Singularity confinement for a class of $m$-th order difference equations of combinatorics

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Abstract

In a recent publication, it was shown that a large class of integrals over the unitary group $U(n)$ satisfy non-linear, non-autonomous difference equations over $n$, involving a finite number of steps; special cases are generating functions appearing in questions of longest increasing subsequences in random permutations and words. The main result of the paper states that these difference equations have the discrete Painlevé property; roughly speaking, this means that, after a finite number of steps, the solution to these difference equations may develop a pole (Laurent solution), depending on the maximal number of free parameters, and immediately after be finite again (“singularity confinement”). The technique used in the proof is based on an intimate relationship between the difference equations (discrete time) and the Toeplitz lattice (continuous time differential equations); the point is that the “Painlevé property” for the discrete relations is inherited from the “Painlevé property” of the (continuous) Toeplitz lattice.

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1 Introduction

In a recent publication (Adler & van Moerbeke 2003), we have shown that a large class of integrals over the unitary group $U(n)$ satisfy non-linear, non-autonomous difference equations over $n$, involving a finite number of steps; these $U(n)$-integrals are motivated by generating functions appearing in questions of longest increasing subsequences in random permutations and words (see Adler & van Moerbeke 2001, 2003, 2004; Baik & Rains 2001, Borodin 2003; Rains 1998; Tracy & Widom 1999, 2001). The main result of this paper, announced in Adler & van Moerbeke 2003, states that these difference equations, which are also recursion relations, have the discrete Painlevé property; roughly speaking, this means that the solution to these difference equations may develop a pole (formal Laurent solution) after a finite number of steps and immediately after be finite again, a fact which had been observed by Borodin 2003 in the very special case of unitary matrix integrals related to longest increasing sequences of random permutations. Moreover, these formal Laurent solutions depend on the maximal number of free parameters, which equals ((order of difference equation) $-1$) $\times$ (dim of phase space), with the poles disappearing after a finite number of steps (“singularity confinement”), due to the non-linear and non-autonomous character of the equations.

The property of singularity confinement was introduced in Grammaticos, Ramani & Papageorgiou 1991 (see also Suris 1989), and further studied in Grammaticos, Nijhoff & Ramani 1999, as a method to find discrete Painlevé systems. They were motivated by the famous Painlevé property for continuous systems (see Ince 1944) that movable (initial condition dependent) singularities be single-valued. They were further motivated to get a classification of discrete Painlevé equations in the style of the situation for the continuous case, which they and others have embarked on and had some success. For instance, Clarkson & Webster 2000 used singularity confinement to get the so-called d-PIII equation, whose particular solutions in the limit go to Painlevé III. It should be pointed out that singularity confinement can fail to produce integrability, as was shown by Hietarinta & Viallet 1998, and further tests for integrability have been proposed, such as using “algebraic entropy” by Bellon & Viallet 1999 or using Nevanlinna theory (Ablowitz, Halburd & Herbst 2000 or Ramani et al. 2003). Thus in discrete systems the situation is more complicated than in the continuous situation, which should come as no surprise.

Nonetheless, singularity confinement is still a stiff requirement for a discrete system to pass and quite often (but not all the time) indicates integrability. In discovering that a large class, related to combinatorics, of integrals over the unitary group satisfy discrete recursion relations, it is natural to ask: “What might be the nature of these recursion relations?”. Moreover, since from the derivation of these relations, they were clearly related to an integrable system called the Toeplitz lattice, it was natural to wonder if these discrete relations were integrable or at least have some “integrable-like property”, especially since two of the relations coming from combinatorics actually possessed invariants, one case being that of McMillan and the other being a generalization of the McMillan case. This paper answers the latter question in the affirmative. Indeed, this huge class of recursion relations coming from unitary integrals and combinatorics possesses the “integrable-like Painlevé property” called singularity confinement as this paper shall demonstrate. It would be worthwhile to compute the algebraic entropy of these examples, as it has been hoped that also requiring that the algebraic entropy is zero would suffice for integrability.

The technique used in the proof is new and is based on an intimate relation between the difference equations (discrete time) and the Toeplitz lattice (continuous time differential equations), introduced in Adler & van Moerbeke 2001; the point is that the the “Painlevé property” for the discrete relations are inherited from the “Painlevé property” of the (continuous) Toeplitz lattice. Before making a more precise statement and describing the technique, recall the basic facts about the Toeplitz lattice and the recursion relations (Adler & van Moerbeke 2001, 2003).
For \( k \in \mathbb{N} \) and \( \epsilon \in \{-1, 0, 1\} \), consider the matrix integrals
\[
\tau_k^\epsilon(t, s) = \int_{U(k)} (\det M)^{\epsilon+\gamma} e^{i \sum_{j=1}^{\infty} \text{Trace}(t_j M^j - s_j M^{-j})} \, dM
\]
where \( dM \) is Haar measure on \( U(k) \), \( t = (t_1, t_2, \ldots) \) and \( s = (s_1, s_2, \ldots) \). Special choices of \( t_j \) and \( s_j \) lead to generating functions in combinatorics (see Adler & van Moerbeke 2003). Set \( \tau := \tau^0 \) and \( \tau^\pm := \tau^{\pm 1} \). In Adler & van Moerbeke 2001, it was shown that the matrix integrals (1) satisfy a
\[
H^{(1)} = \sum_{i=1}^{2} \int \frac{\partial H^{(1)}_i}{\partial x_i} \, dx_i \quad \text{and} \quad H^{(2)} = \sum_{i, j=1}^{2} \int \frac{\partial H^{(2)}_{ij}}{\partial y_{ij}} \, dy_{ij}
\]
where \( i = 1, 2, 3, \ldots \) Moreover, \( \tau_n \) is a polynomial expression in the variables \( x_k \) and \( y_k \) and \( \tau_1 \):
\[
\tau_n = \tau_1^n \prod_{k=1}^{n-1} (1 - x_k y_k)^{n-k}.
\]
The Hamiltonians \( H^{(1)} \) appearing in (2) are given by
\[
H^{(1)}_i = -\frac{1}{i} \text{Trace} L_i, \quad i = 1, 2, 3, \ldots, \quad l = 1, 2,
\]
where the matrices \( L_1 \) and \( L_2 \) are defined by
\[
L_1 := \begin{pmatrix}
-x_1 y_0 & 1 - x_1 y_1 & 0 & 0 \\
-x_2 y_0 & -x_2 y_1 & 1 - x_2 y_2 & 0 \\
-x_3 y_0 & -x_3 y_1 & -x_3 y_2 & 1 - x_3 y_3 \\
-x_4 y_0 & -x_4 y_1 & -x_4 y_2 & -x_4 y_3 \\
& & & \ddots
\end{pmatrix}
\]
and
\[
L_2 := \begin{pmatrix}
-x_0 y_1 & -x_0 y_2 & -x_0 y_3 & -x_0 y_4 \\
1 - x_1 y_1 & -x_1 y_2 & -x_1 y_3 & -x_1 y_4 \\
0 & 1 - x_2 y_2 & -x_2 y_3 & -x_2 y_4 \\
0 & 0 & 1 - x_3 y_3 & -x_3 y_4 \\
& & & \ddots
\end{pmatrix}.
\]
The system admits a reduction, interesting in its own right, obtained by putting \( x_k = y_k \) for all \( k \). We refer to it as the self-dual Toeplitz lattice.

In Adler & van Moerbeke 2001, it was shown that the matrix integrals (1) satisfy a \( \mathfrak{sl}(2, \mathbb{R}) \)-algebra of Virasoro constraints, which combined with the Toeplitz lattice equations, lead to difference
equations for $x_k$ and $y_k$ given in Adler & van Moerbeke 2003, a subset of the cases leading to recursion relations, which we now describe. Given arbitrary polynomials

$$P_1(\lambda) := \sum_{i=1}^{N} \frac{u_i \lambda^i}{i}, \quad \text{and} \quad P_2(\lambda) := \sum_{i=1}^{N} \frac{u_i \lambda^i}{i},$$

the variables

$$x_k(u) := (-1)^k \frac{\tau_k^+(u)}{\tau_k(u)}, \quad y_k(u) := (-1)^k \frac{\tau_k^-(u)}{\tau_k(u)},$$

with

$$\tau_k^+(u) = \int \left( \det M \right)^{e_{\gamma}} \text{Trace}(P_1(M) - P_2(M^{-1})) \, dM,$$

and $u = (u_1, \ldots, u_N, u_{-1}, \ldots, u_{-N})$, satisfy $2N + 1$ step difference equations $\Gamma_k(x, y) = 0 = \tilde{\Gamma}_k(x, y) = 0$, where $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, and where the polynomials $\Gamma_k(x, y)$ and $\tilde{\Gamma}_k(x, y)$ are defined in terms of the matrices $L_1$ and $L_2$ defined above (denote the derivative of the polynomial $P_i$ by $P'_i$)

$$\Gamma_k(x, y) := \frac{1 - x_k y_k}{y_k} \left( - (L_1 P'_1(L_1))_{k+1,k+1} - (L_2 P'_2(L_2))_{k,k} \right) + k x_k = 0,$$

$$\tilde{\Gamma}_k(x, y) := \frac{1 - x_k y_k}{x_k} \left( - (L_1 P'_1(L_1))_{k,k} - (L_2 P'_2(L_2))_{k+1,k+1} \right) + k y_k = 0. \quad (5)$$

Looking closely, one observes that these difference equations $\Gamma_k = 0$ and $\tilde{\Gamma}_k = 0$ are indeed linear in $x_{k+N}$ and $y_{k+N}$, and can thus be solved in terms of $x_{k-N}$, $y_{k-N}$, ..., $x_{k+N-1}$, $y_{k+N-1}$. See the appendix for a proof of this fact.

This paper deals with the difference equations (5) for their own sake, without further reference to the special solution $x_k(t, s)$ and $y_k(t, s)$, given by the unitary matrix integrals above. Moreover, we will consider the bi-infinite Toeplitz lattice, which is defined as in (2), but with $k \in \mathbb{Z}$. The recursion relations are then also considered for $k \in \mathbb{Z}$, with the semi-infinite case obtained by specialization. The bi-infinite Toeplitz lattice will be introduced in section 2, where we also discuss the self-dual Toeplitz lattice and the recursion relations.

It came as a surprise that the generic solutions of these (very general) equations (5) have the singularity confinement property, a fact which had been observed by Borodin 2003 in the very special case of unitary matrix integrals related to longest increasing sequences of random permutations. In this case the recursion relation is only a 3-step relation. We shall see that although the relations inherit the confinement property from their integrable ancestor, the Toeplitz lattice, they need not, as there are many places where they can easily lose this property. The main result of the paper is to show that this large zoo of examples (1.5), indeed possess the singularity property, namely:

**Theorem 1.1 (singularity confinement; general case)** For any $n \in \mathbb{Z}$, the difference equations $\Gamma_k(x, y) = \Gamma_k(x, y) = 0$, $(k \in \mathbb{Z})$ admit a formal Laurent solution $x = (x_k(\lambda))_{k \in \mathbb{Z}}$ and $y = (y_k(\lambda))_{k \in \mathbb{Z}}$ in a parameter $\lambda$, having a (simple) pole at $k = n$ and $\lambda = 0$, and no other singularities. These solutions depend on $4N$ non-zero free parameters

$$\alpha_{n-2N}, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-2N}, \ldots, \beta_{n-2} \text{ and } \lambda.$$
Setting $z_n := (x_n, y_n)$ and $\gamma_i := (\alpha_i, \beta_i)$, and $\gamma := (\gamma_{n-2N}, \ldots, \gamma_{n-2}, \alpha_{n-1})$, the explicit series with coefficients rational in $\gamma$ read as follows:

$$
\begin{align*}
    z_k(\lambda) &= \sum_{i=0}^{\infty} z_k^{(i)}(\gamma)\lambda^i, & k < n - 2N, \\
    z_k(\lambda) &= \gamma_k, & n - 2N \leq k \leq n - 2, \\
    x_{n-1}(\lambda) &= \alpha_{n-1}, \\
    y_{n-1}(\lambda) &= 1/\alpha_{n-1} + \lambda, \\
    z_n(\lambda) &= \frac{1}{\lambda} \sum_{i=0}^{\infty} z_n^{(i)}(\gamma)\lambda^i, \\
    z_k(\lambda, \gamma) &= \sum_{i=0}^{\infty} z_k^{(i)}(\gamma)\lambda^i, & n < k.
\end{align*}
$$

For the self-dual case, the statement reads as follows:

**Theorem 1.2 (singularity confinement: self-dual case)** For any $n \in \mathbb{Z}$, the difference equations $\Gamma_k(x) = 0$, $(k \in \mathbb{Z})$ admit two\(^4\) formal Laurent solution $x = (x_k(\lambda))_{k \in \mathbb{Z}}$ in a parameter $\lambda$, having a (simple) pole at $k = n$ only and $\lambda = 0$. These solutions depend on $2N$ non-zero free parameters $\alpha = (\alpha_{n-2N}, \ldots, \alpha_{n-2})$ and $\lambda$.

Explicitly, these series with coefficients rational in $\alpha$ are given by

$$
\begin{align*}
    x_k(\lambda) &= \sum_{i=0}^{\infty} x_k^{(i)}(\alpha)\lambda^i, & k < n - 2N, \\
    x_k(\lambda) &= \alpha_k, & n - 2N \leq k \leq n - 2, \\
    x_{n-1}(\lambda) &= \varepsilon + \lambda, \\
    x_n(\lambda) &= \frac{1}{\lambda} \sum_{i=0}^{\infty} x_n^{(i)}(\alpha)\lambda^i, \\
    x_{n+1}(\lambda) &= -\varepsilon + \sum_{i=1}^{\infty} x_{n+1}^{(i)}(\alpha)\lambda^i, \\
    x_k(\lambda) &= \sum_{i=0}^{\infty} x_k^{(i)}(\alpha)\lambda^i, & n + 1 < k.
\end{align*}
$$

The proof of theorems 1.1 and 1.2 is by no means direct, but proceeds via the Painlevé analysis for the Toeplitz lattice. As a starting point, the zero locus $M$, of all polynomials $\Gamma_k$ and $\tilde{\Gamma}_k$, form an invariant manifold for the vector field of the Toeplitz lattice with Hamiltonian $H_1^{(1)} - H_2^{(2)}$, by viewing the coefficients of $P_1(\lambda)$ and $P_2(\lambda)$ as constants, except for $u_{\pm 1}$, which moves linearly in time. Explicitly, this vector field is given by

$$
\begin{align*}
    \frac{dx_k}{dt} &= (1 - x_k y_k) (x_{k+1} - x_{k-1}), & k \in \mathbb{Z}. \\
    \frac{dy_k}{dt} &= (1 - x_k y_k) (y_{k+1} - y_{k-1}),
\end{align*}
$$

In the self-dual case, this vector field reduces to

$$
\begin{align*}
    \frac{dx_k}{dt} &= (1 - x_k^2) (x_{k+1} - x_{k-1}), & k \in \mathbb{Z}.
\end{align*}
$$

\(^4\)They are parametrized by $\varepsilon = \pm 1$.  

5
The first idea is then to restrict the principal balances (formal Laurent solutions depending on the maximal number (= dim phase space – 1) of free parameters, besides time) of (6) to these invariant manifolds. We fix n and look for a formal Laurent solution to the Toeplitz lattice that has a (simple) pole for \( x_n \) and \( y_n \) only, and we find a unique such family, as given by the following proposition:

**Proposition 1.3** For arbitrary but fixed \( n \) manifolds. We fix the maximal number (= dim phase space)

\[ \text{The first idea is then to restrict the principal balances (formal Laurent solutions depending on the maximal number (= dim phase space – 1) of free parameters, besides time) of (6) to these invariant manifolds. We fix n and look for a formal Laurent solution to the Toeplitz lattice that has a (simple) pole for } x_n \text{ and } y_n \text{ only, and we find a unique such family, as given by the following proposition:} \]

\[ \text{Proposition 1.3} \]

For arbitrary but fixed \( n \), the first Toeplitz lattice vector field (6) admits the following formal Laurent solutions,

\[
\begin{align*}
x_n(t) &= \frac{1}{(a_{n+1} - a_n)}t \left( a_{n-1}a_{n+1}(1 + at) + O(t^2) \right) \\
y_n(t) &= \frac{1}{(a_{n+1} - a_n)}t \left( -1 + \frac{a_{n+1}a_+ - a_{n-1}a_-}{a_{n+1} - a_n} \right) + O(t^2)
\end{align*}
\]

\[ x_{n\pm1}(t) = a_{n\pm1} + a_{n\pm1}a_{\pm}t + O(t^2) \]
\[ y_{n\pm1}(t) = 1/a_{n\pm1} - a_{\pm}/a_{n\mp1}t + O(t^2) \]

whereas for all remaining \( k \) such that \( |k - n| \geq 2 \),

\[
\begin{align*}
x_k(t) &= a_k + (1 - a_kb_k)(a_{k+1} - a_{k-1})t + O(t^2) \\
y_k(t) &= b_k + (1 - a_kb_k)(b_{k+1} - b_{k-1})t + O(t^2)
\end{align*}
\]

where \( a, a_{\pm}, a_{n\pm1} \) and all \( a_i, b_i \), with \( i \in \mathbb{Z} \setminus \{ n-1, n, n+1 \} \) and with \( b_{n\pm1} = 1/a_{n\pm1} \), are arbitrary free parameters, and with \( (a_{n-1} - a_{n+1})a_{n-1}a_{n+1} \neq 0 \). In the self-dual case it admits the following two formal Laurent solutions, parametrized by \( \varepsilon = \pm 1 \),

\[
\begin{align*}
x_n(t) &= -\frac{\varepsilon}{2t} \left( 1 + (a_+ - a_-)t + O(t^2) \right), \\
x_{n\pm1}(t) &= \varepsilon \left( \mp 1 + 4a_{\pm}t + O(t^2) \right), \\
x_k(t) &= \varepsilon \left( a_k + (1 - a_k^2)(a_{k+1} - a_{k-1})t + O(t^2) \right), \quad |k - n| \geq 2,
\end{align*}
\]

where \( a_+, a_- \) and all \( a_i \), with \( i \in \mathbb{Z} \setminus \{ n-1, n, n+1 \} \) are arbitrary free parameters and \( a_{n-1} = -a_{n+1} = 1 \).

Together with time \( t \) these parameters are in bijection with the phase space variables; we can put for the general Toeplitz lattice for example \( z_k \leftrightarrow (a_k, b_k) \) for \( |k - n| \geq 1 \) and \( x_{n\pm1} \leftrightarrow a_{n\pm1} \) and \( y_{n\pm1}, x_n, y_n \leftrightarrow a_{\pm}, a, t \). Thus, this formal Laurent solution is the natural candidate to work with; see section 3.

It is however, a priori, not clear that these formal Laurent solutions can be restricted to the invariant manifold \( M \). Indeed, upon introducing a proper time-dependence for \( u \) already mentioned, one has that \( \Gamma_k(t) := \Gamma_k(x(t), y(t); u(t)) \) and \( \hat{\Gamma}_k(t) := \hat{\Gamma}_k(x(t), y(t); u(t)) \) satisfy a system of differential equations, as given in the following proposition:

**Proposition 1.4** Upon setting \( \frac{du}{dt} = \delta \), the recursion relations satisfy the following differential equations

\[
\begin{align*}
\frac{d\Gamma_k}{dt} &= (1 - x_ky_k)(\Gamma_{k+1} - \Gamma_{k-1}) + (x_{k+1} - x_{k-1})(x_k\hat{\Gamma}_k - y_k\Gamma_k), \\
\frac{d\hat{\Gamma}_k}{dt} &= (1 - x_ky_k)(\hat{\Gamma}_{k+1} - \hat{\Gamma}_{k-1}) - (y_{k+1} - y_{k-1})(x_k\hat{\Gamma}_k - y_k\Gamma_k),
\end{align*}
\]

which specialize in the self-dual case (7) to

\[
\frac{d\Gamma_k}{dt} = (1 - x_k^2)(\Gamma_{k+1} - \Gamma_{k-1}).
\]
In addition to propositions 1.3 and 1.4, many other arguments are needed to fine-tune the free parameters, when going from the Laurent solutions of the Toeplitz lattice to the existence of formal Laurent solutions to the difference equations, depending on the announced number of free parameters. See section 6. The proof of these facts will be spread over two sections, as the arguments get rather involved; see section 5 for the self-dual case and section 6 for the case of the general Toeplitz lattice.

This ultimately leads to the proof of the main theorems 1.1 and 1.2.

