the rows R^i_α. Recall the dependence of ψ^{-s}_α on s_α as follows

$$\psi^{-s}_\alpha(x) = \psi_\alpha(x) e^{-\sum_{k=1}^\infty s_{\alpha k} x^k},$$

so that, according to the identity (24), when s_α gets replaced by $s_\alpha - [z^{-1}]$, then $\psi^{-s}_\alpha(x)$ gets replaced by $\psi^{-s}_\alpha(x)\left(1 + \frac{x}{z} + \frac{x^2}{z^2} + \cdots\right)$. It follows that

$$R^i_\alpha(s_\alpha - [z^{-1}]) = \left(\left\langle x^i \psi^{-s}_\alpha(x) \left(1 + \frac{x}{z} + \frac{x^2}{z^2} + \cdots\right) \,\middle|\, y^{j_{\beta'}} \varphi^t_{\beta'}(y) \right\rangle\right)_{\substack{1 \le \beta' < p \\ 0 \le j_{\beta'} < n'_{\beta'}}},$$

where we introduce the convenient abbreviation $n'_{\beta'} := n_{\beta'} + \delta_{\beta\beta'} = (n + e_\beta)_{\beta'}$. Notice that

$$R^i_\alpha(s_\alpha - [z^{-1}]) = R^i_\alpha(s_\alpha) + \frac{1}{z} R^{i+1}_\alpha(s_\alpha - [z^{-1}]),$$

for $0 \le i \le m_\alpha - 1$; we stop at $m_\alpha - 1$ because the highest index i for which R^i_α appears in $T_{m+e_\alpha, n+e_\beta}$ is $i = m_\alpha$. By recursively applying this formula we get that for $0 \le i \le m_\alpha - 1$

$$R^i_\alpha(s_\alpha - [z^{-1}]) = R^i_\alpha(s_\alpha) + \text{ lin. comb. of lower rows.}$$

This leads to the first equality in

$$z^{-m_\alpha - 1} \tau_{m+e_\alpha, n+e_\beta}(s_\alpha - [z^{-1}]) \tag{32}$$

$$= \quad z^{-m_\alpha-1}\det\begin{pmatrix} R_1^0 \\ \vdots \\ R_\alpha^{m_\alpha-1} \\ R_\alpha^{m_\alpha}(s_\alpha-\left[z^{-1}\right]) \\ R_{\alpha+1}^0 \\ \vdots \\ R_q^{m_q-1}\end{pmatrix} = \det\begin{pmatrix} R_1^0 \\ \vdots \\ R_\alpha^{m_\alpha-1} \\ \tilde R_\alpha(z) \\ R_{\alpha+1}^0 \\ \vdots \\ R_q^{m_q-1}\end{pmatrix}.$$

For the second equality, in which we have put

$$\tilde R_\alpha(z) := \left(\quad \left(\left\langle \tfrac{\psi_\alpha^{-s}(x)}{z-x} \mid y^{j_1}\varphi_1^t(y)\right\rangle\right)_{0\le j_1<n_1'} \quad \cdots \quad \left(\left\langle \tfrac{\psi_\alpha^{-s}(x)}{z-x} \mid y^{j_p}\varphi_p^t(y)\right\rangle\right)_{0\le j_p<n_p'} \quad\right), \tag{33}$$

it suffices to show that

$$R_\alpha^{m_\alpha}(s_\alpha-\left[z^{-1}\right]) = z^{m_\alpha+1}\tilde R_\alpha(z) + \text{ lin. comb. of higher rows } R_\alpha^i. \tag{34}$$

To do this, compare a typical entry of $R_\alpha^{m_\alpha}(s_\alpha-\left[z^{-1}\right])$, to wit

$$\begin{aligned} &\left\langle x^{m_\alpha}\psi_\alpha^{-s}(x)\left(1+\frac{x}{z}+\frac{x^2}{z^2}+\cdots\right) \mid y^j\varphi_{\beta'}^t(y)\right\rangle \\ &= \quad z^{m_\alpha}\left\langle \left(\frac{x}{z}\right)^{m_\alpha}\psi_\alpha^{-s}(x)\left(1+\frac{x}{z}+\frac{x^2}{z^2}+\cdots\right) \mid y^j\varphi_{\beta'}^t(y)\right\rangle \\ &= \quad z^{m_\alpha}\left\langle \psi_\alpha^{-s}(x)\left(\left(\frac{x}{z}\right)^{m_\alpha}+\left(\frac{x}{z}\right)^{m_\alpha+1}+\cdots\right) \mid y^j\varphi_{\beta'}^t(y)\right\rangle \end{aligned}$$

with the corresponding typical entry of $z^{m_\alpha+1}\tilde R_\alpha(z)$, to wit

$$\begin{aligned} &z^{m_\alpha+1}\left\langle \frac{\psi_\alpha^{-s}(x)}{z-x} \mid y^j\varphi_{\beta'}^t(y)\right\rangle \\ &\quad= \quad z^{m_\alpha}\left\langle \psi_\alpha^{-s}(x)\left(1+\frac{x}{z}+\left(\frac{x}{z}\right)^2+\cdots\right) \mid y^j\varphi_{\beta'}^t(y)\right\rangle. \end{aligned}$$

It leads to the following explicit expression for (34):

$$R_\alpha^{m_\alpha}(s_\alpha-\left[z^{-1}\right]) + \sum_{i=0}^{m_\alpha-1} z^{m_\alpha-i}R_\alpha^i(s_\alpha) = z^{m_\alpha+1}\tilde R_\alpha(z),$$

and hence to the proof of the second equality in (32).

In order to make the connection with mixed mops, we introduce for $\beta' = 1,\ldots,p$ the row $S_\beta^{\beta'}(z)$ of size $|n+e_\beta| = |n|+1$ which has zeroes everywhere, except in its β'-th block, namely[7]

$$S_\beta^{\beta'}(y) = \left(0\ldots0 \ \ \left(y^j\right)_{0\le j<n_{\beta'}'} \ \ 0\ldots0\right).$$

[7] Recall that $n_{\beta'}' = n_{\beta'} + \delta_{\beta\beta'}$.

Notice that with this notation, definition (33) of $\tilde{R}_\alpha(z)$ can be rewritten as

$$\tilde{R}_\alpha(z) = \mathcal{C}_{\psi_\alpha^{-s}} \left(\sum_{\beta'=1}^{p} S_\beta^{\beta'}(z) \varphi_{\beta'}^t(z) \right). \tag{35}$$

It suggests the introduction of the following polynomials (in y)

$$S_{\alpha\beta}^{\beta'}(y) := \det \begin{pmatrix} R_1^0 \\ \vdots \\ R_\alpha^{m_\alpha - 1} \\ S_\beta^{\beta'}(y) \\ R_{\alpha+1}^0 \\ \vdots \\ R_q^{m_q-1} \end{pmatrix}. \tag{36}$$

Expanding this determinant along its $(m_1 + \cdots + m_\alpha + 1)$-th row, which is the (unique) row that contains y, it is clear that if $\beta' \neq \beta$, then $\deg S_{\alpha\beta}^{\beta'}(y) < n'_{\beta'} = n_{\beta'}$. In view of (31) we also have[8]

$$S_{\alpha\beta}^{\beta}(y) = \epsilon_{\alpha\beta}\,(m,n)\,\tau_{mn} y^{n_\beta} + O(y^{n_\beta - 1}).$$

Moreover, for any $\alpha' = 1, \ldots, q$ and $i = 0, \ldots, m_{\alpha'} - 1$, we have by linearity of the determinant

$$\begin{aligned} &\left\langle x^i \psi_{\alpha'}^{-s}(x) \,\middle|\, \sum_{\beta'=1}^{p} S_{\alpha\beta}^{\beta'}(y) \varphi_{\beta'}^t(y) \right\rangle \\ &= \left\langle x^i \psi_{\alpha'}^{-s}(x) \,\middle|\, \det \begin{pmatrix} R_1^0 \\ \vdots \\ R_\alpha^{m_\alpha-1} \\ \sum_{\beta'=1}^{p} S_\beta^{\beta'}(y) \varphi_{\beta'}^t(y) \\ R_{\alpha+1}^0 \\ \vdots \\ R_q^{m_q-1} \end{pmatrix} \right\rangle \end{aligned} \tag{37}$$

[8] See (27) for the definition of $\epsilon_{\alpha\beta}(m,n)$.

$$
= \quad \det \begin{pmatrix} R_1^0 \\ \vdots \\ R_\alpha^{m_\alpha - 1} \\ \left(\left\langle x^i \psi_{\alpha'}^{-s}(x) \,\middle|\, y^j \varphi_{\beta'}^t(y) \right\rangle \right)_{\substack{1 \le \beta' < p \\ 0 \le j < n'_{\beta'}}} \\ R_{\alpha+1}^0 \\ \vdots \\ R_q^{m_q - 1} \end{pmatrix} = \det \begin{pmatrix} R_1^0 \\ \vdots \\ R_\alpha^{m_\alpha - 1} \\ R_{\alpha'}^i \\ R_{\alpha+1}^0 \\ \vdots \\ R_q^{m_q - 1} \end{pmatrix},
$$

which is zero, since the latter matrix has two identical rows ($i < m_{\alpha'}$). This shows that

$$
\frac{\epsilon_{\alpha\beta}(m,n)}{\tau_{mn}} S_{\alpha\beta}^1(y) \,, \ \ldots \ , \ \frac{\epsilon_{\alpha\beta}(m,n)}{\tau_{mn}} S_{\alpha\beta}^p(y)
$$

are type II mixed mops, normalized with respect to φ_β^t. It follows from Proposition 2.2 that

$$
S_{\alpha\beta}^{\beta'}(y) = \epsilon_{\alpha\beta}(m,n)\, \tau_{mn}\, Q_{mn}^{(\beta,\beta')}(y), \tag{38}
$$

for any[9] $\alpha = 1, \ldots, q$. Since $Q_{mn}^{(\beta)}(y) = Q_{mn}^{(\beta,1)}(y)\varphi_1^t(y) + \cdots + Q_{mn}^{(\beta,p)}(y)\varphi_p^t(y)$, it follows from (32), (35), (38) and (20) that

$$
\begin{aligned}
z^{-m_\alpha - 1} \frac{\tau_{m+e_\alpha, n+e_\beta}\left(s_\alpha - \left[z^{-1}\right]\right)}{\tau_{mn}} &= \frac{1}{\tau_{mn}}\, \mathcal{C}_{\psi_\alpha^{-s}} \left(\sum_{\beta'=1}^{p} S_{\alpha\beta}^{\beta'}(z) \varphi_{\beta'}^t(z) \right) \\
&= \epsilon_{\alpha\beta}(m,n)\, \mathcal{C}_{\psi_\alpha^{-s}} \left(\sum_{\beta'=1}^{p} Q_{mn}^{(\beta,\beta')}(z) \varphi_{\beta'}^t(z) \right) \\
&= \epsilon_{\alpha\beta}(m,n)\, \mathcal{C}_{\psi_\alpha^{-s}} Q_{mn}^{(\beta)}(z) \qquad (39)
\end{aligned}
$$

This finishes the proof. □

Observe our proof shows, as a byproduct, that each $Q_{mn}^{(\beta)}(y)$ is expressible naturally as a determinant, like in the classical case, namely

$$
Q_{mn}^{(\beta)}(y) = \frac{\epsilon_{\alpha\beta}(m,n)}{\tau_{mn}} \det \begin{pmatrix} R_1^0 \\ \vdots \\ R_\alpha^{m_\alpha - 1} \\ \displaystyle\sum_{\beta'=1}^{p} S_\beta^{\beta'}(y) \varphi_{\beta'}^t(y) \\ R_{\alpha+1}^0 \\ \vdots \\ R_q^{m_q - 1} \end{pmatrix}. \tag{40}
$$

[9]The formulas for the different values of α are all the same, up to a sign, as they amount to changing the location of a row in the evaluation of a determinant.

