Singularity confinement for a class of m-th order difference equations of combinatorics

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Abstract

In a recent publication, it was shown that a large class of integrals over the unitary group $U(n)$ satisfy non-linear, non-autonomous difference equations over n, involving a finite number of steps; special cases are generating functions appearing in questions of longest increasing subsequences in random permutations and words. The main result of the paper states that these difference equations have the *discrete Painlevé property*; roughly speaking, this means that, after a finite number of steps, the solution to these difference equations may develop a pole (Laurent solution), depending on the maximal number of free parameters, and immediately after be finite again ("*singularity confinement*"). The technique used in the proof is based on an intimate relationship between the difference equations (discrete time) and the Toeplitz lattice (continuous time differential equations); the point is that the "Painlevé property" for the discrete relations is inherited from the "Painlevé property" of the (continuous) Toeplitz lattice.

Contents

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1 Introduction

In a recent publication (Adler & van Moerbeke 2003), we have shown that a large class of integrals over the unitary group $\mathbf{U}(n)$ satisfy non-linear, non-autonomous difference equations over n, involving a finite number of steps; these $\mathbf{U}(n)$-integrals are motivated by generating functions appearing in questions of longest increasing subsequences in random permutations and words (see Adler & van Moerbeke 2001, 2003, 2004; Baik & Rains 2001, Borodin 2003; Rains 1998; Tracy & Widom 1999, 2001). The main result of this paper, announced in Adler & van Moerbeke 2003, states that these difference equations, which are also recursion relations, have the *discrete Painlevé property*; roughly speaking, this means that the solution to these difference equations may develop a pole (formal Laurent solution) after a finite number of steps and immediately after be finite again, a fact which had been observed by Borodin 2003 in the very special case of unitary matrix integrals related to longest increasing sequences of random permutations. Moreover, these formal Laurent solutions depend on the *maximal* number of free parameters, which equals ((order of difference equation) -1) $\times$ (dim of phase space), with the poles disappearing after a finite number of steps ("*singularity confinement*"), due to the non-linear and non-autonomous character of the equations.

The property of singularity confinement was introduced in Grammaticos, Ramani & Papageorgiou 1991 (see also Suris 1989), and further studied in Grammaticos, Nijhoff & Ramani 1999, as a method to find discrete Painlevé systems. They were motivated by the famous Painlevé property for continuous systems (see Ince 1944) that movable (initial condition dependent) singularities be single-valued. They were further motivated to get a classification of discrete Painlevé equations in the style of the situation for the continuous case, which they and others have embarked on and had some success. For instance, Clarkson & Webster 2000 used singularity confinement to get the so-called d-PIII equation, whose particular solutions in the limit go to Painlevé III. It should be pointed out that singularity confinement can fail to produce integrability, as was shown by Hietarinta & Viallet 1998, and further tests for integrability have been proposed, such as using "algebraic entropy" by Bellon & Viallet 1999 or using Nevanlinna theory (Ablowitz, Halburd & Herbst 2000 or Ramani et al. 2003). Thus in discrete systems the situation is more complicated than in the continuous situation, which should come as no surprise.

Nonetheless, singularity confinement is still a stiff requirement for a discrete system to pass and quite often (but not all the time) indicates integrability. In discovering that a large class, related to combinatorics, of integrals over the unitary group satisfy discrete recursion relations, it is natural to ask: "What might be the nature of these recursion relations?". Moreover, since from the derivation of these relations, they were clearly related to an integrable system called the Toeplitz lattice, it was natural to wonder if these discrete relations were integrable or at least have some "integrable-like property", especially since two of the relations coming from combinatorics actually possessed invariants, one case being that of McMillan and the other being a generalization of the McMillan case. This paper answers the latter question in the affirmative. Indeed, this huge class of recursion relations coming from unitary integrals and combinatorics possesses the "integrable-like Painlevé property" called singularity confinement as this paper shall demonstrate. It would be worthwhile to compute the algebraic entropy of these examples, as it has been hoped that also requiring that the algebraic entropy is zero would suffice for integrability.

The technique used in the proof is new and is based on an intimate relation between the difference equations (discrete time) and the Toeplitz lattice (continuous time differential equations), introduced in Adler & van Moerbeke 2001; the point is that the the "Painlevé property" for the discrete relations are inherited from the "Painlevé property" of the (continuous) Toeplitz lattice. Before making a more precise statement and describing the technique, recall the basic facts about the Toeplitz lattice and the recursion relations (Adler & van Moerbeke 2001, 2003).

For $k \in \mathbf{N}$ and $\epsilon \in \{-1, 0, 1\}$, consider the matrix integrals

$$\tau_k^\epsilon(\mathbf{t}, \mathbf{s}) = \int_{\mathbf{U}(k)} (\det M)^{\epsilon+\gamma} e^{\sum_{j=1}^\infty \operatorname{Trace}(t_j M^j - s_j M^{-j})}\, dM \tag{1}$$

where dM is Haar measure on $\mathbf{U}(k)$, $\mathbf{t} = (t_1, t_2, \ldots)$ and $\mathbf{s} = (s_1, s_2, \ldots)$. Special choices of t_j and s_j lead to generating functions in combinatorics (see Adler & van Moerbeke 2003). Set $\tau := \tau^0$ and $\tau^\pm := \tau^{\pm 1}$. In Adler & van Moerbeke 2003 it was shown that the ratios

$$x_k(\mathbf{t}, \mathbf{s}) := (-1)^k \frac{\tau_k^+(\mathbf{t}, \mathbf{s})}{\tau_k(\mathbf{t}, \mathbf{s})}, \qquad y_k(\mathbf{t}, \mathbf{s}) := (-1)^k \frac{\tau_k^-(\mathbf{t}, \mathbf{s})}{\tau_k(\mathbf{t}, \mathbf{s})},$$

with $k \in \mathbf{N}$, satisfy the *Toeplitz lattice*, an integrable Hamiltonian system,

$$\begin{array}{rclcrcl} \dfrac{dx_k}{dt_i} &=& (1 - x_k y_k)\dfrac{\partial H_i^{(1)}}{\partial y_k}, & \qquad & \dfrac{dy_k}{dt_i} &=& -(1 - x_k y_k)\dfrac{\partial H_i^{(1)}}{\partial x_k}, \\[2ex] \dfrac{dx_k}{ds_i} &=& (1 - x_k y_k)\dfrac{\partial H_i^{(2)}}{\partial y_k}, & \qquad & \dfrac{dy_k}{ds_i} &=& -(1 - x_k y_k)\dfrac{\partial H_i^{(2)}}{\partial x_k}, \end{array} \tag{2}$$

where $i = 1, 2, 3, \ldots$ Moreover, τ_n is a polynomial expression in the variables x_k and y_k and τ_1:

$$\tau_n = \tau_1^n \prod_{k=1}^{n-1} (1 - x_k y_k)^{n-k}.$$

The Hamiltonians $H_i^{(l)}$ appearing in (2) are given by

$$H_i^{(l)} = -\frac{1}{i} \operatorname{Trace} L_l^i, \qquad i = 1, 2, 3, \ldots,\ l = 1, 2,$$

where the matrices L_1 and L_2 are defined by

$$L_1 := \begin{pmatrix} -x_1 y_0 & 1 - x_1 y_1 & 0 & 0 & \\ -x_2 y_0 & -x_2 y_1 & 1 - x_2 y_2 & 0 & \\ -x_3 y_0 & -x_3 y_1 & -x_3 y_2 & 1 - x_3 y_3 & \\ -x_4 y_0 & -x_4 y_1 & -x_4 y_2 & -x_4 y_3 & \\ & & & & \ddots \end{pmatrix} \tag{3}$$

and

$$L_2 := \begin{pmatrix} -x_0 y_1 & -x_0 y_2 & -x_0 y_3 & -x_0 y_4 & \\ 1 - x_1 y_1 & -x_1 y_2 & -x_1 y_3 & -x_1 y_4 & \\ 0 & 1 - x_2 y_2 & -x_2 y_3 & -x_2 y_4 & \\ 0 & 0 & 1 - x_3 y_3 & -x_3 y_4 & \\ & & & & \ddots \end{pmatrix}. \tag{4}$$

The system admits a reduction, interesting in its own right, obtained by putting $x_k = y_k$ for all k. We refer to it as the *self-dual Toeplitz lattice*.

In Adler & van Moerbeke 2001, it was shown that the matrix integrals (1) satisfy a $\mathfrak{sl}(2, \mathbf{R})$-algebra of Virasoro constraints, which combined with the Toeplitz lattice equations, lead to difference equations for x_k and y_k given in Adler & van Moerbeke 2003, a subset of the cases leading to recursion relations, which we now describe. Given arbitrary polynomials

$$P_1(\lambda) := \sum_{i=1}^{N} \frac{u_i \lambda^i}{i}, \qquad \text{and} \qquad P_2(\lambda) := \sum_{i=1}^{N} \frac{u_{-i} \lambda^i}{i},$$

the variables

$$x_k(\mathbf{u}) := (-1)^k \frac{\tau_k^+(\mathbf{u})}{\tau_k(\mathbf{u})}, \qquad y_k(\mathbf{u}) := (-1)^k \frac{\tau_k^-(\mathbf{u})}{\tau_k(\mathbf{u})},$$

with

$$\tau_k^\epsilon(\mathbf{u}) = \int_{\mathbf{U}(k)} (\det M)^{\epsilon+\gamma} e^{\mathrm{Trace}(P_1(M) - P_2(M^{-1}))}\, dM,$$

and $\mathbf{u} = (u_1, \ldots, u_N, u_{-1}, \ldots, u_{-N})$, satisfy $2N+1$ step difference equations $\Gamma_k(x,y) = 0 = \tilde{\Gamma}_k(x,y) = 0$, where $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, and where the polynomials $\Gamma_k(x,y)$ and $\tilde{\Gamma}_k(x,y)$ are defined in terms of the matrices L_1 and L_2 defined above (denote the derivative of the polynomial P_i by P_i')

$$\begin{aligned}
\Gamma_k(x,y) &:= \frac{1 - x_k y_k}{y_k} \begin{pmatrix} -(L_1 P_1'(L_1))_{k+1,k+1} - (L_2 P_2'(L_2))_{k,k} \\ +(P_1'(L_1))_{k+1,k} + (P_2'(L_2))_{k,k+1} \end{pmatrix} + k x_k = 0, \\
\tilde{\Gamma}_k(x,y) &:= \frac{1 - x_k y_k}{x_k} \begin{pmatrix} -(L_1 P_1'(L_1))_{k,k} - (L_2 P_2'(L_2))_{k+1,k+1} \\ +(P_1'(L_1))_{k+1,k} + (P_2'(L_2))_{k,k+1} \end{pmatrix} + k y_k = 0.
\end{aligned} \tag{5}$$

Looking closely, one observes that these difference equations $\Gamma_k = 0$ and $\tilde{\Gamma}_k = 0$ are indeed linear in x_{k+N} and y_{k+N}, and can thus be solved in terms of x_{k-N}, $y_{k-N}, \ldots,$ x_{k+N-1}, y_{k+N-1}. See the appendix for a proof of this fact.

This paper deals with the difference equations (5) for their own sake, without further reference to the special solution $x_k(\mathbf{t}, \mathbf{s})$ and $y_k(\mathbf{t}, \mathbf{s})$, given by the unitary matrix integrals above. Moreover, we will consider the bi-infinite Toeplitz lattice, which is defined as in (2), but with $k \in \mathbf{Z}$. The recursion relations are then also considered for $k \in \mathbf{Z}$, with the semi-infinite case obtained by specialization. The bi-infinite Toeplitz lattice will be introduced in section 2, where we also discuss the self-dual Toeplitz lattice and the recursion relations.

It came as a surprise that the generic solutions of these (very general) equations (5) have the *singularity confinement* property, a fact which had been observed by Borodin 2003 in the very special case of unitary matrix integrals related to longest increasing sequences of random permutations. In this case the recursion relation is only a 3-step relation. We shall see that although the relations inherit the confinement property from their integrable ancestor, the Toeplitz lattice, they need not, as there are many places where they can easily loose this property. The main result of the paper is to show that this large zoo of examples (1.5), indeed possess the singularity property, namely:

Theorem 1.1 (singularity confinement: general case) *For any $n \in \mathbf{Z}$, the difference equations $\Gamma_k(x,y) = \tilde{\Gamma}_k(x,y) = 0$, $(k \in \mathbf{Z})$ admit a formal Laurent solution $x = (x_k(\lambda))_{k\in\mathbf{Z}}$ and $y = (y_k(\lambda))_{k\in\mathbf{Z}}$ in a parameter λ, having a (simple) pole at $k = n$ and $\lambda = 0$, and no other singularities. These solutions depend on $4N$ non-zero free parameters*

$$\alpha_{n-2N}, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-2N}, \ldots, \beta_{n-2} \textit{ and } \lambda.$$

Setting $z_n := (x_n, y_n)$ and $\gamma_i := (\alpha_i, \beta_i)$, and $\gamma := (\gamma_{n-2N}, \ldots, \gamma_{n-2},\ \alpha_{n-1})$, the explicit series with coefficients rational in γ read as follows:

$$
\begin{array}{rcll}
z_k(\lambda) & = & \sum_{i=0}^{\infty} z_k^{(i)}(\gamma)\lambda^i, & k < n-2N, \\
z_k(\lambda) & = & \gamma_k, & n-2N \leq k \leq n-2, \\
x_{n-1}(\lambda) & = & \alpha_{n-1}, & \\
y_{n-1}(\lambda) & = & 1/\alpha_{n-1} + \lambda, & \\
z_n(\lambda) & = & \frac{1}{\lambda}\sum_{i=0}^{\infty} z_n^{(i)}(\gamma)\lambda^i, & \\
z_k(\lambda, \gamma) & = & \sum_{i=0}^{\infty} z_k^{(i)}(\gamma)\lambda^i, & n < k.
\end{array}
$$

For the self-dual case, the statement reads as follows:

Theorem 1.2 (singularity confinement: self-dual case) *For any $n \in \mathbf{Z}$, the difference equations $\Gamma_k(x) = 0$, $(k \in \mathbf{Z})$ admit two[4] formal Laurent solution $x = (x_k(\lambda))_{k \in \mathbf{Z}}$ in a parameter λ, having a (simple) pole at $k = n$ only and $\lambda = 0$. These solutions depend on $2N$ non-zero free parameters*

$$\alpha = (\alpha_{n-2N}, \ldots, \alpha_{n-2}) \text{ and } \lambda$$

Explicitly, these series with coefficients rational in α are given by

$$
\begin{array}{rcll}
x_k(\lambda) & = & \sum_{i=0}^{\infty} x_k^{(i)}(\alpha)\lambda^i, & k < n-2N, \\
x_k(\lambda) & = & \alpha_k, & n-2N \leq k \leq n-2, \\
x_{n-1}(\lambda) & = & \varepsilon + \lambda, & \\
x_n(\lambda) & = & \frac{1}{\lambda}\sum_{i=0}^{\infty} x_n^{(i)}(\alpha)\lambda^i, & \\
x_{n+1}(\lambda) & = & -\varepsilon + \sum_{i=1}^{\infty} x_{n+1}^{(i)}(\alpha)\lambda^i, & \\
x_k(\lambda) & = & \sum_{i=0}^{\infty} x_k^{(i)}(\alpha)\lambda^i, & n+1 < k.
\end{array}
$$

The proof of theorems 1.1 and 1.2 is by no means direct, but proceeds via the Painlevé analysis for the Toeplitz lattice. As a starting point, the zero locus $\mathcal{M}$, of all polynomials Γ_k and $\tilde{\Gamma}_k$, form an invariant manifold for the vector field of the Toeplitz lattice with Hamiltonian $H_1^{(1)} - H_2^{(2)}$, by viewing the coefficients of $P_1(\lambda)$ and $P_2(\lambda)$ as constants, except for $u_{\pm 1}$, which moves linearly in time. Explicitly, this vector field is given by

$$
\begin{aligned}
\frac{dx_k}{dt} &= (1 - x_k y_k)(x_{k+1} - x_{k-1}), \\
\frac{dy_k}{dt} &= (1 - x_k y_k)(y_{k+1} - y_{k-1}),
\end{aligned}
\qquad k \in \mathbf{Z}. \tag{6}
$$

In the self-dual case, this vector field reduces to

$$\frac{dx_k}{dt} = (1 - x_k^2)(x_{k+1} - x_{k-1}), \qquad k \in \mathbf{Z}. \tag{7}$$

[4]They are parametrized by $\epsilon = \pm 1$.

The first idea is then to restrict the principal balances (formal Laurent solutions depending on the maximal number (= dim phase space -1) of free parameters, besides time) of (6) to these invariant manifolds. We fix n and look for a formal Laurent solution to the Toeplitz lattice that has a (simple) pole for x_n and y_n only, and we find a unique such family, as given by the following proposition:

Proposition 1.3 *For arbitrary but fixed n, the first Toeplitz lattice vector field (6) admits the following formal Laurent solutions,*

$$\begin{aligned} x_n(t) &= \frac{1}{(a_{n-1}-a_{n+1})t}\Big(a_{n-1}a_{n+1}(1+at)+O(t^2)\Big) \\ y_n(t) &= \frac{1}{(a_{n-1}-a_{n+1})t}\left(-1+\big(a+\frac{a_{n+1}a_+ - a_{n-1}a_-}{a_{n+1}-a_{n-1}}\big)t+O(t^2)\right) \end{aligned}$$

$$\begin{aligned} x_{n\pm1}(t) &= a_{n\pm1}+a_{n\pm1}a_\pm t+O(t^2) \\ y_{n\pm1}(t) &= 1/a_{n\pm1}-a_\pm/a_{n\mp1}t+O(t^2) \end{aligned}$$

whereas for all remaining k such that $|k-n|\geq 2$,

$$x_k(t) = a_k+(1-a_kb_k)(a_{k+1}-a_{k-1})t+O(t^2) \tag{8}$$
$$y_k(t) = b_k+(1-a_kb_k)(b_{k+1}-b_{k-1})t+O(t^2) \tag{9}$$

where a, $a_\pm$, $a_{n\pm1}$ and all a_i, b_i, with $i\in\mathbf{Z}\setminus\{n-1,n,n+1\}$ and with $b_{n\pm1}=1/a_{n\pm1}$, are arbitrary free parameters, and with $(a_{n-1}-a_{n+1})a_{n-1}a_{n+1}\neq 0$. In the self-dual case it admits the following two formal Laurent solutions, parametrized by $\varepsilon=\pm1$,

$$\begin{aligned} x_n(t) &= -\frac{\varepsilon}{2t}\left(1+(a_+-a_-)t+O(t^2)\right), \\ x_{n\pm1}(t) &= \varepsilon\left(\mp1+4a_\pm t+O(t^2)\right), \\ x_k(t) &= \varepsilon\left(a_k+(1-a_k^2)(a_{k+1}-a_{k-1})t+O(t^2)\right), \qquad |k-n|\geq 2, \end{aligned} \tag{10}$$

where a_+,a_- and all a_i, with $i\in\mathbf{Z}\setminus\{n-1,n,n+1\}$ are arbitrary free parameters and $a_{n-1}=-a_{n+1}=1$.