Examples

I. Denote by $P$ the uniform probability on the group $S_k$ of permutations $\pi_k$ and by $L(\pi_k)$ the length of the largest (strictly) increasing subsequence of $\pi_k$. According to an identity, due to Gessel 1990,

$$\int_{U(n)} e^{t \text{Trace}(M+M^{-1})} dM = \sum_{k=0}^{\infty} \frac{t^{2k}}{k!} P(L(\pi_k) \leq n).$$

The quantities defined for $n > 0$ by

$$x_n(t) = (-1)^n \frac{\int_{U(n)} \det M e^{t(M+M^{-1})} dM}{\int_{U(n)} e^{t(M+M^{-1})} dM}$$

satisfy the following 3-step relation, found by Borodin 2003,

$$nx_n + t(1 - x_n^2)(x_{n+1} + x_{n-1}) = 0,$$

possessing the McMillan invariant (McMillan 1971)

$$\Phi_n(x_{n+1}, x_n) = \Phi_n(x_n, x_{n-1})$$

with

$$\Phi_n(y, z) = (1 - y^2)(1 - z^2) - \frac{n}{t} yz.$$

II. According to Rains 1998, respectively Tracy & Widom 1999,

$$\int_{U(n)} e^{s \text{Trace}(M^2+M^{-2})} dM = \sum_{k=0}^{\infty} \frac{(\sqrt{2}s)^{2k}}{k!} P(L(\pi^0_{2k}) \leq n)$$

and

$$\frac{1}{4} \frac{\partial^2}{\partial t^2} \left( \int_{U(n)} e^{t \text{Trace}(t(M+M^{-1}) + s(M^2+M^{-2}))} dM + \int_{U(n)} e^{t \text{Trace}(t(M+M^{-1}) - s(M^2+M^{-2}))} dM \right) |_{t=0}$$

$$= \sum_{k=0}^{\infty} \frac{(\sqrt{2}s)^{2k}}{k!} P(L(\pi^0_{2k+1}) \leq n),$$

where $\pi^0_{2k}$ and $\pi^0_{2k+1}$ are odd permutations of respectively order $2k$ and $2k+1$ acting on $(-k, \ldots, -1, 1, \ldots, k)$ and $(-k, \ldots, -1, 0, 1, \ldots, k)$. Then

$$x_n(s, t) = (-1)^n \frac{\int_{U(n)} \det M e^{t \text{Trace}(t(M+M^{-1}) + s(M^2+M^{-2}))} dM}{\int_{U(n)} e^{t \text{Trace}(t(M+M^{-1}) + s(M^2+M^{-2}))} dM}$$
possessing the invariant complex) linear space of bi-infinite sequences (x)

The semi-infinite Toeplitz lattice is obtained from it by setting (v, v, 0, 0, 0, ..., 0) := 1

The (bi-infinite) Toeplitz lattice is obtained from it by setting (x, y) := 1

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and similarly for $\lambda_1^{(1)}[y_k]$, which leads to the following expression for the vector field $\lambda_1^{(1)}$,

$$
\lambda_1^{(1)} : \begin{cases} 
\frac{dx_k}{dt_i} = -(1 - x_k y_k) \left( L_{i-1}^1 \big| \frac{\partial L_i}{\partial y_k} \right), \\
\frac{dy_k}{dt_i} = (1 - x_k y_k) \left( L_{i-1}^1 \big| \frac{\partial L_i}{\partial x_k} \right).
\end{cases}
$$

(16)

The vector field $\lambda_1^{(2)}$, has the same form, but with $L_1$ replaced by $L_2$. This is a particular case of a phenomenon that we will refer to as duality. Namely, there is a natural automorphism $\sigma$ of our phase space, given by $\sigma : (x_i, y_i) \in \mathbb{Z} \mapsto (y_i, x_i) \in \mathbb{Z}$. It preserves the first Toeplitz vector field (13), it permutes the Hamiltonians $H_i^{(1)} \leftrightarrow H_i^{(2)}$, it permutes the Lax operators as follows: $L_1 \leftrightarrow L_2$ and it reverses the sign of the Poisson structure. The first Toeplitz vector field (13) can be restricted to the fixed point locus $(x_i = y_i) \in \mathbb{Z}$ of $\sigma$, which leads to the self-dual (bi-infinite) Toeplitz lattice,

$$
\frac{dx_k}{dt} = (1 - x_k^2)(x_{k+1} - x_{k-1}), \quad k \in \mathbb{Z}.
$$

(17)

All constructions in this paper will be done for this self-dual lattice first, and then for the general Toeplitz lattice. This is not only for pedagogical reasons: even if the ideas that lead to the proofs are similar in both cases, the self-dual lattice can for our purposes not be treated as a particular case of the general Toeplitz lattice, as we will see.

For $i = 1$, the equations (16) for $\lambda_1^{(1)}$ and for $\lambda_1^{(2)}$ specialize to

$$
\lambda_1^{(1,2)}[x_k] = (1 - x_k y_k)x_{k \pm 1}, \\
\lambda_1^{(1,2)}[y_k] = -(1 - x_k y_k)y_{k \mp 1}.
$$

(18)

Fixing $2N$ constants $u := (u_{-N}, \ldots, u_{-1}, u_1, \ldots, u_N)$, with $u_N \neq 0$ and $u_{-N} \neq 0$, we consider the polynomials

$$
P_1(\lambda) := \sum_{i=1}^N \frac{u_i \lambda^i}{i}, \quad \text{and} \quad P_2(\lambda) := \sum_{i=1}^N \frac{u_{-i} \lambda^i}{i},
$$

(19)

whose derivatives we simply denote by $P_1'$ and $P_2'$. They lead to two strings of polynomials$^5$ $\Gamma_k$ and $\tilde{\Gamma}_k$ in $x_i$, $y_i$ ($i \in \mathbb{Z}$):

$$
\Gamma_k(x, y; u) := \frac{1 - x_k y_k}{y_k} \left( (L_1 P_1'(L_1))_{k+1, k+1} - (L_2 P_2'(L_2))_{k, k} \right) + k x_k, \\
\tilde{\Gamma}_k(x, y; u) := \frac{1 - x_k y_k}{x_k} \left( -(L_1 P_1'(L_1))_{k, k} - (L_2 P_2'(L_2))_{k+1, k+1} \right) + k y_k.
$$

(20)

Notice that the only elements that appear in these polynomials are the diagonal and next-to-diagonal entries of $L_1$ and $L_2$ for $l = 1, \ldots, N$. For fixed $u$ we consider the zero locus of all polynomials $\Gamma_k$ and $\tilde{\Gamma}_k$,

$$
\mathcal{M}_u := \bigcap_{k \in \mathbb{Z}} \left\{ (x_i, y_i) \in \mathbb{Z} \mid \Gamma_k(x, y; u) = 0 \text{ and } \tilde{\Gamma}_k(x, y; u) = 0 \right\}.
$$

(21)

$^5$The structure of the matrices $L_1$ and $L_2$ implies that $\Gamma_k$ and $\tilde{\Gamma}_k$ are indeed polynomials. They are also polynomials (of degree 1) in the variables $u_i$, but we often do not mention this, because we think of these variables as parameters.
In terms of the variables $x_i$ and $y_i$ the leading terms of $\Gamma_k$ and $\tilde{\Gamma}_k$ are given by

$$
\Gamma_k(x, y; u) = u_N x_{k+N} \prod_{i=0}^{N-1} (1 - x_{k+i} y_{k+i}) + \cdots + u_N x_{k-N} \prod_{i=0}^{N-1} (1 - x_{k-i} y_{k-i}),
$$

$$
\tilde{\Gamma}_k(x, y; u) = -u_N y_{k+N} \prod_{i=0}^{N-1} (1 - x_{k+i} y_{k+i}) + \cdots + u_N y_{k-N} \prod_{i=0}^{N-1} (1 - x_{k-i} y_{k-i}).
$$

See the Appendix for a precise statement, a few more terms and a proof. We often write $\Delta_k$ as a shorthand for the vector $(\Gamma_k, \tilde{\Gamma}_k)^\top$ and $z_k$ for $(x_k, y_k)^\top$.

In order to get the corresponding formulas for the self-dual case we put $\sigma(u_i) := u_{-i}$, so that $\sigma$ permutes $P_1$ and $P_2$, as well as $\Gamma_k$ and $\tilde{\Gamma}_k$, hence $P_1 = P_2$ in the self-dual case, and $\Gamma_k = \tilde{\Gamma}_k$. Writing $L := L_1$ and $P := P_1$, the polynomials $\Gamma_k$ and $\tilde{\Gamma}_k$ reduce in that case to

$$
\Gamma_k(x; u) := \frac{1 - x_k^2}{x_k} (2(P'(L))_{k+1,k} - (LP'(L))_{k+1,k+1} - (LP'(L))_{k,k}) + k x_k,
$$

while its leading terms are now given by

$$
\Gamma_k(x; u) = u_N x_{k+N} \prod_{i=0}^{N-1} (1 - x_{k+i}^2) + \cdots + u_N x_{k-N} \prod_{i=0}^{N-1} (1 - x_{k-i}^2).
$$

The zero locus $\mathcal{M}_u$ now takes the simple form

$$
\mathcal{M}_u := \bigcap_{k \in \mathbb{Z}} \{ (x_i)_{i \in \mathbb{Z}} \mid \Gamma_k(x; u) = 0 \}.
$$

Following Adler & van Moerbeke 2003, we show that, upon introducing a proper time dependence, the polynomials $\Gamma_k$ and $\tilde{\Gamma}_k$ satisfy a simple set of differential equations, showing that the zero locus (21) of these polynomials is a (time-dependent) invariant manifold of the first Toeplitz flow (13).

**Proposition 2.1** Let $(x(t), y(t))$ be a solution to the first Toeplitz vector field (13), to wit:

$$
\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A^{(1)} & A^{(2)} \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},
$$

and let $\Gamma(t) := \Gamma(x(t), y(t); u(t))$ and $\tilde{\Gamma}(t) := \tilde{\Gamma}(x(t), y(t); u(t))$, where

$$
u(t) = (u_{-N}, \ldots, u_{-2}, u_{-1} + t, u_1 + t, u_2, \ldots, u_N).
$$

Then $\Gamma(t)$ and $\tilde{\Gamma}(t)$ satisfy the following differential equations:

$$
\frac{d\Gamma_k}{dt} = (1 - x_k y_k)(\Gamma_{k+1} - \Gamma_{k-1}) + (x_k \tilde{\Gamma}_k - y_k \Gamma_k),
$$

$$
\frac{d\tilde{\Gamma}_k}{dt} = (1 - x_k y_k)(\tilde{\Gamma}_{k+1} - \tilde{\Gamma}_{k-1}) - (y_k \tilde{\Gamma}_k - x_k \Gamma_k).
$$

In particular, $\mathcal{M}_{u(t)}$ is a (time-dependent) invariant manifold of the first Toeplitz flow. In the self-dual case, these differential equations specialize to

$$
\frac{d\Gamma_k}{dt} = (1 - x_k^2)(\Gamma_{k+1} - \Gamma_{k-1}).
$$

Then $\mathcal{M}_{u(t)}$ is a (time-dependent) invariant manifold of the first vector field of the self-dual Toeplitz lattice, where $u(t) = (u_1 + t, u_2, \ldots, u_N)$. 

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Proof We first show that
\[ \Gamma_k(x, y; u) = V^u[x_k] + kx_k, \]
\[ \tilde{\Gamma}_k(x, y; u) = -V^u[y_k] + ky_k, \]
where \( V \) is the Hamiltonian vector field
\[ V^u := \sum_{i=1}^N \left( u_i \lambda^{(1)} + u_{i-1} \lambda^{(2)} \right). \]

It suffices to prove that \( \Gamma_k(x, y; u) = V^u[x_k] + kx_k \), the other identity being obtained by duality (indeed, \( \sigma(V^u) = -V^u \) since \( \sigma(\lambda^{(1)}) = -\lambda^{(2)} \)). In view of the Definition (20) of \( \Gamma_k \) this means that we need to prove that
\[
\begin{align*}
\lambda^{(1)}[x_k] &= \frac{1 - x_k y_k}{y_k} \left( (L_{1}^{-1})_{k+1,k} - (L_{1}^{0})_{k+1,k+1} \right), \\
\lambda^{(2)}[x_k] &= \frac{1 - x_k y_k}{y_k} \left( (L_{2}^{-1})_{k,k+1} - (L_{2}^{0})_{k,k} \right).
\end{align*}
\]

According to (16), the first equation amounts to
\[ y_k \left( L_{1}^{-1} | \frac{\partial L_1}{\partial y_k} \right) = (L_{1}^{0})_{k+1,k+1} - (L_{1}^{-1})_{k+1,k}, \]
where we recall that \( \langle A | B \rangle = \text{Trace} AB \). The proof of (30) follows immediately by writing \( (L_{1}^{0})_{k+1,k+1} \) as \( (L_{1}^{-1} L_{1})_{k+1,k+1} \), and the expression (14) for the entries of \( L_{1} \). For the second equation in (29) the proof is similar.

Notice that (28) implies that the time-dependent polynomials \( \Gamma_k(t) \) and \( \tilde{\Gamma}_k(t) \) are given by
\[
\begin{align*}
\Gamma_k(t) &= V^{u(t)}[x_k](t) + kx_k(t), \\
\tilde{\Gamma}_k(t) &= -V^{u(t)}[y_k](t) + ky_k(t),
\end{align*}
\]
where \( V^{u(t)} \) can, in view of (25) be written as
\[ V^{u(t)} = t(\lambda^{(1)} + \lambda^{(2)}) + V^u. \]

Since the vector field \( d/dt \) commutes with all the Hamiltonian vector fields \( \lambda^{(1)} \) and \( \lambda^{(2)} \), it follows from these equations and (18) that
\[
\frac{d\Gamma_k}{dt}(t) = \lambda^{(1)}[x_k](t) + \lambda^{(2)}[x_k](t) + V^{u(t)}[dx_k/dt](t) + k \frac{dx_k}{dt}(t)
= (k + 1)\lambda^{(1)}[x_k](t) - (k - 1)\lambda^{(2)}[x_k](t) + V^{u(t)}[(1 - x_k y_k)(x_{k+1} - x_{k-1})](t)
= (k + 1)(1 - x_k y_k(t))x_{k+1}(t) - (k - 1)(1 - x_k y_k(t))x_{k-1}(t)
+ (1 - x_k y_k(t))V^{u(t)}[x_{k+1} - x_{k-1}](t) - (x_{k+1}(t) - x_{k-1}(t))V^{u(t)}[y_k](t)
= (1 - x_k y_k(t))(\Gamma_{k+1}(t) - \Gamma_{k-1}(t)) + (x_{k+1}(t) - x_{k-1}(t))(x_k(t)\tilde{\Gamma}_k(t) - y_k(t)\Gamma_k(t)).
\]
This yields the first relation in (26). The second equation is obtained by duality.

At points of \( \mathcal{M}_u \) all \( \Gamma_k \) and \( \tilde{\Gamma}_k \) vanish so the right hand sides of (26) vanish. The unique solution to (26) that corresponds to such initial data is the zero solution, \( \Gamma_k(t) = \tilde{\Gamma}_k(t) = 0 \). As a consequence, \( \mathcal{M}_u(t) \) is a time-dependent invariant manifold for the first Toeplitz flow.
3 Painlevé analysis of the first Toeplitz flow

In this section we will show that the first Toeplitz flow admits many families of formal Laurent solutions, a property reminiscent of (finite-dimensional) algebraic completely integrable systems (see Adler, van Moerbeke & Vanhaecke 2004). They will be used in the subsequent sections. We will first consider the self-dual case, which is easier, and then we will consider the full Toeplitz lattice.

3.1 The self-dual Toeplitz lattice

Recall that the first vector field of the self-dual Toeplitz lattice is given by

\[ \frac{dx_k}{dt} = (1 - x_k^2)(x_{k+1} - x_{k-1}), \quad k \in \mathbb{Z}. \]  

Proposition 3.1 For any \( n \in \mathbb{Z} \), the first vector field (31) of the self-dual Toeplitz lattice admits a formal Laurent solution \( x(t) \), with only \( x_n(t) \) having a pole, given by

\[
x_k(t) = \varepsilon \left( a_k + (1 - a_k^2)(a_{k+1} - a_{k-1})t + \frac{1}{2}(1 - a_k^2)(a_{k-2} - (1 - a_{k-1}^2) + a_{k+2}(1 - a_{k+1}^2) - a_k(a_{k+1} - a_{k-1})^2 + 2 - 2a_{k-1}a_{k+1}) + \kappa_k t^2 + O(t^3) \right), \\
x_{n \pm 1}(t) = \varepsilon \left( \mp 1 + 4a_\pm t + 4a_\pm(2a_{n \pm 2} \mp (a_+ + a_-))t^2 + O(t^3) \right), \\
x_n(t) = -\frac{\varepsilon}{2t} \left( 1 + (a_+ - a_-)t + \frac{1}{3}(a_+ - a_-)^2 + 4(a_+a_{n+2} - a_-a_{n-2} + 1 - 2a_+a_-)t^2 + O(t^3) \right),
\]

where \( a_\pm \) and all \( a_i \), with \( i \in \mathbb{Z} \setminus \{ n - 1, n, n + 1 \} \) are arbitrary free parameters; also, \( \varepsilon^2 = 1 \) and \( a_{n-1} = -a_{n+1} = 1 \). When \( |k - n| > 2 \) then \( \kappa_k = 0 \), while \( \kappa_{n \pm 2} = \mp 4a_\pm \).

Proof We look for formal Laurent solutions \( x(t) \) to (31) that have a simple pole for one of the variables (only). To do this, we substitute \( x_n(t) = x_n^{(0)}/t + O(1) \), with \( x_n^{(0)} \neq 0 \), and \( x_j(t) = x_j^{(0)} + O(t) \), \( j \neq n \) into (31) for different values of \( k \). Taking \( k = n \pm 1 \) we find that \( (x_{n \pm 1}^{(0)})^2 = 1 \), in both cases because \( 1 - x_k^2(t) \) needs to cancel the pole coming from \( x_n(t) \). Given this, (31) with \( k = n \) is given by

\[-\frac{x_n^{(0)}}{t^2} + O(1) = \frac{1}{t^2} \left( x_n^{(0)} - x_{n+1}^{(0)} \right)^2 + O(t^{-1}).\]

Since \( x_n^{(0)} \neq 0 \), we deduce from it on the one hand that \( x_{n+1}^{(0)} \) and \( x_{n-1}^{(0)} \) have opposite signs, so that \( x_{n+1}^{(0)} = -x_n^{(0)} \) and that \( x_n^{(0)} = 1/(2x_{n+1}^{(0)}) \). It follows that \( x_n^{(0)}(t) = \mp \varepsilon + O(t) \) and \( x_n(t) = -\varepsilon/(2t) + O(1) \), where \( \varepsilon^2 = 1 \). For \( |k - n| \geq 2 \), the coefficient in \( t^{-1} \) of (31) does not impose any condition on the constant coefficient of \( x_k(t) \), which is therefore a free parameter, which we denote as \( \varepsilon a_k \).

Having determined the first term of the series we suppose that

\[
x_k(t) = \varepsilon \left( a_k + \sum_{i=1}^{r} x_k^{(i)} t^i + x_k^{(r+1)} t^{r+1} \right), \quad |k - n| \geq 2,
\]
\[
x_{n \pm 1}(t) = \varepsilon \left( \mp 1 + \sum_{i=1}^{r} x_{n \pm 1}^{(i)} t^i + x_{n \pm 1}^{(r+1)} t^{r+1} \right),
\]

\[ x_n(t) = -\frac{\varepsilon}{2t} \left( 1 + (a_+ - a_-)t + \frac{1}{3}(a_+ - a_-)^2 + 4(a_+a_{n+2} - a_-a_{n-2} + 1 - 2a_+a_-)t^2 + O(t^3) \right),
\]

\[ x_{n \pm 1}(t) = \varepsilon \left( \mp 1 + \sum_{i=1}^{r} x_{n \pm 1}^{(i)} t^i + x_{n \pm 1}^{(r+1)} t^{r+1} \right),
\]

\[ x_k(t) = \varepsilon \left( a_k + \sum_{i=1}^{r} x_k^{(i)} t^i + x_k^{(r+1)} t^{r+1} \right), \quad |k - n| \geq 2,
\]

\[ x_n(t) = -\frac{\varepsilon}{2t} \left( 1 + (a_+ - a_-)t + \frac{1}{3}(a_+ - a_-)^2 + 4(a_+a_{n+2} - a_-a_{n-2} + 1 - 2a_+a_-)t^2 + O(t^3) \right),
\]

\[ x_{n \pm 1}(t) = \varepsilon \left( \mp 1 + \sum_{i=1}^{r} x_{n \pm 1}^{(i)} t^i + x_{n \pm 1}^{(r+1)} t^{r+1} \right),
\]

\[ x_k(t) = \varepsilon \left( a_k + \sum_{i=1}^{r} x_k^{(i)} t^i + x_k^{(r+1)} t^{r+1} \right), \quad |k - n| \geq 2,
\]

\[ x_n(t) = -\frac{\varepsilon}{2t} \left( 1 + (a_+ - a_-)t + \frac{1}{3}(a_+ - a_-)^2 + 4(a_+a_{n+2} - a_-a_{n-2} + 1 - 2a_+a_-)t^2 + O(t^3) \right),
\]

\[ x_{n \pm 1}(t) = \varepsilon \left( \mp 1 + \sum_{i=1}^{r} x_{n \pm 1}^{(i)} t^i + x_{n \pm 1}^{(r+1)} t^{r+1} \right),
\]
\[ x_n(t) = -\frac{\varepsilon}{2t} \left( 1 + \sum_{i=1}^{r} x_n^{(i)} t^i + x_n^{(r+1)} t^{r+1} \right), \]

where all coefficients \( x_k^{(i)} \), with \( i \leq r \) have been determined. We show that (31) then yields linear relations on the coefficients \( x_k^{(r+1)} \). To see that, pick the coefficient in \( t^r \) in (31) when \( k \neq n \), while taking the coefficient in \( t^{r-1} \) when \( k = n \). This yields the following relations, where “known” means coefficients \( x_k^{(i)} \), with \( i \leq r \):

\[
\begin{align*}
|k-n| \geq 2 & : \varepsilon(r+1)x_k^{(r+1)} = \text{known}, \\
 k = n \pm 1 & : \varepsilon x_n^{(r+1)} = \text{known}, \\
 k = n & : -\frac{\varepsilon}{2}(r+2)x_n^{(r+1)} = -\frac{\varepsilon}{4}(x_{n+1}^{(r+1)} - x_{n-1}^{(r+1)}) = \text{known}. 
\end{align*}
\]

This yields a linear system in the unknowns \( x_k^{(r+1)} \), where \( k \in \mathbb{Z} \), which has upper triangular form when \( x_k^{(r+1)} \) is put at the end. It uniquely determines the coefficients \( x_k^{(r+1)} \), except when \( k = n \pm 1 \) and \( r = 0 \): the corresponding equations both reduce then to 0 = 0, so that \( x_k^{(1)} \) and \( x_k^{(0)} \) are also free parameters; we denote them by \( 4a_{\pm} := x_k^{(1)} \). Then the third equation in (32) implies that \( x_k^{(1)} = a_+ - a_- \); also, the first equation is explicitly given by \( \varepsilon x_k^{(1)} = \varepsilon(1 - a_0^2)(a_{k+1} - a_{k-1}) \), for \( |k-n| \geq 2 \). Since for \( r > 0 \) we can solve uniquely for all \( x_k^{(r+1)} \), we get a formal Laurent solution depending on the free parameters, as indicated. The extra term that is given in the proposition is easily verified.