We now get to the second type of Cauchy transforms which correspond to the Type I mops $P^{(\alpha)}_{mn}(y)$.

Proposition 3.2 *For* $1 \le \alpha, \alpha' \le q$ *the Cauchy transforms of* $P^{(\alpha')}_{mn}(y) = P^{(\alpha',1)}_{mn}(y)\,\varphi^t_1(y) + \cdots + P^{(\alpha',p)}_{mn}(y)\varphi^t_p(y)$ *with respect to* ψ^{-s}_α *can be expressed in terms of tau functions as follows:*

$$\mathcal{C}_{\psi^{-s}_\alpha} P^{(\alpha)}_{mn}(z) = z^{-m_\alpha}\,\frac{\tau_{mn}(s_\alpha - [z^{-1}])}{\tau_{mn}}, \tag{41}$$

$$\mathcal{C}_{\psi^{-s}_\alpha} P^{(\alpha')}_{mn}(z) = \varepsilon_{\alpha'\alpha}(m)\, z^{-1-m_\alpha}\,\frac{\tau_{m+e_\alpha-e_{\alpha'},n}(s_\alpha - [z^{-1}])}{\tau_{mn}}, \qquad \alpha' \neq \alpha. \tag{42}$$

Proof Up to a relabeling of the indices, the shifted tau functions in question were already expressed as polynomials in the previous proof. Let us show how this leads to a quick proof of (41). Shifting the m_α and n_β indices down by 1, it follows from (32) that

$$z^{-m_\alpha}\tau_{mn}(s_\alpha - [z^{-1}]) = \det\begin{pmatrix} R^0_1 \\ \vdots \\ R^{m_\alpha-2}_\alpha \\ \tilde{R}_\alpha(z) \\ R^0_{\alpha+1} \\ \vdots \\ R^{m_q-1}_q \end{pmatrix},$$

while the orthogonality relations (37) become

$$\left\langle x^i \psi^{-s}_{\alpha'}(x) \,\middle|\, \sum_{\beta'=1}^{p} S^{\beta'}_{\alpha\beta}(y)\varphi^t_{\beta'}(y) \right\rangle = \det\begin{pmatrix} R^0_1 \\ \vdots \\ R^{m_\alpha-2}_\alpha \\ R^i_{\alpha'} \\ R^0_{\alpha+1} \\ \vdots \\ R^{m_q-1}_q \end{pmatrix} = \delta_{\alpha\alpha'}\delta_{i,m_\alpha-1}\tau_{mn}. \tag{43}$$

Since $\deg S^{\beta'}_{\alpha\beta} < n_{\beta'}$ for $\beta' = 1, \ldots, p$ this means that the polynomials

$$\frac{1}{\tau_{mn}} S^1_{\alpha\beta}(y)\,,\ \ldots\ ,\ \frac{1}{\tau_{mn}} S^p_{\alpha\beta}(y)$$

are type I mixed mops, normalized with respect to ψ^{-s}_α, so they coincide according to Proposition 2.3 with the polynomials $P^{(\alpha,1)}_{mn}(y), \ldots, P^{(\alpha,p)}_{mn}(y)$. We conclude, as in (39), that

$$z^{-m_\alpha} \frac{\tau_{mn}(s_\alpha - [z^{-1}])}{\tau_{mn}} = \frac{1}{\tau_{mn}} \mathcal{C}_{\psi_\alpha^{-s}} \left(\sum_{\beta'=1}^{p} S_{\alpha\beta}^{\beta'}(z) \varphi_{\beta'}^{t}(z) \right) = \mathcal{C}_{\psi_\alpha^{-s}} P_{mn}^{(\alpha)}(z).$$

Similarly, one obtains (42) from (30) by shifting m'_α and n_β down by 1; the sign in this case is determined (for $\alpha' < \alpha$) from the right hand side of (43) now taking the form

$$\det \begin{pmatrix} R_1^0 \\ \vdots \\ R_{\alpha'}^{m_{\alpha'}-2} \\ R_{\alpha'+1}^0 \\ \vdots \\ R_\alpha^{m_\alpha-1} \\ R_{\alpha''}^i \\ R_{\alpha+1}^0 \\ \vdots \\ R_q^{m_q-1} \end{pmatrix} = \varepsilon_{\alpha'\alpha}(m) \det \begin{pmatrix} R_1^0 \\ \vdots \\ R_{\alpha'}^{m_{\alpha'}-2} \\ R_{\alpha''}^i \\ R_{\alpha'+1}^0 \\ \vdots \\ R_\alpha^{m_\alpha-1} \\ R_{\alpha+1}^0 \\ \vdots \\ R_q^{m_q-1} \end{pmatrix} = \varepsilon_{\alpha'\alpha}(n)\, \delta_{\alpha'\alpha''} \delta_{i,m_{\alpha'}-1} \tau_{mn}.$$

□

4 Duality

By interchanging the rôles of the weights ψ_α^t with the weights φ_β^{-s} we obtain *Type I* $|_{\varphi_\beta^t}$ mixed mops and *Type II* $|_{\psi_\alpha^{-s}}$ mixed mops, expressed in terms of tau functions, leading to a *duality*. As a general rule, in order to dualize a formula one does the following exchanges

$$q \leftrightarrow p, \quad m \leftrightarrow n, \quad \psi \leftrightarrow \varphi, \quad s \leftrightarrow -t, \quad x \leftrightarrow y. \tag{44}$$

At the level of the indices, duality amounts to

$$\alpha \leftrightarrow \beta, \quad i \leftrightarrow j. \tag{45}$$

As for the mixed mops which we have constructed, they will correspond to new mixed mops for which we will use the same letter, but adding a star. Thus,

$$P_{mn}^{(\alpha)} \leftrightarrow P_{nm}^{*(\beta)}, \quad P_{mn}^{(\alpha,\beta)} \leftrightarrow P_{nm}^{*(\beta,\alpha)}, \quad Q_{mn}^{(\beta)} \leftrightarrow Q_{nm}^{*(\alpha)}, \quad Q_{mn}^{(\beta,\beta')} \leftrightarrow Q_{nm}^{*(\alpha,\alpha')}. \tag{46}$$

What happens to the tau functions τ_{mn}? To see this, pick a typical shifted tau function $\tau_{m+e_\alpha-e_{\alpha'},n}(s_\alpha - [z^{-1}])$ and make its dependence on the weights and on all times explicit, writing $\tau_{m+e_\alpha-e_{\alpha'},n}(s-\left[z^{-1}\right]e_\alpha,t;\psi,\varphi)$. According to the above rule it becomes $\tau_{n+e_\beta-e_{\beta'},m}(-t-\left[z^{-1}\right]e_\beta,-s;\varphi,\psi)$ which is equal to $\tau_{m,n+e_\beta-e_{\beta'}}(s,t+\left[z^{-1}\right]e_\beta;\psi,\varphi)$, since transposing the moment matrix has no effect on the determinant, while it permutes the indices in the tau function, it permutes the time-dependence (with signs) and it permutes the weights. Thus,

$$\begin{aligned}
\tau_{mn} &\leftrightarrow \tau_{mn}\\
\tau_{mn}(t_\beta-\left[z^{-1}\right]) &\leftrightarrow \tau_{mn}(s_\alpha+\left[z^{-1}\right])\\
\tau_{m-e_\alpha,n-e_\beta}(t_\beta-\left[z^{-1}\right]) &\leftrightarrow \tau_{m-e_\alpha,n-e_\beta}(s_\alpha+\left[z^{-1}\right])\\
\tau_{m+e_\alpha-e_{\alpha'},n}(s_\alpha-\left[z^{-1}\right]) &\leftrightarrow \tau_{m,n+e_\beta-e_{\beta'}}(t_\beta+\left[z^{-1}\right]),
\end{aligned}$$

and so on. Dualizing Propositions 2.2 and 2.3, we get the following proposition.

Proposition 4.1 *For $\alpha=1,\dots,q$ and $\beta=1,\dots,p$, let*

$$\begin{aligned}
P^{*(\beta)}_{nm}(x) &:= P^{*(\beta,1)}_{nm}(x)\psi_1^{-s}(x)+\cdots+P^{*(\beta,q)}_{nm}(x)\psi_q^{-s}(x),\\
Q^{*(\alpha)}_{nm}(x) &:= Q^{*(\alpha,1)}_{nm}(x)\psi_1^{-s}(x)+\cdots+Q^{*(\alpha,q)}_{nm}(x)\psi_q^{-s}(x),
\end{aligned}\tag{47}$$

where $P^{*(\beta,\alpha)}_{nm}$ and $Q^{*(\alpha,\alpha')}_{nm}$ are the polynomials, defined by

$$P^{*(\beta,\alpha)}_{nm}(z):=\epsilon_{\beta\alpha}(n,m)z^{m_\alpha-1}\frac{\tau_{m-e_\alpha,n-e_\beta}(s_\alpha+\left[z^{-1}\right])}{\tau_{mn}},\tag{48}$$

and

$$\begin{aligned}
Q^{*(\alpha,\alpha)}_{nm}(z) &:= z^{m_\alpha}\frac{\tau_{mn}(s_\alpha+\left[z^{-1}\right])}{\tau_{mn}}\\
Q^{*(\alpha,\alpha')}_{nm}(z) &:= \varepsilon_{\alpha\alpha'}(m)z^{m_{\alpha'}-1}\frac{\tau_{m+e_\alpha-e_{\alpha'},n}(s_{\alpha'}+\left[z^{-1}\right])}{\tau_{mn}}, \qquad \alpha'\neq\alpha.
\end{aligned}\tag{49}$$

Then $P^{*(\beta,1)}_{nm}(x),\dots,P^{*(\beta,q)}_{nm}(x)$ are Type I $_{|\varphi^t_\beta}$ *mixed mops, while $Q^{*(\alpha 1)}_{nm}(x),\dots,Q^{*(\alpha,q)}_{nm}(x)$ are* Type II $_{|\psi^{-s}_\alpha}$ *mixed mops.*

□

Dualizing Definition (29) we get the following definition for the dual Cauchy transform: for any function F and a weight φ we put

$$\mathcal{C}^*_\varphi F(z):=\iint_{\mathbb{R}^2}\frac{\varphi(y)}{z-y}F(x)d\mu(x,y)=\left\langle F(x)\,\middle|\,\frac{\varphi(y)}{z-y}\right\rangle.\tag{50}$$

If we dualize now Propositions 3.1 and 3.2, then we get the following proposition.