Together with time t these parameters are in bijection with the phase space variables; we can put for the general Toeplitz lattice for example $z_k\leftrightarrow(a_k,b_k)$ for $|k-n|\geq 1$ and $x_{n\pm1}\leftrightarrow a_{n\pm1}$ and $y_{n\pm1},x_n,y_n\leftrightarrow a_\pm,a,t$. Thus, this formal Laurent solution is the natural candidate to work with; see section 3.

It is however, a priori, not clear that these formal Laurent solutions can be restricted to the invariant manifold $\mathcal{M}$. Indeed, upon introducing a proper time-dependence for u already mentioned, one has that $\Gamma_k(t):=\Gamma_k(x(t),y(t);u(t))$ and $\tilde\Gamma_k(t):=\tilde\Gamma_k(x(t),y(t);u(t))$ satisfy a system of differential equations, as given in the following proposition:

Proposition 1.4 *Upon setting $\frac{du_{\pm i}}{dt}=\delta_{1i}$, the recursion relations satisfy the following differential equations*

$$\begin{aligned} \frac{d\Gamma_k}{dt} &= (1-x_ky_k)(\Gamma_{k+1}-\Gamma_{k-1})+(x_{k+1}-x_{k-1})(x_k\tilde\Gamma_k-y_k\Gamma_k), \\ \frac{d\tilde\Gamma_k}{dt} &= (1-x_ky_k)(\tilde\Gamma_{k+1}-\tilde\Gamma_{k-1})-(y_{k+1}-y_{k-1})(x_k\tilde\Gamma_k-y_k\Gamma_k), \end{aligned} \tag{11}$$

which specialize in the self-dual case (7) to

$$\frac{d\Gamma_k}{dt} = (1-x_k^2)(\Gamma_{k+1}-\Gamma_{k-1}). \tag{12}$$

In addition to propositions 1.3 and 1.4, many other arguments are needed to fine-tune the free parameters, when going from the Laurent solutions of the Toeplitz lattice to the existence of formal Laurent solutions to the difference equations, depending on the announced number of free parameters. See section 6. The proof of these facts will be spread over two sections, as the arguments get rather involved; see section 5 for the self-dual case and section 6 for the case of the general Toeplitz lattice.

This ultimately leads to the proof of the main theorems 1.1 and 1.2.

Examples

I. Denote by P the uniform probability on the group S_k of permutations π_k and by $L(\pi_k)$ the length of the largest (strictly) increasing subsequence of π_k. According to an identity, due to Gessel 1990,

$$\int_{\mathbf{U}(n)} e^{t\,\mathrm{Trace}(M+M^{-1})}\,dM = \sum_{k=0}^{\infty} \frac{t^{2k}}{k!}\,P(L(\pi_k) \le n).$$

The quantities defined for $n > 0$ by

$$x_n(t) = (-1)^n \frac{\int_{\mathbf{U}(n)} \det M\, e^{t(M+M^{-1})}\,dM}{\int_{\mathbf{U}(n)} e^{t(M+M^{-1})}\,dM}$$

satisfy the following 3-step relation, found by Borodin 2003,

$$nx_n + t(1-x_n^2)(x_{n+1}+x_{n-1}) = 0,$$

possessing the McMillan invariant (McMillan 1971)

$$\Phi_n(x_{n+1}, x_n) = \Phi_n(x_n, x_{n-1})$$

with

$$\Phi_n(y,z) = (1-y^2)(1-z^2) - \frac{n}{t}\,yz.$$

II. According to Rains 1998, respectively Tracy & Widom 1999,

$$\int_{\mathbf{U}(n)} e^{s\,\mathrm{Trace}(M^2+M^{-2})}\,dM = \sum_{k=0}^{\infty} \frac{(\sqrt{2}s)^{2k}}{k!} P(L(\pi^0_{2k}) \le n)$$

and

$$\frac{1}{4}\frac{\partial^2}{\partial t^2}\left(\int_{\mathbf{U}(n)} e^{\mathrm{Trace}(t(M+M^{-1})+s(M^2+M^{-2}))}\,dM + \int_{\mathbf{U}(n)} e^{\mathrm{Trace}(t(M+M^{-1})-s(M^2+M^{-2}))}\,dM\right)_{|t=0}$$

$$= \sum_{k=0}^{\infty} \frac{(\sqrt{2}s)^{2k}}{k!} P(L(\pi^0_{2k+1}) \le n),$$

where π^0_{2k} and π^0_{2k+1} are *odd permutations* of respectively order $2k$ and $2k+1$ acting on $(-k, \dots, -1, 1, \dots, k)$ and $(-k, \dots, -1, 0, 1, \dots, k)$. Then

$$x_n(s,t) = (-1)^n \frac{\int_{\mathbf{U}(n)} \det M\, e^{\mathrm{Trace}(t(M+M^{-1})+s(M^2+M^{-2}))}\,dM}{\int_{\mathbf{U}(n)} e^{\mathrm{Trace}(t(M+M^{-1})+s(M^2+M^{-2}))}\,dM}$$

satisfies a 5-step recursion relation ($v_n := 1 - x_n^2$)

$$nx_n + tv_n(x_{n-1} + x_{n+1}) + 2sv_n(x_{n+2}v_{n+1} + x_{n-2}v_{n-1} - x_n(x_{n+1} + x_{n-1})^2) = 0,$$

possessing the invariant

$$\Phi_n(x_{n-1}, x_n, x_{n+1}, x_{n+2}) = \Phi_n(x_n, x_{n+1}, x_{n+2}, x_{n+3})$$

with

$$\Phi_n(x, y, z, u) = nyz - (1 - y^2)(1 - z^2)(t + 2s(x(u - y) - z(u + y))).$$

2 An invariant manifold $\mathcal{M}$ for the first Toeplitz flow

In this section we introduce the bi-infinite Toeplitz lattice, in analogy with the semi-infinite Toeplitz lattice, introduced in Adler & van Moerbeke 2001. We also recall the basic formulas related to the invariant manifold $\mathcal{M}$ that we will introduce below (see Adler & van Moerbeke 2003).

The (bi-infinite) *Toeplitz lattice* consists of two infinite strings of vector fields on the (real or complex) linear space of bi-infinite sequences $(x_i, y_i)_{i\in\mathbf{Z}}$. The particular vector field that we will be interested in (the "first" Toeplitz vector field) is given by

$$\begin{aligned} \frac{dx_k}{dt} &= (1 - x_k y_k)(x_{k+1} - x_{k-1}), \\ \frac{dy_k}{dt} &= (1 - x_k y_k)(y_{k+1} - y_{k-1}), \end{aligned} \qquad k \in \mathbf{Z}. \tag{13}$$

The semi-infinite Toeplitz lattice is obtained from it by setting $(x_k, y_k) = (0, 0)$ for $k < 0$ and $(x_0, y_0) = (1, 1)$. The invariant polynomials of the matrices L_1 and L_2, defined by

$$\begin{aligned} (L_1)_{ij} &:= \begin{cases} -x_i y_{j-1} + \delta_{i+1,j} & \text{if} \quad j - i \le 1, \\ 0 & \text{if} \quad j - i > 1, \end{cases} \\ (L_2)_{ij} &:= \begin{cases} -y_j x_{i-1} + \delta_{j+1,i} & \text{if} \quad j - i \ge 1, \\ 0 & \text{if} \quad j - i < 1, \end{cases} \end{aligned} \tag{14}$$

provide two infinite strings of constants of motion $H_i^{(1)}$ and $H_i^{(2)}$ ($i \in \mathbf{Z}$) of (13), defined by

$$H_i^{(l)} := -\frac{1}{i}\operatorname{Trace} L_l^i, \qquad i = 1, 2, 3, \ldots, \qquad l = 1, 2. \tag{15}$$

The first Toeplitz vector field (13) is the Hamiltonian vector field that corresponds to

$$H_1 := H_1^{(1)} - H_1^{(2)} = \operatorname{Trace}(L_2 - L_1) = \sum_{i\in\mathbf{Z}} (x_i y_{i-1} - x_{i-1} y_i),$$

with respect to the Poisson structure defined by

$$\{x_i, x_j\} = \{y_i, y_j\} = 0, \qquad \{x_i, y_j\} = (1 - x_i y_i)\delta_{ij},$$

and the functions $H_i^{(1)}$ and $H_i^{(2)}$ are all in involution with respect to $\{\cdot\,,\cdot\}$, as follows from a direct computation. As a corollary, all Hamiltonian vector fields $\mathcal{X}_i^{(1)} := \left\{\cdot\,, H_i^{(1)}\right\}$ and $\mathcal{X}_i^{(2)} := \left\{\cdot\,, H_i^{(2)}\right\}$ commute. If we denote $\langle A \,|\, B\rangle := \operatorname{Trace} AB$, whenever this makes sense, then for $i = 1, 2, \ldots$,

$$\mathcal{X}_i^{(1)}[x_k] = \left\{x_k, -\frac{1}{i}\operatorname{Trace} L_1^i\right\} = -(1 - x_k y_k)\left\langle L_1^{i-1} \,|\, \frac{\partial L_1}{\partial y_k}\right\rangle,$$

and similarly for $\mathcal{X}_i^{(1)}[y_k]$, which leads to the following expression for the vector field $\mathcal{X}_i^{(1)}$,

$$\mathcal{X}_i^{(1)}: \begin{cases} \dfrac{dx_k}{dt_i} &= -(1-x_ky_k)\left\langle L_1^{i-1} \mid \frac{\partial L_1}{\partial y_k}\right\rangle, \\ \dfrac{dy_k}{dt_i} &= (1-x_ky_k)\left\langle L_1^{i-1} \mid \frac{\partial L_1}{\partial x_k}\right\rangle. \end{cases} \tag{16}$$

The vector field $\mathcal{X}_i^{(2)}$, has the same form, but with L_1 replaced by L_2. This is a particular case of a phenomenon that we will refer to as duality. Namely, there is a natural automorphism σ of our phase space, given by $\sigma : (x_i, y_i)_{i\in\mathbf{Z}} \mapsto (y_i, x_i)_{i\in\mathbf{Z}}$. It preserves the first Toeplitz vector field (13), it permutes the Hamiltonians $H_i^{(1)} \leftrightarrow H_i^{(2)}$, it permutes the Lax operators as follows: $L_1 \leftrightarrow L_2^\top$ and it reverses the sign of the Poisson structure. The first Toeplitz vector field (13) can be restricted to the fixed point locus $(x_i = y_i)_{i\in\mathbf{Z}}$ of σ, which leads to the *self-dual* (bi-infinite) Toeplitz lattice,

$$\frac{dx_k}{dt} = (1-x_k^2)(x_{k+1} - x_{k-1}), \qquad k \in \mathbf{Z}. \tag{17}$$

All constructions in this paper will be done for this self-dual lattice first, and then for the general Toeplitz lattice. This is not only for pedagogical reasons: even if the ideas that lead to the proofs are similar in both cases, the self-dual lattice can for our purposes not be treated as a particular case of the general Toeplitz lattice, as we will see.

For $i = 1$, the equations (16) for $\mathcal{X}_i^{(1)}$ and for $\mathcal{X}_i^{(2)}$ specialize to

$$\begin{aligned} \mathcal{X}_1^{(1,2)}[x_k] &= (1-x_ky_k)x_{k\pm1}, \\ \mathcal{X}_1^{(1,2)}[y_k] &= -(1-x_ky_k)y_{k\mp1}. \end{aligned} \tag{18}$$

Fixing $2N$ constants $u := (u_{-N}, \ldots, u_{-1}, u_1, \ldots, u_N)$, with $u_N \neq 0$ and $u_{-N} \neq 0$, we consider the polynomials

$$P_1(\lambda) := \sum_{i=1}^N \frac{u_i\lambda^i}{i}, \qquad \text{and} \qquad P_2(\lambda) := \sum_{i=1}^N \frac{u_{-i}\lambda^i}{i}, \tag{19}$$

whose derivatives we simply denote by P_1' and P_2'. They lead to two strings of polynomials[5] Γ_k and $\tilde\Gamma_k$ in x_i, y_i ($i \in \mathbf{Z}$):

$$\begin{aligned} \Gamma_k(x,y;u) &:= \frac{1-x_ky_k}{y_k}\begin{pmatrix} -(L_1P_1'(L_1))_{k+1,k+1} - (L_2P_2'(L_2))_{k,k} \\ +(P_1'(L_1))_{k+1,k} + (P_2'(L_2))_{k,k+1} \end{pmatrix} + kx_k, \\ \tilde\Gamma_k(x,y;u) &:= \frac{1-x_ky_k}{x_k}\begin{pmatrix} -(L_1P_1'(L_1))_{k,k} - (L_2P_2'(L_2))_{k+1,k+1} \\ +(P_1'(L_1))_{k+1,k} + (P_2'(L_2))_{k,k+1} \end{pmatrix} + ky_k. \end{aligned} \tag{20}$$

Notice that the only elements that appear in these polynomials are the diagonal and next-to-diagonal entries of L_1^l and L_2^l for $l = 1, \ldots, N$. For fixed u we consider the zero locus of all polynomials Γ_k and $\tilde\Gamma_k$,

$$\mathcal{M}_u := \bigcap_{k\in\mathbf{Z}} \left\{(x_i, y_i)_{i\in\mathbf{Z}} \mid \Gamma_k(x,y;u) = 0 \text{ and } \tilde\Gamma_k(x,y;u) = 0\right\}. \tag{21}$$

[5]The structure of the matrices L_1 and L_2 implies that Γ_k and $\tilde\Gamma_k$ are indeed polynomials. They are also polynomials (of degree 1) in the variables u_i, but we often do not mention this, because we think of these variables as parameters.

In terms of the variables x_i and y_i the leading terms of Γ_k and $\tilde\Gamma_k$ are given by

$$\begin{aligned}\Gamma_k(x,y;u) &= u_N x_{k+N}\prod_{i=0}^{N-1}(1-x_{k+i}y_{k+i})+\cdots+u_{-N}x_{k-N}\prod_{i=0}^{N-1}(1-x_{k-i}y_{k-i}),\\ \tilde\Gamma_k(x;y;u) &= u_{-N}y_{k+N}\prod_{i=0}^{N-1}1-x_{k+i}y_{k+i}+\cdots+u_N y_{k-N}\prod_{i=0}^{N-1}(1-x_{k-i}y_{k-i}).\end{aligned}$$

See the Appendix for a precise statement, a few more terms and a proof. We often write Δ_k as a shorthand for the vector $(\Gamma_k,\tilde\Gamma_k)^\top$ and z_k for $(x_k,y_k)^\top$.

In order to get the corresponding formulas for the self-dual case we put $\sigma(u_i):=u_{-i}$, so that σ permutes P_1 and P_2, as well as Γ_k and $\tilde\Gamma_k$, hence $P_1=P_2$ in the self-dual case, and $\Gamma_k=\tilde\Gamma_k$. Writing $L:=L_1$ and $P:=P_1$, the polynomials Γ_k and $\tilde\Gamma_k$ reduce in that case to

$$\Gamma_k(x;u):=\frac{1-x_k^2}{x_k}\Big(2(P'(L))_{k+1,k}-(LP'(L))_{k+1,k+1}-(LP'(L))_{k,k}\Big)+kx_k, \tag{22}$$

while its leading terms are now given by

$$\Gamma_k(x;u)=u_N x_{k+N}\prod_{i=0}^{N-1}(1-x_{k+i}^2)+\cdots+u_N x_{k-N}\prod_{i=0}^{N-1}(1-x_{k-i}^2). \tag{23}$$

The zero locus $\mathcal{M}_u$ now takes the simple form

$$\mathcal{M}_u:=\bigcap_{k\in\mathbf{Z}}\{(x_i)_{i\in\mathbf{Z}}\mid\Gamma_k(x;u)=0\}. \tag{24}$$

Following Adler & van Moerbeke 2003, we show that, upon introducing a proper time dependence, the polynomials Γ_k and $\tilde\Gamma_k$ satisfy a simple set of differential equations, showing that the zero locus (21) of these polynomials is a (time-dependent) invariant manifold of the first Toeplitz flow (13).

Proposition 2.1 *Let $(x(t),y(t))$ be a solution to the first Toeplitz vector field (13), to wit:*

$$\frac{d}{dt}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\left(\mathcal{X}_1^{(1)}-\mathcal{X}_1^{(2)}\right)\begin{pmatrix}x(t)\\ y(t)\end{pmatrix},$$

and let $\Gamma(t):=\Gamma(x(t),y(t);u(t))$ and $\tilde\Gamma(t):=\Gamma(x(t),y(t);u(t))$, where

$$u(t)=(u_{-N},\ldots,u_{-2},u_{-1}+t,u_1+t,u_2,\ldots,u_N). \tag{25}$$

Then $\Gamma(t)$ and $\tilde\Gamma(t)$ satisfy the following differential equations:

$$\begin{aligned}\frac{d\Gamma_k}{dt} &= (1-x_ky_k)(\Gamma_{k+1}-\Gamma_{k-1})+(x_{k+1}-x_{k-1})(x_k\tilde\Gamma_k-y_k\Gamma_k),\\ \frac{d\tilde\Gamma_k}{dt} &= (1-x_ky_k)(\tilde\Gamma_{k+1}-\tilde\Gamma_{k-1})-(y_{k+1}-y_{k-1})(x_k\tilde\Gamma_k-y_k\Gamma_k).\end{aligned} \tag{26}$$

In particular, $\mathcal{M}_{u(t)}$ is a (time-dependent) invariant manifold of the first Toeplitz flow. In the self-dual case, these differential equations specialize to

$$\frac{d\Gamma_k}{dt}=(1-x_k^2)(\Gamma_{k+1}-\Gamma_{k-1}). \tag{27}$$

Then $\mathcal{M}_{u(t)}$ is a (time-dependent) invariant manifold of the first vector field of the self-dual Toeplitz lattice, where $u(t)=(u_1+t,u_2,\ldots,u_N)$.