Notice that under the natural correspondence between the phase variables \( x_k \) (with \( k \neq n \)) and the free parameters \( a_k \) (\( a_{\pm} \) in the case \( k = n \pm 1 \)) we have that the number of free parameters on which the coefficients of the series depend, is one less than the number of phase variables, a property reminiscent of principal balances for (finite-dimensional) algebraic completely integrable systems (see Adler, van Moerbeke & Vanhaecke 2004, Chapter 6). There are of course also formal Laurent solutions that depend on less free parameters (lower balances), but these will not be used here.

For future reference we give the first few terms of the formal Laurent series of \( 1 - x_k^2 \), which is easily computed from the series given in proposition 3.1,

\[
\begin{align*}
1 - x_k^2(t) &= (1 - a_k^2)(1 - 2a_k(a_{k+1} - a_{k-1})t) + O(t^2), \quad |k-n| \geq 2, \\
1 - x_{\pm 1}^2(t) &= \pm 8a_{\pm} t + O(t^2), \\
1 - x_n^2(t) &= -\frac{1}{4t^2}(1 + 2(a_+ - a_-)t + O(t^2)).
\end{align*}
\]

The displayed terms are the only ones that will be needed below.

### 3.2 The full Toeplitz lattice

We will now show that the full Toeplitz lattice also allows such formal Laurent solutions. To make the analogy with the self-dual case transparent we will vectorize the variables and the equations, namely we introduce \( z_k := \begin{pmatrix} x_k \\ y_k \end{pmatrix} \) and \( c_k := \begin{pmatrix} a_k \\ b_k \end{pmatrix} \), for \( k \in \mathbb{Z} \); the variables \( a_k \) and \( b_k \) will be the free parameters in the formal Laurent series. With these notations the first Toeplitz vector field (13) becomes

\[
\frac{dz_k}{dt} = (1 - x_k y_k)(z_{k+1} - z_{k-1}).
\]
Proposition 3.2 For any \( n \in \mathbb{Z} \), the vector field (34) of the (general) Toeplitz lattice admits a formal Laurent solution \( z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \), such that only \( x_n(t) \) and \( y_n(t) \) have a (simple) pole. It is given by
\[
\begin{align*}
  z_k(t) &= c_k + (1 - a_k b_k)(c_{k+1} - c_{k-1})t + O(t^2), \quad |k - n| \geq 2, \\
  z_{n \pm 1}(t) &= \left( \frac{a_n \pm 1 + a_{n \pm 1}a \pm t}{1/a_n \pm 1 - a \pm/a_{n \pm 1}t} \right) + O(t^2) \\
  z_n(t) &= \frac{1}{(a_{n-1} - a_{n+1})t} \left(-1 + \frac{a_{n-1} a_{n+1} (1 + at)}{a_{n+1} a_{n-1} (a_{n+1} - a_{n-1})} \right) + O(t),
\end{align*}
\]
where \( a, a \pm, a_{n \pm 1} \) and all \( c_i = \left( \frac{a_i}{b_i} \right) \), with \( i \in \mathbb{Z} \setminus \{ n-1, n+1 \} \) are arbitrary free parameters, and where \( c_{n \pm 1} = \left( \frac{a_{n \pm 1}}{1/a_{n \pm 1}} \right) \). Precisely, the free parameters \( a_{n \pm 1} \) satisfy the condition \( a_{n+1} a_{n-1} (a_{n+1} - a_{n-1}) \neq 0 \). The parameters on which the next order term in the series \( x(t) \) and \( y(t) \) depend is given in table 1.

Remark 3.3 In section 6 we will need some extra information on these formal Laurent series, namely that the coefficient in \( t^2 \) of \( z_k \), for \( |k - n| \geq 2 \) depends in the following way on \( c_{k+2} \),
\[
  z_k^{(2)} = \frac{1}{2} (1 - a_k b_k)(1 - a_{k+1} b_{k+1}) c_{k+2} + \tilde{z}_k^{(2)}, \tag{35}
\]
where \( z_k^{(2)} \) is independent of \( a_{k+2} \) and \( b_{k+2} \). In particular, \( x_k^{(2)} \) depends linearly on \( a_{k+2} \) and is independent of \( b_{k+2} \), while \( y_k^{(2)} \) depends linearly on \( b_{k+2} \) and is independent of \( a_{k+2} \). This easily follows from the given terms by considering the coefficient of \( t \) in (34).

Proof For fixed \( n \in \mathbb{Z} \), we look for formal Laurent solutions \( z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \), to (34) where \( x_n(t) \) or \( y_n(t) \) have a simple pole, and where none of the other variables \( x_k(t) \) or \( y_k(t) \) have a pole (in \( t \)). Thus, we substitute \( z_n(t) = z_n^{(0)}/t + O(1) \) and \( z_j(t) = z_j^{(0)} + O(t), j \neq n \) into (34) for different values of \( k \). For \( k = n \pm 1 \) we find that \( x_n^{(0)} y_{n \pm 1}^{(0)} = 1 \), because \( 1 - x_n \pm 1 y_{n \pm 1} \) needs to cancel the pole coming from \( x_n \) or from \( y_n \); we put \( a_{n \pm 1} := x_n^{(0)} ; y_n^{(0)} \), so that \( y_n^{(0)} = 1/a_{n \pm 1} \). The parameters \( a_{n \pm 1} \) are free, except that \( a_{n+1} a_{n-1} \neq 0 \). Next, (34) with \( k = n \), yields
\[
\begin{pmatrix}
  x_n^{(0)} \\
  y_n^{(0)}
\end{pmatrix}
= \begin{pmatrix}
  x_n^{(0)} - x_{n-1}^{(0)} \\
  y_n^{(0)} - y_{n-1}^{(0)}
\end{pmatrix}
\begin{pmatrix}
  x_n^{(0)} y_n^{(0)}
\end{pmatrix},
\]
which shows on the one hand that \( x_n^{(0)} \) and \( y_n^{(0)} \) are both different from zero (since at least one of them is supposed to be different from zero), so that also \( a_{n+1} - a_{n-1} \neq 0 \). On the other hand it shows that \( x_n^{(0)} \) and \( y_n^{(0)} \) are expressible in terms of \( a_{n+1} \) and \( a_{n-1} \) as
\[
  x_n^{(0)} = \frac{a_{n+1} a_{n-1}}{a_{n-1} - a_{n+1}}, \quad y_n^{(0)} = \frac{1}{a_{n+1} - a_{n-1}}.
\]
For \( |k - n| \geq 2 \), the coefficient in \( t^{-1} \) of (34) does not impose any condition on the constant coefficient of \( z_k(t) \), yielding free parameters for the constant coefficients of \( x_k \) and of \( y_k \), with \( |k - n| > 1 \). We
Table 1: We list on which free parameters the first few terms of the formal Laurent solutions depend. It is understood that we do not list again the parameters that appear already before, on the same line; for example, \( x^{(i)}_n \) depends only on \( a_{n+1}, a_{n-1} \) and \( a \). The last two lines correspond to the values \( k \) for which \(|k - n| > 2\). For \( k \neq n \), \( x_k^{(i)} \) is the coefficient of \( t^i \) in \( x_k(t) \), while for \( k = n \) it is the coefficient of \( t^{i-1} \) in \( x_n(t) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_n )</td>
<td>( a_{n+1}, a_{n-1} )</td>
<td>( a )</td>
<td>( a_{n+2}, a_{n-2}, b_{n+2}, b_{n-2} )</td>
</tr>
<tr>
<td>( y_n )</td>
<td>( a_{n+1}, a_{n-1} )</td>
<td>( a, a_{n+}, a_{n-} )</td>
<td>( a_{n+2}, b_{n+2}, a_{n-2}, b_{n-2} )</td>
</tr>
<tr>
<td>( x_{n \pm 1} )</td>
<td>( a_{n \pm 1} )</td>
<td>( a_{n \mp 1} )</td>
<td>( a_{n \pm 2}, b_{n \pm 2}, a_{n \mp 1} )</td>
</tr>
<tr>
<td>( y_{n \pm 1} )</td>
<td>( a_{n \pm 1} )</td>
<td>( a_{n \mp 1} )</td>
<td>( a_{n \pm 2}, b_{n \pm 2}, a_{n \mp 1} )</td>
</tr>
<tr>
<td>( x_{n \pm 2} )</td>
<td>( a_{n \pm 2} )</td>
<td>( a_{n \pm 3}, a_{n \mp 1}, b_{n \pm 2} )</td>
<td>( a_{n \pm 4}, b_{n \pm 3}, a_{n \mp 1} )</td>
</tr>
<tr>
<td>( y_{n \pm 2} )</td>
<td>( b_{n \pm 2} )</td>
<td>( b_{n \pm 3}, b_{n \mp 1}, a_{n \pm 2} )</td>
<td>( b_{n \pm 4}, a_{n \pm 3}, a_{n \mp 1} )</td>
</tr>
<tr>
<td>( x_k )</td>
<td>( a_k )</td>
<td>( a_{k+1}, a_{k-1}, b_k )</td>
<td>( a_{k+2}, b_{k+1}, a_{k-2}, b_{k-1} )</td>
</tr>
<tr>
<td>( y_k )</td>
<td>( b_k )</td>
<td>( b_{k+1}, b_{k-1}, a_k )</td>
<td>( b_{k+2}, a_{k+1}, b_{k-2}, a_{k-1} )</td>
</tr>
</tbody>
</table>

We denote these free parameters by \( c_k = \begin{pmatrix} a_k \\ b_k \end{pmatrix} \). Upon specialization, some of the formulas below may contain \( c_{n+1} \) or \( c_{n-1} \); it is understood that these stand for

\[
\frac{a_{n+1}}{1/ a_{n+1}} = \begin{pmatrix} a_{n+1} \\ 1/a_{n+1} \end{pmatrix}.
\]

We can now proceed as in the second part of the proof of proposition 3.1, namely we suppose that

\[
\begin{align*}
    z_k(t) &= c_k + \sum_{i=1}^{r} z_k^{(i)} t^i + z_k^{(r+1)} t^{r+1}, \\
    z_{n \pm 1}(t) &= \begin{pmatrix} a_{n \pm 1} \\ 1/a_{n \pm 1} \end{pmatrix} + \sum_{i=1}^{r} z_{n \pm 1}^{(i)} t^i + z_{n \pm 1}^{(r+1)} t^{r+1}, \\
    z_n(t) &= \frac{1}{(a_{n-1} - a_{n+1})^r} \left( \begin{pmatrix} a_{n-1} a_{n+1} \\ -1 \end{pmatrix} + \sum_{i=1}^{r} z_n^{(i)} t^i + z_n^{(r+1)} t^{r+1} \right),
\end{align*}
\]

where all coefficients \( z_k^{(i)} \), with \( i \leq r \) have been determined. On the coefficients \( z_k^{(r+1)} \), \( k \in \mathbb{Z} \), we find linear relations by substituting the above series into (34). For \( k \) such that \(|n - k| > 1\) it is clear that, as in the self-dual case, \( z_k^{(r+1)} \) is linearly computed in terms of the known coefficients, from the coefficient of \( t^r \), when substituting the series in (34). Therefore, let us concentrate on what happens for \( k \in \{ n-1, n, n+1 \} \). Taking \( k = n \pm 1 \) in (34) the coefficient of \( t^r \) yields

\[
(r + 1) z_{n \pm 1}^{(r+1)} = \pm \begin{pmatrix} x_n y \pm 1 \\ a_{n \pm 1} \end{pmatrix} + \begin{pmatrix} y_n x \pm 1 \end{pmatrix} \begin{pmatrix} \frac{a_{n-1} a_{n+1}}{a_{n-1} - a_{n+1}} \\ -1 \end{pmatrix} \text{ known,}
\]
a linear equation in \( x_{n\pm 1} \) and \( y_{n\pm 1} \), which can be written in the compact form

\[
(L_{\pm} + (r + 1) \text{Id}) z_{n\pm 1}^{(r+1)} = \text{known},
\]

where \( L_{\pm} \) is the matrix that governs the linear problem,

\[
L_{\pm} := \pm \frac{1}{a_{n-1} - a_{n+1}} \begin{pmatrix} -a_{n+1} & -a_{n-1} a_{n+1} a_{n\pm 1} \\ 1/a_{n1} & a_{n\pm 1} \end{pmatrix}.
\]

Since \( \det(L_{\pm} + (r + 1) \text{Id}) = r(r + 1) \) this linear system admits a unique solution, except when \( r = 0 \) (recall that \( r \geq 0 \)). Before analyzing the case \( r = 0 \) further, let us first consider what happens to (34) in the remaining case \( k = n \). As in the self-dual case, we pick the coefficient of \( t^{r-1} \) in (34) to find a linear system that can be written in the compact form

\[
(L_n + r \text{Id}) z_n^{(r+1)} = \text{known},
\]

where the matrix \( L_n \) is given by

\[
L_n := \begin{pmatrix} 1 & -a_{n+1} a_{n-1} \\ -1/(a_{n+1} a_{n-1}) & 1 \end{pmatrix}.
\]

Since \( \det(L_n + r \text{Id}) = r(r+2) \) we have again that \( z_n^{(r+1)} \) is determined uniquely, unless \( r = 0 \). Thus, we are done with \( r \geq 1 \).

As we have seen, a free parameter may appear in \( z_{n+1}^{(1)} \), in \( z_{n-1}^{(1)} \) and in \( z_n^{(1)} \), but one has to check that the corresponding linear equations are consistent. Therefore we substitute

\[
z_k(t) = c_k + z_k^{(1)} t + O(t^2),
\]

\[
z_{n\pm 1}(t) = \left( \frac{a_{n\pm 1}}{1/a_{n\pm 1}} \right) + z_{n\pm 1}^{(1)} t + O(t^2),
\]

\[
z_n(t) = \frac{1}{(a_{n-1} - a_{n+1}) t} \left( \frac{a_{n-1} a_{n+1}}{a_{n\pm 1}} + z_{n\pm 1}^{(1)} t + O(t^2) \right),
\]

in (34), which yields for \( k = n \pm 1 \) and \( t = 0 \) the homogeneous linear system

\[
\begin{pmatrix} x_{n\pm 1}^{(1)} \\ y_{n\pm 1}^{(1)} \end{pmatrix} = \pm \frac{1}{a_{n-1} - a_{n+1}} \begin{pmatrix} x_{n\mp 1}^{(1)} + y_{n\pm 1}^{(1)} a_{n\pm 1} \\ a_{n-1} a_{n+1} \end{pmatrix} \begin{pmatrix} a_{n-1} a_{n+1} \\ -1 \end{pmatrix},
\]

which is equivalent to

\[
x_{n\pm 1}^{(1)} + a_{n-1} a_{n+1} y_{n\pm 1}^{(1)} = 0.
\]

Thus, upon setting \( x_{n\pm 1}^{(1)} = a_{n\pm 1} a_{\pm} \), where \( a_+ \) and \( a_- \) are free parameters, we have that \( y_{n\pm 1}^{(1)} = -a_+/a_{n\pm 1} = -a_+ b_{n\mp 1} \). Similarly, for \( k = n \) the substitution of the series (36) in (34) yields at the level \( t^{-1} \):

\[
\frac{a_{n-1} a_{n+1}}{a_{n-1} - a_{n+1}} (x_{n-1}^{(1)} - x_n^{(1)}) - x_n^{(1)} + a_{n-1} a_{n+1} y_n^{(1)} = 0,
\]

\[
\frac{a_{n-1} a_{n+1}}{a_{n-1} - a_{n+1}} (y_{n+1}^{(1)} - y_n^{(1)}) - y_n^{(1)} + \frac{x_n^{(1)}}{a_{n-1} a_{n+1}} = 0.
\]

These equation are proportional, in view of (37). Thus we have

\[
x_n^{(1)} = a_{n+1} a_{n-1} a,
\]

\[
y_n^{(1)} = a + \frac{a_{n+1} a_+ - a_{n-1} a_-}{a_{n+1} - a_{n-1}}.
\]
where \( a \) is a free parameter.

The first two terms in the series lead at once to the second and third columns of table 1. In order to obtain the last column it suffices to list on which parameters the linear term (resp. the constant term) in the right hand side of \((1 - x_k(t)y_k(t))(z_{k+1}(t) - z_{k-1}(t))\) depends, when \( k \neq n \) (resp. when \( k = n \)). The two leading terms of \( x(t) \) and \( y(t) \) that we computed suffice for doing this.

It is easily verified that the involution \( \sigma \), that permutes \( x_k \) and \( y_k \) extends naturally to an involution on the free parameters, given by

\[
\sigma(a) = b, \quad \sigma(a_{n \pm 1}) = 1/a_{n \pm 1}, \quad \sigma(a_{\pm}) = -a_{\pm}/a_{n \pm 1}, \quad \sigma(a) = -a - \frac{a_{n+1}a_+ - a_{n-1}a_-}{a_{n+1} - a_{n-1}}.
\] (38)

Notice that, altogether, we have besides the free parameters \( a_k, b_k \), for \(|k-n| > 1\), which naturally correspond to the variables \( x_k \) and \( y_k \), five extra free parameters \( a_{n \pm 1}, a_\pm \) and \( a \), that correspond to the remaining six variables \( x_{n \pm 1}, y_{n \pm 1} \) and \( x_n, y_n \), which again yields that the number of free parameters, plus time, is equal to the number of phase variables. This count will be important, and rigorous, when we restrict these formal Laurent solutions to certain finite-dimensional submanifolds.

4 Tangency to \( \mathcal{M} \)

We have seen that the polynomials \( \Gamma_k \) and \( \tilde{\Gamma}_k \), which define an invariant manifold for the first Toeplitz flow, satisfy a non-autonomous system of linear differential equations, where the time-dependence is defined by the latter flow. In a (finite-dimensional) manifold setting, if such differential equations have coefficients that depend smoothly on time, solutions (integral curves) that start out on the invariant manifold will stay on it, by the uniqueness of solutions to differential equations with smooth coefficients and given initial conditions. In the case that we deal with the situation is quite a bit different, because the coefficients develop poles in \( t \), for \( t = 0 \), and of course the solutions are only formal Laurent series. As it turns out, the conditions that assure that the formal Laurent solutions “stay on the invariant manifold” are similar to those in the smooth case for the self-dual Toeplitz lattice, but are different in an essential way for the general Toeplitz lattice.

4.1 Tangency in the self-dual case

We start out with the case of the self-dual Toeplitz lattice.

**Proposition 4.1** Let \( x(t) \) denote the formal Laurent solution that is given by proposition 3.1, and let \( \Gamma(t) := \Gamma(x(t); u(t)) \), where we recall that \( u(t) = (u_1 + t, u_2, \ldots, u_N) \). Then, as formal series in \( t \),

\[
\Gamma_k(t) = \Gamma_k^{(0)} + O(t), \quad k \in \mathbb{Z} \setminus \{n\},
\]

\[
\Gamma_n(t) = \frac{1}{4t} (\Gamma_{n+1}^{(0)} - \Gamma_{n-1}^{(0)}) + \Gamma_n^{(0)} + O(t).
\]

Moreover, \( \Gamma_k(t) = 0 \) as a formal series in \( t \), for all \( k \in \mathbb{Z} \), as soon as \( x(t) \) is such that

\( \Gamma_k^{(0)} = 0 \), for all \( k \in \mathbb{Z} \).

**Proof** According to (23), \( \Gamma_k(x; u) \) involves only the variables \( x_l \) with \(|l - k| \leq N \) (2N + 1 step relation). Since only \( x_n(t) \) has a pole, \( \Gamma_k(t) = O(1) \) as soon as \( \Gamma_k \) does not contain \( x_n \), i.e., if \(|n - k| > N\). But notice that (27) implies

\[
\Gamma_{n-N} = \frac{1}{1 - x_{n-N-1}^2} \frac{d\Gamma_{n-N-1}}{dt} + \Gamma_{n-N-2},
\]

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so that \( \Gamma_{n-N}(t) = O(1) \), as the leading term \( 1 - a_{N-1}^2 \) of \( 1 - x_{n-N-1}^2(t) \) is non-zero (recall that \( a_{N-1} \) is a free parameter). This argument can be repeated to yield \( \Gamma_k(t) = O(1) \) for all \( k < n \), and similarly it is shown that \( \Gamma_k(t) = O(1) \) for all \( k > n \). Since \( \Gamma_n(t) \) satisfies the differential equation (27), for \( k = n \), we have in view of (33) that

\[
\frac{d\Gamma_n}{dt}(t) = (1 - x_n^2(t))(\Gamma_{n+1}(t) - \Gamma_{n-1}(t)) = -\frac{1}{4t^2}(\Gamma^{(0)}_{n+1} - \Gamma^{(0)}_{n-1}) + O(1),
\]

which leads upon integration to (39).