Proposition 4.2 *For $\alpha=1,\dots,q$ and $\beta,\beta'=1,\dots,p$, the Cauchy transforms of $P^{*(\beta')}_{nm}(x)$ with respect to φ^t_β, and of $Q^{*(\alpha)}_{nm}(x)$ with respect to φ^t_β can be expressed in terms of tau functions as follows:*

$$\mathcal{C}^*_{\varphi^t_\beta}P^{*(\beta)}_{nm}(z) = z^{-n_\beta}\frac{\tau_{mn}(t_\beta+\left[z^{-1}\right])}{\tau_{mn}},$$

$$\mathcal{C}^*_{\varphi^t_\beta} P^{*(\beta')}_{nm}(z) \quad = \quad \varepsilon_{\beta'\beta}(n)\, z^{-1-n_\beta}\, \frac{\tau_{m,n+e_\beta-e_{\beta'}}(t_\beta + [z^{-1}])}{\tau_{mn}}, \qquad \beta' \neq \beta,$$

and

$$\mathcal{C}^*_{\varphi^t_\beta} Q^{*(\alpha)}_{nm}(z) = \epsilon_{\beta\alpha}(n,m)\, z^{-n_\beta-1}\, \frac{\tau_{m+e_\alpha,n+e_\beta}(t_\beta + [z^{-1}])}{\tau_{mn}}.$$

□

5 The Riemann-Hilbert matrix and the bilinear identity

Orthogonal polynomials were shown to be characterized by a Riemann-Hilbert problem in [11] and [10]. This was generalized by Daems and Kuijlaars to the case of mixed mops. According to [9][10] the corresponding Riemann-Hilbert matrix is given by the $(p+q)\times(p+q)$ matrix

$$Y_{mn}(z) := \begin{pmatrix} \left(Q^{(\beta,\beta')}_{mn}\right)_{\substack{1\leq\beta\leq p\\ 1\leq\beta'\leq p}} & \left(\mathcal{C}_{\psi^{-s}_\alpha} Q^{(\beta)}_{mn}\right)_{\substack{1\leq\beta\leq p\\ 1\leq\alpha\leq q}} \\ \left(P^{(\alpha,\beta)}_{mn}\right)_{\substack{1\leq\alpha\leq q\\ 1\leq\beta\leq p}} & \left(\mathcal{C}_{\psi^{-s}_\alpha} P^{(\alpha')}_{mn}\right)_{\substack{1\leq\alpha'\leq q\\ 1\leq\alpha\leq q}} \end{pmatrix} =$$

$$\begin{pmatrix} \left(\varepsilon_{\beta\beta'}(n)\frac{\tau_{m,n+e_\beta-e_{\beta'}}(t_{\beta'}-[z^{-1}])}{\tau_{mn}} z^{n_{\beta'}+\delta_{\beta\beta'}-1}\right)_{\substack{1\leq\beta\leq p\\ 1\leq\beta'\leq p}} & \left(\epsilon_{\alpha\beta}(m,n)\frac{\tau_{m+e_\alpha,n+e_\beta}(s_\alpha-[z^{-1}])}{\tau_{mn}} z^{-m_\alpha-1}\right)_{\substack{1\leq\beta\leq p\\ 1\leq\alpha\leq q}} \\ \left(\epsilon_{\alpha\beta}(m,n)\frac{\tau_{m-e_\alpha,n-e_\beta}(t_\beta-[z^{-1}])}{\tau_{mn}} z^{n_\beta-1}\right)_{\substack{1\leq\alpha\leq q\\ 1\leq\beta\leq p}} & \left(\varepsilon_{\alpha'\alpha}(m)\frac{\tau_{m+e_\alpha-e_{\alpha'},n}(s_\alpha-[z^{-1}])}{\tau_{mn}} z^{\delta_{\alpha\alpha'}-1-m_\alpha}\right)_{\substack{1\leq\alpha'\leq q\\ 1\leq\alpha\leq q}} \end{pmatrix}$$

whose inverse transpose matrix is given by

$$Y^*_{mn}(z) = \begin{pmatrix} \left(\mathcal{C}^*_{\varphi^t_\beta} P^{*(\beta')}_{nm}\right)_{\substack{1\leq\beta'\leq p\\ 1\leq\beta\leq p}} & \left(-P^{*(\beta,\alpha)}_{nm}\right)_{\substack{1\leq\beta\leq p\\ 1\leq\alpha\leq q}} \\ \left(-\mathcal{C}^*_{\varphi^t_\beta} Q^{*(\alpha)}_{nm}\right)_{\substack{1\leq\alpha\leq q\\ 1\leq\beta\leq p}} & \left(Q^{*(\alpha,\alpha')}_{nm}\right)_{\substack{1\leq\alpha\leq q\\ 1\leq\alpha'\leq q}} \end{pmatrix} =$$

$$\begin{pmatrix} \left(\varepsilon_{\beta'\beta}(n)\frac{\tau_{m,n+e_\beta-e_{\beta'}}(t_\beta+[z^{-1}])}{\tau_{mn}} z^{\delta_{\beta'\beta}-1-n_\beta}\right)_{\substack{1\leq\beta'\leq p\\ 1\leq\beta\leq p}} & \left(-\epsilon_{\beta\alpha}(n,m)\frac{\tau_{m-e_\alpha,n-e_\beta}(s_\alpha+[z^{-1}])}{\tau_{mn}} z^{m_\alpha-1}\right)_{\substack{1\leq\beta\leq p\\ 1\leq\alpha\leq q}} \\ \left(-\epsilon_{\beta\alpha}(n,m)\frac{\tau_{m+e_\alpha,n+e_\beta}(t_\beta+[z^{-1}])}{\tau_{mn}} z^{-n_\beta-1}\right)_{\substack{1\leq\alpha\leq q\\ 1\leq\beta\leq p}} & \left(\varepsilon_{\alpha\alpha'}(m)\frac{\tau_{m+e_\alpha-e_{\alpha'},n}(s_{\alpha'}+[z^{-1}])}{\tau_{mn}} z^{\delta_{\alpha\alpha'}-1+m_{\alpha'}}\right)_{\substack{1\leq\alpha\leq q\\ 1\leq\alpha'\leq q}} \end{pmatrix}$$

We will obtain bilinear identities for these tau functions from an identity which is satisfied by the Riemann-Hilbert matrix and its adjoint. We define the wave matrix $W_{mn}(z)$ by $Y_{mn}(z)\Delta(z)$, where $\Delta(z)$ is the diagonal matrix[11]

$$\Delta(z) := \operatorname{diag}(e^{\xi(t_1,z)}, \ldots, e^{\xi(t_p,z)},\ e^{\xi(s_1,z)},\ e^{\xi(s_q,z)}),$$

[10]Up to a factor $\operatorname{diag}(I_p, -2\pi\sqrt{-1}I_q)$ which we suppress.

[11]Throughout this section, we set $\xi(t,z) := \sum_1^\infty t_k z^k$.

with adjoint wave matrix $Y^*_{mn}(z)\Delta^{-1}(z)$. In order to make the dependence on the time variables (s,t) explicit, we will write $W_{mn}(z;s,t)$ for $W(z)$ and $W^*_{mn}(z;s,t)$ for $W^*_{mn}(z)$.

Theorem 5.1 *The tau functions τ_{mn} satisfy the following bilinear identities that characterize the tau functions of the $(p+q)$-KP hierarchy (see [13]):*

$$\oint_\infty W_{mn}(z;s,t)W^*_{m^*n^*}(z;s^*,t^*)^\top dz = 0,$$

which is equivalent to the single identity

$$\sum_{\beta=1}^p \oint_\infty (-1)^{\sigma_\beta(n)}\,\tau_{m,n-e_\beta}(t_\beta-[z^{-1}])\tau_{m^*,n^*+e_\beta}(t^*_\beta+[z^{-1}])e^{\xi(t_\beta-t^*_\beta,z)}\,z^{n_\beta-n^*_\beta-2}\,dz =$$
$$\sum_{\alpha=1}^q \oint_\infty (-1)^{\sigma_\alpha(m)}\,\tau_{m+e_\alpha,n}(s_\alpha-[z^{-1}])\tau_{m^*-e_\alpha,n^*}(s^*_\alpha+[z^{-1}])e^{\xi(s_\alpha-s^*_\alpha,z)}\,z^{m^*_\alpha-m_\alpha-2}\,dz, \quad (51)$$

where

$$\sigma_\alpha(m)=\sum_{\alpha'=1}^{\alpha}(m_{\alpha'}-m^*_{\alpha'}) \quad \textit{and} \quad \sigma_\beta(n)=\sum_{\beta'=1}^{\beta}(n_{\beta'}-n^*_{\beta'}). \qquad (52)$$

and $|m^*|=|n^*|+1$ and $|m|=|n|-1$.

Proof For the entry (β',β'') of the product

$$Y\Delta(Y^*\Delta^{-1})^\top = Y\,\mathrm{diag}(e^{\xi(t_1-t_1^*,z)},\ldots,e^{\xi(t_p-t_p^*,z)},\,e^{\xi(s_1-s_1^*,z)},\ldots,e^{\xi(s_q-s_q^*,z)})Y^{*\top},$$

we need to prove that

$$\sum_{\beta=1}^p \oint_\infty Q^{(\beta',\beta)}_{mn}(z)\,\mathcal{C}^*_{\varphi^{t^*}_\beta}P^{*(\beta'')}_{n^*m^*}(z)\,e^{\xi(t_\beta-t^*_\beta,z)}\,dz = \sum_{\alpha=1}^q \oint_\infty \mathcal{C}_{\psi_\alpha^{-s}}Q^{(\beta')}_{mn}(z)\,P^{*(\beta'',\alpha)}_{n^*m^*}(z)\,e^{\xi(s_\alpha-s^*_\alpha,z)}\,dz \qquad (53)$$

where it is understood that all polynomials P^* go with starred times s^* and t^*. Also, the integral stands for (minus) the residue at infinity, and can be computed using the following formal residue identities, with $f(z)=\sum_{i=0}^\infty a_iz^i$,

$$\frac{1}{2\pi\sqrt{-1}}\oint_\infty f(z)\,\mathcal{C}_\psi g(z)\,dz = \langle f(x)\psi(x)\mid g(y)\rangle, \qquad (54)$$
$$\frac{1}{2\pi\sqrt{-1}}\oint_\infty \mathcal{C}^*_\varphi f(z)\,g(z)\,dz = \langle f(x)\mid \varphi(y)g(y)\rangle, \qquad (55)$$

whose proof we defer until the end. Using this, and Definition (47) of the functions $P^{*(\beta)}_{nm}(x)$, the left hand side in (53) becomes (up to a factor $2\pi\sqrt{-1}$)

$$\begin{aligned}\sum_{\beta=1}^p\left\langle P^{*(\beta'')}_{n^*m^*}(x)\,\middle|\,Q^{(\beta',\beta)}_{mn}(y)\,\varphi^{t^*}_\beta(y)\,e^{\xi(t_\beta-t^*_\beta,y)}\right\rangle &= \sum_{\beta=1}^p\left\langle P^{*(\beta'')}_{n^*m^*}(x)\,\middle|\,Q^{(\beta',\beta)}_{mn}(y)\,\varphi^{t}_\beta(y)\right\rangle\\ &= \left\langle P^{*(\beta'')}_{n^*m^*}(x)\,\middle|\,Q^{(\beta')}_{mn}(y)\right\rangle.\end{aligned}$$

Similarly, the right hand side in (53) becomes (up to a factor $2\pi\sqrt{-1}$)

$$\begin{aligned}\sum_{\alpha=1}^{q}\left\langle P^{*(\beta'',\alpha)}_{n^*m^*}(x)\,\psi_\alpha^{-s}(x)\,e^{\xi(s_\alpha-s_\alpha^*,x)}\;\middle|\;Q^{(\beta')}_{mn}(y)\right\rangle &= \sum_{\alpha=1}^{q}\left\langle P^{*(\beta'',\alpha)}_{n^*m^*}(x)\,\psi_\alpha^{-s^*}(x)\;\middle|\;Q^{(\beta')}_{mn}(y)\right\rangle\\ &= \left\langle P^{*(\beta'')}_{n^*m^*}(x)\;\middle|\;Q^{(\beta')}_{mn}(y)\right\rangle.\end{aligned}$$

The three other identities are obtained in the same way.