Proof We first show that

$$\begin{aligned} \Gamma_k(x,y;u) &= \mathcal{V}^u[x_k] + kx_k, \\ \tilde{\Gamma}_k(x,y;u) &= -\mathcal{V}^u[y_k] + ky_k, \end{aligned} \tag{28}$$

where $\mathcal{V}^u$ is the Hamiltonian vector field

$$\mathcal{V}^u := \sum_{i=1}^{N} \left(u_i \mathcal{X}_i^{(1)} + u_{-i} \mathcal{X}_i^{(2)} \right).$$

It suffices to prove that $\Gamma_k(x,y;u) = \mathcal{V}^u[x_k] + kx_k$, the other identity being obtained by duality (indeed, $\sigma(\mathcal{V}^u) = -\mathcal{V}^u$ since $\sigma(\mathcal{X}_i^{(1)}) = -\mathcal{X}_i^{(2)}$). In view of the Definition (20) of Γ_k this means that we need to prove that

$$\begin{aligned} \mathcal{X}_i^{(1)}[x_k] &= \frac{1-x_ky_k}{y_k} \left(\left(L_1^{i-1}\right)_{k+1,k} - \left(L_1^{i}\right)_{k+1,k+1} \right), \\ \mathcal{X}_i^{(2)}[x_k] &= \frac{1-x_ky_k}{y_k} \left(\left(L_2^{i-1}\right)_{k,k+1} - \left(L_2^{i}\right)_{k,k} \right). \end{aligned} \tag{29}$$

According to (16), the first equation amounts to

$$y_k \left\langle L_1^{i-1} \,|\, \frac{\partial L_1}{\partial y_k} \right\rangle = \left(L_1^{i}\right)_{k+1,k+1} - \left(L_1^{i-1}\right)_{k+1,k}, \tag{30}$$

where we recall that $\langle A \,|\, B \rangle = \operatorname{Trace} AB$. The proof of (30) follows immediately by writing $(L_1^i)_{k+1,k+1}$ as $(L_1^{i-1}L_1)_{k+1,k+1}$, and the expression (14) for the entries of L_1. For the second equation in (29) the proof is similar.

Notice that (28) implies that the time-dependent polynomials $\Gamma_k(t)$ and $\tilde{\Gamma}_k(t)$ are given by

$$\begin{aligned} \Gamma_k(t) &= \mathcal{V}^{u(t)}[x_k](t) + kx_k(t), \\ \tilde{\Gamma}_k(t) &= -\mathcal{V}^{u(t)}[y_k](t) + ky_k(t), \end{aligned}$$

where $\mathcal{V}^{u(t)}$ can, in view of (25) be written as

$$\mathcal{V}^{u(t)} = t(\mathcal{X}_1^{(1)} + \mathcal{X}_1^{(2)}) + \mathcal{V}^u.$$

Since the vector field d/dt commutes with all the Hamiltonian vector fields $\mathcal{X}_i^{(1)}$ and $\mathcal{X}_i^{(2)}$, it follows from these equations and (18) that

$$\begin{aligned} \frac{d\Gamma_k}{dt}(t) &= \mathcal{X}_1^{(1)}[x_k](t) + \mathcal{X}_1^{(2)}[x_k](t) + \mathcal{V}^{u(t)}[dx_k/dt](t) + k\,\frac{dx_k}{dt}(t) \\ &= (k+1)\mathcal{X}_1^{(1)}[x_k](t) - (k-1)\mathcal{X}_1^{(2)}[x_k](t) + \mathcal{V}^{u(t)}\left[(1-x_ky_k)(x_{k+1}-x_{k-1})\right](t) \\ &= (k+1)(1-x_k(t)y_k(t))x_{k+1}(t) - (k-1)(1-x_k(t)y_k(t))x_{k-1}(t) \\ &\quad + (1-x_k(t)y_k(t))\mathcal{V}^{u(t)}\left[x_{k+1}-x_{k-1}\right](t) - (x_{k+1}(t)-x_{k-1}(t))\mathcal{V}^{u(t)}\left[x_ky_k\right](t) \\ &= (1-x_k(t)y_k(t))(\Gamma_{k+1}(t)-\Gamma_{k-1}(t)) + (x_{k+1}(t)-x_{k-1}(t))(x_k(t)\tilde{\Gamma}_k(t) - y_k(t)\Gamma_k(t)). \end{aligned}$$

This yields the first relation in (26). The second equation is obtained by duality.

At points of $\mathcal{M}_u$ all Γ_k and $\tilde{\Gamma}_k$ vanish so the right hand sides of (26) vanish. The unique solution to (26) that corresponds to such initial data is the zero solution, $\Gamma_k(t) = \tilde{\Gamma}_k(t) = 0$. As a consequence, $\mathcal{M}_{u(t)}$ is a time-dependent invariant manifold for the first Toeplitz flow.

3 Painlevé analysis of the first Toeplitz flow

In this section we will show that the first Toeplitz flow admits many families of formal Laurent solutions, a property reminiscent of (finite-dimensional) algebraic completely integrable systems (see Adler, van Moerbeke & Vanhaecke 2004). They will be used in the subsequent sections. We will first consider the self-dual case, which is easier, and then we will consider the full Toeplitz lattice.

3.1 The self-dual Toeplitz lattice

Recall that the first vector field of the self-dual Toeplitz lattice is given by

$$\frac{dx_k}{dt} = (1-x_k^2)(x_{k+1}-x_{k-1}), \qquad k \in \mathbf{Z}. \tag{31}$$

Proposition 3.1 *For any $n \in \mathbf{Z}$, the first vector field (31) of the self-dual Toeplitz lattice admits a formal Laurent solution $x(t)$, with only $x_n(t)$ having a pole, given by*

$$\begin{aligned}
x_k(t) &= \varepsilon\Big(a_k + (1-a_k^2)(a_{k+1}-a_{k-1})t + \frac{1}{2}(1-a_k^2)(a_{k-2}(1-a_{k-1}^2)+ \\
&\qquad a_{k+2}(1-a_{k+1}^2) -\ a_k((a_{k+1}-a_{k-1})^2+2-2a_{k-1}a_{k+1})+\kappa_k)t^2 + +O(t^3)\Big)\,, \\
&\qquad\qquad |k-n| \geq 2, \\
x_{n\pm1}(t) &= \varepsilon\left(\mp 1 + 4a_\pm t + 4a_\pm(2a_{n\pm2} \mp (a_- + a_+))t^2 + O(t^3)\right), \\
x_n(t) &= -\frac{\varepsilon}{2t}\Big(1 + (a_+ - a_-)t + \frac{1}{3}((a_+ - a_-)^2 \\
&\qquad + 4(a_+a_{n+2} - a_-a_{n-2} + 1 - 2a_+a_-))t^2 + O(t^3)\Big)\,,
\end{aligned}$$

where a_+, a_- and all a_i, with $i \in \mathbf{Z} \setminus \{n-1, n, n+1\}$ are arbitrary free parameters; also, $\varepsilon^2 = 1$ and $a_{n-1} = -a_{n+1} = 1$. When $|k-n| > 2$ then $\kappa_k = 0$, while $\kappa_{n\pm2} = \mp 4a_\pm$.

Proof We look for formal Laurent solutions $x(t)$ to (31) that have a simple pole for one of the variables (only). To do this, we substitute $x_n(t) = x_n^{(0)}/t + O(1)$, with $x_n^{(0)} \neq 0$, and $x_j(t) = x_j^{(0)} + O(t)$, $j \neq n$ into (31) for different values of k. Taking $k = n \pm 1$ we find that $\left(x_{n\pm1}^{(0)}\right)^2 = 1$, in both cases because $1 - x_k^2(t)$ needs to cancel the pole coming from $x_n(t)$. Given this, (31) with $k = n$ is given by

$$-\frac{x_n^{(0)}}{t^2} + O(1) = -\frac{\left(x_n^{(0)}\right)^2}{t^2}(x_{n+1}^{(0)} - x_{n-1}^{(0)}) + O(t^{-1}).$$

Since $x_n^{(0)} \neq 0$, we deduce from it on the one hand that $x_{n+1}^{(0)}$ and $x_{n-1}^{(0)}$ have opposite signs, so that $x_{n+1}^{(0)} = -x_{n-1}^{(0)}$ and that $x_n^{(0)} = 1/(2x_{n+1}^{(0)})$. It follows that $x_{n\pm1}(t) = \mp\varepsilon + O(t)$ and $x_n(t) = -\varepsilon/(2t) + O(1)$, where $\varepsilon^2 = 1$. For $|k-n| \geq 2$, the coefficient in t^{-1} of (31) does not impose any condition on the constant coefficient of $x_k(t)$, which is therefore a free parameter, which we denote as εa_k.

Having determined the first term of the series we suppose that

$$\begin{aligned}
x_k(t) &= \varepsilon\left(a_k + \sum_{i=1}^{r} x_k^{(i)} t^i + x_k^{(r+1)} t^{r+1}\right), \quad |k-n| \geq 2, \\
x_{n\pm1}(t) &= \varepsilon\left(\mp 1 + \sum_{i=1}^{r} x_{n\pm1}^{(i)} t^i + x_{n\pm1}^{(r+1)} t^{r+1}\right),
\end{aligned}$$

$$x_n(t) \;=\; -\frac{\varepsilon}{2t}\left(1+\sum_{i=1}^{r} x_n^{(i)}t^i + x_n^{(r+1)}t^{r+1}\right),$$

where all coefficients $x_k^{(i)}$, with $i \leq r$ have been determined. We show that (31) then yields linear relations on the coefficients $x_k^{(r+1)}$. To see that, pick the coefficient in t^r in (31) when $k \neq n$, while taking the coefficient in t^{r-1} when $k = n$. This yields the following relations, where "known" means coefficients $x_k^{(i)}$, with $i \leq r$:

$$\begin{aligned} |k-n| \geq 2 \quad &: \quad \varepsilon(r+1)x_k^{(r+1)} = \text{known}, \\ k = n \pm 1 \quad &: \quad \varepsilon r x_{n\pm1}^{(r+1)} = \text{known}, \\ k = n \quad &: \quad -\frac{\varepsilon}{2}(r+2)x_n^{(r+1)} = -\frac{\varepsilon}{4}(x_{n+1}^{(r+1)} - x_{n-1}^{(r+1)}) + \text{known}. \end{aligned} \tag{32}$$

This yields a linear system in the unknowns $x_k^{(r+1)}$, where $k \in \mathbf{Z}$, which has upper triangular form when $x_n^{(r+1)}$ is put at the end. It uniquely determines the coefficients $x_k^{(r+1)}$, except when $k = n \pm 1$ and $r = 0$: the corresponding equations both reduce then to $0 = 0$, so that $x_{n+1}^{(1)}$ and $x_{n-1}^{(1)}$ are also free parameters; we denote them by $4a_\pm := x_{n\pm1}^{(1)}$. Then the third equation in (32) implies that $x_n^{(1)} = a_+ - a_-$; also, the first equation is explicitly given by $\varepsilon x_k^{(1)} = \varepsilon(1-a_k^2)(a_{k+1} - a_{k-1})$, for $|k-n| \geq 2$. Since for $r > 0$ we can solve uniquely for all x_k^{r+1}, we get a formal Laurent solution depending on the free parameters, as indicated. The extra term that is given in the proposition is easily verified.

Notice that under the natural correspondence between the phase variables x_k (with $k \neq n$) and the free parameters a_k ($a_\pm$ in the case $k = n \pm 1$) we have that the number of free parameters on which the coefficients of the series depend, is one less than the number of phase variables, a property reminiscent of principal balances for (finite-dimensional) algebraic completely integrable systems (see Adler, van Moerbeke & Vanhaecke 2004, Chapter 6). There are of course also formal Laurent solutions that depend on less free parameters (lower balances), but these will not be used here.

For future reference we give the first few terms of the formal Laurent series of $1 - x_k^2$, which is easily computed from the series given in proposition 3.1,

$$\begin{aligned} 1 - x_k^2(t) &= (1-a_k^2)(1 - -2a_k(a_{k+1} - a_{k-1})t) + O(t^2), \qquad |k-n| \geq 2, \\ 1 - x_{n\pm1}^2(t) &= \pm 8a_\pm t + O(t^2), \\ 1 - x_n^2(t) &= -\frac{1}{4t^2}(1 + 2(a_+ - a_-)t + O(t^2)). \end{aligned} \tag{33}$$

The displayed terms are the only ones that will be needed below.

3.2 The full Toeplitz lattice

We will now show that the full Toeplitz lattice also allows such formal Laurent solutions. To make the analogy with the self-dual case transparent we will vectorize the variables and the equations, namely we introduce $z_k := \begin{pmatrix} x_k \\ y_k \end{pmatrix}$ and $c_k := \begin{pmatrix} a_k \\ b_k \end{pmatrix}$, for $k \in \mathbf{Z}$; the variables a_k and b_k will be the free parameters in the formal Laurent series. With these notations the first Toeplitz vector field (13) becomes

$$\frac{dz_k}{dt} = (1 - x_k y_k)(z_{k+1} - z_{k-1}). \tag{34}$$

Proposition 3.2 *For any $n \in \mathbf{Z}$, the vector field (34) of the (general) Toeplitz lattice admits a formal Laurent solution $z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, such that only $x_n(t)$ and $y_n(t)$ have a (simple) pole. It is given by*

$$\begin{aligned}
z_k(t) &= c_k + (1 - a_k b_k)(c_{k+1} - c_{k-1})t + O(t^2), \qquad |k-n| \geq 2,\\
z_{n\pm1}(t) &= \begin{pmatrix} a_{n\pm1} + a_{n\pm1}a_{\pm}t \\ 1/a_{n\pm1} - a_{\pm}/a_{n\mp1}t \end{pmatrix} + O(t^2)\\
z_n(t) &= \frac{1}{(a_{n-1} - a_{n+1})t} \begin{pmatrix} a_{n-1}a_{n+1}(1 + at) \\ -1 + \frac{a_{n+1}(a_+ + a) - a_{n-1}(a_- + a)}{a_{n+1} - a_{n-1}} t \end{pmatrix} + O(t),
\end{aligned}$$

where a, $a_{\pm}$, $a_{n\pm1}$ and all $c_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix}$, with $i \in \mathbf{Z} \setminus \{n-1, n, n+1\}$ are arbitrary free parameters, and where $c_{n\pm1} = \begin{pmatrix} a_{n\pm1} \\ 1/a_{n\pm1} \end{pmatrix}$. Precisely, the free parameters $a_{n\pm1}$ satisfy the condition $a_{n+1}a_{n-1}(a_{n+1} - a_{n-1}) \neq 0$. The parameters on which the next order term in the series $x(t)$ and $y(t)$ depend is given in table 1.

Remark 3.3 *In section 6 we will need some extra information on these formal Laurent series, namely that the coefficient in t^2 of z_k, for $|k-n| \geq 2$ depends in the following way on c_{k+2},*

$$z_k^{(2)} = \frac{1}{2}(1 - a_k b_k)(1 - a_{k+1} b_{k+1}) c_{k+2} + \tilde{z}_k^{(2)}, \tag{35}$$

where $\tilde{z}_k^{(2)}$ is independent of a_{k+2} and of b_{k+2}. In particular, $x_k^{(2)}$ depends linearly on a_{k+2} and is independent of b_{k+2}, while $y_k^{(2)}$ depends linearly on b_{k+2} and is independent of a_{k+2}. This easily follows from the given terms by considering the coefficient of t in (34).

Proof For fixed $n \in \mathbf{Z}$, we look for formal Laurent solutions $z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, to (34) where $x_n(t)$ or $y_n(t)$ have a simple pole, and where none of the other variables $x_k(t)$ or $y_k(t)$ have a pole (in t). Thus, we substitute $z_n(t) = z_n^{(0)}/t + O(1)$ and $z_j(t) = z_j^{(0)} + O(t)$, $j \neq n$ into (34) for different values of k. For $k = n \pm 1$ we find that $x_{n\pm1}^{(0)} y_{n\pm1}^{(0)} = 1$, because $1 - x_{n\pm1}y_{n\pm1}$ needs to cancel the pole coming from x_n or from y_n; we put $a_{n\pm1} := x_{n\pm1}^{(0)}$, so that $y_{n\pm1}^{(0)} = 1/a_{n\pm1}$. The parameters $a_{n\pm1}$ are free, except that $a_{n+1}a_{n-1} \neq 0$. Next, (34) with $k = n$, yields

$$\begin{pmatrix} x_n^{(0)} \\ \\ y_n^{(0)} \end{pmatrix} = \begin{pmatrix} x_{n+1}^{(0)} - x_{n-1}^{(0)} \\ \\ y_{n+1}^{(0)} - y_{n-1}^{(0)} \end{pmatrix} x_n^{(0)} y_n^{(0)}$$

which shows on the one hand that $x_n^{(0)}$ and $y_n^{(0)}$ are both different from zero (since at least one of them is supposed to be different from zero), so that also $a_{n+1} - a_{n-1} \neq 0$. On the other hand it shows that $x_n^{(0)}$ and $y_n^{(0)}$ are expressible in terms of a_{n+1} and a_{n-1} as

$$x_n^{(0)} = \frac{a_{n+1}a_{n-1}}{a_{n-1} - a_{n+1}}, \qquad y_n^{(0)} = \frac{1}{a_{n+1} - a_{n-1}}.$$

For $|k-n| \geq 2$, the coefficient in t^{-1} of (34) does not impose any condition on the constant coefficient of $z_k(t)$, yielding free parameters for the constant coefficients of x_k and of y_k, with $|k-n| > 1$. We

Table 1: We list on which free parameters the first few terms of the formal Laurent solutions depend. It is understood that we do not list again the parameters that appear already before, on the same line; for example, $x_n^{(1)}$ depends only on a_{n+1}, a_{n-1} and a. The last two lines correspond to the values k for which $|k-n|>2$. For $k\neq n$, $x_k^{(i)}$ is the coefficient of t^i in $x_k(t)$, while for $k=n$ it is the coefficient of t^{i-1} in $x_n(t)$.