Suppose now that \( x(t) \) is such that \( \Gamma^{(0)}_k = 0 \) for all \( k \in \mathbb{Z} \). In view of the first part of the proof, we have that \( \Gamma_k(t) = O(t) \) for all \( k \in \mathbb{Z} \). We show that this implies that \( \Gamma_k(t) = 0 \) as a formal series in \( t \), for all \( k \in \mathbb{Z} \). We do this by induction on \( r \in \mathbb{N}^* \): assuming that \( \Gamma_k(t) = O(t^r) \) for \( k \in \mathbb{Z} \) we show that \( \Gamma_k(t) = O(t^{r+1}) \) for \( k \in \mathbb{Z} \). Notice that in the case \( r = 1 \) the assumption holds. For \( k \notin \{n-1, n, n+1\} \) the right hand side of (27) is \( O(t^r) \), by (33) and by the assumption, so that \( \frac{d\Gamma_k}{dt}(t) = O(t^r) \), hence \( \Gamma_k(t) = O(t^{r+1}) \), by integration. For \( k = n \pm 1 \) we have from (33) that \( 1 - x_{n\pm 1}^2(t) = O(t) \), so that (27) yields for \( k = n \pm 1 \) that \( \frac{d\Gamma_{n\pm 1}}{dt}(t) = O(t^{r+1}) \), i.e., \( \Gamma_{n\pm 1}(t) = O(t^{r+2}) \). For \( k = n \) we have that \( 1 - x_n^2(t) = 1 - x_n^2(t) \) has a double pole, but since we have just shown that \( \Gamma_{n+1}(t) - \Gamma_{n-1}(t) = O(t^{r+2}) \) the differential equation (27) for \( k = n \) leads to \( \frac{d\Gamma_n}{dt}(t) = O(t^r) \) and we conclude that \( \Gamma_n(t) = O(t^{r+1}) \), as was to be shown.

### 4.2 Tangency in the general case

For the full Toeplitz lattice the tangency condition is rather similar, yet is different in some detail that will turn out to be crucial in the next section. We recall that the differential equations that are satisfied by the polynomials \( \Gamma_k \) and \( \tilde{\Gamma}_k \) are given by

\[
\frac{d\Gamma_k}{dt} = (1 - x_k y_k)(\Gamma_{k+1} - \Gamma_{k-1}) + (x_{k+1} - x_{k-1})(x_k \tilde{\Gamma}_k - y_k \Gamma_k),
\]

\[
\frac{d\tilde{\Gamma}_k}{dt} = (1 - x_k y_k)(\tilde{\Gamma}_{k+1} - \tilde{\Gamma}_{k-1}) - (y_{k+1} - y_{k-1})(x_k \tilde{\Gamma}_k - y_k \Gamma_k).
\]

**Proposition 4.2** Let \((x(t), y(t))\) denote the formal Laurent solution that is given by proposition 3.2, and let \( \Gamma(t) := \Gamma(x(t), y(t); u(t)) \), where \( u(t) \) is given by (25). Then, as a formal series in \( t \), \( \Gamma_k(t) = \Gamma_k^{(0)} + O(t) \) and \( \tilde{\Gamma}_k(t) = \tilde{\Gamma}_k^{(0)} + O(t) \) for \( k \in \mathbb{Z} \setminus \{n\} \). Also

\[
\Gamma_n(t) = \frac{a_{n+1}^2}{a - (a_{n-1} - a_{n+1})^2 t^2} \left( \Gamma_{n-1}^{(0)} - a_{n-1}^2 \tilde{\Gamma}_{n-1}^{(0)} \right) + \frac{1}{t} \tilde{\Gamma}_n^{(-1)} + O(1),
\]

\[
\tilde{\Gamma}_n(t) = \frac{a_{n+1} a_{n-1}}{a - (a_{n-1} - a_{n+1})^2 t^2} \left( \Gamma_{n-1}^{(0)} / a_{n-1}^2 - \tilde{\Gamma}_{n-1}^{(0)} \right) + \frac{1}{t} \Gamma_n^{(-1)} + O(1),
\]

where \( \Gamma_n^{(-1)} \) and \( \tilde{\Gamma}_n^{(-1)} \) are both linear combinations of \( \Gamma_{n\pm 1}^{(0)} \) and \( \tilde{\Gamma}_{n\pm 1}^{(0)} \) (for the explicit formula, see (47)); moreover, the latter coefficients are related in the following way:

\[
a - \left( \frac{1}{a_{n+1}^2} \Gamma_{n+1}^{(0)} - \frac{1}{a_{n-1}^2} \tilde{\Gamma}_{n+1}^{(0)} \right) = a + \left( \frac{1}{a_{n+1}^2} \Gamma_{n-1}^{(0)} - \tilde{\Gamma}_{n-1}^{(0)} \right).
\]

**Proof** As in the self-dual case, the polynomials \( \Gamma_k(x; u) \) and \( \tilde{\Gamma}_k(x; u) \) define \( 2N + 1 \) step relations, so they depend only on the variables \( x \) and \( y \) with \( |l - k| \leq N \). Only \( x_n(t) \) and \( y_n(t) \) have a pole,
so that \( \Gamma_k(t) = O(1) \) and \( \hat{\Gamma}_k(t) = O(1) \) for \(|n - k| > N \). Writing (40) for \( k \to k - 1 \) as

\[
\Gamma_k = \frac{1}{1 - x_{k-1}y_{k-1}} \left( \frac{d\Gamma_{k-1}}{dt} - (x_k - x_{k-2})(x_{k-1}\hat{\Gamma}_{k-1} - y_{k-1}\Gamma_{k-1}) \right) + \Gamma_{k-2},
\]

\[
\hat{\Gamma}_k = \frac{1}{1 - x_{k-1}y_{k-1}} \left( \frac{d\hat{\Gamma}_{k-1}}{dt} + (y_k - y_{k-2})(x_{k-1}\hat{\Gamma}_{k-1} - y_{k-1}\Gamma_{k-1}) \right) + \hat{\Gamma}_{k-2},
\]

(43)

and taking as consecutive values \( k := n - N, \ldots, n - 1 \) in (43) we find that \( \Gamma_k(t) = O(1) \) and \( \hat{\Gamma}_k(t) = O(1) \) for all \( k \leq n - 1 \), since \( 1 - x_k(t)y_k(t) \) does not vanish for \( t = 0 \) when \( k \neq n \pm 1 \). Similarly \( \Gamma_k(t) = O(1) \) and \( \hat{\Gamma}_k(t) = O(1) \) when \( k \geq n + 1 \). So we have that \( \Gamma_k(t) = O(1) \) and \( \hat{\Gamma}_k(t) = O(1) \) when \( k \neq n \) and we are left with the case \( k = n \).

In order to deal with the case \( k = n \) we write (40) as an equation for \( \Gamma_n \) and \( \hat{\Gamma}_n \) in two different ways:

\[
\Gamma_n = \mp \frac{1}{1 - x_{n+1}y_{n+1}} \left( \frac{d\Gamma_{n\pm 1}}{dt} \mp (x_n - x_{n\pm 2})(x_{n\pm 1}\hat{\Gamma}_{n\pm 1} - y_{n\pm 1}\Gamma_{n\pm 1}) \right) + \Gamma_{n\pm 2},
\]

\[
\hat{\Gamma}_n = \mp \frac{1}{1 - x_{n+1}y_{n+1}} \left( \frac{d\hat{\Gamma}_{n\pm 1}}{dt} \mp (y_n - y_{n\pm 2})(x_{n\pm 1}\hat{\Gamma}_{n\pm 1} - y_{n\pm 1}\Gamma_{n\pm 1}) \right) + \hat{\Gamma}_{n\pm 2}.
\]

(44)

Either of them implies that \( \Gamma_n(t) = O(t^{-2}) \) and that \( \hat{\Gamma}_n(t) = O(t^{-2}) \), so we write

\[
\Gamma_n(t) = \frac{1}{t^2} \left( \Gamma_n^{(-2)} + \Gamma_n^{(-1)}t + \Gamma_n^{(0)}t^2 + O(t^3) \right),
\]

and similarly for \( \hat{\Gamma}_n(t) \). In fact, as \( 1 - x_{n+1}(t)y_{n+1}(t) \) and \( 1 - x_{n-1}(t)y_{n-1}(t) \) have a simple zero, while \( x_n(t) \) and \( y_n(t) \) have a simple pole, the coefficient of \( t^{-2} \) in (44), leads to the following linear equations

\[
\Gamma_n^{(-2)} = -x_n^{(0)} \left( x_n^{(0)}\hat{\Gamma}_n^{(0)} - y_n^{(0)}\Gamma_n^{(0)} \right) / \zeta_n, \]

\[
\hat{\Gamma}_n^{(-2)} = -\Gamma_n^{(-2)} y_n^{(0)} / x_n^{(0)},
\]

(45)

where we have written \( 1 - x_{n\pm 1}(t)y_{n\pm 1}(t) = \zeta_n t + O(t^2) \), so that

\[
\zeta_n = \pm a_n \frac{a_{n+1} - a_{n-1}}{a_{n\mp 1}}.
\]

It suffices now to substitute \( x_n^{(0)} = a_{n\pm 1} = 1/y_n^{(0)} \) and \( x_n^{(0)} = a_{n-1}a_{n+1} / (a_{n-1} - a_{n+1}) = -a_{n-1}a_{n+1}y_n^{(0)} \) in (45) to find the coefficient of \( t^{-2} \) in (41). Actually, the latter corresponds to taking the lower sign; equating the two expressions for \( \Gamma_n^{(-2)} \) in (45) that correspond to the two signs leads to (42); notice that this is also the expression that is obtained from the two expressions of \( \hat{\Gamma}_n^{(-2)} \) in (45).

It remains to compute \( \Gamma_n^{(-1)} \) and \( \hat{\Gamma}_n^{(-1)} \), which can be done from the coefficient of \( t^{-2} \) in \( \frac{dx_n}{dt}(t) \) and in \( \frac{d\hat{x}_n}{dt}(t) \), computed from their differential equations

\[
\frac{dx_n}{dt} = (1 - x_n y_n)(\Gamma_{n+1} - \Gamma_n) + (x_{n+1} - x_{n-1})(x_n\hat{\Gamma}_n - y_n\Gamma_n),
\]

\[
\frac{d\hat{x}_n}{dt} = (1 - x_n y_n)(\hat{\Gamma}_{n+1} - \hat{\Gamma}_n) - (y_{n+1} - y_{n-1})(x_n\hat{\Gamma}_n - y_n\Gamma_n).
\]

(46)

Since \( 1 - x_n(t)y_n(t) \) has a double pole, while \( \Gamma_{n\pm 1}(t) \) and \( \hat{\Gamma}_{n\pm 1}(t) \) have no pole, the contribution of the first term to the coefficient in \( t^2 \) will be linear in \( \Gamma_n^{(0)} \) and in \( \hat{\Gamma}_n^{(0)} \). Since \( x_n(t) \) and \( y_n(t) \)
have a simple pole, while $\Gamma_n(t)$ and $\tilde{\Gamma}_n(t)$ have a double pole, the contribution of the second term will yield a linear combination of on the one hand $\Gamma_n(-2)$ and $\tilde{\Gamma}_n(-2)$ which, as we have seen, are themselves linear combinations of $\Gamma_{n \pm 1}^{(0)}$ and in $\tilde{\Gamma}_{n \pm 1}^{(0)}$; on the other hand, $\Gamma_n(-1)$ and $\tilde{\Gamma}_n(-1)$, which are the unknowns. Explicitly, this linear system is given by

$$\begin{pmatrix}
a_{n+1}a_{n-1}^2 \tilde{\Gamma}_n^{(-1)} \\
1/(a_{n+1}a_{n-1}) \Gamma_n^{(-1)}
\end{pmatrix} = \frac{a_{n+1}a_{n-1}}{(a_{n+1} - a_{n-1})^2} \begin{pmatrix}
\Gamma_n^{(0)}_n - \Gamma_n^{(0)}_{n+1} \\
\Gamma_n^{(0)}_{n+1} - \Gamma_n^{(0)}_{n-1}
\end{pmatrix} + \begin{pmatrix}
1 \\
1/a_{n+1}a_{n-1}
\end{pmatrix} \left(\Gamma_n^{(-2)}(a) + \tilde{\Gamma}_n^{(-2)}(a) a_{n+1}a_{n-1} \right).$$

(47)

Since $\Gamma_n^{(-2)}$ and $\tilde{\Gamma}_n^{(-2)}$ are linear combinations of $\Gamma_{n \pm 1}^{(0)}$ and $\tilde{\Gamma}_{n \pm 1}^{(0)}$ it follows that each of $\Gamma_n^{(-1)}$ and $\tilde{\Gamma}_n^{(-1)}$ is a linear combination of $\Gamma_{n \pm 1}^{(0)}$ and $\tilde{\Gamma}_{n \pm 1}^{(0)}$, as we asserted.

**Proposition 4.3** Suppose that $(x(t), y(t))$ is a formal Laurent solution of the first vector field of the Toeplitz lattice, such that $\Gamma_k(t) = O(t)$ and $\tilde{\Gamma}_k(t) = O(t)$ for all $k$ with $k \neq n + 1$, and such that, as formal Laurent solutions in $t$, $\Gamma_{n-1}(t) = O(t^2)$ and $\Gamma_{n+1}(t) = O(t)$. Then, as formal Laurent series, $\Gamma_k(t) = O(t)$ for all $k \in \mathbb{Z}$.

**Proof** According to (42), the hypothesis imply that $\tilde{\Gamma}_{n+1}(t) = O(t)$. In view of proposition 4.2, we have that $\Gamma_k(t) = O(t)$ and $\tilde{\Gamma}_k(t) = O(t)$ for every $k \in \mathbb{Z}$. We will now proceed by induction on $r \in \mathbb{N}^*$, but in a different way than in the self-dual case: assuming that $\Gamma_k(t) = O(t^r)$ and $\tilde{\Gamma}_k(t) = O(t^r)$ for $k \neq n \pm 1$, as well as $\Gamma_{n \pm 1}(t) = O(t^{r+1})$ and $\tilde{\Gamma}_{n \pm 1}(t) = O(t^{r+1})$, we show that $\Gamma_k(t) = O(t^{r+1})$ and $\tilde{\Gamma}_k(t) = O(t^{r+1})$ for $k \neq n \pm 1$, as well as $\Gamma_{n \pm 1}(t) = O(t^{r+2})$ and $\tilde{\Gamma}_{n \pm 1}(t) = O(t^{r+2})$. Notice that the $r = 1$ induction assumption needs to be shown at the end of the proof, as only part of it is in the actual hypothesis of the theorem.

For $k$ such that $|k - n| \geq 2$ the differential equations (40) yield that $\frac{d\Gamma_k}{dt}(t) = O(t^r)$ and $\frac{d\tilde{\Gamma}_k}{dt}(t) = O(t^r)$, so that $\Gamma_k(t) = O(t^{r+1})$ and $\tilde{\Gamma}_k(t) = O(t^{r+1})$, by integration. So we are left with $k \in \{n - 1, n, n + 1\}$. Let us write

$$\begin{align*}
\Gamma_n &= \gamma_n t^r + O(t^{r+1}), \\
\tilde{\Gamma}_n &= \tilde{\gamma}_n t^r + O(t^{r+1}), \\
\Gamma_k &= \gamma_k t^{r+1} + O(t^{r+2}), \\
\tilde{\Gamma}_k &= \tilde{\gamma}_k t^{r+1} + O(t^{r+2}), \quad k \neq n,
\end{align*}$$

which we substitute in

$$\begin{align*}
\frac{d\Gamma_{n \pm 1}}{dt} &= \mp (1 - x_{n \pm 1} y_{n \pm 1}) (\Gamma_{n} - \Gamma_{n \pm 2}) \pm (x_{n \pm 2} - x_n)\gamma_n (x_{n \pm 1} \tilde{\Gamma}_{n \pm 1} - y_{n \pm 1} \Gamma_{n \pm 1}), \\
\frac{d\tilde{\Gamma}_{n \pm 1}}{dt} &= \mp (1 - x_{n \pm 1} y_{n \pm 1}) (\tilde{\Gamma}_{n \pm 1} - \tilde{\Gamma}_{n}) \mp (y_{n \pm 2} - y_n)\tilde{\gamma}_n (x_{n \pm 1} \Gamma_{n \pm 1} - y_{n \pm 1} \tilde{\Gamma}_{n \pm 1}).
\end{align*}$$

(48)

Remembering that $1 - x_{n \pm 1}(t) y_{n \pm 1}(t) = O(t)$ we pick the coefficient of $t^r$ in (48), which leads to the following linear system,

$$\begin{align*}
(r + 1) \gamma_{n \pm 1} &= \mp \frac{a_{n - 1} a_{n + 1}}{\gamma_{n \pm 1} - a_{n \pm 1} \gamma_{n \pm 1}} \\
(r + 1) \tilde{\gamma}_{n \pm 1} &= \mp \frac{1}{\gamma_{n \pm 1} - a_{n \pm 1} \gamma_{n \pm 1}}.
\end{align*}$$

(49)

Since

$$\begin{vmatrix}
(r + 1) \mp \frac{a_{n \pm 1}}{\gamma_{n \pm 1} - a_{n \pm 1} \gamma_{n \pm 1}} & \mp \frac{a_{n - 1} a_{n + 1}}{\gamma_{n \pm 1} - a_{n \pm 1} \gamma_{n \pm 1}} \\
(r + 1) + \frac{a_{n \pm 1}}{\gamma_{n \pm 1} - a_{n \pm 1} \gamma_{n \pm 1}} & \mp \frac{1}{\gamma_{n \pm 1} - a_{n \pm 1} \gamma_{n \pm 1}}
\end{vmatrix} = (r + 1)^2 - (r + 1) = r(r + 1),$$

20
it follows, since \( r \geq 1 \), that \( \gamma_{n\pm1} = \tilde{\gamma}_{n\pm1} = 0 \), and hence that \( \Gamma_{n\pm1}(t) = O(t^{r+2}) \) and \( \tilde{\Gamma}_{n\pm1}(t) = O\left(t^{r+2}\right) \). It follows that, if we substitute the series in

\[
\begin{align*}
\frac{d\Gamma_n}{dt} &= (1-x_ny_n)(\Gamma_{n+1} - \Gamma_{n-1}) + (x_{n+1} - x_{n-1})(x_n\tilde{\Gamma}_n - y_n\Gamma_n), \\
\frac{d\tilde{\Gamma}_n}{dt} &= (1-x_ny_n)(\tilde{\Gamma}_{n+1} - \tilde{\Gamma}_{n-1}) - (y_{n+1} - y_{n-1})(x_n\tilde{\Gamma}_n - y_n\Gamma_n),
\end{align*}
\]

(50)

then the coefficient of \( t^{r-1} \) is simply given by

\[
\begin{align*}
r\gamma_n &= -(a_{n-1}a_{n+1}\tilde{\gamma}_n + \gamma_n), \\
r\tilde{\gamma}_n &= -\frac{1}{a_{n-1}a_{n+1}}(a_{n-1}a_{n+1}\tilde{\gamma}_n + \gamma_n).
\end{align*}
\]

Since

\[
\det \begin{pmatrix} r + 1 & a_{n-1}a_{n+1} \\ \frac{1}{a_{n-1}a_{n+1}} & r + 1 \end{pmatrix} = (r + 1)^2 - 1 \neq 0,
\]

we have that \( \gamma_n = \tilde{\gamma}_n = 0 \), so that \( \Gamma_n(t) = O(t^{r+1}) \) and \( \tilde{\Gamma}_n(t) = O(t^{r+1}) \), as was to be shown.

We finally check that our assumptions imply that for \( r = 1 \) the induction hypothesis is valid. According to proposition 4.2, we have that \( \Gamma(t) = O(t) \) and \( \tilde{\Gamma}(t) = O(t) \). Let us write \( \Gamma_{n\pm1} = \gamma_{n\pm1}t + O(t^2) \) and \( \tilde{\Gamma}_{n\pm1} = \tilde{\gamma}_{n\pm1}t + O(t^2) \). Then we need to show that \( \gamma_{n\pm1} = \tilde{\gamma}_{n\pm1} = 0 \). From (49), which is also valid for \( r = 0 \), we conclude that \( \gamma_{n\pm1} = a_{n-1}a_{n+1}\tilde{\gamma}_{n\pm1} \). It was assumed that \( \Gamma_n^{-1}(t) = O(t^2) \), i.e., that \( \gamma_{n-1} = 0 \), so that we can conclude that \( \tilde{\gamma}_{n-1} = 0 \). In order to obtain a second relation between \( \gamma_n \) and \( \gamma_{n+1} \) we consider the residue in the first\(^6 \) equation in (50), which reduces to

\[
0 = a_{n-1}a_{n+1}\gamma_{n+1}/(a_{n-1} - a_{n+1})^2,
\]

since \( \Gamma_n^{(0)} = \tilde{\Gamma}_n^{(0)} = 0 \). Thus, \( \gamma_{n+1} = \tilde{\gamma}_{n+1} = 0 \), as was to be shown.