In terms of tau functions, it means that we have shown that for any m, n, m^*, n^*, β' and β'', with $|m| = |n|$ and $|m^*| = |n^*|$ the following bilinear identities hold:

$$\sum_{\beta=1}^{p}\oint_\infty \clubsuit\,\tau_{m,n+e_{\beta'}-e_\beta}(t_\beta-[z^{-1}])\tau_{m^*,n^*+e_\beta-e_{\beta''}}(t^*_\beta+[z^{-1}])e^{\xi(t_\beta-t^*_\beta,z)}\,dz =$$
$$\sum_{\alpha=1}^{q}\oint_\infty \spadesuit\,\tau_{m+e_\alpha,n+e_{\beta'}}(s_\alpha-[z^{-1}])\tau_{m^*-e_\alpha,n^*-e_{\beta''}}(s^*_\alpha+[z^{-1}])e^{\xi(s_\alpha-s^*_\alpha,z)}\,dz,$$

where

$$\begin{aligned}\clubsuit &= \varepsilon_{\beta'\beta}(n)\varepsilon_{\beta''\beta}(n^*)z^{n_\beta-n^*_\beta-2+\delta_{\beta\beta'}+\delta_{\beta\beta''}},\\ \spadesuit &= \epsilon_{\alpha\beta'}(m,n)\epsilon_{\beta''\alpha}(n^*,m^*)z^{m^*_\alpha-m_\alpha-2}.\end{aligned}$$

For different values of β' and β'' this yields the same identity, up to a relabeling of n and n^*. Namely, replace in the bilinear identity $n+e_{\beta'}$ by n and $n^*-e_{\beta''}$ by n^* and multiply by $(-1)^{n_1+\cdots+n_{\beta'}}(-1)^{n^*_1+\cdots+n^*_{\beta''}}$ to find the following symmetric expression for the identity, that is independent of β' and β'':

$$\sum_{\beta=1}^{p}\oint_\infty (-1)^{\sigma_\beta(n)}\,\tau_{m,n-e_\beta}(t_\beta-[z^{-1}])\tau_{m^*,n^*+e_\beta}(t^*_\beta+[z^{-1}])e^{\xi(t_\beta-t^*_\beta,z)}\,z^{n_\beta-n^*_\beta-2}\,dz =$$
$$\sum_{\alpha=1}^{q}\oint_\infty (-1)^{\sigma_\alpha(m)}\,\tau_{m+e_\alpha,n}(s_\alpha-[z^{-1}])\tau_{m^*-e_\alpha,n^*}(s^*_\alpha+[z^{-1}])e^{\xi(s_\alpha-s^*_\alpha,z)}\,z^{m^*_\alpha-m_\alpha-2}\,dz,$$

where $\sigma_\alpha(m)$ and $\sigma_\beta(n)$ are given by (52). Notice that, due to the shift, one must have in this symmetric form that $|m| = |n|-1$ and $|m^*| = |n^*|+1$. The other three identities also yield the above identity, up to relabeling.

Finally, to prove (54), compute

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}}\oint_\infty f(z)\left\langle \frac{\psi(x)}{z-x}\mid g(y)\right\rangle dz &= \frac{1}{2\pi\sqrt{-1}}\oint_\infty \sum_{i=0}^{\infty}a_iz^i\sum_{j=0}^{\infty}\frac{1}{z^{j+1}}\left\langle x^j\psi(x)\mid g(y)\right\rangle\\ &= \sum_{i=0}^{\infty}a_i\left\langle x^i\psi(x)\mid g(y)\right\rangle = \left\langle \sum_{i=0}^{\infty}a_ix^i\,\psi(x)\mid g(y)\right\rangle = \left\langle f(x)\psi(x)\mid g(y)\right\rangle,\end{aligned}$$

and similarly for (55), completing the proof. □

6 Consequences of the bilinear identities

In this section we will derive from the bilinear identities (51) a series of PDE's for the tau functions τ_{mn}. In order to keep the formulas transparant we will use the following simplification in the notation. Recall that we have time variables $s_\alpha = (s_{\alpha 1},\, s_{\alpha 2}, \ldots)$ and $t_\beta = (t_{\beta 1},\, t_{\beta 2}, \ldots)$, where $\alpha = 1, \ldots, q$ and $\beta = 1, \ldots, p$. In the bilinear identities (51) we consider in each term a shift in t_α, for a single α, or in s_β, for a single β; we will denote this t_α or s_β by v (so v is an infinite vector $v = (v_1, v_2, \ldots)$ and we assemble all the other $r := p + q - 1$ series of time variables in $w = (w_1, w_2, \ldots, w_r)$, where $w_1 = (w_{11}, w_{12}, \ldots)$ and so on. Moreover, precisely like in the bilinear identities we will want to consider an independent collection of all these variables, in fact we will consider here (v', w') and (v'', w'') besides (v, w). We use the Hirota symbol, which takes in our case the following form

$$P(\partial_v, \partial_w)\, F \circ G = P(\partial_{v'}, \partial_{w'})\, F(v+v', w+w')\, G(v-v', w-w')\Big|_{v'=w'=0}. \quad (56)$$

The elementary Schur polynomials $S_\ell(v)$ are defined by

$$e^{\sum_{k=1}^\infty v_k z^k} = \sum_{k=0}^\infty S_k(v) z^k, \quad (57)$$

for $\ell \geq 0$ and $S_\ell(v) := 0$ otherwise. In particular, if we put degree $v_i := i$, then

$$S_0 = 1, \qquad S_1(v) = v_1, \qquad S_\ell(v) = v_\ell + \text{ degree } \ell \text{ in } v_1, \ldots, v_{\ell-1}. \quad (58)$$

We also use the standard notation

$$\tilde{\partial}_v = \left(\frac{\partial}{\partial v_1}, \frac{1}{2}\frac{\partial}{\partial v_2}, \frac{1}{3}\frac{\partial}{\partial v_3}, \ldots \right).$$

We first give an identity which will allow us to compute the formal residues which appear in (51) in terms of derivatives of the tau function.

Lemma 6.1 *For any $n \in \mathbf{Z}$ we have the following formal residue identity*

$$\oint_\infty F(v'' + \left[z^{-1}\right], w'')\, G(v' - \left[z^{-1}\right], w')\, e^{\sum_{\ell=0}^\infty (v'_\ell - v''_\ell) z^\ell}\, z^n \frac{dz}{2\pi\sqrt{-1}}$$

$$= \sum_{j \geq 0} S_{j-1-n}(-2a)\, S_j(\tilde{\partial}_v)\, e^{\sum_{\ell=1}^\infty (a_\ell \frac{\partial}{\partial v_\ell} + \sum_{\gamma=1}^r b_{\gamma\ell} \frac{\partial}{\partial w_{\gamma\ell}})}\, F(v,w) \circ G(v,w), \quad (59)$$

where

$$v' = v - a, \quad v'' = v + a, \qquad w'_i = w_i - b_i, \quad w''_i = w_i + b_i,$$

$$a = (a_1, a_2, a_3, \ldots), \qquad b_i = (b_{i1},\, b_{i2},\, b_{i3}, \ldots),$$

for $1 \leq i \leq r$.

Proof The proof is an immediate, but tricky, consequence of Definition (57) of the Schur functions and of the following two properties of the Hirota symbol:

$$F(v+\left[z^{-1}\right],w)\,G(v-\left[z^{-1}\right],w)=\sum_{j=0}^{\infty}z^{-j}S_j(\tilde{\partial}_v)\,F\circ G,$$
$$F(v+a,w+b)\,G(v-a,w-b)=e^{\sum_{\ell=0}^{\infty}(a_\ell\frac{\partial}{\partial v_\ell}+\sum_{\gamma=1}^{r}b_{\gamma\ell}\frac{\partial}{\partial w_{\gamma\ell}})}\,F\circ G.$$

□

Proposition 6.2 *The bilinear equations imply, upon specialization, that the tau functions τ_{mn}, with $|m|=|n|$ satisfy the following PDE's expressed in terms of the Hirota symbol:*

$$\begin{aligned}
\tau_{mn}^2\frac{\partial^2}{\partial t_{\beta,\ell+1}\partial t_{\beta',1}}\ln\tau_{mn} &= S_{\ell+2\delta_{\beta\beta'}}(\tilde{\partial}_{t_\beta})\tau_{m,n+e_\beta-e_{\beta'}}\circ\tau_{m,n+e_{\beta'}-e_\beta} &(60)\\
\tau_{mn}^2\frac{\partial^2}{\partial s_{\alpha,\ell+1}\partial s_{\alpha',1}}\ln\tau_{mn} &= S_{\ell+2\delta_{\alpha\alpha'}}(\tilde{\partial}_{s_\alpha})\tau_{m+e_{\alpha'}-e_\alpha,n}\circ\tau_{m+e_\alpha-e_{\alpha'},n} &(61)\\
-\tau_{mn}^2\frac{\partial^2}{\partial s_{\alpha,1}\partial t_{\beta,\ell+1}}\ln\tau_{mn} &= S_\ell(\tilde{\partial}_{t_\beta})\tau_{m+e_\alpha,n+e_\beta}\circ\tau_{m-e_\alpha,n-e_\beta} &(62)\\
-\tau_{mn}^2\frac{\partial^2}{\partial t_{\beta,1}\partial s_{\alpha,\ell+1}}\ln\tau_{mn}. &= S_\ell(\tilde{\partial}_{s_\alpha})\tau_{m-e_\alpha,n-e_\beta}\circ\tau_{m+e_\alpha,n+e_\beta} &(63)
\end{aligned}$$

Equations (60) resp. (61) for $\beta'=\beta$ (resp. for $\alpha'=\alpha$) yield a solution to the KP hierarchy in t_β (resp. in s_α), while for $\beta'\neq\beta$ and $\alpha'\neq\alpha$, (60) — (63) yields

$$\begin{aligned}
\frac{\partial^2}{\partial t_{\beta,1}\partial t_{\beta',1}}\ln\tau_{mn} &= \frac{\tau_{m,n+e_\beta-e_{\beta'}}\tau_{m,n+e_{\beta'}-e_\beta}}{\tau_{mn}^2} &(64)\\
\frac{\partial^2}{\partial s_{\alpha,1}\partial s_{\alpha',1}}\ln\tau_{mn} &= \frac{\tau_{m+e_{\alpha'}-e_\alpha,n}\tau_{m+e_\alpha-e_{\alpha'},n}}{\tau_{mn}^2} &(65)\\
\frac{\partial^2}{\partial s_{\alpha,1}\partial t_{\beta,1}}\ln\tau_{mn} &= -\frac{\tau_{m+e_\alpha,n+e_\beta}\tau_{m-e_\alpha,n-e_\beta}}{\tau_{mn}^2} &(66)\\
\frac{\partial}{\partial t_{\beta,1}}\ln\frac{\tau_{m,n+e_\beta-e_{\beta'}}}{\tau_{m,n+e_{\beta'}-e_\beta}} &= \frac{\frac{\partial^2}{\partial t_{\beta,2}\partial t_{\beta',1}}\ln\tau_{mn}}{\frac{\partial^2}{\partial t_{\beta,1}\partial t_{\beta',1}}\ln\tau_{mn}} &(67)\\
\frac{\partial}{\partial s_{\alpha,1}}\ln\frac{\tau_{m-e_\alpha+e_{\alpha'},n}}{\tau_{m-e_{\alpha'}+e_\alpha,n}} &= \frac{\frac{\partial^2}{\partial s_{\alpha,2}\partial s_{\alpha',1}}\ln\tau_{mn}}{\frac{\partial^2}{\partial s_{\alpha,1}\partial t_{\alpha',1}}\ln\tau_{mn}} &(68)\\
\frac{\partial}{\partial t_{\beta,1}}\ln\frac{\tau_{m+e_\alpha,n+e_\beta}}{\tau_{m-e_\alpha,n-e_\beta}} &= \frac{\frac{\partial^2}{\partial t_{\beta,2}\partial s_{\alpha,1}}\ln\tau_{mn}}{\frac{\partial^2}{\partial t_{\beta,1}\partial s_{\alpha,1}}\ln\tau_{mn}} &(69)\\
\frac{\partial}{\partial s_{\alpha,1}}\ln\frac{\tau_{m-e_\alpha,n-e_\beta}}{\tau_{m+e_\alpha,n+e_\beta}} &= \frac{\frac{\partial^2}{\partial s_{\alpha,2}\partial t_{\beta,1}}\ln\tau_{mn}}{\frac{\partial^2}{\partial s_{\alpha,1}\partial t_{\beta,1}}\ln\tau_{mn}}. &(70)
\end{aligned}$$