	$x^{(0)}, y^{(0)}$	$x^{(1)}, y^{(1)}$	$x^{(2)}, y^{(2)}$
x_n	a_{n+1}, a_{n-1}	a	$a_-, a_+, a_{n+2}, b_{n+2}, a_{n-2}, b_{n-2}$
y_n	a_{n+1}, a_{n-1}	a, a_+, a_-	$a_{n+2}, b_{n+2}, a_{n-2}, b_{n-2}$
$x_{n\pm1}$	$a_{n\pm1}$	$a_\pm$	$a_{n\pm2}, b_{n\pm2}, a_\mp, a, a_{n\mp1}$
$y_{n\pm1}$	$a_{n\pm1}$	$a_{n\mp1}, a_\pm$	$a_{n\pm2}, b_{n\pm2}, a_\mp, a$
$x_{n\pm2}$	$a_{n\pm2}$	$a_{n\pm3}, a_{n\pm1}, b_{n\pm2}$	$a_{n\pm4}, b_{n\pm3}, a_\pm$
$y_{n\pm2}$	$b_{n\pm2}$	$b_{n\pm3}, b_{n\pm1}, a_{n\pm2}$	$b_{n\pm4}, a_{n\pm3}, a_\pm, a_{n\mp1}$
x_k	a_k	a_{k+1}, a_{k-1}, b_k	$a_{k+2}, b_{k+1}, a_{k-2}, b_{k-1}$
y_k	b_k	b_{k+1}, b_{k-1}, a_k	$b_{k+2}, a_{k+1}, b_{k-2}, a_{k-1}$

denote these free parameters by $c_k=\begin{pmatrix} a_k\\ b_k\end{pmatrix}$. Upon specialization, some of the formulas below may contain c_{n+1} or c_{n-1}; it is understood that these stand for

$$c_{n\pm1}=\begin{pmatrix} a_{n\pm1}\\ b_{n\pm1}\end{pmatrix}=\begin{pmatrix} a_{n\pm1}\\ 1/a_{n\pm1}\end{pmatrix}.$$

We can now proceed as in the second part of the proof of proposition 3.1, namely we suppose that

$$\begin{aligned}
z_k(t) &= c_k+\sum_{i=1}^r z_k^{(i)}t^i+z_k^{(r+1)}t^{r+1},\\
z_{n\pm1}(t) &= \begin{pmatrix} a_{n\pm1}\\ 1/a_{n\pm1}\end{pmatrix}+\sum_{i=1}^r z_{n\pm1}^{(i)}t^i+z_{n\pm1}^{(r+1)}t^{r+1},\\
z_n(t) &= \frac{1}{(a_{n-1}-a_{n+1})t}\left(\begin{pmatrix} a_{n-1}a_{n+1}\\ -1\end{pmatrix}+\sum_{i=1}^r z_n^{(i)}t^i+z_n^{(r+1)}t^{r+1}\right),
\end{aligned}$$

where all coefficients $z_k^{(i)}$, with $i\leq r$ have been determined. On the coefficients $z_k^{(r+1)}$, $k\in\mathbf{Z}$, we find linear relations by substituting the above series into (34). For k such that $|n-k|>1$ it is clear that, as in the self-dual case, $z_k^{(r+1)}$ is linearly computed in terms of the known coefficients, from the coefficient of t^r, when substituting the series in (34). Therefore, let us concentrate on what happens for $k\in\{n-1,n,n+1\}$. Taking $k=n\pm1$ in (34) the coefficient of t^r yields

$$(r+1)z_{n\pm1}^{(r+1)}=\pm\left(\frac{x_{n\pm1}^{(r+1)}}{a_{n\pm1}}+y_{n\pm1}^{(r+1)}a_{n\pm1}\right)\begin{pmatrix} \frac{a_{n-1}a_{n+1}}{a_{n-1}-a_{n+1}}\\ \frac{-1}{a_{n-1}-a_{n+1}}\end{pmatrix}+\text{known},$$

a linear equation in $x_{n\pm1}$ and $y_{n\pm1}$, which can be written in the compact form

$$(\mathcal{L}_\pm + (r+1)\,\mathrm{Id})\, z_{n\pm1}^{(r+1)} = \text{known},$$

where $\mathcal{L}_\pm$ is the matrix that governs the linear problem,

$$\mathcal{L}_\pm := \pm\frac{1}{a_{n-1}-a_{n+1}}\begin{pmatrix} -a_{n\mp1} & -a_{n-1}a_{n+1}a_{n\pm1} \\ 1/a_{n\pm1} & a_{n\pm1}\end{pmatrix}.$$

Since $\det(\mathcal{L}_\pm+(r+1)\,\mathrm{Id}) = r(r+1)$ this linear system admits a unique solution, except when $r=0$ (recall that $r\geq 0$). Before analyzing the case $r=0$ further, let us first consider what happens to (34) in the remaining case $k=n$. As in the self-dual case, we pick the coefficient of t^{r-1} in (34) to find a linear system that can be written in the compact form

$$(\mathcal{L}_n + r\,\mathrm{Id})\, z_n^{(r+1)} = \text{known},$$

where the matrix $\mathcal{L}_n$ is given by

$$\mathcal{L}_n := \begin{pmatrix} 1 & -a_{n+1}a_{n-1} \\ -1/(a_{n+1}a_{n-1}) & 1\end{pmatrix}.$$

Since $\det(\mathcal{L}_n+r\,\mathrm{Id}) = r(r+2)$ we have again that $z_n^{(r+1)}$ is determined uniquely, unless $r=0$. Thus, we are done with $r\geq 1$.

As we have seen, a free parameter may appear in $z_{n+1}^{(1)}$, in $z_{n-1}^{(1)}$ and in $z_n^{(1)}$, but one has to check that the corresponding linear equations are consistent. Therefore we substitute

$$\begin{aligned}
z_k(t) &= c_k + z_k^{(1)}t + O(t^2),\\
z_{n\pm1}(t) &= \begin{pmatrix} a_{n\pm1} \\ 1/a_{n\pm1}\end{pmatrix} + z_{n\pm1}^{(1)}t + O(t^2), \qquad (36)\\
z_n(t) &= \frac{1}{(a_{n-1}-a_{n+1})t}\left(\begin{pmatrix} a_{n-1}a_{n+1} \\ -1\end{pmatrix} + z_n^{(1)}t + O(t^2)\right),
\end{aligned}$$

in (34), which yields for $k=n\pm1$ and $t=0$ the homogeneous linear system

$$\begin{pmatrix} x_{n\pm1}^{(1)} \\ y_{n\pm1}^{(1)}\end{pmatrix} = \pm\frac{1}{a_{n-1}-a_{n+1}}\left(\frac{x_{n\pm1}^{(1)}}{a_{n\pm1}} + y_{n\pm1}^{(1)}a_{n\pm1}\right)\begin{pmatrix} a_{n-1}a_{n+1} \\ -1\end{pmatrix},$$

which is equivalent to

$$x_{n\pm1}^{(1)} + a_{n-1}a_{n+1}y_{n\pm1}^{(1)} = 0. \qquad (37)$$

Thus, upon setting $x_{n\pm1}^{(1)} = a_{n\pm1}a_\pm$, where a_+ and a_- are free parameters, we have that $y_{n\pm1}^{(1)} = -a_\pm/a_{n\mp1} = -a_\pm b_{n\mp1}$. Similarly, for $k=n$ the substitution of the series (36) in (34) yields at the level t^{-1}:

$$\frac{a_{n-1}a_{n+1}}{a_{n-1}-a_{n+1}}(x_{n+1}^{(1)} - x_{n-1}^{(1)}) - x_n^{(1)} + a_{n-1}a_{n+1}y_n^{(1)} = 0,$$

$$\frac{a_{n-1}a_{n+1}}{a_{n-1}-a_{n+1}}(y_{n+1}^{(1)} - y_{n-1}^{(1)}) - y_n^{(1)} + \frac{x_n^{(1)}}{a_{n-1}a_{n+1}} = 0.$$

These equation are proportional, in view of (37). Thus we have

$$\begin{aligned}
x_n^{(1)} &= a_{n+1}a_{n-1}a,\\
y_n^{(1)} &= a + \frac{a_{n+1}a_+ - a_{n-1}a_-}{a_{n+1}-a_{n-1}},
\end{aligned}$$

where a is a free parameter.

The first two terms in the series lead at once to the second and third columns of table 1. In order to obtain the last column it suffices to list on which parameters the linear term (resp. the constant term) in the right hand side of $(1-x_k(t)y_k(t))(z_{k+1}(t)-z_{k-1}(t))$ depends, when $k\neq n$ (resp. when $k=n$). The two leading terms of $x(t)$ and $y(t)$ that we computed suffice for doing this.

It is easily verified that the involution σ, that permutes x_k and y_k extends naturally to an involution on the free parameters, given by

$$\begin{gathered}\sigma(a_k)=b_k,\ \sigma(a_{n\pm 1})=1/a_{n\pm 1},\ \sigma(a_\pm)=-a_\pm a_{n\pm 1}/a_{n\mp 1},\\ \sigma(a)=-a-\frac{a_{n+1}a_+-a_{n-1}a_-}{a_{n+1}-a_{n-1}}.\end{gathered}\tag{38}$$

Notice that, altogether, we have besides the free parameters a_k,b_k, for $|k-n|>1$, which naturally correspond to the variables x_k and y_k, five extra free parameters $a_{n\pm 1}$, $a_\pm$ and a, that correspond to the remaining six variables $x_{n\pm 1}$, $y_{n\pm 1}$ and x_n, y_n, which again yields that the number of free parameters, plus time, is equal to the number of phase variables. This count will be important, and rigorous, when we restrict these formal Laurent solutions to certain finite-dimensional submanifolds.

4 Tangency to $\mathcal{M}$

We have seen that the polynomials Γ_k and $\tilde\Gamma_k$, which define an invariant manifold for the first Toeplitz flow, satisfy a non-autonomous system of linear differential equations, where the time-dependence is defined by the latter flow. In a (finite-dimensional) manifold setting, if such differential equations have coefficients that depend smoothly on time, solutions (integral curves) that start out on the invariant manifold will stay on it, by the uniqueness of solutions to differential equations with smooth coefficients and given initial conditions. In the case that we deal with the situation is quite a bit different, because the coefficients develop poles in t, for $t=0$, and of course the solutions are only formal Laurent series. As it turns out, the conditions that assure that the formal Laurent solutions "stay on the invariant manifold" are similar to those in the smooth case for the self-dual Toeplitz lattice, but are different in an essential way for the general Toeplitz lattice.

4.1 Tangency in the self-dual case

We start out with the case of the self-dual Toeplitz lattice.

Proposition 4.1 *Let $x(t)$ denote the formal Laurent solution that is given by proposition 3.1, and let $\Gamma(t):=\Gamma(x(t);u(t))$, where we recall that $u(t)=(u_1+t,u_2,\ldots,u_N)$. Then, as formal series in t,*

$$\begin{aligned}\Gamma_k(t)&=\Gamma_k^{(0)}+O(t), && k\in\mathbf{Z}\setminus\{n\},\\ \Gamma_n(t)&=\frac{1}{4t}(\Gamma_{n+1}^{(0)}-\Gamma_{n-1}^{(0)})+\Gamma_n^{(0)}+O(t).\end{aligned}\tag{39}$$

Moreover, $\Gamma_k(t)=0$ as a formal series in t, for all $k\in\mathbf{Z}$, as soon as $x(t)$ is such that

$$\Gamma_k^{(0)}=0,\ \textit{for all } k\in\mathbf{Z}.$$

Proof According to (23), $\Gamma_k(x;u)$ involves only the variables x_l with $|l-k|\leq N$ ($2N+1$ step relation). Since only $x_n(t)$ has a pole, $\Gamma_k(t)=O(1)$ as soon as Γ_k does not contain x_n, i.e., if $|n-k|>N$. But notice that (27) implies

$$\Gamma_{n-N}=\frac{1}{1-x_{n-N-1}^2}\frac{d\Gamma_{n-N-1}}{dt}+\Gamma_{n-N-2},$$

so that $\Gamma_{n-N}(t)=O(1)$, as the leading term $1-a_{n-N-1}^2$ of $1-x_{n-N-1}^2(t)$ is non-zero (recall that a_{n-N-1} is a free parameter). This argument can be repeated to yield $\Gamma_k(t)=O(1)$ for all $k<n$, and similarly it is shown that $\Gamma_k(t)=O(1)$ for all $k>n$. Since $\Gamma_n(t)$ satisfies the differential equation (27), for $k=n$, we have in view of (33) that

$$\frac{d\Gamma_n}{dt}(t)=(1-x_n^2(t))(\Gamma_{n+1}(t)-\Gamma_{n-1}(t))=-\frac{1}{4t^2}(\Gamma_{n+1}^{(0)}-\Gamma_{n-1}^{(0)})+O(1),$$

which leads upon integration to (39).

Suppose now that $x(t)$ is such that $\Gamma_k^{(0)}=0$ for all $k\in\mathbf{Z}$. In view of the first part of the proof, we have that $\Gamma_k(t)=O(t)$ for all $k\in\mathbf{Z}$. We show that this implies that $\Gamma_k(t)=0$ as a formal series in t, for all $k\in\mathbf{Z}$. We do this by induction on $r\in\mathbf{N}^*$: assuming that $\Gamma_k(t)=O(t^r)$ for $k\in\mathbf{Z}$ we show that $\Gamma_k(t)=O(t^{r+1})$ for $k\in\mathbf{Z}$. Notice that in the case $r=1$ the assumption holds. For $k\notin\{n-1,n,n+1\}$ the right hand side of (27) is $O(t^r)$, by (33) and by the assumption, so that $\frac{d\Gamma_k}{dt}(t)=O(t^r)$, hence $\Gamma_k(t)=O(t^{r+1})$, by integration. For $k=n\pm1$ we have from (33) that $1-x_{n\pm1}^2(t)=O(t)$, so that (27) yields for $k=n\pm1$ that $\frac{d\Gamma_{n\pm1}}{dt}(t)=O(t^{r+1})$, i.e., $\Gamma_{n\pm1}(t)=O(t^{r+2})$. For $k=n$ we have that $1-x_n^2(t)=1-x_n^2(t)$ has a double pole, but since we have just shown that $\Gamma_{n+1}(t)-\Gamma_{n-1}(t)=O(t^{r+2})$ the differential equation (27) for $k=n$ leads to $\frac{d\Gamma_n}{dt}(t)=O(t^r)$ and we conclude that $\Gamma_n(t)=O(t^{r+1})$, as was to be shown.

4.2 Tangency in the general case

For the full Toeplitz lattice the tangency condition is rather similar, yet is different in some detail that will turn out to be crucial in the next section. We recall that the differential equations that are satisfied by the polynomials Γ_k and $\tilde\Gamma_k$ are given by

$$\begin{aligned}\frac{d\Gamma_k}{dt}&=(1-x_ky_k)(\Gamma_{k+1}-\Gamma_{k-1})+(x_{k+1}-x_{k-1})(x_k\tilde\Gamma_k-y_k\Gamma_k),\\ \frac{d\tilde\Gamma_k}{dt}&=(1-x_ky_k)(\tilde\Gamma_{k+1}-\tilde\Gamma_{k-1})-(y_{k+1}-y_{k-1})(x_k\tilde\Gamma_k-y_k\Gamma_k).\end{aligned}\tag{40}$$

Proposition 4.2 *Let $(x(t),y(t))$ denote the formal Laurent solution that is given by proposition 3.2, and let $\Gamma(t):=\Gamma(x(t),y(t);u(t))$, where $u(t)$ is given by (25). Then, as a formal series in t, $\Gamma_k(t)=\Gamma_k^{(0)}+O(t)$ and $\tilde\Gamma_k(t)=\tilde\Gamma_k^{(0)}+O(t)$ for $k\in\mathbf{Z}\setminus\{n\}$. Also*

$$\begin{aligned}\Gamma_n(t)&=\frac{a_{n+1}^2}{a_-(a_{n-1}-a_{n+1})^2t^2}\left(\Gamma_{n-1}^{(0)}-a_{n-1}^2\tilde\Gamma_{n-1}^{(0)}\right)+\frac1t\Gamma_n^{(-1)}+O(1),\\ \tilde\Gamma_n(t)&=\frac{a_{n+1}a_{n-1}}{a_-(a_{n-1}-a_{n+1})^2t^2}\left(\Gamma_{n-1}^{(0)}/a_{n-1}^2-\tilde\Gamma_{n-1}^{(0)}\right)+\frac1t\tilde\Gamma_n^{(-1)}+O(1),\end{aligned}\tag{41}$$

where $\Gamma_n^{(-1)}$ and $\tilde\Gamma_n^{(-1)}$ are both linear combinations of $\Gamma_{n\pm1}^{(0)}$ and $\tilde\Gamma_{n\pm1}^{(0)}$ (for the explicit formula, see (47)); moreover, the latter coefficients are related in the following way:

$$a_-\left(\tilde\Gamma_{n+1}^{(0)}-\frac{1}{a_{n+1}^2}\Gamma_{n+1}^{(0)}\right)=a_+\left(\frac{1}{a_{n-1}^2}\Gamma_{n-1}^{(0)}-\tilde\Gamma_{n-1}^{(0)}\right).\tag{42}$$

Proof As in the self-dual case, the polynomials $\Gamma_k(x;u)$ and $\tilde\Gamma_k(x;u)$ define $2N+1$ step relations, so they depend only on the variables x_l and y_l with $|l-k|\le N$. Only $x_n(t)$ and $y_n(t)$ have a pole, so that $\Gamma_k(t)=O(1)$ and $\tilde\Gamma_k(t)=O(1)$ for $|n-k|>N$. Writing (40) for $k\to k-1$ as

$$\begin{aligned}\Gamma_k &= \frac{1}{1-x_{k-1}y_{k-1}}\left(\frac{d\Gamma_{k-1}}{dt}-(x_k-x_{k-2})(x_{k-1}\tilde\Gamma_{k-1}-y_{k-1}\Gamma_{k-1})\right)+\Gamma_{k-2},\\ \tilde\Gamma_k &= \frac{1}{1-x_{k-1}y_{k-1}}\left(\frac{d\tilde\Gamma_{k-1}}{dt}+(y_k-y_{k-2})(x_{k-1}\tilde\Gamma_{k-1}-y_{k-1}\Gamma_{k-1})\right)+\tilde\Gamma_{k-2},\end{aligned}\tag{43}$$

and taking as consecutive values $k:=n-N,\dots,n-1$ in (43) we find that $\Gamma_k(t)=O(1)$ and $\tilde\Gamma_k(t)=O(1)$ for all $k\le n-1$, since $1-x_k(t)y_k(t)$ does not vanish for $t=0$ when $k\ne n\pm1$. Similarly $\Gamma_k(t)=O(1)$ and $\tilde\Gamma_k(t)=O(1)$ when $k\ge n+1$. So we have that $\Gamma_k(t)=O(1)$ and $\tilde\Gamma_k(t)=O(1)$ when $k\ne n$ and we are left with the case $k=n$.