5 Restricting the formal Laurent solutions: the self-dual case

We have seen conditions on \( \Gamma(t) = \Gamma(x(t); u(t)) \) that guarantee that solutions \( x(t) \) to the self-dual Toeplitz lattice that start out in the invariant manifold \( \mathcal{M}_{u(t)} \) stay in it, formally speaking. In this section we show how these conditions can be translated into conditions on the formal Laurent solution \( x(t) \) to the first vector field of the self-dual Toeplitz lattice.

5.1 Structure of the polynomials \( \Gamma_k \)

The polynomials \( \Gamma_k \), which define the invariant manifolds \( \mathcal{M} \) depend on the variable \( x_n \) in a special way, that we will analyze by using the fact that \( \Gamma_k \) remains pole free (for \( k \neq n \)) when the formal Laurent series \( x(t) \) is substituted in them, as we have seen in proposition 4.1. Let us denote by \( \mathcal{A} \) the algebra of polynomials in all variables \( x_k \), where \( k \in \mathbb{Z} \) and by \( \mathcal{A}_n \) the subalgebra of those polynomials that are independent of \( x_n \). Also, let us denote by \( \mathcal{A}'_n \) the subalgebra of \( \mathcal{A} \) that consists of those elements that can be written as polynomials in \( w_1, w_2 \) and \( x_k \), with \( k \neq n \), where

\[
w_1 := x_n(x_{n+1} + x_{n-1}), \quad w_2 := x_n(1 + x_{n+1}x_{n-1}).
\]

\(^{6}\)Taking the second equation would lead to the same result.
Thus, elements of \( \mathcal{A}'_n \) may depend only on \( x_n \) through \( w_1 \) and \( w_2 \). For future use, we give the first few terms of the formal Laurent series of the generators of \( \mathcal{A}'_n \), as obtained by substituting the series from proposition 3.1 in (51):

\[
\begin{align*}
    w_1(t) &= -2(a_+ + a_- + 2(a_+a_{n+2} + a_-a_{n-2})t + O(t^2)), \\
    w_2(t) &= -2\varepsilon(a_+ - a_- + 2(a_+a_{n+2} - a_-a_{n-2})t + O(t^2)), \\
    x_k(t) &= \varepsilon(a_k + (1 - a_k^2)(a_{k+1} - a_{k-1})t + O(t^2)), \quad k \neq n.
\end{align*}
\]

It follows that \( G(x(t)) = O(1) \), for any \( G \in \mathcal{A}'_n \). Notice that the polynomials \( w_{\pm} := (1 - x_{n \pm 1}^2)x_n \), which both have the property \( w_{\pm}(t) = O(1) \), belong to \( \mathcal{A}'_n \), since

\[
(1 - x_{n \pm 1}^2)x_n = w_2 - x_{n \pm 1}w_1.
\]

The following proposition generalizes this statement.

**Proposition 5.1** For \( G \in \mathcal{A} \), let \( G(t) := G(x(t)) \), where \( x(t) \) is the formal Laurent solution to the first vector field of the self-dual Toeplitz lattice, constructed in proposition 3.1. If \( G(t) = O(1) \) then \( G \in \mathcal{A}'_n \), i.e., \( G \) is a polynomial in

\[
x_n(x_{n+1} - x_{n-1}), x_n(1 + x_{n+1}x_{n-1}), \text{ and } x_k \quad (k \neq n).
\]

**Proof** We suppose that \( G \in \mathcal{A} \) is such that \( G(t) = O(1) \), where \( G(t) := G(x(t)) \). We write \( G \) as a polynomial in \( x_n \) with coefficients in \( \mathcal{A}'_n \),

\[
G = G_l x_n^l + G_{l-1} x_n^{l-1} + \cdots + G_1 x_n + G_0,
\]

where \( G_0, \ldots, G_l \in \mathcal{A}'_n \). If \( l = 0 \) then we are done. Let us suppose therefore that \( l \) is minimal, but \( l > 0 \). We will show that this leads to a contradiction. Since each coefficient \( G_i \) belongs to \( \mathcal{A}'_n \), we have that \( G_i(t) = O(1) \). Thus, the pole that \( x_n(t) \) has, needs to be compensated by a zero in \( G_l(t) \), i.e., \( G_l(t) = O(t) \). We show that this implies that \( G_l x_n \in \mathcal{A}'_n \). By Euclidean division in \( \mathcal{A}'_n \) we can write \( G_l \) as

\[
G_l = (1 - x_{n+1}^2)K_1 + (1 - x_{n-1}^2)K_2 + K_3,
\]

where \( K_1, K_2 \) and \( K_3 \) belong to \( \mathcal{A}'_n \), and where \( K_3 \) is of degree 1 at most in \( x_{n+1} \) and \( x_{n-1} \): we can write \( K_3 \) as

\[
K_3 = \kappa_1(x_{n+1} + x_{n-1}) + \kappa_2(1 + x_{n+1}x_{n-1}) + \kappa_3 x_{n+1} + \kappa_4
\]

where \( \kappa_1, \ldots, \kappa_4 \) are elements of \( \mathcal{A}'_n \), that are independent of \( x_{n+1} \) and \( x_{n-1} \). Since \( G_l(t) = O(1) \) and \( 1 = x_{n \pm 1}^2(t) = O(1) \) it follows from (54) that \( K_3(t) = O(t) \), and so that the leading terms \( \kappa_3^{(0)} \) and \( \kappa_4^{(0)} \) of \( K_3(t) \) and \( \kappa_4(t) \) satisfy \( \kappa_4^{(0)} = \varepsilon \kappa_3^{(0)} \). Since the leading terms \( \varepsilon a_k \) of all \( x_k(t) \), with \( k \in \mathbb{Z} \setminus \{n - 1, n, n + 1\} \), and the leading terms of \( w_1(t) \) and \( w_2(t) \) are all independent, even modulo \( \varepsilon \), it follows that \( \kappa_4^{(0)} = \kappa_3^{(0)} = 0 \), as \( \kappa_4 \) and \( \kappa_3 \) are independent of \( x_{n \pm 1} \). Using (53) it follows that

\[
G_l x_n = (1 - x_{n+1}^2)x_n K_1 + (1 - x_{n-1}^2)x_n K_2 + \kappa_1 w_1 + \kappa_2 w_2
\]

\[
= (w_2 - x_{n+1}w_1)K_1 + (w_2 - x_{n-1}w_1)K_2 + \kappa_1 w_1 + \kappa_2 w_2,
\]

where \( K_1, K_2, \kappa_1, \kappa_2 \in \mathcal{A}'_n \), showing that \( G_l x_n = G_l' \in \mathcal{A}'_n \), as promised. Then,

\[
G = (G_l' + G_{l-1}) x_n^{l-1} + \cdots + G_1 x_n + G_0,
\]

with \( G_l' + G_{l-1} \in \mathcal{A}'_n \). This contradicts the minimality of \( l \).
Lemma 5.2 For \( k \neq n \), \( \Gamma_k(t) := \Gamma_k(x(t); u(t)) \) is of the form
\[
\Gamma_k(t) = \mathcal{F}(a_{k-N}, a_{k-N+1}, \ldots, a_{k+N}, a_+, a_-) + O(t),
\] (55)
i.e., the constant term in \( \Gamma_k(t) \) is a polynomial in the variables\(^7\) \( a_{k-N}, a_{k-N+1}, \ldots, a_{k+N}, a_+ \) and \( a_- \) only.

Proof According to (23), \( \Gamma_k \) depends on \( x_{k-N}, \ldots, x_{k+N} \) only. For \( k \neq n \) we know from proposition 4.1 that \( \Gamma_k(t) = O(1) \), so that proposition 5.1 yields that \( \Gamma_k \) depends on \( x_n \) through \( w_1 \) and \( w_2 \) only, i.e., \( \Gamma_k \) is a polynomial in \( w_1, w_2 \) and the \( x_i \) with \( |k - l| \leq N \) and \( l \neq n \). Each of these variables is \( O(1) \), so the constant term in \( \Gamma_k \) is a polynomial in their leading terms, which are the parameters \( a_{k-N}, a_{k-N+1}, \ldots, a_{k+N}, a_+ \) and \( a_- \) (see (52)).

It is clear that when \( |k - n| > N \) then \( \Gamma_k(0) \) is independent of \( a_+ \) and \( a_- \), as it cannot contain \( w_1 \) or \( w_2 \). The following lemma deals with the case of \( \Gamma_n(t) \), which is slightly harder because \( \Gamma_n(t) \) develops a pole.

Lemma 5.3 \( \Gamma_n(t) := \Gamma_n(x(t); u(t)) \) is of the form
\[
\Gamma_n(t) = \frac{\Gamma_n(0) - \Gamma_n(0)}{4t} + \mathcal{F}(a_{n-N-1}, \ldots, a_{n+N+1}, a_+, a_-) + O(t)
\]
where \( \mathcal{F} \) is a polynomial in all its arguments, with \( a_{n+N+1} \) and \( a_{n-N-1} \) present (linearly).

Proof Consider the following alternative ways of writing \( \Gamma_n = \Gamma_n(x; u) \),
\[
\Gamma_n(x; u) = (1 - x_n^2)H_n(x; u) + nx_n = x_nG_n(x; u) + H_n(x; u).
\] (56)
\( H_n \) is a polynomial in \( x = (x_i)_{i \in \mathbb{Z}} \), because (28) implies that \( H_n(x; u) = \sum u[x_n] \), and because \( \partial x_n/\partial t_i = \{x_n, H_i\} \) is always divisible by \( 1 - x_n^2 \), see (16). Also, we have put \( G_n(x; u) := n - x_nH_n(x; u) \) to obtain the second equality. The first equation in (56) implies that \( H_n(x; u(t)) = O(t) \), since \( \Gamma_n(x(t); u(t)) = O(t^{-1}) \) and \( x_n(t) = O(t^{-1}) \), while \( 1 - x_n^2(t) = -1/(4t^2) + O(t^{-1}) \). The second equation in (56) then allows us to conclude that \( G_n(x(t); u(t)) = O(1) \), and hence also that \( G_n(x(t); u) = O(1) \), since \( u \) is an arbitrary vector of constants. Thus, \( G_n \) is, by proposition 5.1, an element of \( A_n \), depending (linearly) on the parameters \( u_i \).

Summarizing, the constant term in \( \Gamma_n(t) \) will be given by the constant term in \( x_n(t)G_n(t) \), hence will depend only on the first two terms \( \varepsilon(1 + (a_+ - a_-)t)/(2t) \) of \( x_n(t) \) and on the first two terms of \( G_n(t) \), where \( G_n \in A_n' \). The latter first two terms can depend only on the first two terms of the variables \( x_{n-N}, \ldots, x_{n+N} \), \( w_1 \) and \( w_2 \) that appear in \( G_n \); the first two terms of their series can be read off from (52), yielding that the constant term in \( \Gamma_n(t) \) can only depend on \( a_{n-N-1}, \ldots, a_{n+N+1}, a_+, a_- \). Notice that the only dependence on \( a_{n-N-1} \) can come from the presence of \( x_{n-N} \), but (23) tells us that \( x_{n-N} \) appears linearly in \( \Gamma_n \), and with a non-zero coefficient. Therefore, the parameter \( a_{n-N-1} \) is indeed present in the constant term in \( \Gamma_n \); similarly, \( a_{n+N+1} \) is also present. The leading term of \( \Gamma_n(t) \) was already determined in proposition 4.1.

5.2 Parameter restriction

We now show that we can tune the free parameters in the formal Laurent solution \( x(t) \) of the self-dual Toeplitz lattice in such a way that \( \Gamma_k(t) = 0 \) for all \( k \in \mathbb{Z} \), as a formal series in \( t \). As it turns out, it will be possible to keep \( 2N - 1 \) parameters arbitrary, and the other ones are determined rationally in terms of these. Together with time it means that the constructed solution depends on \( 2N \) free parameters, which is the maximum one can hope for in an \( 2N + 1 \) step relation.

\(^7\)Recall that \( a_{n+1} = \mp 1 \) and that \( a_n \) does not exist; so \( a_{n+1} \) and \( a_n \) may be thought of as being absent in the list. Thus, \( a_\pm \) is the natural substitute for \( a_{n\pm 1} \).
we consider all of them (exactly once); it is understood that steps (6)–(8) are absent when \( k \) are the extremal terms, are actually present in \( \Gamma \) which yields the third column. It is important to point out that the two written variables, which so that \( \Gamma (0) \) Proposition 5.4 Keeping the parameters in the series \( x(t) \), given by proposition 3.1, can be chosen as rational functions of these parameters, so that \( \Gamma_k(t) = 0 \), as a formal series in \( t \), for all \( k \in \mathbb{Z} \).

**Proof** In this proof we will assume that \( N > 1 \). See Remark 5.5 below for the adaption to the case \( N = 1 \). According to proposition 4.1, it suffices to determine the parameters in the series \( x(t) \) so that \( \Gamma_k(t) \), the constant term in \( \Gamma_k(t) \), is zero, for all \( k \in \mathbb{Z} \). Thus, we need to write \( \Gamma_k(0) \) in terms of the parameters in the series \( x(t) \). We do this for the different values of \( k \) in a very specific order, as indicated in table 2. The second column indicates which \( \Gamma_k \) we consider; it is easy to see that we consider all of them (exactly once); it is understood that steps (6)–(8) are absent when \( N = 2 \). We know from (23) that for any \( k \in \mathbb{Z} \), \( \Gamma_k \) depends only on the variables \( x_{k-N}, x_{k-N+1}, \ldots, x_{k+N} \), which yields the third column. It is important to point out that the two written variables, which are the extremal terms, are actually present in \( \Gamma_k \), and that these two variables appear linearly (see proposition 8.2 in the appendix).

<table>
<thead>
<tr>
<th>step</th>
<th>( \Gamma_k )</th>
<th>( \Gamma_k ) polynomial in</th>
<th>( \Gamma_k(0) ) polynomial in</th>
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</thead>
<tbody>
<tr>
<td>(1)</td>
<td>( \Gamma_{N-1} )</td>
<td>( x_{n-2N+1}, \ldots, x_n )</td>
<td>( a_{n-2N+1}, \ldots, a_n = 1 )</td>
</tr>
<tr>
<td>(2)</td>
<td>( \Gamma_{N-2} )</td>
<td>( x_{n-2N+2}, \ldots, x_n )</td>
<td>( a_{n-2N+2}, \ldots, a_n )</td>
</tr>
<tr>
<td>(3)</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>(4)</td>
<td>( \Gamma_N )</td>
<td>( x_{n-N-1}, \ldots, x_n )</td>
<td>( a_{n-N-1}, \ldots, a_n, a_1 )</td>
</tr>
<tr>
<td>(5)</td>
<td>( \Gamma_{N+1} )</td>
<td>( x_{n-N-2}, \ldots, x_{n+1} )</td>
<td>( a_{n-N-2}, \ldots, a_{n+1} )</td>
</tr>
<tr>
<td>(6)</td>
<td>( \Gamma_{N+2} )</td>
<td>( x_{n-N-3}, \ldots, x_{n+2} )</td>
<td>( a_{n-N-3}, \ldots, a_{n+2} )</td>
</tr>
<tr>
<td>(7)</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>(8)</td>
<td>( \Gamma_{N-1} )</td>
<td>( x_{n-N-4}, \ldots, x_{n+3} )</td>
<td>( a_{n-N-4}, \ldots, a_{n+3} )</td>
</tr>
<tr>
<td>(9)</td>
<td>( \Gamma_{N+1} )</td>
<td>( x_{n-N-5}, \ldots, x_{n+4} )</td>
<td>( a_{n-N-5}, \ldots, a_{n+4} )</td>
</tr>
<tr>
<td>(10)</td>
<td>( \Gamma_N )</td>
<td>( x_{n-N-6}, \ldots, x_{n+5} )</td>
<td>( a_{n-N-6}, \ldots, a_{n+5} )</td>
</tr>
<tr>
<td>(11)</td>
<td>( \Gamma_{N+2} )</td>
<td>( x_{n-N-7}, \ldots, x_{n+6} )</td>
<td>( a_{n-N-7}, \ldots, a_{n+6} )</td>
</tr>
<tr>
<td>(12)</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>
The delicate step is in obtaining the last column; the information displayed in it contains the parameters\(^8\) that may appear in \(\Gamma_k^{(0)}\), where the underlined term actually does appear, and it appears linearly. Before validating this column in each of the steps, let us first point out how the proposition follows from it. Precisely, we can in each step solve for one of the underlined parameters in terms of the nonunderlined parameters, as the underlined parameter appears linearly in the equation \(\Gamma_k^{(0)} = 0\). Using the previous steps, this yields (using the previous steps) inductively a rational formula for each of the parameters, in terms of \(a_{n-2N}, \ldots, a_{n-2}\), which remain free. In fact, the variables \(a_{n-2N-i}\), with \(i > 0\) are determined in steps (1)–(3); \(a_n-1 = -a_{n+1} = 1\) while \(a_n\) does not exist; the variables \(a_{n+i+1}\) with \(i > 0\) are determined in steps (6)–(12); the only other variables are \(a_–\) and \(a_+\), which are determined in steps (4) and (5).

We now show that in each step the parameters that are indicated in the fourth column of the table appear indeed (linearly) in \(\Gamma_k^{(0)}\). This is done by carefully using the leading terms of \(\Gamma_k\), as given by proposition 8.2. As a general remark, notice that (23) implies that \(\Gamma_k\) contains the variables \(x_{k-N}\) and \(x_{k+N}\) linearly, but that the behaviour of its coefficients \(\prod_{i=0}^{N-1} (1 - x_{k+i}^2)\) and \(\prod_{i=0}^{N-1} (1 - x_{k-i}^2)\), evaluated at \(t\), depends on \(k\), as given (33).

For step (1) we have that \(x_{n-2N-1}(t), \ldots, x_{n-1}(t)\) have no pole in \(t\), so that only their leading coefficients, the parameters \(a_{n-2N-1}, \ldots, a_{n-2}, a_{n-1} = 1\), can appear. Since \(x_{n-2N-1}\) appears (linearly) in \(\Gamma_{n-N-1}\), with a coefficient \(u_N \prod_{i=1}^{N-1} (1 - x_{n-N-i}^2)\) that is non-vanishing for \(t = 0\), namely \(\prod_{i=1}^{N} (1 - x_{n-N-i}^2(0)) = \prod_{i=1}^{N} (1 - a_{n-N-i}^2)\), the parameter \(a_{n-2N-1}\) appears (linearly) in \(\Gamma_{n-N-1}^{(0)}\). The same argument works in steps (2) and (3). Step (4) is more interesting because it involves \(x_n\) (linearly). However, \(x_n\) appears only in the leading term of \(\Gamma_{n-N}\), which we can write, using \(w_- = x_n(1 - x_{n-1}^2)\), as

\[
u_Nx_n \prod_{i=0}^{N-1} (1 - x_{n-N-i}^2) = u_Nw_- \prod_{i=0}^{N-2} (1 - x_{n-N-i}^2), \quad u_N \neq 0. \tag{57}\]

Now \(w_-(t) = 4\varepsilon a_- + O(t)\), and the other factors in (57) are finite, non-vanishing, which yields the proposed dependence on the parameters in step (4). For step (5), \(x_n\) may be present in other terms than the leading term in \(\Gamma_{n-N+1}\), but in view of proposition 5.1, \(\Gamma_{n-N+1} \in A'_n\) is a polynomial in \(x_{n-2N+1}, \ldots, x_{n-1}, x_{n+1}\) and in \(w_1\) and \(w_2\) only. Since their series do not have a pole for \(t = 0\), we get an eventual dependence on \(a_+\) and \(a_-\), besides the parameters \(a_{n-2N+1}, \ldots, a_{n-2}\). Let us show that \(a_+\) actually appears. The leading term in \(\Gamma_{n-N+1}\) is, according to (23),

\[
u_Nx_n(1 - x_n^2)(1 - x_{n-1}^2) \prod_{i=n-N+1}^{n-2} (1 - x_i^2).\]

Since it is the only term in \(\Gamma_{n-N+1}\) that contains \(x_{n+1}\) we can write \(\Gamma_{n-N+1} = P_1 + P_2\), where

\[
P_1 = u_N (x_{n+1} + x_n)(1 - x_n^2)(1 - x_{n-1}^2) \prod_{i=n-N+1}^{n-2} (1 - x_i^2),
\]

and \(P_2\) is independent of \(x_{n+1}\), so \(P_2\) depends only on \(x_{n-2N+1}, \ldots, x_n\). Now \(P_1(t) = O(1)\), since \(x_{n+1}(t) + x_n(t) = O(t), \quad 1 - x_n^2(t) = O(t^2), \quad 1 - x_{n-1}^2(t) = O(t)\), while the other factors \(1 - x_i^2(t)\) that appear in \(P_1(t)\) are \(O(1)\). Since \(\Gamma_{n-N+1}(t) = O(1)\) this implies that \(P_2(t) = O(1)\), so that \(P_2\) satisfies the hypothesis of proposition 5.1; since \(P_2\) is independent

---

\(^8\)Besides the constants \(u_1, \ldots, u_N\) that define \(P\).
of \( x_{n+1} \) we may conclude, as in step (4), that \( P_2 \) is independent of \( a_+ \). On the other hand \( P_1(0) \) depends (linearly) on \( a_+ \), as

\[
(x_{n+1}(t) + x_{n-1}(t))(1 - x_n^2(t))(1 - x_{n-1}^2(t)) = 8\varepsilon a_-(a_- + a_+) + O(t).
\]

The conclusion is that \( \Gamma_n^{(0)} = P_1(0) + P_2(0) \) depends (linearly) on \( a_+ \).