It leads to the following $\binom{p+q}{2}$ *PDE's for* $\ln \tau_{mn}$ *involving not just one* s_α *or* t_β*, but a few of them*

$$\frac{\partial}{\partial t_{\beta',1}}\left(\frac{\frac{\partial^2}{\partial t_{\beta,2}\partial t_{\beta',1}}\ln\tau_{mn}}{\frac{\partial^2}{\partial t_{\beta,1}\partial t_{\beta',1}}\ln\tau_{mn}}\right)+\frac{\partial}{\partial t_{\beta,1}}\left(\frac{\frac{\partial^2}{\partial t_{\beta',2}\partial t_{\beta,1}}\ln\tau_{mn}}{\frac{\partial^2}{\partial t_{\beta',1}\partial t_{\beta,1}}\ln\tau_{mn}}\right) = 0, \tag{71}$$

$$\frac{\partial}{\partial s_{\alpha',1}}\left(\frac{\frac{\partial^2}{\partial s_{\alpha,2}\partial s_{\alpha',1}}\ln\tau_{mn}}{\frac{\partial^2}{\partial s_{\alpha,1}\partial s_{\alpha',1}}\ln\tau_{mn}}\right)+\frac{\partial}{\partial s_{\alpha,1}}\left(\frac{\frac{\partial^2}{\partial s_{\alpha',2}\partial s_{\alpha,1}}\ln\tau_{mn}}{\frac{\partial^2}{\partial s_{\alpha',1}\partial s_{\alpha,1}}\ln\tau_{mn}}\right) = 0, \tag{72}$$

$$\frac{\partial}{\partial s_{\alpha,1}}\left(\frac{\frac{\partial^2}{\partial t_{\beta,2}\partial s_{\alpha,1}}\ln\tau_{mn}}{\frac{\partial^2}{\partial t_{\beta,1}\partial s_{\alpha,1}}\ln\tau_{mn}}\right)+\frac{\partial}{\partial t_{\beta,1}}\left(\frac{\frac{\partial^2}{\partial s_{\alpha,2}\partial t_{\beta,1}}\ln\tau_{mn}}{\frac{\partial^2}{\partial s_{\alpha,1}\partial t_{\beta,1}}\ln\tau_{mn}}\right) = 0. \tag{73}$$

Proof Let us denote for $a = (a_1, \dots, a_q)$ and $b = (b_1, \dots, b_q)$ by $\Omega(a, b)$ the differential operator

$$\Omega(a,b) := \sum_{\ell=1}^{\infty}\left(\sum_{\alpha'=1}^{q} a_{\alpha'\ell}\frac{\partial}{\partial s_{\alpha'\ell}}+\sum_{\beta'=1}^{p} b_{\beta'\ell}\frac{\partial}{\partial t_{\beta'\ell}}\right). \tag{74}$$

Using Lemma (6.1), rewrite the bilinear identity[12] (51):

$$\begin{aligned}&\sum_{\beta=1}^{p}(-1)^{\sigma_\beta(n)}\sum_{k=0}^{\infty}S_{n^*_\beta-n_\beta+1+k}(-2b_\beta)S_k(\tilde\partial_{t_\beta})e^{\Omega(a,b)}\tau_{m^*,n^*+e_\beta}\circ\tau_{m,n-e_\beta}\\ -&\sum_{\alpha=1}^{q}(-1)^{\sigma_\alpha(m)}\sum_{k=0}^{\infty}S_{m_\alpha-m^*_\alpha+1+k}(-2a_\alpha)S_k(\tilde\partial_{s_\alpha})e^{\Omega(a,b)}\tau_{m^*-e_\alpha,n^*}\circ\tau_{m+e_\alpha,n}=0.\end{aligned} \tag{75}$$

Note that all infinite vectors a_α and b_β can be chosen completely arbitrary. We set all components of a and b equal to zero, *except* $b_{\beta,\ell+1} = B \neq 0$ (for some fixed β and ℓ), and we set $m^* = m$ and $n^* - n = -2e_{\beta'}$ (for some fixed β'). Then only the first term in (75) survives, the signs $\sigma_\beta(n)$ are all 1 (see (52)) and, in view of (58), the identity (75) becomes

$$\begin{aligned}0 &= \sum_{\beta''=1}^{p}\sum_{k=0}^{\infty}S_{1+k-2\delta_{\beta'\beta''}}(-2b_{\beta''})S_k(\tilde\partial_{t_{\beta''}})e^{B\frac{\partial}{\partial t_{\beta,\ell+1}}}\tau_{m,n+e_{\beta''}-2e_{\beta'}}\circ\tau_{m,n-e_{\beta''}}\\ &= B\left(-2S_{\ell+2\delta_{\beta\beta'}}(\tilde\partial_{t_\beta})\tau_{m,n+e_\beta-2e_{\beta'}}\circ\tau_{m,n-e_\beta}+\frac{\partial^2}{\partial t_{\beta',1}\partial t_{\beta,\ell+1}}\tau_{m,n-e_{\beta'}}\circ\tau_{m,n-e_{\beta'}}\right)+O(B^2).\end{aligned}$$

Expressing that the coefficient of B in this expression must vanish we get (60), upon relabeling $n-e_{\beta'} \mapsto n$ and upon using the following property of the Hirota symbol, valid for f depending on (time-) variables s and t:

$$\frac{\partial^2}{\partial t\partial s}F\circ F = 2F^2\frac{\partial^2}{\partial t\partial s}\ln F. \tag{76}$$

[12] Recall that in this form of the bilinear identity $|m^*| = |n^*| + 1$ and $|m| = |n| - 1$.

(61) follows from (60) by duality, using $P(-\partial_s)\, F \circ G = P(\partial_s)\, G \circ F$. In order to obtain (62) we consider again (75), with $b_{\beta,\ell+1} = B \neq 0$ and all other components of a and b equal to zero, but we set now $n^* = n$ and $m - m^* = -2e_\alpha$. Then (75) becomes

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} S_{k+1}(-2b_\beta) S_k(\tilde{\partial}_\beta)\, e^{B \frac{\partial}{\partial t_{\beta,\ell+1}}}\, \tau_{m+2e_\alpha, n+e_\beta} \circ \tau_{m,n-e_\beta} \\ &\quad - \sum_{k=0}^{\infty} S_{k-1}(0) S_k(\tilde{\partial}_{s_\alpha})\, e^{B \frac{\partial}{\partial t_{\beta,\ell+1}}}\, \tau_{m+e_\alpha,n} \circ \tau_{m+e_\alpha,n} \\ &= -B \left(2S_\ell(\tilde{\partial}_{t_\beta}) \tau_{m+2e_\alpha,n+e_\beta} \circ \tau_{m,n-e_\beta} + \frac{\partial^2}{\partial s_{\alpha,1} \partial t_{\beta,\ell+1}} \tau_{m+e_\alpha,n} \circ \tau_{m+e_\alpha,n} \right) + O(B^2). \end{aligned}$$

The nullity of the coefficient of B in this expression, rewritten by using (76), leads at once to (62), upon doing the relabeling $m + e_\alpha \mapsto m$. From it, (63) follows by duality. Equations (64) — (66) follow from (60) — (62) by setting $\beta' \neq \beta$, $\alpha' \neq \alpha$ and $\ell = 0$. Equations (67) — (70) follow from (60) — (63) by setting $\beta' \neq \beta$, $\alpha' \neq \alpha$ and forming in each equation the ratio of the cases $\ell = 0$ and $\ell = 1$, and using the following property of the Hirota symbol, valid for F and G depending on a (time-) variable t:

$$\frac{\partial}{\partial t}\, F \circ G = FG\, \frac{\partial}{\partial t} \left(\ln \frac{F}{G} \right).$$

Equations (71) — (73) are just respectively the compatibility equations between (67) and $(67)_{\beta \leftrightarrow \beta'}$, between (68) and $(68)_{\alpha \leftrightarrow \alpha'}$, and between between (69) and (70). $\square$

Corollary 6.3 *The tau functions τ_{mn} and the polynomials $P^{*(\beta'')}_{n^*m^*}(x) = P^{*(\beta'')}_{n^*m^*}(x, s^*, t^*)$, $Q^{(\beta')}_{mn}(y) = Q^{(\beta')}_{mn}(y, s, t)$ appearing in Y^* and Y respectively, satisfy the following 4 formal series identities ($\delta_{\beta'\beta''}(n, n^*) = (-1)^{n_1+n_2+\cdots+n_{\beta'}+n_1^*+n_2^*+\cdots+n^*_{\beta''}}$):*

$$\begin{aligned} 0 &= \delta_{\beta'\beta''}(n,n^*) \tau_{mn}(s,t) \tau_{m^*n^*}(s^*,t^*) \left\langle P^{*(\beta'')}_{n^*m^*}(x) \,\middle|\, Q^{(\beta')}_{mn}(y) \right\rangle \Bigg|_{\substack{m^* \mapsto m \\ n \mapsto n - e_{\beta'} + e_{\hat\beta},\ n^* \mapsto n + e_{\beta''} - e_{\hat\beta} \\ s^*_\alpha \mapsto s_\alpha,\ t_\beta \mapsto t_\beta - b_\beta,\ t^*_\beta \mapsto t_\beta + b_\beta}} \\ &= \sum_{\beta=1}^{p} \sum_{\ell=0}^{\infty} b_{\beta,\ell+1} \begin{pmatrix} -2S_{\ell+2\delta_{\beta\hat\beta}}(\tilde{\partial}_{t_\beta}) \tau_{m,n+e_\beta-e_{\hat\beta}} \circ \tau_{m,n-e_\beta+e_{\hat\beta}} \\ + \dfrac{\partial^2}{\partial t_{\hat\beta,1} \partial t_{\beta,\ell+1}} \tau_{mn} \circ \tau_{mn} \end{pmatrix} + O(b^2), \end{aligned}$$

$$\begin{aligned} 0 &= \delta_{\beta'\beta''}(n,n^*) \tau_{mn}(s,t) \tau_{m^*n^*}(s^*,t^*) \left\langle P^{*(\beta'')}_{n^*m^*}(x) \,\middle|\, Q^{(\beta')}_{mn}(y) \right\rangle \Bigg|_{\substack{m \mapsto m - e_{\hat\alpha},\ m^* \mapsto m + e_{\hat\alpha} \\ n \mapsto n - e_{\beta'},\ n^* \mapsto n + e_{\beta''} \\ s_\alpha \mapsto s_\alpha - a_\alpha,\ s^*_\alpha \mapsto s_\alpha + a_\alpha,\ t^*_\beta \mapsto t_\beta}} \\ &= \sum_{\alpha=1}^{q} \sum_{\ell=0}^{\infty} a_{\alpha,\ell+1} \begin{pmatrix} 2S_{\ell+2\delta_{\alpha\hat\alpha}}(\tilde{\partial}_{s_\alpha}) \tau_{m+e_{\hat\alpha}-e_\alpha,n} \circ \tau_{m-e_{\hat\alpha}+e_\alpha,n} \\ - \dfrac{\partial^2}{\partial s_{\hat\alpha,1} \partial s_{\alpha,\ell+1}} \tau_{mn} \circ \tau_{mn} \end{pmatrix} + O(a^2), \end{aligned}$$

$$\delta_{\beta'\beta''}(n,n^*)\tau_{mn}(s,t)\tau_{m^*n^*}(s^*,t^*)\left\langle P^{*(\beta'')}_{n^*m^*}(x)\,\Big|\,Q^{(\beta')}_{mn}(y)\right\rangle\Bigg|_{\substack{m\mapsto m-e_{\hat\alpha},\ m^*\mapsto m+e_{\hat\alpha}\\ n\mapsto n-e_{\beta'},\ n^*\mapsto n+e_{\beta''}\\ s^*_\alpha\mapsto s_\alpha,\ t_\beta\mapsto t_\beta-b_\beta,\ t^*_\beta\mapsto t_\beta+b_\beta}}$$

$$= \quad -2\sum_{\beta=1}^{p}\sum_{\ell=0}^{\infty} b_{\beta,\ell+1}S_\ell(\tilde\partial_{t_\beta})\tau_{m+e_{\hat\alpha},n+e_\beta}\circ\tau_{m-e_{\hat\alpha},n-e_\beta}+O(b^2)$$

and

$$=\sum_{\beta=1}^{p}\sum_{\ell=0}^{\infty} b_{\beta,\ell+1}\frac{\partial^2}{\partial s_{\hat\alpha,1}\partial t_{\beta,\ell+1}}\tau_{mn}\circ\tau_{mn}+O(b^2),$$

$$\delta_{\beta'\beta''}(n,n^*)\tau_{mn}(s,t)\tau_{m^*n^*}(s^*,t^*)\left\langle P^{*(\beta'')}_{n^*m^*}(x)\,\Big|\,Q^{(\beta')}_{mn}(y)\right\rangle\Bigg|_{\substack{m^*\mapsto m\\ n\mapsto n-e_{\beta'}+e_{\hat\beta},\ n^*\mapsto n+e_{\beta''}-e_{\hat\beta}\\ s_\alpha\mapsto s_\alpha-a_\alpha,\ s^*_\alpha\mapsto s_\alpha+a_\alpha,\ t^*_\beta\mapsto t_\beta}}$$

$$= \quad \sum_{\alpha=1}^{q}\sum_{\ell=0}^{\infty} a_{\alpha,\ell+1}\frac{\partial^2}{\partial t_{\hat\beta,1}\partial s_{\alpha,\ell+1}}\tau_{mn}\circ\tau_{mn}+O(a^2),$$

and

$$=-2\sum_{\alpha=1}^{q}\sum_{\ell=0}^{\infty} a_{\alpha,\ell+1}S_\ell(\tilde\partial_{s_\alpha})\tau_{m-e_\alpha,n-e_{\hat\beta}}\circ\tau_{m+e_\alpha,n+e_\beta}+O(a^2).$$