In order to deal with the case $k=n$ we write (40) as an equation for Γ_n and $\tilde\Gamma_n$ in two different ways:

$$\begin{aligned}\Gamma_n &= \mp\frac{1}{1-x_{n\pm1}y_{n\pm1}}\left(\frac{d\Gamma_{n\pm1}}{dt}\pm(x_n-x_{n\pm2})(x_{n\pm1}\tilde\Gamma_{n\pm1}-y_{n\pm1}\Gamma_{n\pm1})\right)+\Gamma_{n\pm2},\\ \tilde\Gamma_n &= \mp\frac{1}{1-x_{n\pm1}y_{n\pm1}}\left(\frac{d\tilde\Gamma_{n\pm1}}{dt}\mp(y_n-y_{n\pm2})(x_{n\pm1}\tilde\Gamma_{n\pm1}-y_{n\pm1}\Gamma_{n\pm1})\right)+\tilde\Gamma_{n\pm2}.\end{aligned}\tag{44}$$

Either of them implies that $\Gamma_n(t)=O(t^{-2})$ and that $\tilde\Gamma_n(t)=O(t^{-2})$, so we write

$$\Gamma_n(t)=\frac{1}{t^2}\left(\Gamma_n^{(-2)}+\Gamma_n^{(-1)}t+\Gamma_n^{(0)}t^2+O(t^3)\right),$$

and similarly for $\tilde\Gamma_n(t)$. In fact, as $1-x_{n+1}(t)y_{n+1}(t)$ and $1-x_{n-1}(t)y_{n-1}(t)$ have a simple zero, while $x_n(t)$ and $y_n(t)$ have a simple pole, the coefficient of t^{-2} in (44), leads to the following linear equations

$$\begin{aligned}\Gamma_n^{(-2)} &= -x_n^{(0)}\left(x_{n\pm1}^{(0)}\tilde\Gamma_{n\pm1}^{(0)}-y_{n\pm1}^{(0)}\Gamma_{n\pm1}^{(0)}\right)/\zeta_\pm,\\ \tilde\Gamma_n^{(-2)} &= -\Gamma_n^{(-2)}y_n^{(0)}/x_n^{(0)},\end{aligned}\tag{45}$$

where we have written $1-x_{n\pm1}(t)y_{n\pm1}(t)=\zeta_\pm t+O(t^2)$, so that

$$\zeta_\pm=\pm a_\pm\frac{a_{n+1}-a_{n-1}}{a_{n\mp1}}.$$

It suffices now to substitue $x_{n\pm1}^{(0)}=a_{n\pm1}=1/y_{n\pm1}^{(0)}$ and $x_n^{(0)}=a_{n-1}a_{n+1}/\,(a_{n-1}-a_{n+1})=-a_{n-1}a_{n+1}y_n^{(0)}$ in (45) to find the coefficient of t^{-2} in (41). Actually, the latter corresponds to taking the lower sign; equating the two expressions for $\Gamma_n^{(-2)}$ in (45) that correspond to the two signs leads to (42); notice that this is also the expression that is obtained from the two expressions of $\tilde\Gamma_n^{(-2)}$ in (45).

It remains to compute $\Gamma_n^{(-1)}$ and $\tilde\Gamma_n^{(-1)}$, which can be done from the coefficient of t^{-2} in $\frac{d\Gamma_n}{dt}(t)$ and in $\frac{d\tilde\Gamma_n}{dt}(t)$, computed from their differential equations

$$\begin{aligned}\tfrac{d\Gamma_n}{dt} &= (1-x_ny_n)(\Gamma_{n+1}-\Gamma_{n-1})+(x_{n+1}-x_{n-1})(x_n\tilde\Gamma_n-y_n\Gamma_n),\\ \tfrac{d\tilde\Gamma_n}{dt} &= (1-x_ny_n)(\tilde\Gamma_{n+1}-\tilde\Gamma_{n-1})-(y_{n+1}-y_{n-1})(x_n\tilde\Gamma_n-y_n\Gamma_n).\end{aligned}\tag{46}$$

Since $1-x_n(t)y_n(t)$ has a double pole, while $\Gamma_{n\pm1}(t)$ and $\tilde\Gamma_{n\pm1}(t)$ have no pole, the contribution of the first term to the coefficient in t^2 will be linear in $\Gamma_{n\pm1}^{(0)}$ and in $\tilde\Gamma_{n\pm1}^{(0)}$. Since $x_n(t)$ and $y_n(t)$

have a simple pole, while $\Gamma_n(t)$ and $\tilde{\Gamma}_n(t)$ have a double pole, the contribution of the second term will yield a linear combination of on the one hand $\Gamma_n^{(-2)}$ and $\tilde{\Gamma}_n^{(-2)}$ which, as we have seen, are themselves linear combinations of $\Gamma_{n\pm1}^{(0)}$ and in $\tilde{\Gamma}_{n\pm1}^{(0)}$; on the other hand, $\Gamma_n^{(-1)}$ and $\tilde{\Gamma}_n^{(-1)}$, which are the unknowns. Explicitly, this linear system is given by

$$\begin{pmatrix} a_{n+1}a_{n-1}\tilde{\Gamma}_n^{(-1)} \\ 1/(a_{n+1}a_{n-1})\Gamma_n^{(-1)} \end{pmatrix} = \frac{a_{n-1}a_{n+1}}{(a_{n+1}-a_{n-1})^2}\begin{pmatrix} \Gamma_{n+1}^{(0)}-\Gamma_{n-1}^{(0)} \\ \tilde{\Gamma}_{n+1}^{(0)}-\tilde{\Gamma}_{n-1}^{(0)} \end{pmatrix} - \begin{pmatrix} 1 \\ 1/a_{n+1}a_{n-1} \end{pmatrix}\left(\Gamma_n^{(-2)}\sigma(a)+\tilde{\Gamma}_n^{(-2)}aa_{n+1}a_{n-1}\right). \tag{47}$$

Since $\Gamma_n^{(-2)}$ and $\tilde{\Gamma}_n^{(-2)}$ are linear combinations of $\Gamma_{n\pm1}^{(0)}$ and $\tilde{\Gamma}_{n\pm1}^{(0)}$ it follows that each of $\Gamma_n^{(-1)}$ and $\tilde{\Gamma}_n^{(-1)}$ is a linear combination of $\Gamma_{n\pm1}^{(0)}$ and $\tilde{\Gamma}_{n\pm1}^{(0)}$, as we asserted.

Proposition 4.3 *Suppose that $(x(t),y(t))$ is a formal Laurent solution of the first vector field of the Toeplitz lattice, such that $\Gamma_k(t)=O(t)$ and $\tilde{\Gamma}_k(t)=O(t)$ for all k with $k\neq n+1$, and such that, as formal Laurent solutions in t, $\Gamma_{n-1}(t)=O(t^2)$ and $\Gamma_{n+1}(t)=O(t)$. Then, as formal Laurent series, $\Gamma_k(t)=0=\tilde{\Gamma}_k(t)$ for all $k\in\mathbf{Z}$.*

Proof According to (42), the hypothesis imply that $\tilde{\Gamma}_{n+1}(t)=O(t)$. In view of proposition 4.2, we have that $\Gamma_k(t)=O(t)$ and $\tilde{\Gamma}_k(t)=O(t)$ for every $k\in\mathbf{Z}$. We will now proceed by induction on $r\in\mathbf{N}^*$, but in a different way than in the self-dual case: assuming that $\Gamma_k(t)=O(t^r)$ and $\tilde{\Gamma}_k(t)=O(t^r)$ for $k\neq n\pm1$, as well as $\Gamma_{n\pm1}(t)=O(t^{r+1})$ and $\tilde{\Gamma}_{n\pm1}(t)=O(t^{r+1})$, we show that $\Gamma_k(t)=O(t^{r+1})$ and $\tilde{\Gamma}_k(t)=O(t^{r+1})$ for $k\neq n\pm1$, as well as $\Gamma_{n\pm1}(t)=O(t^{r+2})$ and $\tilde{\Gamma}_{n\pm1}(t)=O(t^{r+2})$. Notice that the $r=1$ induction assumption needs to be shown at the end of the proof, as only part of it is in the actual hypothesis of the theorem.

For k such that $|k-n|\geq 2$ the differential equations (40) yield that $\frac{d\Gamma_k}{dt}(t)=O(t^r)$ and $\frac{d\tilde{\Gamma}_k}{dt}(t)=O(t^r)$, so that $\Gamma_k(t)=O(t^{r+1})$ and $\tilde{\Gamma}_k(t)=O(t^{r+1})$, by integration. So we are left with $k\in\{n-1,n,n+1\}$. Let us write

$$\begin{aligned} \Gamma_n &= \gamma_n t^r+O(t^{r+1}), & \tilde{\Gamma}_n &= \tilde{\gamma}_n t^r+O(t^{r+1}), \\ \Gamma_k &= \gamma_k t^{r+1}+O(t^{r+2}), & \tilde{\Gamma}_k &= \tilde{\gamma}_k t^{r+1}+O(t^{r+2}), \quad k\neq n, \end{aligned}$$

which we substitute in

$$\begin{aligned} \frac{d\Gamma_{n\pm1}}{dt} &= \mp(1-x_{n\pm1}y_{n\pm1})(\Gamma_n-\Gamma_{n\pm2})\pm(x_{n\pm2}-x_n)(x_{n\pm1}\tilde{\Gamma}_{n\pm1}-y_{n\pm1}\Gamma_{n\pm1}), \\ \frac{d\tilde{\Gamma}_{n\pm1}}{dt} &= \mp(1-x_{n\pm1}y_{n\pm1})(\tilde{\Gamma}_n-\tilde{\Gamma}_{n\pm2})\mp(y_{n\pm2}-y_n)(x_{n\pm1}\tilde{\Gamma}_{n\pm1}-y_{n\pm1}\Gamma_{n\pm1}). \end{aligned} \tag{48}$$

Remembering that $1-x_{n\pm1}(t)y_{n\pm1}(t)=O(t)$ we pick the coefficient of t^r in (48), which leads to the following linear system,

$$\begin{aligned} (r+1)\gamma_{n\pm1} &= \mp\tfrac{a_{n-1}a_{n+1}}{a_{n-1}-a_{n+1}}\left(a_{n\pm1}\tilde{\gamma}_{n\pm1}-\tfrac{1}{a_{n\pm1}}\gamma_{n\pm1}\right), \\ (r+1)\tilde{\gamma}_{n\pm1} &= \mp\tfrac{1}{a_{n-1}-a_{n+1}}\left(a_{n\pm1}\tilde{\gamma}_{n\pm1}-\tfrac{1}{a_{n\pm1}}\gamma_{n\pm1}\right). \end{aligned} \tag{49}$$

Since

$$\begin{vmatrix} r+1\mp\frac{a_{n\mp1}}{a_{n-1}-a_{n+1}} & \pm\frac{a_{n-1}a_{n+1}a_{n\pm1}}{a_{n-1}-a_{n+1}} \\ \mp\frac{1}{(a_{n-1}-a_{n+1})a_{n\pm1}} & r+1\pm\frac{a_{n\pm1}}{a_{n-1}-a_{n+1}} \end{vmatrix} = (r+1)^2-(r+1)=r(r+1),$$

it follows, since $r \geq 1$, that $\gamma_{n\pm1} = \tilde{\gamma}_{n\pm1} = 0$, and hence that $\Gamma_{n\pm1}(t) = O(t^{r+2})$ and $\tilde{\Gamma}_{n\pm1}(t) = O(t^{r+2})$. It follows that, if we substitute the series in

$$\begin{aligned} \frac{d\Gamma_n}{dt} &= (1 - x_n y_n)(\Gamma_{n+1} - \Gamma_{n-1}) + (x_{n+1} - x_{n-1})(x_n\tilde{\Gamma}_n - y_n\Gamma_n), \\ \frac{d\tilde{\Gamma}_n}{dt} &= (1 - x_n y_n)(\tilde{\Gamma}_{n+1} - \tilde{\Gamma}_{n-1}) - (y_{n+1} - y_{n-1})(x_n\tilde{\Gamma}_n - y_n\Gamma_n), \end{aligned} \tag{50}$$

then the coefficient of t^{r-1} is simply given by

$$\begin{aligned} r\gamma_n &= -(a_{n-1}a_{n+1}\tilde{\gamma}_n + \gamma_n), \\ r\tilde{\gamma}_n &= -\frac{1}{a_{n-1}a_{n+1}}(a_{n-1}a_{n+1}\tilde{\gamma}_n + \gamma_n). \end{aligned}$$

Since

$$\det\begin{pmatrix} r+1 & a_{n-1}a_{n+1} \\ \frac{1}{a_{n+1}a_{n-1}} & r+1 \end{pmatrix} = (r+1)^2 - 1 \neq 0,$$

we have that $\gamma_n = \tilde{\gamma}_n = 0$, so that $\Gamma_n(t) = O(t^{r+1})$ and $\tilde{\Gamma}_n(t) = O(t^{r+1})$, as was to be shown.

We finally check that our assumptions imply that for $r = 1$ the induction hypothesis is valid. According to proposition 4.2, we have that $\Gamma(t) = O(t)$ and $\tilde{\Gamma}(t) = O(t)$. Let us write $\Gamma_{n\pm1} = \gamma_{n\pm1}t + O(t^2)$ and $\tilde{\Gamma}_{n\pm1} = \tilde{\gamma}_{n\pm1}t + O(t^2)$. Then we need to show that $\gamma_{n\pm1} = \tilde{\gamma}_{n\pm1} = 0$. From (49), which is also valid for $r = 0$, we conclude that $\gamma_{n\pm1} = a_{n-1}a_{n+1}\tilde{\gamma}_{n\pm1}$. It was assumed that $\Gamma_{n-1}(t) = O(t^2)$, i.e., that $\gamma_{n-1} = 0$, so that we can conclude that $\tilde{\gamma}_{n-1} = 0$. In order to obtain a second relation between γ_{n+1} and $\tilde{\gamma}_{n+1}$ we consider the residue in the first[6] equation in (50), which reduces to $0 = a_{n-1}a_{n+1}\gamma_{n+1}/(a_{n-1} - a_{n+1})^2$, since $\Gamma_n^{(0)} = \tilde{\Gamma}_n^{(0)} = 0$. Thus, $\gamma_{n+1} = \tilde{\gamma}_{n+1} = 0$, as was to be shown.

5 Restricting the formal Laurent solutions: the self-dual case

We have seen conditions on $\Gamma(t) = \Gamma(x(t); u(t))$ that guarantee that solutions $x(t)$ to the self-dual Toeplitz lattice that start out in the invariant manifold $\mathcal{M}_{u(t)}$ stay in it, formally speaking. In this section we show how these conditions can be translated into conditions on the formal Laurent solution $x(t)$ to the first vector field of the self-dual Toeplitz lattice.

5.1 Structure of the polynomials Γ_k

The polynomials Γ_k, which define the invariant manifolds $\mathcal{M}$ depend on the variable x_n in a special way, that we will analyze by using the fact that Γ_k remains pole free (for $k \neq n$) when the formal Laurent series $x(t)$ are substituted in them, as we have seen in proposition 4.1. Let us denote by $\mathcal{A}$ the algebra of polynomials in all variables x_k, where $k \in \mathbf{Z}$ and by $\mathcal{A}_n$ the subalgebra of those polynomials that are independent of x_n. Also, let us denote by $\mathcal{A}'_n$ the subalgebra of $\mathcal{A}$ that consists of those elements that can be written as polynomials in w_1, w_2 and x_k, with $k \neq n$, where

$$w_1 := x_n(x_{n+1} + x_{n-1}), \ w_2 := x_n(1 + x_{n+1}x_{n-1}). \tag{51}$$

[6] Taking the second equation would lead to the same result.

Thus, elements of $\mathcal{A}'_n$ may depend only on x_n through w_1 and w_2. For future use, we give the first few terms of the formal Laurent series of the generators of $\mathcal{A}'_n$, as obtained by substituting the series from proposition 3.1 in (51):

$$\begin{aligned} w_1(t) &= -2(a_+ + a_- + 2(a_+a_{n+2} + a_-a_{n-2})t + O(t^2)), \\ w_2(t) &= -2\varepsilon(a_+ - a_- + 2(a_+a_{n+2} - a_-a_{n-2})t + O(t^2)), \\ x_k(t) &= \varepsilon(a_k + (1-a_k^2)(a_{k+1} - a_{k-1})t + O(t^2)), \quad k \neq n. \end{aligned} \tag{52}$$

It follows that $G(x(t)) = O(1)$, for any $G \in \mathcal{A}'_n$. Notice that the polynomials $w_\pm := (1 - x_{n\pm1}^2)x_n$, which both have the property $w_\pm(t) = O(1)$, belong to $\mathcal{A}'_n$, since

$$(1 - x_{n\pm1}^2)x_n = w_2 - x_{n\pm1}w_1. \tag{53}$$

The following proposition generalizes this statement.