We are at step (6). Skip this step and steps (7) and (8) when \( N = 2 \). Proposition 5.1 implies that \( \Gamma_n^{(0)} \) can only depend on the proposed parameters, and that the dependence comes from the constant terms of the series in (52). The dependence of \( \Gamma_n^{(0)} \) on \( a_{n+2} \) comes only from the leading term \( u_N x_{n+2} (1 - x_{n+1}^2)(1 - x_n^2) (1 - x_{n-1}^2) \prod_{i=0}^{N-1} (1 - x_{n+2+i}^2) \) which, at \( t \), is \( O(1) \), since \( (1 - x_{n+1}^2)(1 - x_n^2)(1 - x_{n-1}^2) = O(1) \) and non-vanishing. It follows that \( \Gamma_n^{(0)} \) depends on \( a_{n+2} \) (linearly). The same happens in steps (7) and (8), as the leading term will always contain the product \( (1 - x_{n+1}^2)(1 - x_n^2)(1 - x_{n-1}^2) \) which is finite and non-zero for \( t = 0 \).

A new phenomenon arises in step (9). Notice that we have moved to \( \Gamma_n^{(0)} \), keeping \( \Gamma_n \) for step (10). The leading term of \( \Gamma_n^{(0)} \) is

\[
u_N x_{n+N+1} \prod_{i=1}^{N} (1 - x_{n+i}^2),
\]

which does not contribute to \( \Gamma_n^{(0)} \), since \( 1 - x_{n+1}^2 = O(t) \), while all other factors in this term are finite in \( t \). Therefore, \( \Gamma_n^{(0)} \) is independent of \( a_{n+N+1} \). To show that \( \Gamma_n^{(0)} \) depends on \( a_{n+N} \) we need to investigate the next term in \( \Gamma_n^{(0)} \), the one that contains \( x_{n+N} \), because it is the only one that might lead to a dependence on \( a_{n+N} \). According to proposition 8.2, this term consists of the following three pieces,

\[
u_N x_{n+N} \prod_{i=0}^{N-2} (1 - x_{n+1+i}^2) - u_N x_{n+N} x_{n+N-1} \prod_{i=0}^{N-2} (1 - x_{n+1+i}^2)
- 2u_N x_{n+N} \prod_{i=0}^{N-2} (1 - x_{n+1+i}^2) \sum_{j=0}^{N-2} x_{n+j+1} x_{n+j}.
\]

The two terms on the first line of (58) do not contribute to \( \Gamma_n^{(0)} \), again because both terms contain \( 1 - x_{n+1}^2 \), and all other terms are finite for \( t = 0 \). The third term however does contribute, when \( j = 0 \), as \( x_n(t)(1 - x_{n+1}^2) \sim a_+ + O(t) \); moreover, this term is the only one that involves \( a_{n+N} \), so that the latter parameter appears (linearly) in \( \Gamma_n^{(0)} \). For step (10) the presence of \( a_{n+N+1} \) was established in lemma 5.3. Starting from step (11) the leading coefficients do not contain \( 1 - x_{n+1}^2 \) or \( 1 - x_n^2 \) anymore, so that everything goes smoothly.

**Remark 5.5** When \( N = 1 \) the polynomial that defines the recursion relation reduces to

\[
\Gamma_k = k x_k + u_1 (1 - x_{k+1}^2)(x_{k+1} + x_{k-1}).
\]

Steps (4)–(9) then get replaced by two steps in which we consider \( \Gamma_{n\pm 1} \), which allows us to determine \( a_\pm \). Indeed, substituting the series \( x(t) \) in \( \Gamma_{n\pm 1} \) yields for the leading term \( (t = 0) \):

\[
(n \pm 1) + 4u_1 a_\pm = 0.
\]

The other parameters are determined as in the general case.
6 Restricting the formal Laurent solutions: the general case

In this section we will do a similar analysis as the one that has been done for the case of the self-dual Toeplitz lattice in section 5.

6.1 Structure of the polynomials $\Gamma_k$ and $\tilde{\Gamma}_k$

We first investigate on which parameters the leading term(s) in the polynomials $\Gamma_k$ and $\tilde{\Gamma}_k$ depends on the free parameters. We denote by $\mathcal{A}$ the algebra of all polynomials in the variables $x_i$ and $y_i$, where $i \in \mathbb{Z}$, while $\mathcal{A}_n$ stands for the subalgebra of $\mathcal{A}$ that consists of all polynomials that do not depend on $x_n$ and on $y_n$. Consider the following four polynomials\(^9\)

\[
\begin{align*}
    w_1 &= x_n y_{n-1} + y_n x_{n+1}, \\
    w_1' &= y_n x_{n-1} + x_n y_{n+1}, \\
    w_2 &= x_n + x_{n-1} y_n x_{n+1}, \\
    w_2' &= y_n + y_{n-1} x_n y_{n+1}.
\end{align*}
\] (59)

For future use, observe that these polynomials are linked by the following identity:

\[
x_n (w_2' - y_{n-1} w_1') = y_n (w_2 - x_{n-1} w_1),
\] (60)

in fact both expressions in (60) are equal to $x_n y_{n-1} - x_{n-1} y_{n-1}$. We denote by $\mathcal{A}_n'$ the subalgebra of $\mathcal{A}$ that consists of all polynomials that can be written in terms of these four polynomials, besides all $x_i$ and $y_i$, with $i \neq n$. The polynomials $w$ have the following series in $t$, when the first few\(^10\) terms of the series $x_i(t)$ and $y_i(t)$ that are constructed in proposition 3.2, are substituted in them.

\[
\begin{align*}
    w_1(t) &= \Omega b_{n-1} - a_- + (a_+ a_{n+2} b_{n-1} - a_- a_{n-1} b_{n-2}) t + O(t^2), \\
    w_2(t) &= \Omega + (a_+ a_{n+2} + a_- a_{n-2}) t + O(t^2),
\end{align*}
\] (61)

where

\[
\Omega := \frac{a_{n-1} a_{n+1}}{(a_{n+1} - a_{n-1})^2} (a_{n-1} (2a_+ - a_-) - a_{n+1} (2a_- - a_+)).
\]

The formal Laurent series for the other polynomials in (59) is found from it by using the automorphism $\sigma$ (see (38)), which yields in particular

\[
\sigma(\Omega) = \Omega b_{n-1} b_{n+1} + a_+ b_{n-1} - a_- b_{n+1}.
\] (62)

It follows that if $G \in \mathcal{A}_n'$ then $G(t) = O(1)$, where $G(t) := G(x(t), y(t))$, with $x(t)$ and $y(t)$ as above. We will show that the converse is also true, so that the algebra $\mathcal{A}_n'$ plays in the general case a similar rôle as in the self-dual case. For this we need the following lemma.

**Lemma 6.1** Let $G$ be a polynomial in $\mathcal{A}_n'$ that is independent of $w_2$ and none of whose terms contains $x_{n+1} y_{n+1}$ or $x_{n-1} y_{n-1}$. If $G(t) = O(t)$ then $G = 0$, as a formal series in $t$.

**Proof** It follows from (61) that

\[
\begin{pmatrix}
    w_1(0) \\
    w_1'(0) \\
    w_2'(0)
\end{pmatrix}
= T \begin{pmatrix}
    a(a_{n-1} - a_{n+1}) \\
    a a_{n+1} \\
    a_- a_{n-1}
\end{pmatrix}
\]

\(^9\)Recall that $\sigma$ denotes the involution that permutes all $x_i \leftrightarrow y_i$.

\(^{10}\)A priori, one needs to compute an extra term in the series $z_k(t)$ (see proposition 3.2) in order to find the shown terms in (61). After proposition 6.2 we will however show how such a cumbersome can be avoided.

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fields, which allows us to replace any term in $G_{l}y_{n}$ that contains $w_{2}$, or a power of it, by a term of lower degree in $w_{2}$, at the cost of changing $G_{l-1}$, so that we can eventually remove $w_{2}$ entirely from the leading coefficient $G_{l}$. Assuming that $G_{l}$ does not depend on $w_{2}$ we perform an Euclidean division in $\mathcal{A}'_{n}$,

$$G_{l} = (1 - x_{n-1}y_{n-1})K_{1} + (1 - x_{n+1}y_{n+1})K_{2} + K_{3},$$

(63)

where $K_{1}$, $K_{2}$ and $K_{3}$ belong to $\mathcal{A}'_{n}$, with $K_{3}$ independent of $w_{2}$ and not containing $x_{n-1}y_{n-1}$ or $x_{n+1}y_{n+1}$.

Assume now that $G(t) = O(1)$. Since all $G_{i}(t)$ are $O(1)$, as $G_{l} \in \mathcal{A}'_{n}$, we must have that $G_{l}(t) = O(t)$, as $y_{n}(t)$ has a pole. Then (63) implies that $K_{3}(t) = O(t)$, since $1 - x_{n+1}(t)y_{n+1}(t) = O(t)$. This means that $K_{3}$ satisfies the conditions of lemma 6.1, hence that $K_{3} = 0$. The identities

$$(1 - x_{n-1}y_{n-1})y_{n} = w_{2}^\sigma - y_{n-1}w_{1}^\sigma \in \mathcal{A}'_{n}$$

$$(1 - x_{n+1}y_{n+1})y_{n} = w_{2}^\sigma - y_{n+1}w_{1} \in \mathcal{A}'_{n}$$

then imply that $G_{l}y_{n} \in \mathcal{A}'_{n}$, as was to be shown.

As a first application of this proposition, we show how the shown terms in (61) can easily be computed. Since $w_{i}(t) = O(1)$ we also have $\frac{dw_{i}}{dt}(t) = O(1)$ for $i = 1, 2$. By proposition 6.2, $\frac{dw_{1}}{dt} \in \mathcal{A}'_{n}$, in fact

$$\frac{dw_{1}}{dt} = \frac{d}{dt}(x_{n}y_{n-1} + y_{n}x_{n+1})$$

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\[
\frac{dw_2}{dt} = \frac{d}{dt} \left( \sum_{i=1}^N x_{i+1} \right) - \sum_{i=1}^N y_{i+1} + x_{n+1} + y_{n+1} - x_{n+1},
\]

where \( w_\pm := (1 - x_{n+1}y_{n+1}) y_n \), with \( w_\pm(0) = \pm a_\pm b_{n+1} + O(t) \). Since \( x_{n+1}y_{n+1} = 1 \), it follows that

\[
\begin{align*}
\frac{dw_1}{dt}(0) &= b_{n-2}a_{n-1}a_{n+1}w_\mp(0) + a_{n+2}w_\mp(0) = a_+ a_{n+1} b_{n-1} - a_+ a_{n-1} b_{n-2}, \\
\frac{dw_2}{dt}(0) &= a_{n+2}b_{n-1}w_\mp(0) - a_{n-2}a_{n+1}w_\mp(0) = a_{n+2}a_+ + a_{n-2}a_-
\end{align*}
\]

which yield after integration the linear terms in \( \sigma \). The same formulas can be used to show that \( w_1^{(2)} \) and \( w_2^{(2)} \), which are the \( t^2 \) terms in \( w_1(t) \) and \( w_2(t) \), depend only on the parameters \( c_{n-3}, c_{n+3}, a_+, a_- \) and \( a \); the precise formula will not be needed, except that they depend on \( c_{n+3} \) as follows:

\[
\begin{align*}
w_1^{(2)} &= x_{n+1}w_\mp(0)/2 + \cdots = a_{n+3}a_+ b_{n-1} (1 - a_{n+1} b_{n+2})/2 + \cdots, \\
w_2^{(2)} &= x_{n+2}w_\mp(0)/2 + \cdots = a_{n+3}a_+ (1 - b_{n+2})/2 + \cdots
\end{align*}
\]

where the dots are independent of \( a_{n+3} \) (and of \( b_{n+3} \)).

The following lemma is the analog of lemma 5.2 and is proven in exactly the same way.

**Lemma 6.3** If \( k \neq n \), then the series \( \Gamma_k(t) := \Gamma_k(x(t), y(t); u(t)) \) and \( \bar{\Gamma}_k(t) := \bar{\Gamma}_k(x(t), y(t); u(t)) \) are of the form

\[
\begin{align*}
\Gamma_k(t) &= \mathcal{F}(a_{k-N}, c_{k-N+1}, \ldots, a_{k+N}, a_\pm, a) + O(t), \\
\bar{\Gamma}_k(t) &= \mathcal{\bar{F}}(b_{k-N}, c_{k-N+1}, \ldots, b_{k+N}, a_\pm, a) + O(t),
\end{align*}
\]

where we recall that \( c_i = (a_i, b_i) \) and that \( a_{n+1} b_{n+1} = 1 \), and \( \mathcal{F}, \mathcal{\bar{F}} \) are polynomials in their arguments.

For \( k = n \) the corresponding result is more complicated and the method of proof is different from the one in the self-dual case (lemma 5.3).

**Lemma 6.4** The constant terms \( \Gamma_n^{(0)} \) and \( \bar{\Gamma}_n^{(0)} \) are of the form

\[
\begin{pmatrix}
\Gamma_n^{(0)} \\
\bar{\Gamma}_n^{(0)}
\end{pmatrix} = A \begin{pmatrix}
a_{n+N+1} \\
b_{n+N+1}
\end{pmatrix} + \mathcal{F}(c_{n-N-1}, \ldots, c_{n+N}, a_\pm, a),
\]

where \( A \) is an invertible \( 2 \times 2 \) matrix and \( \mathcal{F} \) is a polynomial 2-vector that depends on the listed free parameters only. See proposition 4.2 for the leading terms of \( \Gamma_n(t) \) and \( \bar{\Gamma}_n(t) \).

**Proof** We will assume in our proof that \( N > 2 \), see Remark 6.5 below. The proof is based on the explicit expression for \( \Gamma_n \) that is given in proposition 8.2 (see the appendix), which we write in the form \( \Gamma_n = (1 - x_n y_n) H_n + n x_n \), where

\[
\begin{align*}
H_n &= u_N x_{n+N} \sum_{i=1}^{N-1} (1 - x_{n+i} y_{n+i}) - u_{N^2} x_{n+N}^2 \sum_{i=1}^{N-1} (1 - x_{n+i} y_{n+i}) \\
&- u_N x_{n+N-1} (x_n y_{n-1} + 2 \sum_{j=1}^{N-2} x_{n+j} y_{n+j-1}) \sum_{i=1}^{N-2} (1 - x_{n+i} y_{n+i}) \\
&+ (u_{N-1} x_{n+N-1} - u_{-N} y_{n+N-1} x_{n-1}) \sum_{i=1}^{N-2} (1 - x_{n+i} y_{n+i}) \\
&+ \mathcal{F}(x_{n-N+1}, \ldots, x_{n+N-2}, y_{n-N+1}, \ldots, y_{n+N-2}) \\
&- (u_N x_{n+1} y_{n+N+1} - u_{-N} x_{n-N} (1 - x_{n-N+1} y_{n-N+1})) \sum_{i=1}^{N-2} (1 - x_{n+i} y_{n+i}).
\end{align*}
\]
Our first claim is that \( F \in \mathcal{A}' \). Since \( \Gamma_n(t) \) and \( 1 - x_n(t)y_n(t) \) have a double pole, while \( x_n(t) \) has a simple pole, \( H_n(t) = O(1) \). The terms in the above expression that do not involve \( x_n \) or \( y_n \) are also \( O(1) \), because \( x_k(t) = O(1) \) and \( y_k(t) = O(1) \) for \( k \neq n \). There are a few terms that contain \( x_n \) or \( y_n \) (linearly), but they are all of the form \( x_n(1 - x_n \pm y_n) \), \( y_n(1 - x_n \pm y_n) \) or \( x_n(1 - x_n \pm y_n) \), which are both \( O(1) \). It follows that \( F(t; u(t)) = O(1) \), and hence that \( F(t; u) = O(1) \). Thinking of \( u \) as constants we have, in view of proposition 6.2, that \( F \in \mathcal{A}' \).

Since \( 1 - x_n(t)y_n(t) \) has a double pole, only the first three terms of \( 1 - x_n(t)y_n(t) \) and of \( F(t) \) can contribute to the constant term in \( (1 - x_n(t)y_n(t))F(t) \); in view of table 1, this contribution can only yield a dependence on the parameters \( c_{n-N-1}, \ldots, c_{n+N}, a_{\pm} \) and \( a \).

We now turn to the other terms in \( H_n \) and we use their explicit form to show that they only depend on the listed parameters. Let us first consider the following terms that do not involve \( x_n \) or \( y_n \),

\[
- \left( u_N x_n^2 x_n x_n + 2u_N x_n x_n + \sum_{j=2}^{N-2} x_n y_n + j - 1 \right) \prod_{i=1}^{N-2} (1 - x_n y_n + i) + u_N x_n x_n - N \prod_{i=1}^{N-1} (1 - x_n y_n - i). \tag{65}
\]

Since \( 1 - x_n \pm y_n \) has a simple zero for \( i = 1 \) and is \( O(1) \) for \( i > 1 \) we have that \( \prod_{i=1}^{N-2} (1 - x_n y_n + i) \) and \( \prod_{i=1}^{N-1} (1 - x_n y_n - i) \) have a simple zero, so we only need to look for the parameters that appear in the first two terms of the coefficients. The former add nothing new to the above parameter list. For the coefficients of the first one for example, we read off from table 1 that the constant and linear terms of \( x_n y_n - 1(t) \) depend on \( a_{N+1}, c_{n+N-1}, c_{n+N-2} \) and \( b_{n+N-3} \), which falls inside the proposed limits. Notice in particular that neither \( a_{n+N+1} \) nor \( b_{n+N+1} \) appear in this term. We arrive similarly at the same conclusion for the other three terms in (65). Notice that the lowest free parameter that appears is \( a_{n-1} \); it comes from the last term in (65).

We now get to the terms that contain \( x_n \) or \( y_n \). As we already noticed these terms always come with \( 1 - x_n y_n \) or \( 1 - x_n y_n \). As \( x_n(t)(1 - x_n(t)y_n(t)) = O(1) \) we must investigate the first three terms in the remaining factors. For the term

\[
- u_N x_n (1 - x_n y_n - 1) x_n y_n - N + 1 \prod_{i=2}^{N-2} (1 - x_n y_n - i)
\]

we need to look at \( x_n y_n - N + 1 \prod_{i=2}^{N-2} (1 - x_n y_n - i) \), which yields terms with a low index, the lowest coming from the coefficient in \( t^2 \) in \( y_n - N + 1(t) \), to wit \( b_{n-N-1} \) and \( a_{n-N} \). The other three terms that involve \( x_n \) or \( y_n \) can be written as

\[
B := - \left( x_n (1 - x_n y_n + 1) (u_N x_n y_n + 1) + u_N y_n + 1 x_n - 1 \right) \prod_{i=2}^{N-2} (1 - x_n y_n + i).
\]

Again, since \( 1 - x_n y_n \) has a double pole the first three terms in \( B(t) = B_1(t) + B_2(t^2) + O(t^3) \) will contribute to the constant term in \( (1 - x_n y_n) B(t) \). It is clear that \( B_2 \) will contain \( a_{n+N+1} \), coming from \( x_n y_n - 1 \) and \( b_{n+N+1} \), coming from \( y_n + 1 \). To know the precise value, it suffices to substitute the relevant coefficients of the formal Laurent series \( x(t), y(t) \) in the following part of \( B_2 \),

\[
- \left( x_n (1 - x_n y_n + 1)^0 (u_N x_n y_n + 1)^0 + u_N y_n + 1 x_n - 1^0 \right) \prod_{i=2}^{N-2} (1 - x_n y_n + i)^0,
\]

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which gives, by using proposition 3.2, and in particular 
\[-(y_n(1 - x_{n+1}y_{n+1}))^{(0)} = a_+a_{n+1} \]
and
\[-(y_n(1 - x_{n+1}y_{n+1}))^{(0)} = a_+b_{n-1}, \]

\[-\frac{a_+a_{n+1}}{2}(u_Na_{n+N+1}b_{n-1} - u_Na_{n-1}b_{n+N+1}) \prod_{i=2}^{N} (1 - a_{n+i}b_{n+i}) + \cdots, \tag{66} \]

where the dots are independent of \(a_{n+N+1}\) and \(b_{n+N+1}\). There remains one term in \(H_n\), namely the leading term \(C := u_Nx_{n+N} \prod_{i=1}^{N-1}(1 - x_{n+i}y_{n+i})\). It does not involve \(x_n\) but does involve \(1 - x_{n+1}y_{n+1}\), which will also lead to a dependence on \(a_{n+N+1}\). Writing \(C(t) = C_1t + C_2t^2 + O(t^3)\) we have that

\[ C_2 = u_Na_{n+N+1}a_+(a_{n+1} - a_{n-1})b_{n-1} \prod_{i=2}^{N} (1 - a_{n+i}b_{n+i}) + \cdots, \]

where the dots are again independent of \(a_{n+N+1}\) and \(b_{n+N+1}\). Summing up, we have that the leading terms in \(\tilde{\Gamma}_n^{(0)}\) are given by

\[-\frac{a_+(1 - x_ny_n)^{(0)}}{2a_{n-1}}(u_N(a_{n+1} - 2a_{n-1})b_{n-1}a_{n+N+1} + u_Na_{n+1}a_{n-1}b_{n+N+1}) \prod_{i=2}^{N} (1 - a_{n+i}b_{n+i}). \]

By duality, the leading terms in \(\Gamma_n^{(0)}\) are given by

\[-\frac{a_+(1 - x_ny_n)^{(0)}}{2a_{n+1}}(u_{-N}(a_{n+1} - 2a_{n-1})b_{n+N+1} + u_Nb_{n-1}a_{n+N+1}) \prod_{i=2}^{N} (1 - a_{n+i}b_{n+i}). \]

We may conclude that

\[
\begin{pmatrix}
\tilde{\Gamma}_n^{(0)} \\
\Gamma_n^{(0)}
\end{pmatrix}
= A \begin{pmatrix}
a_{n+N+1} \\
b_{n+N+1}
\end{pmatrix} + \mathcal{F}(c_{n-N-1}, \ldots, c_{n+N}, a_+, a), \tag{67}
\]

where

\[ A = \frac{a_+a_{n+1}}{2(a_{n-1} - a_{n+1})^2} \begin{pmatrix}
(a_{n+1} - 2a_{n-1})u_N & a_{n+1}a_{n-1}^2u_{-N} \\
-\frac{u_N}{a_{n-1}} & (2a_{n+1} - a_{n-1})u_{-N}
\end{pmatrix} \prod_{i=2}^{N} (1 - a_{n+i}b_{n+i}). \]

Since

\[
\det A = \frac{u_Nu_{-N}}{2} \left( \frac{a_+a_{n+1}}{a_{n+1} - a_{n-1}} \prod_{i=2}^{N} (1 - a_{n+i}b_{n+i}) \right)^2,
\]

\(A\) is invertible.