Proof ¿From the proof of Theorem 5.1 and (59) it follows that

$$\delta_{\beta'\beta''}(n,n^*)\tau_{mn}(s,t)\tau_{m^*n^*}(s^*,t^*)\left\langle P^{*(\beta'')}_{n^*m^*}(x)\,\Big|\,Q^{(\beta')}_{mn}(y)\right\rangle\Bigg|_{\substack{n\mapsto n-e_{\beta'},\ n^*\mapsto n^*+e_{\beta''}\\ s_\alpha\mapsto s_\alpha-a_\alpha,\ s^*_\alpha\mapsto s_\alpha+a_\alpha\\ t_\beta\mapsto t_\beta-b_\beta,\ t^*_\beta\mapsto t_\beta+b_\beta}}$$

$$= \quad \sum_{\beta=1}^{p}(-1)^{\sigma_\beta(n)}\sum_{k=0}^{\infty}S_{n^*_\beta-n_\beta+1+k}(-2b_\beta)S_k(\tilde\partial_{t_\beta})e^{\Omega(a,b)}\tau_{m^*,n^*+e_\beta}\circ\tau_{m,n-e_\beta}$$

and

$$=\sum_{\alpha=1}^{q}(-1)^{\sigma_\alpha(m)}\sum_{k=0}^{\infty}S_{m_\alpha-m^*_\alpha+1+k}(-2a_\alpha)S_k(\tilde\partial_{s_\alpha})e^{\Omega(a,b)}\tau_{m^*-e_\alpha,n^*}\circ\tau_{m+e_\alpha,n}$$

and so if we just follow the 4 specializations leading to (60) – (63), in order, we find the 4 equations of the corollary, in their given order. □

7 Examples

7.1 Biorthogonal polynomials ($p = q = 1$)

Given the (not necessarily symmetric) inner product with regard to the weight $\rho(x,y)$ on $\mathbb{R}^2$,

$$\langle f(x)\mid g(y)\rangle := \iint_{\mathbb{R}^2} f(x)g(y)\rho(x,y)\,dx\,dy$$

and the deformed weight

$$\rho_{t,s}(x,y) = e^{\sum_1^\infty (t_k y^k - s_k x^k)} \rho(x,y).$$

Setting $p = q = 1,\ m = m_1,\ n = n_1$, with $m = n$, implies that the indices m, n in τ_{mn} can be replaced by one single index; namely, set $\tau_n := \tau_{mn}$, where

$$\tau_n(t,s) = \det\left(\left\langle x^i e^{-\sum_1^\infty s_k x^k} \,\middle|\, y^j e^{\sum_1^\infty t_k y^k} \right\rangle\right)_{0 \le i,j \le n-1}.$$

Moreover, set $\psi_1 = \varphi_1 = 1$ and define the monic polynomials $p_n^{(1)}(y) := p_n^{(1)}(t,s;y)$ and $p_n^{(2)}(x) := p_n^{(2)}(t,s;x)$ (with h_{n-1}^{-1} the leading coefficient of $P_{nm}^{*(1,1)}(x)$) by

$$\begin{aligned} p_n^{(1)}(y) &:= Q_{mn}^{(1,1)}(y) = y^n + \cdots \\ h_{n-1}^{-1} p_{n-1}^{(2)}(x) &:= P_{nm}^{*(1,1)}(x) = h_{n-1}^{-1} x^{n-1} + \cdots \end{aligned}$$

The orthogonality conditions (4) and (6) imply

$$\begin{aligned} \left\langle x^i e^{-\sum_1^\infty s_k x^k} \,\middle|\, p_n^{(1)}(y) e^{\sum_1^\infty t_k y^k} \right\rangle &= 0 \quad \text{for} \quad 0 \le i \le n-1 \\ \left\langle h_n^{-1} p_n^{(2)}(x) e^{-\sum_1^\infty s_k x^k} \,\middle|\, y^j e^{\sum_1^\infty t_k y^k} \right\rangle &= 0 \quad \text{for} \quad 0 \le j \le n-1 \\ &= 1 \quad \text{for} \quad j = n. \end{aligned}$$

for all $n \ge 0$, from which the bi-orthogonality can be deduced[13]

$$\iint_{\mathbb{R}^2} p_n^{(2)}(x) p_m^{(1)}(y) \rho_{t,s}(x,y)\,dx\,dy = \delta_{nm} h_n.$$

¿From (3), (5), (7) and (8) and from $h_n = \tau_{n+1}/\tau_n$, it follows that

$$\begin{aligned} z^n \frac{\tau_n(t - [z^{-1}], s)}{\tau_n(t,s)} &= p_n^{(1)}(z) \\ z^n \frac{\tau_n(t, s + [z^{-1}])}{\tau_n(t,s)} &= p_n^{(2)}(z) \\ z^{-n-1} \frac{\tau_{n+1}(t + [z^{-1}], s)}{\tau_n(t,s)} &= \iint_{\mathbb{R}^2} \frac{p_n^{(2)}(x)}{z - y} \rho_{t,s}(x,y)\,dx\,dy \\ z^{-n-1} \frac{\tau_{n+1}(t, s - [z^{-1}])}{\tau_n(t,s)} &= \iint_{\mathbb{R}^2} \frac{p_n^{(1)}(y)}{z - x} \rho_{t,s}(x,y)\,dx\,dy. \end{aligned} \tag{77}$$

and from (51), the bilinear identity becomes

$$\begin{aligned} &\oint_{z=\infty} \tau_{n-1}(t - [z^{-1}], s)\, \tau_{m+1}(t' + [z^{-1}], s')\, e^{\sum_1^\infty (t_i - t_i') z^i} z^{n-m-2} dz \\ &\quad = \oint_{z=\infty} \tau_n(t, s - [z^{-1}])\, \tau_m(t', s' + [z^{-1}])\, e^{\sum_1^\infty (s_i - s_i') z^i} z^{m-n} dz, \end{aligned}$$

[13]It turns out that $h_n = \tau_{n+1}(t,s)/\tau_n(t,s)$.

which characterizes the τ-functions for the *2-component KP hierarchy*. Equations (77) and the bilinear identity were obtained in [1]. Indicating the dependence on t,s in the polynomials, the following inner product can be computed in two different ways, leading to[14]

$$
\begin{aligned}
&\tau_n(t,s)\tau_{n+1}(t',s')\iint_{\mathbb{R}^2} dxdy\; p^{(2)}_{n+1}(t',s';x)p^{(1)}_n(t,s;y)e^{\sum_1^\infty (t_k y^k - s'_k x^k)}\rho(x,y)\Bigg|_{\substack{t\mapsto t-a\\ t'\mapsto t'+a\\ s'=s}}\\
&=\left(\sum_{j=0}^\infty -2a_{j+1}S_j(\tilde\partial_t)\tau_{n+2}\circ\tau_n + O(a^2)\right)\\
&=\left(\sum_{k=1}^\infty a_k\frac{\partial^2}{\partial t_k\partial s_1}\tau_{n+1}\circ\tau_{n+1}+O(a^2)\right).
\end{aligned}
\tag{78}
$$

Identifying the coefficients of a_{j+1} in both expressions and shifting $n\mapsto n-1$ yield a first identity; then redoing the calculation above for $s\mapsto s-b,\ s'\mapsto s'+b$ and $t'=t$ leads to a second one. All in all we find

$$
\begin{aligned}
S_j(\tilde\partial_t)\tau_{n+1}\circ\tau_{n-1} &= -\tau_n^2\frac{\partial^2}{\partial s_1\partial t_{j+1}}\ln\tau_n,\\
S_j(\tilde\partial_s)\tau_{n-1}\circ\tau_{n+1} &= -\tau_n^2\frac{\partial^2}{\partial t_1\partial s_{j+1}}\ln\tau_n.
\end{aligned}
$$

Specializing the identity (73) leads to an identity, which can be expressed as a sum of two Wronskians[15] and which involves a single tau function:

$$
\left\{\frac{\partial^2\ln\tau_n}{\partial t_1\partial s_2},\frac{\partial^2\ln\tau_n}{\partial t_1\partial s_1}\right\}_{t_1}+\left\{\frac{\partial^2\ln\tau_n}{\partial s_1\partial t_2},\frac{\partial^2\ln\tau_n}{\partial t_1\partial s_1}\right\}_{s_1}=0. \tag{79}
$$

The computation (2.2) was at the origin of the crucial argument (Theorem 5.1) in this paper. It illustrates in a simple way what is being done in this paper. These equations are used, when computing the PDE for the Dyson, Airy and Sine processes ([3]).

7.2 Orthogonal polynomials

Given a weight $\rho(z)$ on $\mathbb{R}$, the symmetric inner product

$$
\langle f(x)\mid g(x)\rangle=\int_{\mathbb{R}} f(x)g(x)\rho(x)\,dx,
$$

and the formal deformation by means of an exponential $\rho_t(x):=\rho(x)e^{\sum_1^\infty t_k z^k}$. This is a special case of the previous example, where the deformation only

[14] This integral is $\neq 0$, unless $t=t'$

[15] in terms of the Wronskian $\{f,g\}_t=\frac{\partial f}{\partial t}g-f\frac{\partial g}{\partial t}$.

depends on $t-s$; thus $t-s$ can be replaced by t. Then $\tau_n(t)$ is the determinant of the moment matrix depending on $t=(t_1,t_2,\ldots)$,

$$\tau_n(t) := \det\Bigl(\int_{\mathbb{R}} z^{i+j} e^{\sum_1^\infty t_k z^k} \rho_t(z)dz\Bigr)_{0\le i,j\le n-1}.$$

Then, from (77), it follows at once that the orthogonal polynomials $p_n(x) := p_n(t;x)$ are given by

$$\begin{aligned} z^n \frac{\tau_n(t-[z^{-1}])}{\tau_n(t)} &= p_n(z) \\ z^{-n-1}\frac{\tau_{n+1}(t+[z^{-1}])}{\tau_n(t)} &= \int_{\mathbb{R}} \frac{p_n(x)}{z-x}\rho_t(x)dx. \end{aligned}$$

Moreover, the integral below can be computed in two different ways: on the one hand, it is automatically zero, because $p_n(z)$ is perpendicular to any polynomial of lower degree; on the other hand, for t and t' close to each other, the integral can also be developed, using the technique of Proposition 6.2, in $t'-t=2y$, yielding the following formula

$$\begin{aligned} 0 &= \tau_n(t)\tau_n(t') \int_{\mathbb{R}} p_n(t;z)p_{n-1}(t',z)\rho_t(z)dz \Big|_{\substack{t\mapsto t-y \\ t'\mapsto t+y}} \\ &= \sum_3^\infty y_k \left(\frac{\partial^2}{\partial t_1 \partial t_k} - 2S_{k+1}\Bigl(\tilde{\partial}_t\Bigr)\right)\tau_n \circ \tau_n + O(y^2), \end{aligned}$$

showing that $\tau_n(t)$ satisfies the *KP hierarchy*.