Proposition 5.1 *For $G \in \mathcal{A}$, let $G(t) := G(x(t))$, where $x(t)$ is the formal Laurent solution to the first vector field of the self-dual Toeplitz lattice, constructed in proposition 3.1. If $G(t) = O(1)$ then $G \in \mathcal{A}'_n$, i.e., G is a polynomial in*

$$x_n(x_{n+1} - x_{n-1}),\ x_n(1 + x_{n+1}x_{n-1}),\ \textit{and } x_k\ (k \neq n).$$

Proof We suppose that $G \in \mathcal{A}$ is such that $G(t) = O(1)$, where $G(t) := G(x(t))$. We write G as a polynomial in x_n with coefficients in $\mathcal{A}'_n$,

$$G = G_l x_n^l + G_{l-1}x_n^{l-1} + \cdots + G_1 x_n + G_0,$$

where $G_0, \dots, G_l \in \mathcal{A}'_n$. If $l = 0$ then we are done. Let us suppose therefore that l is minimal, but $l > 0$. We will show that this leads to a contradiction. Since each coefficient G_i belongs to $\mathcal{A}'_n$, we have that $G_i(t) = O(1)$. Thus, the pole that $x_n(t)$ has, needs to be compensated by a zero in $G_l(t)$, i.e., $G_l(t) = O(t)$. We show that this implies that $G_l x_n \in \mathcal{A}'_n$. By Euclidean division in $\mathcal{A}'_n$ we can write G_l as

$$G_l = (1 - x_{n+1}^2)K_1 + (1 - x_{n-1}^2)K_2 + K_3, \tag{54}$$

where K_1, K_2 and K_3 belong to $\mathcal{A}'_n$, and where K_3 is of degree 1 at most in x_{n+1} and x_{n-1}: we can write K_3 as

$$K_3 = \kappa_1(x_{n+1} + x_{n-1}) + \kappa_2(1 + x_{n+1}x_{n-1}) + \kappa_3 x_{n+1} + \kappa_4$$

where $\kappa_1, \dots, \kappa_4$ are elements of $\mathcal{A}'_n$ that are independent of x_{n+1} and x_{n-1}. Since $G_l(t) = O(t)$ and $1 - x_{n\pm1}^2(t) = O(t)$ it follows from (54) that $K_3(t) = O(t)$, and so that the leading terms $\kappa_3^{(0)}$ and $\kappa_4^{(0)}$ of $\kappa_3(t)$ and $\kappa_4(t)$ satisfy $\kappa_4^{(0)} = \varepsilon\kappa_3^{(0)}$. Since the leading terms εa_k of all $x_k(t)$, with $k \in \mathbf{Z} \setminus \{n-1, n, n+1\}$, and the leading terms of $w_1(t)$ and $w_2(t)$ are all independent, even modulo ε, it follows that $\kappa_4^{(0)} = \kappa_3^{(0)} = 0$, as κ_4 and κ_3 are independent of $x_{n\pm1}$. Using (53) it follows that

$$\begin{aligned} G_l x_n &= (1 - x_{n+1}^2)x_n K_1 + (1 - x_{n-1}^2)x_n K_2 + \kappa_1 w_1 + \kappa_2 w_2 \\ &= (w_2 - x_{n+1}w_1)K_1 + (w_2 - x_{n-1}w_1)K_2 + \kappa_1 w_1 + \kappa_2 w_2, \end{aligned}$$

where $K_1, K_2, \kappa_1, \kappa_2 \in \mathcal{A}'_n$, showing that $G_l x_n = G'_l \in \mathcal{A}'_n$, as promised. Then,

$$G = (G'_l + G_{l-1})x_n^{l-1} + \cdots + G_1 x_n + G_0,$$

with $G'_l + G_{l-1} \in \mathcal{A}'_n$. This contradicts the minimality of l.

Lemma 5.2 *For $k \neq n$, $\Gamma_k(t) := \Gamma_k(x(t); u(t))$ is of the form*

$$\Gamma_k(t) = \mathcal{F}(a_{k-N}, a_{k-N+1}, \ldots, a_{k+N}, a_+, a_-) + O(t), \tag{55}$$

i.e., the constant term in $\Gamma_k(t)$ is a polynomial in the variables[7] *a_{k-N}, $a_{k-N+1}, \ldots$, a_{k+N}, a_+ and a_- only.*

Proof According to (23), Γ_k depends on $x_{k-N}, \ldots, x_{k+N}$ only. For $k \neq n$ we know from proposition 4.1 that $\Gamma_k(t) = O(1)$, so that proposition 5.1 yields that Γ_k depends on x_n through w_1 and w_2 only, i.e., Γ_k is a polynomial in w_1, w_2 and the x_l with $|k - l| \leq N$ and $l \neq n$. Each of these variables is $O(1)$, so the constant term in Γ_k is a polynomial in their leading terms, which are the parameters a_{k-N}, $a_{k-N+1}, \ldots$, a_{k+N}, a_+ and a_- (see (52)).
It is clear that when $|k - n| > N$ then $\Gamma_k(0)$ is independent of a_+ and a_-, as it cannot contain w_1 or w_2. The following lemma deals with the case of $\Gamma_n(t)$, which is slightly harder because $\Gamma_n(t)$ develops a pole.

Lemma 5.3 *$\Gamma_n(t) := \Gamma_n(x(t); u(t))$ is of the form*

$$\Gamma_n(t) = \frac{\Gamma_{n+1}^{(0)} - \Gamma_{n-1}^{(0)}}{4t} + \mathcal{F}(a_{n-N-1}, \ldots, a_{n+N+1}, a_+, a_-) + O(t)$$

where $\mathcal{F}$ is a polynomial in all its arguments, with a_{n+N+1} and a_{n-N-1} present (linearly).

Proof Consider the following alternative ways of writing $\Gamma_n = \Gamma_n(x; u)$,

$$\Gamma_n(x; u) = (1 - x_n^2) H_n(x; u) + n x_n = x_n G_n(x; u) + H_n(x; u). \tag{56}$$

H_n is a polynomial in $x = (x_i)_{i \in \mathbf{Z}}$, because (28) implies that $H_n(x; u) = \mathcal{V}^u[x_n]$, and because $\partial x_n / \partial t_i = \{x_n, H_i\}$ is always divisible by $1 - x_n^2$, see (16). Also, we have put $G_n(x; u) := n - x_n H_n(x; u)$ to obtain the second equality. The first equation in (56) implies that $H_n(x(t); u(t)) = O(t)$, since $\Gamma_n(x(t); u(t)) = O(t^{-1})$ and $x_n(t) = O(t^{-1})$, while $1 - x_n^2(t) = -1/(4t^2) + O(t^{-1})$. The second equation in (56) then allows us to conclude that $G_n(x(t); u(t)) = O(1)$, and hence also that $G_n(x(t); u) = O(1)$, since u is an arbitrary vector of constants. Thus, G_n is, by proposition 5.1, an element of $\mathcal{A}'_n$, depending (linearly) on the parameters u_i.

Summarizing, the constant term in $\Gamma_n(t)$ will be given by the constant term in $x_n(t) G_n(t)$, hence will depend only on the first two terms $\varepsilon(1 + (a_+ - a_-)t)/(2t)$ of $x_n(t)$ and on the first two terms of $G_n(t)$, where $G_n \in \mathcal{A}'_n$. The latter first two terms can depend only on the first two terms of the variables $x_{n-N}, \ldots, x_{n+N}$, w_1 and w_2 that appear in G_n; the first two terms of their series can be read off from (52), yielding that the constant term in $\Gamma_n(t)$ can only depend on $a_{n-N-1}, \ldots, a_{n+N+1}, a_+, a_-$. Notice that the only dependence on a_{n-N-1} can come from the presence of x_{n-N}, but (23) tells us that x_{n-N} appears linearly in Γ_n, and with a non-zero coefficient. Therefore, the parameter a_{n-N-1} is indeed present in the constant term in Γ_n; similarly, a_{n+N+1} is also present. The leading term of $\Gamma_n(t)$ was already determined in proposition 4.1.

5.2 Parameter restriction

We now show that we can tune the free parameters in the formal Laurent solution $x(t)$ of the self-dual Toeplitz lattice in such a way that $\Gamma_k(t) = 0$ for all $k \in \mathbf{Z}$, as a formal series in t. As it turns out, it will be possible to keep $2N - 1$ parameters arbitrary, and the other ones are determined rationally in terms of these. Together with time it means that the constructed solution depends on $2N$ free parameters, which is the maximum one can hope for in an $2N + 1$ step relation.

[7] Recall that $a_{n\pm1} = \mp 1$ and that a_n does not exist; so $a_{n\pm1}$ and a_n may be thought of as being absent in the list. Thus, $a_\pm$ is the natural substitute for $a_{n\pm1}$.

Table 2: Setting $\Gamma_k(0) = 0$ in the given order allows us to solve for all free parameters in the formal Laurent series, except for the $2N-1$ parameters $a_{n-2N}, \dots, a_{n-2}$, that can be taken arbitrarily. We solve (linearly) for the underlined terms. The fact that Γ_{n+1} incidentally does not depend on the crossed out term a_{n+N+1} allows us to solve $\Gamma_{n+1} = 0$ for a_{n+N}.

step	Γ_k	Γ_k polynomial in	$\Gamma_k^{(0)}$ polynomial in
(1)	Γ_{n-N-1}	$x_{n-2N-1}, \dots, x_{n-1}$	$\underline{a_{n-2N-1}}, \dots, a_{n-1} = 1$
(2)	Γ_{n-N-2}	$x_{n-2N-2}, \dots, x_{n-2}$	$\underline{a_{n-2N-2}}, \dots, a_{n-2}$
(3)	$\vdots$	$\vdots$	$\vdots$
(4)	Γ_{n-N}	$x_{n-2N}, \dots, x_n$	$a_{n-2N}, \dots, a_{n-2}, \underline{a_-}$
(5)	Γ_{n-N+1}	$x_{n-2N+1}, \dots, x_{n+1}$	$a_{n-2N+1}, \dots, a_{n-2}, a_-, \underline{a_+}$
(6)	Γ_{n-N+2}	$x_{n-2N+2}, \dots, x_{n+2}$	$a_{n-2N+2}, \dots, a_{n-2}, a_\pm, \underline{a_{n+2}}$
(7)	$\vdots$	$\vdots$	$\vdots$
(8)	Γ_{n-1}	$x_{n-N-1}, \dots, x_{n+N-1}$	$a_{n-N-1}, \dots, a_{n-2}, a_\pm,$ $a_{n+2}, \dots, \underline{a_{n+N-1}}$
(9)	Γ_{n+1}	$x_{n-N+1}, \dots, x_{n+N+1}$	$a_{n-N+1}, \dots, a_{n-2}, a_\pm$ $a_{n+2}, \dots, \underline{a_{n+N}}, \cancel{a_{n+N+1}}$
(10)	Γ_n	$x_{n-N}, \dots, x_{n+N}$	$a_{n-N-1}, \dots, a_{n-2}, a_\pm$ $a_{n+2}, \dots, \underline{a_{n+N+1}}$
(11)	Γ_{n+2}	$x_{n-N+2}, \dots, x_{n+N+2}$	$a_{n-N+2}, \dots, a_{n-2}, a_\pm$ $a_{n+2}, \dots, \underline{a_{n+N+2}}$
(12)	$\vdots$	$\vdots$	$\vdots$

Proposition 5.4 *Keeping the $2N-1$ parameters $a_{n-2N}, \dots, a_{n-2}$ arbitrary, the other parameters in the formal Laurent series $x(t)$, given by proposition 3.1, can be chosen as rational functions of these parameters, so that $\Gamma_k(t) = 0$, as a formal series in t, for all $k \in \mathbf{Z}$.*

Proof In this proof we will assume that $N > 1$. See Remark 5.5 below for the adaption to the case $N = 1$. According to proposition 4.1, it suffices to determine the parameters in the series $x(t)$ so that $\Gamma_k^{(0)}$, the constant term in $\Gamma_k(t)$, is zero, for all $k \in \mathbf{Z}$. Thus, we need to write $\Gamma_k^{(0)}$ in terms of the parameters in the series $x(t)$. We do this for the different values of k in a very specific order, as indicated in table 2. The second column indicates which Γ_k we consider; it is easy to see that we consider all of them (exactly once); it is understood that steps (6)–(8) are absent when $N = 2$. We know from (23) that for any $k \in \mathbf{Z}$, Γ_k depends only on the variables $x_{k-N}, x_{k-N+1}, \dots, x_{k+N}$, which yields the third column. It is important to point out that the two written variables, which are the extremal terms, are actually present in Γ_k, and that these two variables appear linearly (see proposition 8.2 in the appendix).

The delicate step is in obtaining the last column; the information displayed in it contains the parameters[8] that may appear in $\Gamma_k^{(0)}$, where the underlined term actually does appear, and it appears linearly. Before validating this column in each of the steps, let us first point out how the proposition follows from it. Precisely, we can in each step solve for one of the underlined parameters in terms of the nonunderlined parameters, as the underlined parameter appears linearly in the equation $\Gamma_k^{(0)} = 0$. Using the previous steps, this yields (using the previous steps) inductively a rational formula for each of the parameters, in terms of $a_{n-2N}, \ldots, a_{n-2}$, which remain free. In fact, the variables a_{n-2N-i}, with $i > 0$ are determined in steps (1) – (3); $a_{n-1} = -a_{n+1} = 1$ while a_n does not exist; the variables a_{n+i+1} with $i > 0$ are determined in steps (6) – (12); the only other variables are a_- and a_+, which are determined in steps (4) and (5).

We now show that in each step the parameters that are indiciated in the fourth column of the table appear indeed (linearly) in $\Gamma_k^{(0)}$. This is done by carefully using the leading terms of Γ_k, as given by proposition 8.2. As a general remark, notice that (23) implies that Γ_k contains the variables x_{k-N} and x_{k+N} linearly, but that the behaviour of its coefficients $\prod_{i=0}^{N-1}(1-x_{k+i}^2)$ and $\prod_{i=0}^{N-1}(1-x_{k-i}^2)$, evaluated at t, depends on k, as given in (33).

For step (1) we have that $x_{n-2N-1}(t), \ldots, x_{n-1}(t)$ have no pole in t, so that only their leading coefficients, the parameters $a_{n-2N-1}, \ldots, a_{n-2}$, $a_{n-1} = 1$, can appear. Since x_{n-2N-1} appears (linearly) in Γ_{n-N-1}, with a coefficient $u_N \prod_{i=1}^{N}(1-x_{n-N-i}^2)$ that is non-vanishing for $t = 0$, namely $\prod_{i=1}^{N}(1-x_{n-N-i}^2(0)) = \prod_{i=1}^{N}(1-a_{n-N-i}^2)$, the parameter a_{n-2N-1} appears (linearly) in $\Gamma_{n-N-1}^{(0)}$. The same argument works in steps (2) and (3). Step (4) is more interesting because it involves x_n (linearly). However, x_n appears only in the leading term of Γ_{n-N}, which we can write, using $w_- = x_n(1-x_{n-1}^2)$, as

$$u_N x_n \prod_{i=0}^{N-1}(1-x_{n-N+i}^2) = u_N w_- \prod_{i=0}^{N-2}(1-x_{n-N+i}^2), \qquad u_N \neq 0. \tag{57}$$

Now $w_-(t) = 4\varepsilon a_- + O(t)$, and the other factors in (57) are finite, non-vanishing, which yields the proposed dependence on the parameters in step (4). For step (5), x_n may be present in other terms than the leading term in Γ_{n-N+1}, but in view of proposition 5.1, $\Gamma_{n-N+1} \in \mathcal{A}'_n$ is a polynomial in $x_{n-2N+1}, \ldots, x_{n-1}, x_{n+1}$ and in w_1 and w_2 only. Since their series do not have a pole for $t = 0$, we get an eventual dependence on a_+ and a_-, besides the parameters $a_{n-2N+1}, \ldots, a_{n-2}$. Let us show that a_+ actually appears. The leading term in Γ_{n-N+1} is, according to (23),

$$u_N x_{n+1}(1-x_n^2)(1-x_{n-1}^2) \prod_{i=n-N+1}^{n-2}(1-x_i^2).$$

Since it is the only term in Γ_{n-N+1} that contains x_{n+1} we can write $\Gamma_{n-N+1} = P_1 + P_2$, where

$$P_1 = u_N(x_{n+1}+x_{n-1})(1-x_n^2)(1-x_{n-1}^2) \prod_{i=n-N+1}^{n-2}(1-x_i^2),$$

and P_2 is independent of x_{n+1}, so P_2 depends only on $x_{n-2N+1}, \ldots, x_n$. Now $P_1(t) = O(1)$, since

$$x_{n+1}(t) + x_{n-1}(t) = O(t), \qquad 1-x_n^2(t) = O(t^{-2}), \qquad 1-x_{n-1}^2(t) = O(t),$$

while the other factors $1-x_i^2(t)$ that appear in $P_1(t)$ are $O(1)$. Since $\Gamma_{n-N+1}(t) = O(1)$ this implies that $P_2(t) = O(1)$, so that P_2 satisfies the hypothesis of proposition 5.1; since P_2 is independent

[8] Besides the constants $u_1, \ldots, u_N$ that define P.

of x_{n+1} we may conclude, as in step (4), that P_2 is independent of a_+. On the other hand $P_1(0)$ depends (linearly) on a_+, as

$$(x_{n+1}(t) + x_{n-1}(t))(1 - x_n^2(t))(1 - x_{n-1}^2(t)) = 8\varepsilon a_-(a_- + a_+) + O(t).$$

The conclusion is that $\Gamma^{(0)}_{n-N+1} = P_1(0) + P_2(0)$ depends (linearly) on a_+.

We are at step (6). Skip this step and steps (7) and (8) when $N = 2$. Proposition 5.1 implies that $\Gamma^{(0)}_{n-N+2}$ can only depend on the proposed parameters, and that the dependence comes from the constant terms of the series in (52). The dependence of $\Gamma^{(0)}_{n-N+2}$ on a_{n+2} comes only from the leading term $u_N x_{n+2}(1 - x_{n+1}^2)(1 - x_n^2)(1 - x_{n-1}^2)\prod_{i=0}^{N-4}(1 - x_{n-N+2+i}^2)$ which, at t, is $O(1)$, since $(1 - x_{n+1}^2(t))(1 - x_n^2(t))(1 - x_{n-1}^2(t)) = O(1)$ and non-vanishing. It follows that $\Gamma^{(0)}_{n-N+2}$ depends on a_{n+2} (linearly). The same happens in steps (7) and (8), as the leading term will always contain the product $(1 - x_{n+1}^2)(1 - x_n^2)(1 - x_{n-1}^2)$ which is finite and non-zero for $t = 0$.