**Remark 6.5** The above proof breaks down at several places when \(N = 2\). The polynomial \(H_n\) then reduces to

\[ H_n = u_2(x_{n+2}(1 - x_{n+1}y_{n+1}) - x_{n+1}w_1) + u_1x_{n+1} + u_2(x_{n+2}(1 - x_{n+1}y_{n+1}) - x_{n+1}w_1^2) + u_1x_{n+1}. \tag{68} \]

Using (64) and proposition 3.2 we find that \(H_n\) depends in the following way on \(a_{n+3}\) and \(b_{n+3}\).

\[ u_2(x_{n+2}(1 - x_{n+1}y_{n+1})^{(1)}) - x_{n+1}w_1^{(2)} \]

\[ = \frac{a_+(1 - a_{n+2}b_{n+2})}{2a_{n-1}}(u_2(a_{n+1} - 2a_{n-1})a_{n+3} + u_2a_{n+1}a_{n-1}^2b_{n+3}). \]

It leads as in the case \(N > 2\) to (67), with precisely the same matrix \(A\).
6.2 Parameter restriction

The parameter restriction works more or less like in the self-dual case, the main difference coming from the fact that in the self-dual case we had to put all $\Gamma_k^{(0)} = 0$, while in the general case the tangency condition is equivalent to

1. $\Gamma_k(t) = O(t)$ and $\tilde{\Gamma}_k(t) = O(t)$ for all $k$ with $k \neq n + 1$;
2. $\Gamma_{n-1}(t) = O(t^2)$;
3. $\Gamma_{n+1}(t) = O(t)$.

In a sense, the condition $\Gamma_n(t) = O(t)$ replaces the condition $\tilde{\Gamma}_{n+1}(t) = O(t)$, which is redundant because it is a consequence of the other conditions (see proposition 4.3).

**Proposition 6.6** Keeping the $4N - 1$ parameters$^{11}$ $c_{n-2N}, \ldots, c_{n-2}, a_{n-1}$ arbitrary, the other parameters in the formal Laurent series $(x(t), y(t))$, given by proposition 3.2, can be chosen as rational functions of these parameters, so that $\Gamma_k(t) = 0$ and $\tilde{\Gamma}_k(t) = 0$, identically in $t$, for all $k \in \mathbb{Z}$.

**Proof** We give the proof in the case $N > 1$ only, leaving the case $N = 1$ to the reader (see Remark 5.5 for the self-dual case). As in the self-dual case, we summarize the order in which we treat the different equations in a table (see table 3). The second column shows which $\Delta_k = (\Gamma_k, \tilde{\Gamma}_k)$ we consider. For $k \neq n \pm 1$ it is clear that each $\Delta_k$ appears (precisely once). The fact that $\Gamma_{n-1}$ appears on line (9a), while $\Delta_{n-1}$ already appears on line (8) comes from the fact that we consider in line (9a) the coefficient in $t$ of $\Gamma_{n-1}(t)$ (rather than the coefficient in $t^0$); similarly, $\tilde{\Gamma}_{n+1}$ is absent because the nullity of $\tilde{\Gamma}_{n+1}(0)$ is a consequence of the nullity of the other $\Delta_k(0)$ (proposition 4.3).

We know from proposition 8.2 that for any $k \in \mathbb{Z}$,

$$
\Gamma_k(x, y; u) \in \mathbb{R}[x_{k-N}, \ldots, x_{k+N}, y_{k-N+1}, \ldots, y_{k+N-1}],
\tilde{\Gamma}_k(x, y; u) \in \mathbb{R}[x_{k-N+1}, \ldots, x_{k+N-1}, y_{k-N}, \ldots, y_{k+N}]
$$

so that

$$
\Delta_k(x, y; u) \in \mathbb{R}[z_{k-N}, \ldots, z_{k+N}].
$$

This leads, with no effort, to the third column of the table. For future use, let us recall that $\Gamma_k$ depends (linearly) on $x_{k-N}$ and on $x_{k+N}$, while $\tilde{\Gamma}_k$ depends (linearly) on $y_{k-N}$ and on $y_{k+N}$.

Let us now turn, line by line, to the last column, which demands a careful inspection of the polynomials $\Gamma_k$ and $\tilde{\Gamma}_k$. In particular, we show that these polynomials depend on the underlined parameter(s) (linearly), in such a way that one can solve for them. In steps (1) – (3) we have that $z_n$ is absent, so that $\Delta_{n-N-k}(0) (k \geq 1)$ depends on $z_{n-2N-k}(0), \ldots, z_{n-k}(0)$ only, i.e., on $c_{n-2N-k}, \ldots, c_{n-k}$. Now $\Gamma_{n-N-k}$ depends on $x_{n-2N-k}$ (linearly), but not on $y_{n-2N-k}$, while the opposite is true for $\tilde{\Gamma}_{n-N-k}$, so that we can solve the equation $\Gamma_{n-N-k}(0) = 0$ linearly for $a_{n-2N-k}$, and similarly $\tilde{\Gamma}_{n-N-k}(0) = 0$ can be solved linearly for $b_{n-2N-k}$ in terms of $c_{n-2N-k+1}, \ldots, c_{n-k}$. For $k = 1$ this gives $a_{n-2N-1} (resp. b_{n-2N-1})$ in terms of the 4N – 1 parameters $c_{n-2N}, \ldots, c_{n-2}, a_{n-1}$, so that by taking $k = 2, 3, \ldots$, we get recursively $c_{n-2N-k}$ in terms of these parameters, for all $k \geq 1$.

We now get to step (4) which is different because $\Delta_{n-N}$ involves $x_n$ and $y_n$. As for $\Gamma_n$, according to proposition 8.2, $x_n$ appears only in the leading term of $\Gamma_n$, which we can write as

$$
\begin{align*}
N \sum_{i=0}^{N-1} (1 - x_{n-N+i}y_{n-N+i}) = u_N w_n \prod_{i=n-N}^{n-2} (1 - x_iy_i), \quad u_N \neq 0,
\end{align*}
$$

$^{11}$Recall that $c_k = (a_k, b_k)$ and that $a_{n \pm 1}b_{n \pm 1} = 1$. 

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Table 3: The tangency condition allows us to solve for all free parameters in the formal Laurent series, except for the $4N - 1$ parameters $c_{n-2N}, \ldots, c_{n-2}, a_{n-1}$, that can be taken arbitrarily. The equations can be solved linearly for the underlined terms.

<table>
<thead>
<tr>
<th>step</th>
<th>$\Delta_k$</th>
<th>$\Delta_k$ polynomial in</th>
<th>$\Delta_k^{(0)}$, $\Gamma_{n-1}^{(1)}$, $\Gamma_{n+1}^{(0)}$ polynomial in</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>$\Delta_{n-N-1}$</td>
<td>$z_{n-2N-1}, \ldots, z_{n-1}$</td>
<td>$c_{n-2N-1}, \ldots, c_{n-1}$</td>
</tr>
<tr>
<td>(2)</td>
<td>$\Delta_{n-N-2}$</td>
<td>$z_{n-2N-2}, \ldots, z_{n-2}$</td>
<td>$c_{n-2N-2}, \ldots, c_{n-2}$</td>
</tr>
<tr>
<td>(3)</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>(4)</td>
<td>$\Delta_{n-N}$</td>
<td>$z_{n-2N}, \ldots, z_{n}$</td>
<td>$c_{n-2N}, \ldots, c_{n-1}, \underline{a}, a_{n+1}$</td>
</tr>
<tr>
<td>(5)</td>
<td>$\Delta_{n-N+1}$</td>
<td>$z_{n-2N+1}, \ldots, z_{n+1}$</td>
<td>$c_{n-2N+1}, \ldots, c_{n}, \underline{a}, a_{n+1}$</td>
</tr>
<tr>
<td>(6)</td>
<td>$\Delta_{n-N+2}$</td>
<td>$z_{n-2N+2}, \ldots, z_{n+2}$</td>
<td>$c_{n-2N+2}, \ldots, c_{n+1}, \underline{a}, a_{n+2}$</td>
</tr>
<tr>
<td>(7)</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>(8)</td>
<td>$\Delta_{n-1}$</td>
<td>$z_{n-N-1}, \ldots, z_{n+N-1}$</td>
<td>$c_{n-N-1}, \ldots, c_{n-2}, a_{n+1}$</td>
</tr>
<tr>
<td>(9a)</td>
<td>$\Gamma_{n-1}$</td>
<td>$x_{n-N-1}, z_{n-N}, \ldots, z_{n+N-2}, x_{n+N-1}$</td>
<td>$a_{n-N-2}, c_{n-N-1}, \underline{a}, a_{n+1}$</td>
</tr>
<tr>
<td>(9b)</td>
<td>$\Gamma_{n+1}$</td>
<td>$x_{n-N+1}, z_{n-N+2}, \ldots, z_{n+N}, x_{n+N+1}$</td>
<td>$a_{n-N+1}, c_{n-N}, \underline{a}, a_{n+1}$</td>
</tr>
<tr>
<td>(10)</td>
<td>$\Delta_{n}$</td>
<td>$z_{n-N}, \ldots, z_{n+N}$</td>
<td>$c_{n-N}, \ldots, a_{n+1}, a_{n+1}$</td>
</tr>
<tr>
<td>(11)</td>
<td>$\Delta_{n+2}$</td>
<td>$z_{n-N+2}, \ldots, z_{n+N+2}$</td>
<td>$c_{n-N+2}, \ldots, a_{n+2}, a_{n+2}$</td>
</tr>
<tr>
<td>(12)</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>
where \( w_- := x_n(1 - x_{n-1}y_{n-1}) \in \mathcal{A}'_n \), as \( w_- (t) = a_- a_{n-1} + O(t) \). Therefore, using (69),

\[
\Gamma_{n-N}(0) = u_N a_- a_{n-1} \prod_{i=n-N}^{n-2} (1 - a_i b_i) + \mathcal{F}(a_{n-2N}, c_{n-2N+1}, \ldots, c_{n-1}),
\]

which can be solved linearly for \( a_- \) in terms of the previous parameters \((1 - a_i b_i \neq 0 \text{ for } n - N \leq i \leq n - 2)\). Using the automorphism \( \sigma \) (see (38)),

\[
\hat{\Gamma}_{n-N}(0) = u_N \frac{-a_-}{a_{n+1}} \prod_{i=n-N}^{n-2} (1 - a_i b_i) + \mathcal{F}(b_{n-2N}, c_{n-2N+1}, \ldots, c_{n-1}),
\]

so that \( \hat{\Gamma}_{n-N}(0) = 0 \) can be solved linearly for \( b_{n+1} = 1/a_{n+1} \).

For step (5), \( x_n \) and \( y_n \) may be present in several terms in \( \Delta_{n-N+1} \), but in view of proposition 6.2, \( \Gamma_{n-N+1} \) and \( \hat{\Gamma}_{n-N+1} \) are polynomials in \( z_{n-2N+1}, \ldots, z_{n-1}, z_{n+1} \) and in \( w_1 \) and \( w_2 \) and their \( \sigma \) analogs only. Thus, \( \Gamma_{n-N+1}(0) \) and \( \hat{\Gamma}_{n-N+1}(0) \) depend on their leading terms only, to wit \( c_{n-2N+1}, \ldots, c_{n-1}, a_{n+1} \) and \( a, a_+ \). It follows that the only new parameters that appear at step (5) are \( a_+ \) and \( a \). Let us show that they appear in such a way that we can solve for them (linearly) in terms of the other parameters. We do this as in the self-dual case by isolating the leading term in \( \Gamma_{n-N+1} \) as given in proposition 8.2, namely we write \( \Gamma_{n-N+1} \) as

\[
\Gamma_{n-N+1} = -u_N (x_n w_1 - x_{n+1})(1 - x_{n-1} y_{n-1}) \prod_{i=n-N+1}^{n-2} (1 - x_i y_i) + \mathcal{F}(z_{n-2N+2}, \ldots, z_n), \tag{70}
\]

The relation (70) was obtained by writing the leading term

\[x_{n+1}(1 - x_n y_n) = x_{n+1} - (x_n w_1 - x_n^2 y_{n-1}),\]

and throwing the \( x_n^2 y_{n-1} \) term into \( \mathcal{F} \). Since \( \Gamma_{n-N+1}(t) = O(1) \) and since the first two terms in (70) belong to \( \mathcal{A}'_n \), the last term in (70) is also \( O(1) \) in \( t \); since in addition this term does not contain \( z_{n+1} \), by proposition 6.2 and (59) \( x_n \) and \( y_n \) can only appear in it multiplied by \( 1 - x_{n-1} y_{n-1} \), and so by proposition 3.2 we may conclude that the contribution from this term in \( \Gamma_{n-N+1}(0) \) will not involve \( a_+ \) or \( a \). Also, the second term in (70), \( u_N x_{n+1}(1 - x_{n-1} y_{n-1}) \prod_{i=n-N+1}^{n-2} (1 - x_i y_i) \) does not contribute to \( \Gamma_{n-N+1}(0) \) since \( 1 - x_{n-1}(t) y_{n-1}(t) = O(t) \) while all other factors are \( O(1) \). Thus, the dependence on \( a_+ \) and \( a \) in \( \Gamma_{n-N+1}(0) \) comes entirely from the first term in (70), which in view of proposition 3.2 and (61) is given by

\[
\Gamma_{n-N+1}(0) = -u_N a_- \Omega \prod_{i=n-N+1}^{n-2} (1 - a_i b_i) + \text{previous parameters.}
\]

By duality,

\[
\hat{\Gamma}_{n-N+1}(0) = u_N \frac{a_- a_{n-1}}{a_{n+1}} \sigma(\Omega) \prod_{i=n-N+1}^{n-2} (1 - a_i b_i) + \text{previous parameters},
\]

where \( \sigma(\Omega) \) was given in (62). Since \( \Omega \) and \( \sigma(\Omega) \) are linearly independent, as linear functions of \( a_+ \) and \( a \), we can indeed solve \( \hat{\Gamma}_{n-N+1}(0) = 0 \) and \( \hat{\Gamma}_{n-N+1}(0) = 0 \) linearly for \( a_+ \) and \( a \) in terms of the other parameters.
Steps\(^{12}\) (6) – (8) are easy, the point being that by proposition 8.2, for \(2 \leq k \leq N - 1\)
\[
\Delta_{n-N+k}(0) = \left( \begin{array}{c} u_N \\ u_{-N} \end{array} \right) c_{n+k}((1 - x_{n-1}y_{n-1})(1 - x_ny_n)(1 - x_{n+1}y_{n+1}))^{(0)} \prod_{\substack{i = n-N+k \\ i \neq n-1, n+1}}^{n+k-1} (1 - a_i b_i) + \text{known.}
\]

Let us concentrate on the next steps, which are more exciting. In step (9a) we need to compute the linear term in \(\Gamma_{n-1}(t)\), where we recall from propositions 4.2 and 6.2 that \(\Gamma_{n-1} \in A'_n\), hence that this linear term only depends on the constant and linear terms of the elements of \(\Gamma_{n-1} \in A'_n\). Since \(\Gamma_{n-1} \in R[x_{n-N-1}, \ldots, x_{n+N-1}, y_{n-N}, \ldots, y_{n+N-2}]\), with leading term
\[
\Gamma_{n-1} = u_N x_{n+N-1} \prod_{i=0}^{N-1} (1 - x_{n+i-1}y_{n+i-1}) + \cdots,
\]
we have from proposition 3.2 that
\[
\Gamma_{n-1}(t) = \Gamma^{(0)}_{n-1} + \left( u_N a_{n+N}((1 - x_{n-1}y_{n-1})(1 - x_ny_n)(1 - x_{n+1}y_{n+1}))^{(0)} \prod_{i=3}^{N} (1 - a_{n+i-1} b_{n+i-1}) + \cdots \right) t + O(t^2),
\]
where the dots only involve previous parameters. Therefore we may solve \(\Gamma^{(1)}_{n-1} = 0\) (linearly) for \(a_{n+N}\). Step (9b) is similar to step (9) in the self-dual case; notice that we postpone again \(\Delta_{n}\) to the next step. First of all \(\Gamma_{n+1}(t) = O(1)\) and so \(\Gamma_{n+1} \in A'_n\). The leading term in \(\Gamma_{n+1}\), namely the term \(u_N x_{n+N+1}(1 - x_{n+1}y_{n+1}) \prod_{i=1}^{N-1} (1 - x_{n+i+1}y_{n+i+1})\) cannot contribute to \(\Gamma_{n+1}(0)\) because it is \(O(t)\), which explains the absence of \(a_{n+N+1}\) in \(\Gamma_{n+1}(0)\). By proposition 8.2, \(b_{n+N}\) can come only from \(y_{n+N}\), which appears only once, namely in
\[
- u_{-N} y_{n+N} x_n x_{n+1} \prod_{i=0}^{N-2} (1 - x_{n+1+i}y_{n+1+i}) = - u_{-N} y_{n+N} x_n x_{n+1} (x_n(1 - x_{n+1}y_{n+1})) \prod_{i=n+2}^{n+N-1} (1 - x_i y_i),
\]
yielding at \(t = 0\) a non-zero linear term in \(b_{n+N}\), as \(x_n(t)(1 - x_{n+1}(t)y_{n+1}(t)) = O(1)\).

Step (10) is the hardest one, but we dealt with it in lemma 6.4. Notice that after this step we have that \(\Delta_{n}(t) = O(t)\) since the nullity of the previous \(\Delta_k(0)\) already implies that \(\Delta_{n}(t) = O(1)\) (proposition 4.2). Starting from step (11) everything goes smoothly, as \(\Delta_k(t) = O(1)\) for \(k > n + 1\) and the leading term of \(\Gamma_k(0)\), resp. \(\Gamma_k(0)\) will produce precisely the new parameter \(a_{k+N}\), resp. \(b_{k+N}\) (linearly).

7 Singularity confinement

We have constructed in the previous sections formal Laurent series for the Toeplitz lattice (self-dual and general case) solving the recursion relations \(\Gamma_k(x(t); u(t)) = 0\) (\(\Delta_k(x(t), y(t); u(t)) = 0\) in the general case). We will now transform these into solutions of the recursion relations \(\Gamma_k(x; u) = 0\) (resp. \(\Delta_k(x, y; u) = 0\)), depending on a certain number of free parameters, and blowing up for only one (resp. two) variables. We will mainly concentrate on the self-dual case, as the general case is dealt with in precisely the same way.

\(^{12}\)Skip these steps if \(N = 2\).
The main tool to do this transformation is a formal version of the implicit function theorem, which we explain in the case of one variable, the scalar case. Suppose that we have a formal series in $t$,

$$x(t; a) = a + f_1(a)t + f_2(a)t^2 + \cdots; \quad (71)$$

one may think for example of $x(t; a)$ as a formal solution of a vector field (differential equation $\frac{dx}{dt} = F(x)$) on the real line, with initial condition $x(0; a) = a$. In our case the functions $f_i$ will be rational. We wish solve the equation $x(t; a) = \alpha$ formally, namely we wish to construct the formal series in $t$

$$a(t; \alpha) = \alpha + g_1(\alpha)t + g_2(\alpha)t^2 + \cdots$$

with the property that $x(t; a(t; \alpha)) = \alpha$, as a formal t-series identity. Precisely, we claim that there exist for any $s \in \mathbb{N}$ unique (rational) functions $g_1(\alpha), \ldots, g_s(\alpha)$, such that

$$x(t; \alpha + g_1(\alpha)t + g_2(\alpha)t^2 + \cdots + g_s(\alpha)t^s) - \alpha = O(t^{s+1}),$$

where $x(t; \alpha)$ is given by (71). This is a trivial consequence of a formal version of Taylor’s Theorem. For example, for $s = 1$ we neglect all terms in $t^2$ and the condition on $g_1$ becomes

$$x(t; \alpha + g_1(\alpha)t) - \alpha + O(t^2) = g_1(\alpha)t + f_1(\alpha + g_1(\alpha)t)t + O(t^2)$$

so that $g_1(\alpha) = -f_1(\alpha)$. For $s = 2$ we neglect the terms in $t^3$, giving

$$x(t; \alpha - f_1(\alpha)t + g_2(\alpha)t^2) - \alpha + O(t^3) = -f_1(\alpha)t + g_2(\alpha)t^2 + f_1(\alpha - f_1(\alpha)t)t + f_2(\alpha)t^2 + O(t^3)$$

which has $g_2(\alpha) := f_1(\alpha)f'_1(\alpha) - f_2(\alpha)$ as a unique solution. Continuing in this way it is clear that $g_i(\alpha)$ equals $-f_j(\alpha)$, up to a differential polynomial in the $f_j(\alpha)$, with $j < i$. Notice that when all $f_i(\alpha)$ are rational function the same will be true for all $g_j(\alpha)$.