7.3 Orthogonal polynomials on the circle

Consider the inner product on the circle between analytic functions on S^1:

$$\langle f(z) \mid g(z)\rangle = \oint_{S^1} \frac{dz}{2\pi\sqrt{-1}z} f(z^{-1})g(z)$$

and the determinant of moment matrices

$$\begin{aligned} \tau_n(t,s) &:= \det\Bigl(\Bigl\langle z^k e^{-\sum_1^\infty s_i z^i} \Bigm| z^\ell e^{\sum_1^\infty t_i z^i}\Bigr\rangle\Bigr)_{0\le k,\ell\le n-1} \\ &= \det\left(\oint_{S^1} \frac{dz}{2\pi\sqrt{-1}z} z^{-k+\ell} e^{\sum_1^\infty (t_i z^i - s_i z^{-i})}\right)_{0\le k,\ell\le n-1} \end{aligned}$$

Then it follows that

$$\begin{aligned} z^n \frac{\tau_n(t-[z^{-1}],s)}{\tau_n(t,s)} &= p_n^{(1)}(z) \\ z^n \frac{\tau_n(t,s+[z^{-1}])}{\tau_n(t,s)} &= p_n^{(2)}(z) \end{aligned}$$

$$
\begin{aligned}
z^{-n-1}\frac{\tau_{n+1}(t+[z^{-1}],s)}{\tau_n(t,s)} &= \oint_{S^1}\frac{du}{2\pi\sqrt{-1}u}\frac{p_n^{(2)}(u^{-1})}{z-u}e^{\sum_1^\infty(t_iu^i-s_iu^{-i})}\\
z^{-n-1}\frac{\tau_{n+1}(t,s-[z^{-1}])}{\tau_n(t,s)} &= \oint_{S^1}\frac{du}{2\pi iu}\frac{p_n^{(1)}(u)}{z-u^{-1}}e^{\sum_1^\infty(t_iu^i-s_iu^{-i})},
\end{aligned}
$$

with $p_n^{(1)}(z)$ and $p_m^{(2)}(z^{-1})$ monic orthogonal polynomials on the circle:

$$
\oint_{S^1}\frac{dz}{2\pi iz}p_n^{(1)}(z)p_m^{(2)}(z^{-1}) = \delta_{nm}h_n, \quad \text{with } h_n = \frac{\tau_{n+1}}{\tau_n}.
$$

The nature of the inner product implies some extra-relationship between the orthogonal polynomials

$$
\begin{aligned}
p_{n+1}^{(1)}(z) - zp_n^{(1)}(z) &= p_{n+1}^{(1)}(0)z^np_n^{(2)}(z^{-1})\\
p_{n+1}^{(2)}(z) - zp_n^{(2)}(z) &= p_{n+1}^{(2)}(0)z^np_n^{(1)}(z^{-1}).
\end{aligned}
$$

leading to (in the notation of footnote 8)

$$
\begin{aligned}
&\left(\frac{h_n}{h_{m+1}}\right)^2\left(1-\frac{h_{n+1}}{h_n}\right)\left(1-\frac{h_{m+1}}{h_m}\right)\\
&\quad = \frac{1}{\tau_{m+2}^2\tau_n^2}\left(S_{n-m}(\tilde\partial_t)\tau_{m+2}\circ\tau_n\right)\ .\ \left(S_{n-m}(-\tilde\partial_s)\tau_{m+2}\circ\tau_n\right)\ .
\end{aligned}
$$

In particular, for $m = n-1$,

$$
\left(1-\frac{h_{n+1}}{h_n}\right)\left(1-\frac{h_n}{h_{n-1}}\right) = -\frac{\partial}{\partial t_1}\ln h_n\frac{\partial}{\partial s_1}\ln h_n.
$$

7.4 Non-intersecting Brownian motions

Consider N non-intersecting Brownian motions $x_1(t),\dots,x_N(t)$ in $\mathbb{R}$, leaving from distinct points $\alpha_1 < \dots < \alpha_N$ and forced to end up at distinct points $\beta_1 < \dots < \beta_N$. From the Karlin-McGregor formula (see [12]), the probability that all $x_i(t)$ belong to $E \subset \mathbb{R}$ can be expressed in terms of the Gaussian $p(t,x,y) = e^{-(x-y)^2/2t}/\sqrt{2\pi t}$, as follows $(0<t<1)$

$$
\begin{aligned}
&\mathbb{P}_\alpha^\beta\,(\text{all } x_i(t)\in E)\\
&\quad := \mathbb{P}_\alpha^\beta\left(\text{all } x_i(t)\in E \;\middle|\; \begin{array}{l}(x_1(0),\dots,x_N(0)) = (\alpha_1,\dots,\alpha_N)\\ (x_1(1),\dots,x_N(1)) = (\beta_1,\dots,\beta_N)\end{array}\right)\\
&\quad = \frac{1}{Z_N}\int_{E^N}\det[p(t,\alpha_i,x_j)]_{1\le i,j\le N}\det[p(1-t,x_i,\beta_j)]_{1\le i,j\le N}\prod_{i=1}^N dx_i\\
&\quad = \frac{1}{Z_N'}\int_{E^N}\prod_{i=1}^N e^{\frac{-x_i^2}{2t(1-t)}}dx_i\det\left[e^{\frac{\alpha_ix_j}{t}}\right]_{1\le i,j\le N}\det\left[e^{\frac{\beta_ix_j}{1-t}}\right]_{1\le i,j\le N} \qquad (80)
\end{aligned}
$$

The limiting case where several points α and β coincide has been the object of many interesting studies. It is obtained by taking appropriate limits of the formulae above. Just to fix the notation, consider

$$\begin{aligned}(\alpha_1,\dots,\alpha_N) &= (\overbrace{a_1,a_1,\dots,a_1}^{m_1},\overbrace{a_2,a_2,\dots,a_2}^{m_2},\dots,\overbrace{a_q,a_q,\dots,a_q}^{m_q})\\ (\beta_1,\dots,\beta_N) &= (\overbrace{b_1,b_1,\dots,b_1}^{n_1},\overbrace{b_2,b_2,\dots,b_2}^{n_2},\dots,\overbrace{b_p,b_p,\dots,b_p}^{n_p}),\end{aligned}$$

where $\sum_{\alpha=1}^{q} a_\alpha = \sum_{\beta=1}^{p} b_\beta = 0$ and

$$a_1 < a_2 < \cdots a_q, \quad b_1 < b_2 < \ldots < b_p, \quad \sum_{\alpha=1}^{q} m_\alpha = \sum_{\beta=1}^{p} n_\beta = N.$$

Then, take the limit of (80), make a change of variables in the second equality, use the standard matrix identity in the third equality

$$\sum_{\sigma\in S_n} \det\left(a_{i,\sigma(j)}\ b_{j,\sigma(j)}\right)_{1\le i,j\le n} = \det\left(a_{ik}\right)_{1\le i,k\le n} \det\left(b_{ik}\right)_{1\le i,k\le n},$$

and distribute the integral and the Gaussian over the different columns; this yields

$$\begin{aligned}
&\mathbb{P}_\alpha^\beta\left(\text{all } x_i(t)\in E\right)\\
&= \frac{1}{Z_N}\int_{E^N}\prod_{i=1}^{N} e^{-\frac{x_i^2}{2t(1-t)}}dx_i\\
&\qquad \times \det\begin{pmatrix} \left(x_j^i e^{\frac{a_1x_j}{t}}\right)_{\substack{0\le i<m_1\\1\le j\le N}}\\ \vdots \\ \left(x_j^i e^{\frac{a_qx_j}{t}}\right)_{\substack{0\le i<m_q\\1\le j\le N}}\end{pmatrix}\cdot\det\begin{pmatrix} \left(x_j^i e^{\frac{b_1x_j}{1-t}}\right)_{\substack{0\le i<n_1\\1\le j\le N}}\\ \vdots \\ \left(x_j^i e^{\frac{b_qx_j}{1-t}}\right)_{\substack{0\le i<n_p\\1\le j\le N}}\end{pmatrix}\\
&= \frac{1}{Z'_N}\int_{\tilde E^N}\prod_{i=1}^{N} e^{-\frac{y_i^2}{2}}dy_i\\
&\qquad \times \det\begin{pmatrix} \left(y_j^i e^{\tilde a_1 y_j}\right)_{\substack{0\le i<m_1\\1\le j\le N}}\\ \vdots \\ \left(y_j^i e^{\tilde a_q y_j}\right)_{\substack{0\le i<m_q\\1\le j\le N}}\end{pmatrix}\cdot\det\begin{pmatrix} \left(y_j^i e^{\tilde b_1 y_j}\right)_{\substack{0\le i<n_1\\1\le j\le N}}\\ \vdots \\ \left(y_j^i e^{\tilde b_q y_j}\right)_{\substack{0\le i<n_p\\1\le j\le N}}\end{pmatrix}\Bigg|_{\substack{\tilde E=\frac{E}{\sqrt{t(1-t)}}\\ \tilde a_i=\sqrt{\frac{1-t}{t}}a_i\\ \tilde b_i=\sqrt{\frac{t}{1-t}}b_i}}
\end{aligned}$$

$$= \quad \frac{N!}{Z'_N} \det \left(\left(\int_{\tilde E} dy \; e^{-\frac{y^2}{2}} y^{i+j} e^{(\tilde a_\alpha + \tilde b_\beta) y} \right)_{\substack{0 \le i < m_\alpha \\ 0 \le j < n_\beta}} \right)_{\substack{1 \le \alpha \le q \\ 1 \le \beta \le p}} .$$

The numerator of this probability has exactly the form (1) evaluated at $s_\alpha = t_\beta = 0$, with the inner product given by (2)

$$\left\langle x^i \psi_\alpha(x) \;\middle|\; y^j \varphi_\beta(y) \right\rangle = \int_{\tilde E} dy \; e^{-\frac{y^2}{2}} y^{i+j} e^{(\tilde a_\alpha + \tilde b_\beta) y},$$

upon setting $\psi_\alpha(x) = e^{\tilde a_\alpha x}$, $\varphi_\beta(y) = e^{\tilde b_\beta y}$ and $d\mu(x,y) = \delta(x-y) e^{-y^2/2} \chi_{\tilde E}(y)\, dx$. By multiplying each of the exponentials $e^{\tilde a_\alpha y}$ and $e^{\tilde b_\beta y}$ by $e^{-\sum_1^\infty s_{\alpha,k} y^k}$ and $e^{\sum_1^\infty t_{\beta,k} y^k}$ respectively, it follows that both the numerator and the denominator of the probability above,

$$\begin{aligned} &\tau_{mn}(t_1, \ldots, t_p; s_1, \ldots, s_q) \\ &\quad = \det \left(\left(\int_{\tilde E} dy \; e^{-\frac{y^2}{2}} y^{i+j} e^{(\tilde a_\alpha + \tilde b_\beta) y + \sum_1^\infty (t_{\beta,k} - s_{\alpha,k}) y^k} \right)_{\substack{0 \le i < m_\alpha \\ 0 \le j < n_\beta}} \right)_{\substack{1 \le \alpha \le q \\ 1 \le \beta \le p}} \\ &\quad = \det \left(\left(\left\langle x^i \psi_\alpha^{-s}(x) \;\middle|\; y^j \varphi_\beta^t(y) \right\rangle \right)_{\substack{0 \le i < m_\alpha \\ 0 \le j < n_\beta}} \right)_{\substack{1 \le \alpha \le q \\ 1 \le \beta \le p}} \end{aligned}$$

and the same expression for $E = \mathbb{R}$, satisfy the bilinear identity for $p+q$-component KP and, in particular, all the general relations and identities, mentioned in this paper, namely (12) and (71), (72), (73). Note the equations are independent of the set E.