A new phenomenon arises in step (9). Notice that we have moved to Γ_{n+1}, keeping Γ_n for step (10). The leading term of Γ_{n+1} is

$$u_N x_{n+N+1} \prod_{i=1}^{N} (1 - x_{n+i}^2),$$

which does not contribute to $\Gamma^{(0)}_{n+1}$, since $1 - x_{n+1}^2(t) = O(t)$, while all other factors in this term are finite in t. Therefore, $\Gamma^{(0)}_{n+1}$ is independent of a_{n+N+1}. To show that $\Gamma^{(0)}_{n+1}$ depends on a_{n+N} we need to investigate the next term in Γ_{n+1}, the one that contains x_{n+N}, because it is the only one that might lead to a dependence on a_{n+N}. According to proposition 8.2, this term consists of the following three pieces,

$$\begin{aligned} u_{N-1} x_{n+N} \prod_{i=0}^{N-2} (1 - x_{n+1+i}^2) - u_N x_{n+N}^2 x_{n+N-1} \prod_{i=0}^{N-2} (1 - x_{n+1+i}^2) \\ -2u_N x_{n+N} \prod_{i=0}^{N-2} (1 - x_{n+1+i}^2) \sum_{j=0}^{N-2} x_{n+j+1} x_{n+j}. \end{aligned} \tag{58}$$

The two terms on the first line of (58) do not contribute to $\Gamma^{(0)}_{n+1}$, again because both terms contain $1 - x_{n+1}^2$, and all other terms are finite for $t = 0$. The third term however does contribute, when $j = 0$, as $x_n(t)(1 - x_{n+1}^2(t)) \sim a_+ + O(t)$; moreover, this term is the only one that involves a_{n+N}, so that the latter parameter appears (linearly) in $\Gamma^{(0)}_{n+1}$. For step (10) the presence of a_{n+N+1} was established in lemma 5.3. Starting from step (11) the leading coefficients do not contain $1 - x_{n\pm1}^2$ or $1 - x_n^2$ anymore, so that everything goes smoothly.

Remark 5.5 *When $N = 1$ the polynomial that defines the recursion relation reduces to*

$$\Gamma_k = kx_k + u_1(1 - x_k^2)(x_{k+1} + x_{k-1}).$$

Steps (4)–(9) then get replaced by two steps in which we consider $\Gamma_{n\pm1}$, which allows us to determine $a_\pm$. Indeed, substituting the series $x(t)$ in $\Gamma_{n\pm1}$ yields for the leading term ($t = 0$):

$$(n \pm 1) + 4u_1 a_\pm = 0.$$

The other parameters are determined as in the general case.

6 Restricting the formal Laurent solutions: the general case

In this section we will do a similar analysis as the one that has been done for the case of the self-dual Toeplitz lattice in section 5.

6.1 Structure of the polynomials Γ_k and $\tilde{\Gamma}_k$

We first investigate on which parameters the leading term(s) in the polynomials Γ_k and $\tilde{\Gamma}_k$ depends on the free parameters. We denote by $\mathcal{A}$ the algebra of all polynomials in the variables x_i and y_i, where $i \in \mathbf{Z}$, while $\mathcal{A}_n$ stands for the subalgebra of $\mathcal{A}$ that consists of all polynomials that do not depend on x_n and on y_n. Consider the following four polynomials[9]

$$\begin{array}{rclcrcl} w_1 & = & x_n y_{n-1} + y_n x_{n+1}, & \qquad & w_2 & = & x_n + x_{n-1} y_n x_{n+1}, \\ w_1^\sigma & = & y_n x_{n-1} + x_n y_{n+1}, & \qquad & w_2^\sigma & = & y_n + y_{n-1} x_n y_{n+1}. \end{array} \tag{59}$$

For future use, observe that these polynomials are linked by the following identity:

$$x_n(w_2^\sigma - y_{n-1} w_1^\sigma) = y_n(w_2 - x_{n-1} w_1), \tag{60}$$

in fact both expressions in (60) are equal to $x_n y_n 1 - x_{n-1} y_{n-1}$. We denote by $\mathcal{A}_n'$ the subalgebra of $\mathcal{A}$ that consists of all polynomials that can be written in terms of these four polynomials, besides all x_i and y_i, with $i \neq n$. The polynomials w have the following series in t, when the first few[10] terms of the series $x_i(t)$ and $y_i(t)$ that are constructed in proposition 3.2, are substituted in them.

$$\begin{aligned} w_1(t) &= \Omega b_{n-1} - a_- + (a_+ a_{n+2} b_{n-1} - a_- a_{n-1} b_{n-2}) t + O(t^2), \\ w_2(t) &= \Omega + (a_+ a_{n+2} + a_- a_{n-2}) t + O(t^2), \end{aligned} \tag{61}$$

where

$$\Omega := \frac{a_{n-1} a_{n+1}}{(a_{n+1} - a_{n-1})^2} \left(a_{n-1}(2a - a_+) - a_{n+1}(2a - a_-) \right).$$

The formal Laurent series for the other polynomials in (59) is found from it by using the automorphism σ (see (38)), which yields in particular

$$\sigma(\Omega) = \Omega b_{n-1} b_{n+1} + a_+ b_{n-1} - a_- b_{n+1}. \tag{62}$$

It follows that if $G \in \mathcal{A}_n'$ then $G(t) = O(1)$, where $G(t) := G(x(t), y(t))$, with $x(t)$ and $y(t)$ as above. We will show that the converse is also true, so that the algebra $\mathcal{A}_n'$ plays in the general case a similar rôle as in the self-dual case. For this we need the following lemma.

Lemma 6.1 *Let G be a polynomial in $\mathcal{A}_n'$ that is independent of w_2 and none of whose terms contains $x_{n+1} y_{n+1}$ or $x_{n-1} y_{n-1}$. If $G(t) = O(t)$ then $G = 0$, as a formal series in t.*

Proof It follows from (61) that

$$\begin{pmatrix} w_1(0) \\ w_1^\sigma(0) \\ w_2^\sigma(0) \end{pmatrix} = \frac{T}{(a_{n+1} - a_{n-1})^2} \begin{pmatrix} a(a_{n-1} - a_{n+1}) \\ a_+ a_{n+1} \\ a_- a_{n-1} \end{pmatrix}$$

[9] Recall that σ denotes the involution that permutes all $x_i \leftrightarrow y_i$.

[10] A priori, one needs to compute an extra term in the series $z_k(t)$ (see proposition 3.2) in order to find the shown terms in (61). After proposition 6.2 we will however show how such a cumbersome can be avoided.

where

$$T := \begin{pmatrix} 2a_{n+1} & -a_{n-1} & 2a_{n+1} - a_{n-1} \\ 2a_{n-1} & a_{n+1} - 2a_{n-1} & a_{n+1} \\ 2 & \frac{1}{a_{n-1}}(a_{n+1} - 2a_{n-1}) & \frac{1}{a_{n+1}}(2a_{n+1} - a_{n-1}) \end{pmatrix}.$$

T is an invertible matrix, since $\det T = -2(a_{n-1} - a_{n+1})^4/(a_{n-1}a_{n+1})$. Let G be a polynomial in $\mathcal{A}'_n$ that is independent of w_2 and suppose that $G(0) = 0$. We write $G = \sum_{ijk} g_{ijk} w_1^i (w_1^\sigma)^j (w_2^\sigma)^k$, where g_{ijk} is a polynomial in the variables x_k and y_k with $k \neq n$ only. Notice that $g_{ijk}(0)$ is independent of a, a_+ and a_-. Therefore, the fact that T is invertible and that a, a_+ and a_- are independent free variables implies that $g_{ijk}(t) = O(t)$ for any i, j, k. If we assume now in addition that g_{ijk} does not contain either product $x_{n+1}y_{n+1}$ or $x_{n-1}y_{n-1}$ then it is clear that $g_{ijk} = 0$ since the leading terms a_k of x_k and b_k of y_k are independent ($k \neq n$), except that $a_{n+1}b_{n+1} = 1 = a_{n-1}b_{n-1}$.

Proposition 6.2 *For $G \in \mathcal{A}$, let $G(t) := G(x(t), y(t))$, where $(x(t), y(t))$ is the formal Laurent solution to the first vector field of the Toeplitz lattice, constructed in proposition 3.2. If $G(t) = O(1)$ then $G \in \mathcal{A}'_n$, i.e., G depends only on x_n and y_n through the polynomials w_1, w_2, w_1^σ and w_2^σ.*

Proof Given $G \in \mathcal{A}$ we may write G as a polynomial in x_n and y_n, with coefficients in $\mathcal{A}'_n$; in fact, writing $x_n = w_2 - x_{n-1}y_nx_{n+1}$ we may assume that G is independent of x_n and we write

$$G = G_l y_n^l + G_{l-1} y_n^{l-1} + \cdots + G_1 y_n + G_0,$$

where $G_0, \ldots, G_l \in \mathcal{A}'_n$. We suppose that this is done in such a way that l is minimal. If $l = 0$ then $G \in \mathcal{A}'_n$ and we are done; assume therefore that $l > 1$. We will show that $G_l y_n \in \mathcal{A}'_n$, which is in contradiction with the minimality of l, like in the self-dual case. We first show that we may assume that w_2 is absent in $G_l y_n$. If we substitute $x_n = w_2 - x_{n-1}y_nx_{n+1}$ in the identity (60) then we find

$$y_n w_2 = w_2(w_2^\sigma - y_{n-1}w_1^\sigma) + y_n(w_1 x_{n-1} + x_{n-1}x_{n+1}(y_{n-1}w_1^\sigma - w_2^\sigma)),$$

which allows us to replace any term in $G_l y_n$ that contains w_2, or a power of it, by a term of lower degree in w_2, at the cost of changing G_{l-1}, so that we can eventually remove w_2 entirely from the leading coefficient G_l. Assuming that G_l does not depend on w_2 we perform an Euclidean division in $\mathcal{A}'_n$,

$$G_l = (1 - x_{n-1}y_{n-1})K_1 + (1 - x_{n+1}y_{n+1})K_2 + K_3, \tag{63}$$

where K_1, K_2 and K_3 belong to $\mathcal{A}'_n$, with K_3 independent of w_2 and not containing $x_{n-1}y_{n-1}$ or $x_{n+1}y_{n+1}$.

Assume now that $G(t) = O(1)$. Since all $G_i(t)$ are $O(1)$, as $G_i \in \mathcal{A}'_n$, we must have that $G_l(t) = O(t)$, as $y_n(t)$ has a pole. Then (63) implies that $K_3(t) = O(t)$, since $1 - x_{n\pm1}(t)y_{n\pm1}(t) = O(t)$. This means that K_3 satisfies the conditions of lemma 6.1, hence that $K_3 = 0$. The identities

$$\begin{aligned}(1 - x_{n-1}y_{n-1})y_n &= w_2^\sigma - y_{n-1}w_1^\sigma \in \mathcal{A}'_n \\ (1 - x_{n+1}y_{n+1})y_n &= w_2^\sigma - y_{n+1}w_1 \in \mathcal{A}'_n\end{aligned}$$

then imply that $G_l y_n \in \mathcal{A}'_n$, as was to be shown.

As a first application of this proposition, we show how the shown terms in (61) can easily be computed. Since $w_i(t) = O(1)$ we also have $\frac{dw_i}{dt}(t) = O(1)$ for $i = 1, 2$. By proposition 6.2, $\frac{dw_i}{dt} \in \mathcal{A}'_n$, in fact

$$\frac{dw_1}{dt} = \frac{d}{dt}(x_n y_{n-1} + y_n x_{n+1})$$

$$
\begin{aligned}
&= y_{n-2}x_{n-1}x_{n+1}w_-^\sigma + x_{n+2}w_+^\sigma - y_{n-2}(1-x_{n-1}y_{n-1})w_2 + x_{n+1}y_{n+1} - x_{n-1}y_{n-1},\\
\frac{dw_2}{dt} &= \frac{d}{dt}(x_n + x_{n-1}y_nx_{n+1})\\
&= x_{n+2}x_{n-1}w_+^\sigma - x_{n-2}x_{n+1}w_-^\sigma + x_{n+1}(1-x_{n-1}y_{n-1}) - x_{n-1}(1-x_{n+1}y_{n+1}),
\end{aligned}
$$

where $w_\pm^\sigma := (1-x_{n\pm1}y_{n\pm1})y_n$, with $w_\pm^\sigma(0) = \pm a_\pm b_{n\mp1} + O(t)$. Since $x_{n\pm1}y_{n\pm1} = 1$, it follows that

$$
\begin{aligned}
\frac{dw_1}{dt}(0) &= b_{n-2}a_{n-1}a_{n+1}w_-^\sigma(0) + a_{n+2}w_+^\sigma(0) = a_+a_{n+2}b_{n-1} - a_-a_{n-1}b_{n-2},\\
\frac{dw_2}{dt}(0) &= a_{n+2}a_{n-1}w_+^\sigma(0) - a_{n-2}a_{n+1}w_-^\sigma(0) = a_{n+2}a_+ + a_{n-2}a_-,
\end{aligned}
$$

which yield after integration the linear terms in (61). The same formulas can be used to show that $w_1^{(2)}$ and $w_2^{(2)}$, which are the t^2 terms in $w_1(t)$ and in $w_2(t)$, depend only on the parameters $c_{n-3},\dots,c_{n+3}$, a_+, a_- and a; the precise formula will not be needed, except that they depend on c_{n+3} as follows:

$$
\begin{aligned}
w_1^{(2)} &= x_{n+2}^{(1)}w_+^\sigma(0)/2 + \cdots = a_{n+3}a_+b_{n-1}(1-a_{n+2}b_{n+2})/2 + \cdots,\\
w_2^{(2)} &= x_{n+2}^{(1)}x_{n-1}(0)w_+^\sigma(0)/2 + \cdots = a_{n+3}a_+(1-a_{n+2}b_{n+2})/2 + \cdots,
\end{aligned}
\tag{64}
$$

where the dots are independent of a_{n+3} (and of b_{n+3}).

The following lemma is the analog of lemma 5.2 and is proven in exactly the same way.

Lemma 6.3 *If $k \neq n$, then the series $\Gamma_k(t) := \Gamma_k(x(t),y(t);u(t))$ and $\tilde\Gamma_k(t) := \tilde\Gamma_k(x(t),y(t);u(t))$ are of the form*

$$
\begin{aligned}
\Gamma_k(t) &= \mathcal{F}(a_{k-N}, c_{k-N+1},\dots,c_{k+N-1}, a_{k+N}, a_\pm, a) + O(t),\\
\tilde\Gamma_k(t) &= \tilde{\mathcal{F}}(b_{k-N}, c_{k-N+1},\dots,c_{k+N-1}, b_{k+N}, a_\pm, a) + O(t),
\end{aligned}
$$

where we recall that $c_i = (a_i, b_i)$ and that $a_{n\pm1}b_{n\pm1} = 1$, and $\mathcal{F}, \tilde{\mathcal{F}}$ are polynomials in their arguments.

For $k = n$ the corresponding result is more complicated and the method of proof is different from the one in the self-dual case (lemma 5.3).

Lemma 6.4 *The constant terms $\Gamma_n^{(0)}$ and $\tilde\Gamma_n^{(0)}$ are of the form*

$$
\begin{pmatrix} \Gamma_n^{(0)} \\ \tilde\Gamma_n^{(0)} \end{pmatrix} = A \begin{pmatrix} a_{n+N+1} \\ b_{n+N+1} \end{pmatrix} + \mathcal{F}(c_{n-N-1},\dots,c_{n+N}, a_\pm, a),
$$

where A is an invertible 2×2 matrix and $\mathcal{F}$ is a polynomial 2-vector that depends on the listed free parameters only. See proposition 4.2 for the leading terms of $\Gamma_n(t)$ and $\tilde\Gamma_n(t)$.

Proof We will assume in our proof that $N > 2$, see Remark 6.5 below. The proof is based on the explicit expression for Γ_n that is given in proposition 8.2 (see the appendix), which we write in the form $\Gamma_n = (1-x_ny_n)H_n + nx_n$, where

$$
\begin{aligned}
H_n = {} & u_Nx_{n+N}\prod_{i=1}^{N-1}(1-x_{n+i}y_{n+i}) - u_Nx_{n+N-1}^2y_{n+N-2}\prod_{i=1}^{N-2}(1-x_{n+i}y_{n+i})\\
& -u_Nx_{n+N-1}\left(x_ny_{n-1} + 2\sum_{j=1}^{N-2}x_{n+j}y_{n+j-1}\right)\prod_{i=1}^{N-2}(1-x_{n+i}y_{n+i})\\
& +(u_{N-1}x_{n+N-1} - u_{-N}y_{n+N-1}x_{n-1}x_n)\prod_{i=1}^{N-2}(1-x_{n+i}y_{n+i})\\
& + \mathcal{F}(x_{n-N+1},\dots,x_{n+N-2}, y_{n-N+2},\dots,y_{n+N-2})\\
& - (u_Nx_nx_{n+1}y_{n-N+1} - u_{-N}x_{n-N}(1-x_{n-N+1}y_{n-N+1}))\prod_{i=1}^{N-2}(1-x_{n-i}y_{n-i}).
\end{aligned}
$$

Our first claim is that $\mathcal{F} \in \mathcal{A}'_n$. Since $\Gamma_n(t)$ and $1 - x_n(t)y_n(t)$ have a double pole, while $x_n(t)$ has a simple pole, $H_n(t) = O(1)$. The terms in the above expression that do not involve x_n or y_n are also $O(1)$, because $x_k(t) = O(1)$ and $y_k(t) = O(1)$ for $k \neq n$. There are a few terms that contain x_n or y_n (linearly), but they are all of the form $x_n(1 - x_{n+1}y_{n+1})$, $y_n(1 - x_{n+1}y_{n+1})$ or $x_n(1 - x_{n-1}y_{n-1})$, which are both $O(1)$. It follows that $\mathcal{F}(t; u(t)) = O(1)$, and hence that $\mathcal{F}(t; u) = O(1)$. Thinking of u as constants we have, in view of proposition 6.2, that $\mathcal{F} \in \mathcal{A}'_n$.