Let us apply this to the formal Laurent series that we have constructed for the self-dual Toeplitz lattice, and that yield formal solutions to the recursion relations $\Gamma_k(t) := \Gamma_k(x(t); u(t)) = 0$, where $k \in \mathbb{Z}$. Recall from proposition 5.4 that these formal Laurent solutions $x_k(t)$ depend on $2N - 1$ parameters $a_{n-2N}, \ldots, a_{n-2}$, which are the leading coefficients of $x_{n-2N}, \ldots, x_{n-2}$, namely

$$x_k(t) = a_k + O(t), \quad k = n - 2N, \ldots, n - 2, \quad (72)$$

where the higher order terms are rational functions of the parameters $a_{n-2N}, \ldots, a_{n-2}$. Besides the parameters $a_k$ these functions also depend (polynomially) on the parameters $u = (u_1, \ldots, u_N)$ that define the recursion relations, namely $x_k(t) = x_k(t; a_{n-2N}, \ldots, a_{n-2}; u)$, for $n - 2N \leq k \leq n - 2$.

The formal implicit function theorem then leads to the following proposition.

**Proposition 7.1** There exist for $k = n - 2N, \ldots, k = n - 2$ rational functions

$$a_k^{(i)} = a_k^{(i)}(a_{n-2N}, \ldots, a_{n-2}; u_1, \ldots, u_N)$$

such that $a_k := \sum_{i=0}^{\infty} a_k^{(i)} t^i$, $k = n - 2N, \ldots, n - 2$ formally inverts (72), i.e.,

$$x_k \left( t; \sum_{i=0}^{\infty} a_{n-2N}^{(i)} t^i, \ldots, \sum_{i=0}^{\infty} a_{n-2}^{(i)} t^i; u \right) = a_k,$$

for $k = n - 2N, \ldots, n - 2$, with $a_k^{(0)} = \alpha_k$. 

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We can use these series to replace the free parameters \(a_{n-2N}, \ldots, a_{n-2}\) in the series \(x_k(t), k \in \mathbb{Z}\), by \(\alpha := (a_{n-2N}, \ldots, a_{n-2})\), where we think of the latter as (partial) initial conditions to the recursion relation. To do this, one simply substitutes \(a_k = \sum_{i=0}^\infty a_k^{(i)} t^i\) for \(k = n-2N, \ldots, n-2\) in each of the series \(x_k(t) = x_k(t; a_{n-2N}, \ldots, a_{n-2}; u)\), and rewrites this as a series in \(t\); by construction, this simply gives \(x_k(t) = \alpha_k\) for \(k = n-2N, \ldots, k = n-2\). For \(k = n-1\), this yields

\[
x_{n-1}(t) = \varepsilon + \sum_{i=1}^\infty x_{n-1}^{(i)}(a; u)t^i = \varepsilon + \sum_{i=1}^\infty \xi_{n-1}^{(i)}(\alpha; u)t^i,
\]

where we recall that \(\varepsilon^2 = 1\). The functions \(\xi_{n-1}^{(i)}\) are rational in \(\alpha\) and \(u\). We will now use the formal implicit\(^{13}\) function theorem again, but in a form which is different from the one explained above: putting \(x_{n-1}(t) = \varepsilon + \lambda(t)\), i.e., we put

\[
\lambda := \sum_{i=1}^\infty \xi^{(i)}(\alpha; u)t^i,
\]

which we solve for \(t\) as a formal series in \(\lambda\),

\[
t(\lambda) = \sum_{i=1}^\infty t^{(i)}(\alpha; u)\lambda^i, \quad (73)
\]

where it is important to note that the constant term in this series is absent. Indeed, let us first substitute (73) in the series for \(a_k\) that was obtained in proposition (7.1), to get \(a_k = a_k(\alpha; \lambda; u)\).

Then, the latter and \(t(\lambda)\) are substituted in all \(x_k(t)\), to yield series in \(\lambda\) whose coefficients are rational functions of \(\alpha = (\alpha_{n-2N}, \ldots, \alpha_{n-2})\) (and of \(u = (u_1, \ldots, u_N)\)), which take the following form.

\[
x_k(\lambda, \alpha; u) = \sum_{i=0}^\infty \chi_k^{(i)}(\alpha; u)\lambda^i, \quad k < n-2N,
\]

\[
x_k(\lambda, \alpha; u) = \alpha_k, \quad n-2N \leq k < n-1,
\]

\[
x_{n-1}(\lambda, \alpha; u) = \varepsilon + \lambda,
\]

\[
x_n(\lambda, \alpha; u) = \frac{1}{\lambda} \sum_{i=0}^\infty \chi_{n}^{(i)}(\alpha; u)\lambda^i,
\]

\[
x_{n+1}(\lambda, \alpha; u) = -\varepsilon + \sum_{i=1}^\infty \chi_{n+1}^{(i)}(\alpha; u)\lambda^i,
\]

\[
x_k(\lambda, \alpha; u) = \sum_{i=0}^\infty \chi_k^{(i)}(\alpha; u)\lambda^i, \quad n+1 < k.
\]

It may seem that we have reached the final result, but we should not forget that these series are constructed from solutions \(x = x(t)\) to the recursion relations \(\Gamma_k(x; u(t))\), where \(u(t) = (u_1 + t, u_2, \ldots, u_N)\). However, letting \(U = (U_1, \ldots, U_n) := u(t)\), and using (73) to get rid of \(t\), we have that

\[
x_k(\lambda, \alpha; (U_1 - t(\lambda), U_2, \ldots, U_N), k \in \mathbb{Z}\) solves \(\Gamma_k(x; U) = 0, k \in \mathbb{Z}\).
\]

Notice that, when it is all worked out, the \(x_k\) are formal power series in \(\lambda\) (except \(x_n\) which has a simple pole in \(\lambda\)), and their coefficients are rational functions of the initial conditions \(\alpha_{n-2N}, \ldots, \alpha_{n-2}\) and of the parameters \(U_1, \ldots, U_n\). Writing

\[
x_k(\lambda, \alpha; (U_1 - t(\lambda), U_2, \ldots, U_N)) = \sum_{i=0}^\infty x_k^{(i)}(\alpha; U)\lambda^i, \quad k \in \mathbb{Z} \setminus \{n\}
\]

\[
x_n(\lambda, \alpha; (U_1 - t(\lambda), U_2, \ldots, U_N)) = \sum_{i=1}^\infty x_n^{(i)}(\alpha; U)\lambda^i
\]

\(^{13}\)Call this the formal inverse function theorem, if you wish.
leads to our final result.

**Theorem 7.2** The recursion relations $\Gamma_k(x; U) = 0$, $k \in \mathbb{Z}$ admit for any $n \in \mathbb{Z}$ two\(^{14}\) formal Laurent solution $x = (x_k(\alpha, \lambda; U))_{k \in \mathbb{Z}}$, depending on $2N$ free parameters $\alpha = (\alpha_{n-2N}, \ldots, \alpha_{n-2})$ and $\lambda$ with $x_n$ having a (simple) pole for $\lambda \to 0$, and no other singularities. Explicitly, these series with coefficients rational in $\alpha$ are given by

$$x_k(\lambda, \alpha; U) = \sum_{i=0}^{\infty} x_k^{(i)}(\alpha; U) \lambda^i,$$

$$x_k(\lambda, \alpha; U) = \alpha_k,$$

$$x_{n-1}(\lambda, \alpha; U) = \varepsilon + \lambda,$$

$$x_n(\lambda, \alpha; U) = \frac{1}{\lambda} \sum_{i=0}^{\infty} x_n^{(i)}(\alpha; U) \lambda^i,$$

$$x_{n+1}(\lambda, \alpha; U) = -\varepsilon + \sum_{i=1}^{\infty} x_{n+1}^{(i)}(\alpha; U) \lambda^i,$$

$$x_k(\lambda, \alpha; U) = \sum_{i=0}^{\infty} x_k^{(i)}(\alpha; U) \lambda^i,$$  

$n + 1 < k$.

The corresponding theorem for the recursion relations $\Delta_k = 0$, which was formulated in the introduction (Theorem 1.1) follows in the same way, using the formal Laurent solutions $z(t)$ that solve the recursion relations.

**8 Appendix**

In this appendix we obtain the leading terms of the polynomials $\Gamma_k$ and $\hat{\Gamma}_k$, which are needed in sections 5 and 6. The notations are as in the body of the paper, namely $P_1$ and $P_2$ are polynomials of degree $N$ (see (19)), the matrices $L_1$ and $L_2$ are defined by (14) and the polynomials $\Gamma_k$ and $\hat{\Gamma}_k$ are defined by (20). Since $\Gamma_k$ is given by

$$\Gamma_k(x, y; a) := \frac{1}{y_k} \left( -\left( L_1 P_1(L_1) \right)_{k+1, k+1} - \left( L_2 P_2(L_2) \right)_{k, k} \right) + k x_k,$$  

we need, by duality, only to determine the leading terms of $(L_1^s)_{k,k}$ and of $(L_1^s)_{k+1, k}$, for $s, k \in \mathbb{Z}$, with $s \geq 2$, which will be done in the following lemma. Notice that the leading terms of $\hat{\Gamma}_k$ will also follow from it, by duality.

**Lemma 8.1** For $k \in \mathbb{Z}$ and $s \in \mathbb{N}$, with $s \geq 2$, the diagonal and first subdiagonal entries of the Toeplitz matrices $L_1$ and $L_2$, defined in (14), are polynomials in the following variables,

$$(L_1^s)_{k,k} \in \mathbb{R}[x_{k-s+1}, \ldots, x_{k+s-1}, y_{k-s}, \ldots, y_{k-s-2}],$$

$$(L_1^s)_{k+1,k} \in \mathbb{R}[x_{k-s+1}, \ldots, x_{k+s}, y_{k-s}, \ldots, y_{k+s-1}].$$

More precisely\(^{15}\),

$$(L_1^s)_{k,k} = -x_{k+s-1} y_{k-1} \prod_{i=1}^{s-1} (1 - x_{k+i-1} y_{k+i-1}) + x_{k+s-2}^2 y_{k+s-3} y_{k-1} \prod_{i=1}^{s-2} (1 - x_{k+i-1} y_{k+i-1}).$$

---

\(^{14}\)parametrized by $\epsilon = \pm 1$.

\(^{15}\)We give in each case the terms that will be used, no more, no less. When $s = 2$ only the first two lines survive; the term on fourth line coincides with the first term on the second line and should only be counted once.
\[ -x_{k+s-2} \left( y_{k-2}(1 - x_{k-1}y_{k-1}) - 2y_{k-1} \sum_{j=1}^{s-2} x_{k+j-1}y_{k+j-2} \right) \prod_{i=1}^{s-2} (1 - x_{k+i-1}y_{k+i-1}) \]

\[ + \mathcal{F}_1(x_{k-s+2}, \ldots, x_{k+s-3}, y_{k-s+1}, \ldots, y_{k+s-3}) \]

\[ -x_ky_{k-s} \prod_{i=1}^{s-1} (1 - x_{k-i}y_{k-i}) \]

and

\[ (L^s_1)_{k+1,k} = -x_{k+s}y_{k-1} \prod_{i=1}^{s-1} (1 - x_{k+i}y_{k+i}) - x_{k+1}y_{k-s} \prod_{i=1}^{s-1} (1 - x_{k-i}y_{k-i}) \]

\[ + \mathcal{F}_2(x_{k-s+2}, \ldots, x_{k+s-1}, y_{k-s+1}, \ldots, y_{k+s-2}) \]

where \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are polynomials in their arguments.

**Proof** The following notation is useful for obtaining formulas of this type. To the bi-infinite vector \( x \) we associate, for any matrix, with entries \( \Delta_{ij} \) where \( F \) and \( L \) are polynomials in their arguments.

\[ L^1(x) = x \]

The diagonal entries of (75) are obtained by taking \( i = s - 1 \), which yields

\[ \left( -\left( V(0)\Delta \right)^{s-1}X^{(0)}Y^{(s-1)}\Delta^{-s+1} \right)_{kk} = -\left( V(0)\ldots V^{(s-2)}X^{(s-1)}Y^{(1)} \right)_{kk} \]

\[ = -x_{k+s-2}y_{k-1} \prod_{i=1}^{s-1} (1 - x_{k+i-1}y_{k+i-1}). \]

Notice that this leading term already contains \( x_{k+s-2} \), and that it yields, through the factor \( 1 - x_{k+s-2}y_{k+s-2} \), the single term that contains \( y_{k+s-2} \), which is the highest \( y \) variable that appears in \( (L^1_1)_{kk} \).
In order to get the other terms in $L_1^s$ that lead to $x_{k+s-2}$ we need $\Delta^{s-2}$ in front of $X^{(0)}$, i.e., we need $s - 2$ copies of $V^{(0)}\Delta$ (not necessarily consecutive), on the left of $- \sum_{i \geq 0} X^{(i)} Y^{(-i-1)} \Delta^{-i}$. For the remaining factor we can have another copy of $V^{(0)}\Delta$ or of $- \sum_{i \geq 0} X^{(i)} Y^{(-i-1)} \Delta^{-i}$, inserted at an arbitrary place inside the product $-(V^{(0)}\Delta)^{s-2}\sum_{i \geq 0} X^{(i)} Y^{(-i-1)} \Delta^{-i}$. This leads to three possible types of terms. For the first one, we put another $V^{(0)}\Delta$ at the end

$$-(V^{(0)}\Delta)^{s-2}\sum_{i \geq 0} X^{(i)} Y^{(-i-1)} \Delta^{-i}(V^{(0)}\Delta),$$

and we get the $k, k$ diagonal term by taking $i = s - 1$, which gives

$$\left(-(V^{(0)}\Delta)^{s-2} X^{(0)} Y^{(-s)} \Delta^{1-s} V^{(0)}\Delta\right)_{kk} = -x_{k+s-2}y_{k-2} \prod_{i=0}^{s-2}(1 - x_{k+i-1}y_{k+i-1}).$$

For the second one we put another $- \sum_{j \geq 0} X^{(0)} Y^{(-j-1)} \Delta^{-j}$ at the end,

$$(V^{(0)}\Delta)^{s-2}\sum_{i \geq 0} X^{(i)} Y^{(-i-1)} \Delta^{-i}\sum_{j \geq 0} X^{(j)} Y^{(-j-1)} \Delta^{-j},$$

its diagonal terms are given by taking $i + j = s - 2$, i.e., from

$$(V^{(0)}\Delta)^{s-2}\sum_{j=0}^{s-2} X^{(0)} Y^{(j+s)} X^{(j-s+2)} Y^{(-s+1)} \Delta^{2-s},$$

whose $k, k$ term is given by

$$y_{k-1} \left(x_{k+s-2}^2 y_{k+s-3} + x_{k+s-2} \sum_{j=0}^{s-3} x_{k+j} y_{k+j-1}\right) \prod_{i=1}^{s-2}(1 - x_{k+i-1}y_{k+i-1}).$$

The third term is obtained by inserting the constant term $- X^{(0)} Y^{(-1)}$ of $- \sum_{j \geq 0} X^{(0)} Y^{(-j-1)} \Delta^{-j}$ at all possible places in the product $(V^{(0)}\Delta)^{s-2}$, namely from

$$\sum_{j=0}^{s-3}(V^{(0)}\Delta)^j (X^{(0)} Y^{(-1)}) (V^{(0)}\Delta)^{s-j-2}\sum_{i \geq 0} X^{(i)} Y^{(-i-1)} \Delta^{-i},$$

with $i = s - 2$, so that its $k, k$ term is given by

$$\left(y_{k-1} x_{k+s-2} \sum_{j=0}^{s-3} x_{k+j} y_{k+j-1}\right) \prod_{i=1}^{s-2}(1 - x_{k+i-1}y_{k+i-1}),$$

which, combined with the first two terms, yields the leading terms of $(L_1^s)_{kk}$. Using the first formula for $L_1$, the lowest term in $y$ of the diagonal terms of $L_1^s$ is gotten from

$$-\Delta^{s+1} X^{(s-1)} Y^{(-1)} (\Delta V^{(-1)})^{s-1} = -X^{(0)} Y^{(-s)} V^{(-s+1)} \cdots V^{(-1)},$$

whose $k, k$ entry is $-x_k y_{k-s} \prod_{i=1}^{s-1}(1 - x_{k-i}y_{k-i})$. It contains the lowest term in $x$, through the factor $1 - x_{k-s+1}y_{k-s+1}$.
One obtains similarly the entries of \((L_i^+)_{k+1,k}\) by selecting the terms in \(L_i^+\) that contain precisely \(\Delta^{-1}\). Notice in this respect that if \(M\) is a bi-infinite diagonal matrix then \((M\Delta^{-1})_{k+1,k} = M_{k+1,k+1}\). It follows that the leading term in \(x\) of \((L_i^+)_{k+1,k}\), which contains also the leading term in \(y\), is obtained from the product \((75)\), with \(i = s\), yielding

\[- \left( Y^{(0)} \ldots Y^{(s-2)} X^{(s-1)} Y^{(s-2)} \right)_{k+1,k} = -x_{k+s} y_{k-1} \prod_{i=1}^{s-1} (1 - x_{k+i} y_{k+i}) \]

The lowest term in \(y\), which contains the lowest term in \(x\), is obtained in the same way. The above lemma and \((74)\) lead by direct substitution to the following proposition.

**Proposition 8.2** For \(k \in \mathbb{Z}\), the polynomials \(\Gamma_k\) and \(\hat{\Gamma}_k\) depend on the following variables \(x_i\) and \(y_i\):

\[
\Gamma_k(x, y; u) \in \mathbb{R}[x_{k-N}, \ldots, x_{k+N}, y_{k-N+1}, \ldots, y_{k+N}], \\
\hat{\Gamma}_k(x, y; u) \in \mathbb{R}[x_{k-N+1}, \ldots, x_{k+N-1}, y_{k-N}, \ldots, y_{k+N}].
\]

More precisely\(^{16}\),

\[
\Gamma_k(x, y; u) = u_N x_{k+N} \prod_{i=0}^{N-1} (1 - x_{k+i} y_{k+i}) - u_N x_{k+N-1} y_{k+N-2} \prod_{i=0}^{N-2} (1 - x_{k+i} y_{k+i}) \\
-u_N x_{k+N-1} \left( x_k y_{k-1} + 2 \sum_{j=1}^{N-2} x_{k+j} y_{k+j-1} \right) \prod_{i=0}^{N-2} (1 - x_{k+i} y_{k+i}) \\
+ (u_{N-1} x_{k+N-1} - u_N y_{k+N-1} x_{k-1}) \prod_{i=0}^{N-2} (1 - x_{k+i} y_{k+i}) \\
+ (1 - x_k y_k) F(x_{k-N+1}, \ldots, x_{k+N-2}, y_{k-N+2}, \ldots, y_{k+N-2}) + kx_k \\
- (u_N x_k x_{k+1} y_{k-N+1} - u_{N-1} x_k y_{k-N} (1 - x_{k-N+1} y_{k-N+1})) \prod_{i=0}^{N-2} (1 - x_{k+i} y_{k+i}),
\]

where \(F\) is a polynomial in its arguments, with a similar statement for \(\hat{\Gamma}_k\) gotten by duality. In the self-dual case, \(\Gamma_k\) takes the simpler form

\[
\Gamma_k(x; u) = u_N x_{k+N} \prod_{i=0}^{N-1} (1 - x_{k+i} y_{k+i}) + u_{N-1} x_{k+N-1} \prod_{i=0}^{N-2} (1 - x_{k+i} y_{k+i}) \\
-u_N x_{k+N-1} \left( x_k y_{k-1} + 2 \sum_{j=0}^{N-2} x_{k+j} y_{k+j-1} \right) \prod_{i=0}^{N-2} (1 - x_{k+i} y_{k+i}) \\
+ (1 - x_k y_k) F(x_{k-N+1}, \ldots, x_{k+N-2}) + kx_k \\
- u_N (x_k x_{k+1} y_{k-N+1} - x_k y_{k-N} (1 - x_{k-N+1} y_{k-N+1})) \prod_{i=0}^{N-2} (1 - x_{k+i} y_{k+i}).
\]

\(^{16}\)As in the case of lemma 8.1, when \(N = 2\) then the term \(-u_2 x_k x_{k+1} y_{k-1} (1 - x_k y_k)\), which appears twice, should only be taken into account once.
References