In particular, for n non-intersecting Brownian motions, departing from the origin, with n_1 paths forced to end up at $-a$ and n_2 paths forced to end up at a, we have for $0 < t < 1$,

$$\mathbb{P}_0^{\pm a}(\text{all } x(t) \in \tilde E) = \frac{1}{Z_n} \det \begin{pmatrix} (\mu_{ij}^{+})_{0 \le i \le n_1 - 1,\ 0 \le j \le n_1 + n_2 - 1} \\ \\ (\mu_{ij}^{-})_{0 \le i \le n_2 - 1,\ 0 \le j \le n_1 + n_2 - 1} \end{pmatrix}$$

where

$$\mu_{ij}^{\pm} := \int_E x^{i+j} e^{-\frac{x^2}{2} \pm \alpha x} dx, \tag{81}$$

with the change of variables

$$\alpha = a \sqrt{\frac{2t}{1-t}} \quad \text{and} \quad E = \tilde E \sqrt{\frac{2}{t(1-t)}}.$$

In a similar way, for several times $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1$,

$$\mathbb{P}_0^{\pm a}(\text{all } x_i(t_1) \in \tilde E_1, \ldots, \text{all } x_i(t_m) \in \tilde E_m) = \frac{1}{Z_n} \det \begin{pmatrix} (\mu_{ij}^{+})_{0 \le i \le n_1 - 1,\ 0 \le j \le n_1 + n_2 - 1} \\ \\ (\mu_{ij}^{-})_{0 \le i \le n_2 - 1,\ 0 \le j \le n_1 + n_2 - 1} \end{pmatrix},$$

where

$$\mu_{ij}^{\pm} = \int_{\prod_1^m E_k} (x_1)^{j-1}(x_m)^{i-1} e^{-\frac{1}{2}\sum_{\ell=1}^m x_\ell^2 \pm \alpha x_m + \sum_{\ell=1}^{m-1} c_\ell x_\ell x_{\ell+1}} \prod_{\ell=1}^m dx_\ell, \qquad (82)$$

with the change of variables

$$\alpha = a\sqrt{\frac{2(t_m - t_{m-1})}{(1-t_m)(1-t_{m-1})}}, \quad E_\ell = \tilde{E}_\ell \sqrt{\frac{2(t_{\ell+1} - t_{\ell-1})}{(t_{\ell+1}-t_\ell)(t_\ell - t_{\ell-1})}},$$

and

$$c_j = \sqrt{\frac{(t_{j+2}-t_{j+1})(t_j - t_{j-1})}{(t_{j+2}-t_j)(t_{j+1}-t_{j-1})}}, \quad \text{for } 1 \leq j \leq m-1.$$

We now introduce the inner products

$$\langle f \mid g \rangle_1 = \int_E f(x) g(x) F_1(x) dx \quad \text{with} \quad F_1(x) = e^{-\frac{x^2}{2}}$$

and ($m \geq 2$)

$$\langle f \mid g \rangle_m = \int_{\prod_1^m E_k} f(x_1) g(x_m) F_m(x_1, \ldots, x_m) dx_1 \ldots dx_m,$$

with

$$F_m(x_1, \ldots, x_m) := \left(\prod_1^m e^{-\frac{x_\ell^2}{2}} \right) e^{\sum_{p,q \geq 1} \sum_{\ell=1}^{m-1} c_{pq}^{(\ell)} x_\ell^p x_{\ell+1}^q + \sum_{\ell=2}^{m-1} \sum_{r=1}^{\infty} \gamma_r^{(\ell)} x_\ell^r}.$$

The precise form of F_m does not matter very much for the purpose of this paper, but does play a crucial role in satisfying the Virasoro constraints.

In these two sets of moments (81) and (82), we insert extra time-parameters, as follows, which can then be identified with the moments appearing in (1),

$$\begin{aligned} \mu_{ij}^{\pm}(s,u,v) &= \int_E x^{i+j} e^{-\frac{x^2}{2} \pm ax \pm \beta x^2} e^{\sum_1^\infty (s_k - \binom{u_k}{v_k}) x^k} dx \\ &= \left\langle x^i e^{-\sum_1^\infty s_k x^k} \,\middle|\, x^j e^{\sum_1^\infty \binom{u_k}{v_k} x^k} e^{\pm \alpha x \pm \beta x^2} \right\rangle_1, \end{aligned}$$

and

$$\begin{aligned} \mu_{ij}^{\pm}(s,u,v) &= \int_{\prod_1^m E_k} x_1^i x_m^j F_m(x_1, \ldots, x_m) e^{\sum_{k=1}^\infty \left(s_k x_1^k - \binom{u_k}{v_k} x_m^k \right)} \prod_{\ell=1}^m dx_\ell \\ &= \left\langle x^i e^{-\sum_1^\infty s_k x^k} \,\middle|\, x^j e^{\sum_1^\infty \binom{u_k}{v_k} x^k} e^{\pm \alpha x \pm \beta x^2} \right\rangle_m \end{aligned}$$

In both cases, we have $q = 1$ and $p = 2$, and $m := n_1 + n_2$, leading to the introduction of three sets of times $s_i := -s_{1i}$, $u_i := -t_{1i}$ and $v_i := -t_{2i}$.Thus, from the general theory, the numerator of both probabilities,

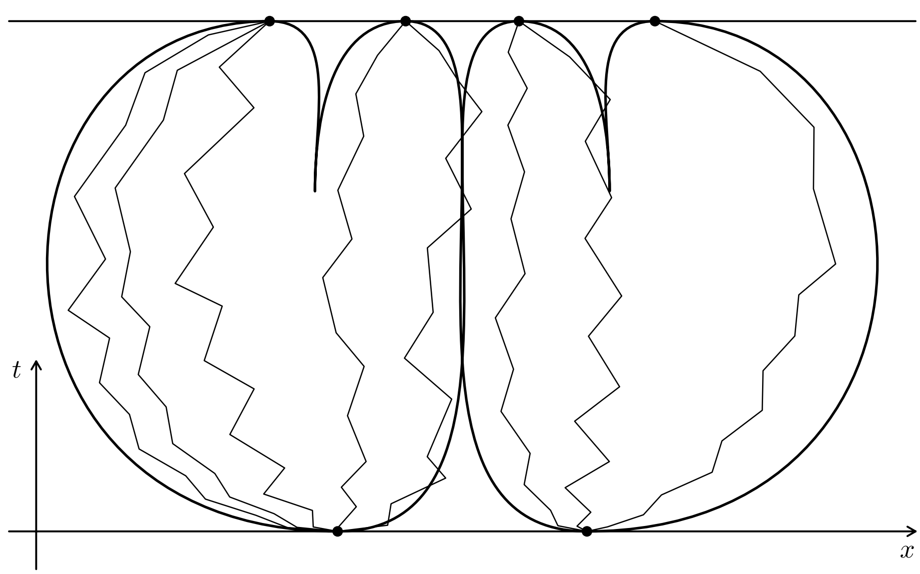
$$\tau_{n_1,n_2} = \det \begin{pmatrix} (\mu^{+}_{ij})_{0\le i\le n_1-1,\ 0\le j\le n_1+n_2-1} \\ \\ (\mu^{-}_{ij})_{0\le i\le n_2-1,\ 0\le j\le n_1+n_2-1} \end{pmatrix}$$

satisfies the bilinear identity for the 3-component KP and, in particular, the PDE's and thus τ_{n_1,n_2} satisfies the single PDE

$$\begin{aligned} \frac{\partial}{\partial s_1} \ln \frac{\tau_{n_1+1,n_2}}{\tau_{n_1-1,n_2}} &= \frac{\frac{\partial^2}{\partial s_2 \partial u_1} \ln \tau_{n_1,n_2}}{\frac{\partial^2}{\partial s_1 \partial u_1} \ln \tau_{n_1,n_2}} \\ -\frac{\partial}{\partial u_1} \ln \frac{\tau_{n_1+1,n_2}}{\tau_{n_1-1,n_2}} &= \frac{\frac{\partial^2}{\partial s_1 \partial u_2} \ln \tau_{n_1,n_2}}{\frac{\partial^2}{\partial s_1 \partial u_1} \ln \tau_{n_1,n_2}}. \end{aligned}$$

$$\frac{\partial}{\partial u_1} \frac{\frac{\partial^2}{\partial s_2 \partial u_1} \ln \tau_{n_1,n_2}}{\frac{\partial^2}{\partial s_1 \partial u_1} \ln \tau_{n_1,n_2}} + \frac{\partial}{\partial s_1} \frac{\frac{\partial^2}{\partial s_1 \partial u_2} \ln \tau_{n_1,n_2}}{\frac{\partial^2}{\partial s_1 \partial u_1} \ln \tau_{n_1,n_2}} = 0$$

and the same PDE with u_i replaced by v_i. These PDE's play a crucial role in establishing the PDE for the Pearcey process; see [4].



The methods developed in this paper should enable one to study more complicated situations of non-intersecting Brownian motions, as indicated in the figure above. The curves in the (x, t)-plane are the boundary of the equilibrium measure as a function of time. When two curves meet, one expects to see a new infinite-dimensional diffusion in that neighborhood, beyond the Pearcey process.

References

[1] M. Adler and P. van Moerbeke. String-orthogonal polynomials, string equations and 2-Toda symmetries, *Comm. Pure & Appl. Math.*, **50**, 241–290 (1997). (arXiv: hep-th/9706182)

[2] M. Adler and P. van Moerbeke. Generalized orthogonal polynomials, discrete KP and Riemann-Hilbert problems. *Comm. Math. Phys.*, 207(3):589–620, 1999.

[3] M. Adler and P. van Moerbeke. PDEs for the joint distributions of the Dyson, Airy and sine processes. *Ann. Probab.*, 33(4):1326–1361, 2005.

[4] M. Adler, P. van Moerbeke. PDE's for the gaussian ensemble with external source and the Pearcey distribution. *arXiv:math.PR/0509047*, 2006.

[5] A. Aptekarev: Multiple orthogonal polynomials, *J. Comp. Appl. Math.*, **99**, 423–447 (1998).

[6] A. Aptekarev, A. Branquinho and W. Van Assche: Multiple orthogonal polynomials for classical weights, *Trans. Amer. Math. Soc.*, **355**, 3887–3914 (2003)

[7] P. Bleher and A. Kuijlaars: Large n limit of Gaussian random matrices with external source, Part I, *Comm. Math. Phys.*, **252**, 43–76 (2004).

[8] P. Bleher and A. Kuijlaars: Random matrices with external source and multiple orthogonal polynomials. *Int. Math. Res. Not.* **3** (2004) 109–129.

[9] E. Daems and A. Kuijlaars. Multiple orthogonal polynomials of mixed type and non-intersection brownian motions. *arXiv:math.CA/0511470*, 2005.

[10] P. Deift and X. Zhou: A steepest descent method for oscillatory Riemann-Hilbert problems. *Ann. of Math.* **137**, 295–368 (1993).

[11] A. S. Fokas, A. R. Its, and A. V. Kitaev. The isomonodromy approach to matrix models in 2D quantum gravity. *Comm. Math. Phys.*, 147(2):395–430, 1992.

[12] S. Karlin and J. McGregor. Coincidence probabilities. *Pacific J. Math.*, 9:1141–1164, 1959.

[13] K. Ueno and K. Takasaki. Toda lattice hierarchy. In Group representations and systems of differential equations (Tokyo, 1982), volume 4 of *Adv. Stud. Pure Math.*, pages 1–95. North-Holland, Amsterdam, 1984.

[14] W. Van Assche and E. Coussement: Some classical multiple orthogonal polynomials, *J. Comp. Appl. Math.*, **127**, 317–347 (2001).

[15] W. Van Assche, J.S. Geronimo and A. Kuijlaars: Riemann-Hilbert problems for multiple orthogonal polynomials, *Special functions 2000: Current Perspectives and Future directions* (J. Bustoz et al., eds) Kluwer, Dordrecht, 23–59 (2001).