Since $1 - x_n(t)y_n(t)$ has a double pole, only the first three terms of $1 - x_n(t)y_n(t)$ and of $\mathcal{F}(t)$ can contribute to the constant term in $(1 - x_n(t)y_n(t))\mathcal{F}(t)$; in view of table 1, this contribution can only yield a dependence on the parameters $c_{n-N-1}, \ldots, c_{n+N}, a_\pm$ and a.

We now turn to the other terms in H_n and we use their explicit form to show that they only depend on the listed parameters. Let us first consider the following terms that do not involve x_n or y_n,

$$\begin{aligned} -\Big(u_N x_{n+N-1}^2 y_{n+N-2} &+ 2u_N x_{n+N-1} \textstyle\sum_{j=2}^{N-2} x_{n+j}y_{n+j-1} \\ &-u_{N-1}x_{n+N-1}\Big) \textstyle\prod_{i=1}^{N-2}(1 - x_{n+i}y_{n+i}) + u_{-N}x_{n-N} \prod_{i=1}^{N-1}(1 - x_{n-i}y_{n-i}). \end{aligned} \tag{65}$$

Since $1 - x_{n\pm i}y_{n\pm i}$ has a simple zero for $i = 1$ and is $O(1)$ for $i > 1$ we have that $\prod_{i=1}^{N-2}(1 - x_{n+i}y_{n+i})$ and $\prod_{i=1}^{N-1}(1 - x_{n-i}y_{n-i})$ have a simple zero, so we only need to look for the parameters that appear in the first two terms of the coefficients. The former add nothing new to the above parameter list. For the coefficients of the first one for example, we read off from table 1 that the constant and linear terms of $x_{n+N-1}^2(t)y_{n+N-2}(t)$ only depend on $a_{n+N}, c_{n+N-1}, c_{n+N-2}$ and b_{n+N-3}, which falls inside the proposed limits. Notice in particular that neither a_{n+N+1} nor b_{n+N+1} appear in this term. We arrive similarly at the same conclusion for the other three terms in (65). Notice that the lowest free parameter that appears is a_{n-N-1}; it comes from the last term in (65).

We now get to the terms that contain x_n or y_n. As we already noticed these terms always come with $1 - x_{n+1}y_{n+1}$ or $1 - x_{n-1}y_{n-1}$. As $x_n(t)(1 - x_{n\pm1}(t)y_{n\pm1}(t)) = O(1)$ we must investigate the first three terms in the remaining factors. For the term

$$-u_N x_n(1 - x_{n-1}y_{n-1})x_{n+1}y_{n-N+1} \prod_{i=2}^{N-2}(1 - x_{n-i}y_{n-i})$$

we need to look at $x_{n+1}y_{n-N+1}\prod_{i=2}^{N-2}(1 - x_{n-i}y_{n-i})$, which yields terms with a low index, the lowest coming from the coefficient in t^2 in $y_{n-N+1}(t)$, to wit b_{n-N-1} and a_{n-N}. The other three terms that involve x_n or y_n can be written as

$$B := -\left(\begin{array}{c} x_n(1 - x_{n+1}y_{n+1})\,(u_N x_{n+N-1}y_{n-1} + u_{-N}y_{n+N-1}x_{n-1}) \\ +2u_N y_n(1 - x_{n+1}y_{n+1})x_{n+N-1}x_{n+1} \end{array}\right) \prod_{i=2}^{N-2}(1 - x_{n+i}y_{n+i}).$$

Again, since $1 - x_n y_n$ has a double pole the first three terms in $B(t) = B + B_1 t + B_2 t^2 + O(t^3)$ will contribute to the constant term in $(1 - x_n(t)y_n(t))B(t)$. It is clear that B_2 will contain a_{n+N+1}, coming from $x_{n+N-1}^{(2)}$ and b_{n+N+1}, coming from y_{n+N-1}. To know the precise value, it suffices to substitute the relevant coefficients of the formal Laurent series $x(t), y(t)$ in the following part of B_2,

$$-\left(\begin{array}{c} (x_n(1 - x_{n+1}y_{n+1}))^{(0)}\left(u_N x_{n+N-1}^{(2)}y_{n-1}^{(0)} + u_{-N}y_{n+N-1}^{(2)}x_{n-1}^{(0)}\right) \\ +2u_N(y_n(1 - x_{n+1}y_{n+1}))^{(0)}x_{n+N-1}^{(2)}x_{n+1}^{(0)} \end{array}\right) \prod_{i=2}^{N-2}(1 - x_{n+i}y_{n+i})^{(0)},$$

which gives, by using proposition 3.2, and in particular $-(x_n(1-x_{n+1}y_{n+1}))^{(0)} = a_+a_{n+1}$ and $-(y_n(1-x_{n+1}y_{n+1}))^{(0)} = -a_+b_{n-1}$,

$$-\frac{a_+a_{n+1}}{2}(u_Na_{n+N+1}b_{n-1} - u_{-N}a_{n-1}b_{n+N+1})\prod_{i=2}^{N}(1-a_{n+i}b_{n+i}) + \cdots, \tag{66}$$

where the dots are independent of a_{n+N+1} and b_{n+N+1}. There remains one term in H_n, namely the leading term $C := u_Nx_{n+N}\prod_{i=1}^{N-1}(1-x_{n+i}y_{n+i})$. It does not involve x_n but does involve $1-x_{n+1}y_{n+1}$, which will also lead to a dependence on a_{n+N+1}. Writing $C(t) = C_1t + C_2t^2 + O(t^3)$ we have that

$$C_2 \quad = \quad u_Na_{n+N+1}a_+(a_{n+1}-a_{n-1})b_{n-1}\prod_{i=2}^{N}(1-a_{n+i}b_{n+i}) + \cdots,$$

where the dots are again independent of a_{n+N+1} and b_{n+N+1}. Summing up, we have that the leading terms in $\Gamma_n^{(0)}$ are given by

$$-\frac{a_+(1-x_ny_n)^{(0)}}{2}\left(u_N(a_{n+1}-2a_{n-1})b_{n-1}a_{n+N+1} + u_{-N}a_{n+1}a_{n-1}b_{n+N+1}\right)\prod_{i=2}^{N}(1-a_{n+i}b_{n+i}).$$

By duality, the leading terms in $\tilde{\Gamma}_n^{(0)}$ are given by

$$\frac{a_+(1-x_ny_n)^{(0)}}{2a_{n-1}}\left(u_{-N}(a_{n-1}-2a_{n+1})b_{n+N+1} + u_Nb_{n-1}a_{n+N+1}\right)\prod_{i=2}^{N}(1-a_{n+i}b_{n+i}).$$

We may conclude that

$$\begin{pmatrix} \Gamma_n^{(0)} \\ \tilde{\Gamma}_n^{(0)} \end{pmatrix} = A\begin{pmatrix} a_{n+N+1} \\ b_{n+N+1} \end{pmatrix} + \mathcal{F}(c_{n-N-1},\ldots,c_{n+N},a_\pm,a), \tag{67}$$

where

$$A = \frac{a_+a_{n+1}}{2(a_{n-1}-a_{n+1})^2}\begin{pmatrix} (a_{n+1}-2a_{n-1})u_N & a_{n+1}a_{n-1}^2u_{-N} \\ -\dfrac{u_N}{a_{n-1}} & (2a_{n+1}-a_{n-1})u_{-N} \end{pmatrix}\prod_{i=2}^{N}(1-a_{n+i}b_{n+i}).$$

Since

$$\det A = \frac{u_Nu_{-N}}{2}\left(\frac{a_+a_{n+1}}{a_{n+1}-a_{n-1}}\prod_{i=2}^{N}(1-a_{n+i}b_{n+i})\right)^2,$$

A is invertible.

Remark 6.5 *The above proof breaks down at several places when $N=2$. The polynomial H_n then reduces to*

$$\begin{aligned} H_n \quad &= \quad u_2(x_{n+2}(1-x_{n+1}y_{n+1}) - x_{n+1}w_1) + u_1x_{n+1} \\ &\quad + u_{-2}(x_{n-2}(1-x_{n-1}y_{n-1}) - x_{n-1}w_1^\sigma) + u_{-1}x_{n-1}. \end{aligned} \tag{68}$$

Using (64) and proposition 3.2 we find that H_n depends in the following way on a_{n+3} and b_{n+3},

$$\begin{aligned} &u_2(x_{n+2}^{(1)}(1-x_{n+1}y_{n+1})^{(1)} - x_{n+1}^{(0)}w_1^{(2)}) - u_{-2}x_{n-1}^{(0)}w_1^{\sigma(2)} \\ &\qquad = \quad \frac{a_+(1-a_{n+2}b_{n+2})}{2a_{n-1}}(u_2(a_{n+1}-2a_{n-1})a_{n+3} + u_{-2}a_{n+1}a_{n-1}^2b_{n+3}). \end{aligned}$$

It leads as in the case $N>2$ to (67), with precisely the same matrix A.

6.2 Parameter restriction

The parameter restriction works more or less like in the self-dual case, the main difference coming from the fact that in the self-dual case we had to put all $\Gamma_k^{(0)} = 0$, while in the general case the tangency condition is equivalent to

1. $\Gamma_k(t) = O(t)$ and $\tilde{\Gamma}_k(t) = O(t)$ for all k with $k \neq n+1$;
2. $\Gamma_{n-1}(t) = O(t^2)$;
3. $\Gamma_{n+1}(t) = O(t)$.

In a sense, the condition $\Gamma_{n-1}(t) = O(t^2)$ replaces the condition $\tilde{\Gamma}_{n+1}(t) = O(t)$, which is redundant because it is a consequence of the other conditions (see proposition 4.3).

Proposition 6.6 *Keeping the $4N-1$ parameters*[11] $c_{n-2N}, \ldots, c_{n-2}, a_{n-1}$ *arbitrary, the other parameters in the formal Laurent series $(x(t), y(t))$, given by proposition 3.2, can be chosen as rational functions of these parameters, so that $\Gamma_k(t) = 0$ and $\tilde{\Gamma}_k(t) = 0$, identically in t, for all $k \in \mathbf{Z}$.*

Proof We give the proof in the case $N > 1$ only, leaving the case $N = 1$ to the reader (see Remark 5.5 for the self-dual $N = 1$ case). As in the self-dual case, we summarize the order in which we treat the different equations in a table (see table 3). The second column shows which $\Delta_k = (\Gamma_k, \tilde{\Gamma}_k)$ we consider. For $k \neq n \pm 1$ it is clear that each Δ_k appears (precisely once). The fact that Γ_{n-1} appears on line ($9a$), while Δ_{n-1} already appears on line (8) comes from the fact that we consider in line ($9a$) the coefficient in t of $\Gamma_{n-1}(t)$ (rather than the coefficient in t^0); similarly, $\tilde{\Gamma}_{n+1}$ is absent because the nullity of $\tilde{\Gamma}_{n+1}(0)$ is a consequence of the nullity of the other $\Delta_k(0)$ (proposition 4.3). We know from proposition 8.2 that for any $k \in \mathbf{Z}$,

$$\begin{aligned} \Gamma_k(x, y; u) &\in \mathbf{R}[x_{k-N}, \ldots, x_{k+N}, y_{k-N+1}, \ldots, y_{k+N-1}], \\ \tilde{\Gamma}_k(x, y; u) &\in \mathbf{R}[x_{k-N+1}, \ldots, x_{k+N-1}, y_{k-N}, \ldots, y_{k+N}], \end{aligned} \tag{69}$$

so that

$$\Delta_k(x, y; u) \in \mathbf{R}[z_{k-N}, \ldots, z_{k+N}].$$

This leads, with no effort, to the third column of the table. For future use, let us recall that Γ_k depends (linearly) on x_{k-N} and on x_{k+N}, while $\tilde{\Gamma}_k$ depends (linearly) on y_{k-N} and on y_{k+N}.

Let us now turn, line by line, to the last column, which demands a careful inspection of the polynomials Γ_k and $\tilde{\Gamma}_k$. In particular, we show that these polynomials depend on the underlined parameter(s) (linearly), in such a way that one can solve for them. In steps (1) – (3) we have that z_n is absent, so that $\Delta_{n-N-k}(0)$ ($k \geq 1$) depends on $z_{n-2N-k}(0), \ldots, z_{n-k}(0)$ only, i.e., on $c_{n-2N-k}, \ldots, c_{n-k}$. Now Γ_{n-N-k} depends on x_{n-2N-k} (linearly), but not on y_{n-2N-k}, while the opposite is true for $\tilde{\Gamma}_{n-N-k}$, so that we can solve the equation $\Gamma_{n-N-k}(0) = 0$ linearly for a_{n-2N-k}, and similarly $\tilde{\Gamma}_{n-N-k}(0) = 0$ can be solved linearly for b_{n-2N-k} in terms of $c_{n-2N-k+1}, \ldots, c_{n-k}$. For $k = 1$ this gives a_{n-2N-1} (resp. b_{n-2N-1}) in terms of the $4N-1$ parameters $c_{n-2N}, \ldots, c_{n-2}, a_{n-1}$, so that by taking $k = 2, 3, \ldots$, we get recursively c_{n-2N-k} in terms of these parameters, for all $k \geq 1$.

We now get to step (4) which is different because Δ_{n-N} involves x_n and y_n. As for Γ_n, according to proposition 8.2, x_n appears only in the leading term of Γ_{n-N}, which we can write as

$$u_N x_n \prod_{i=0}^{N-1} (1 - x_{n-N+i} y_{n-N+i}) = u_N w_- \prod_{i=n-N}^{n-2} (1 - x_i y_i), \qquad u_N \neq 0,$$

[11] Recall that $c_k = (a_k, b_k)$ and that $a_{n\pm1} b_{n\pm1} = 1$.

Table 3: The tangency condition allows us to solve for all free parameters in the formal Laurent series, except for the $4N-1$ parameters $c_{n-2N}, \ldots, c_{n-2}, a_{n-1}$, that can be taken arbitrarily. The equations can be solved linearly for the underlined terms.

step	Δ_k	Δ_k polynomial in	$\Delta_k^{(0)}, \Gamma_{n-1}^{(1)}, \Gamma_{n+1}^{(0)}$ polynomial in
(1)	Δ_{n-N-1}	$z_{n-2N-1}, \ldots, z_{n-1}$	$\underline{c_{n-2N-1}}, \ldots, c_{n-1}$
(2)	Δ_{n-N-2}	$z_{n-2N-2}, \ldots, z_{n-2}$	$\underline{c_{n-2N-2}}, \ldots, c_{n-2}$
(3)	$\vdots$	$\vdots$	$\vdots$
(4)	Δ_{n-N}	$z_{n-2N}, \ldots, z_n$	$c_{n-2N}, \ldots, c_{n-1}, \underline{a_-}, \underline{a_{n+1}}$
(5)	Δ_{n-N+1}	$z_{n-2N+1}, \ldots, z_{n+1}$	$c_{n-2N+1}, \ldots, c_{n+1}, a_-, \underline{a_+}, \underline{a}$
(6)	Δ_{n-N+2}	$z_{n-2N+2}, \ldots, z_{n+2}$	$c_{n-2N+2}, \ldots, c_{n+2}, a_\pm, a, \underline{c_{n+2}}$
(7)	$\vdots$	$\vdots$	$\vdots$
(8)	Δ_{n-1}	$z_{n-N-1}, \ldots, z_{n+N-1}$	$c_{n-N-1}, \ldots, c_{n-2}, a_\pm, a$
			$c_{n+2}, \ldots, \underline{c_{n+N-1}}$
(9a)	Γ_{n-1}	$x_{n-N-1}, z_{n-N}, \ldots$	$a_{n-N-2}, c_{n-N-1}, \ldots$
		$\ldots, z_{n+N-2}, x_{n+N-1}$	$\ldots, c_{n+N-1}, \underline{a_{n+N}}$
(9b)	Γ_{n+1}	$x_{n-N+1}, z_{n-N+2}, \ldots$	$a_{n-N+1}, c_{n-N}, \ldots$
		$\ldots, z_{n+N}, x_{n+N+1}$	$\ldots, c_{n+N-1}, \underline{b_{n+N}}, \cancel{a_{n+N+1}}$
(10)	Δ_n	$z_{n-N}, \ldots, z_{n+N}$	$c_{n-N-1}, \ldots, a_{n-2}, a_\pm$
			$a_{n+2}, \ldots, \underline{c_{n+N+1}}$
(11)	Δ_{n+2}	$z_{n-N+2}, \ldots, z_{n+N+2}$	$c_{n-N+2}, \ldots, a_{n-2}, a_\pm$
			$a_{n+2}, \ldots, \underline{c_{n+N+2}}$
(12)	$\vdots$	$\vdots$	$\vdots$

where $w_- := x_n(1-x_{n-1}y_{n-1}) \in \mathcal{A}'_n$, as $w_-(t) = a_-a_{n-1} + O(t)$. Therefore, using (69),

$$\Gamma_{n-N}(0) = u_N a_- a_{n-1} \prod_{i=n-N}^{n-2} (1-a_ib_i) + \mathcal{F}(a_{n-2N}, c_{n-2N+1}, \ldots, c_{n-1}),$$

which can be solved linearly for a_- in terms of the previous parameters ($1 - a_ib_i \neq 0$ for $n-N \leq i \leq n-2$). Using the automorphism σ (see (38)),

$$\tilde{\Gamma}_{n-N}(0) = u_{-N} \frac{-a_-}{a_{n+1}} \prod_{i=n-N}^{n-2} (1-a_ib_i) + \mathcal{F}(b_{n-2N}, c_{n-2N+1}, \ldots, c_{n-1}),$$

so that $\tilde{\Gamma}_{n-N}(0) = 0$ can be solved linearly for $b_{n+1} = 1/a_{n+1}$.

For step (5), x_n and y_n may be present in several terms in Δ_{n-N+1}, but in view of proposition 6.2, Γ_{n-N+1} and $\tilde{\Gamma}_{n-N+1}$ are polynomials in $z_{n-2N+1}, \ldots, z_{n-1}, z_{n+1}$ and in w_1 and w_2 and their σ analogs only. Thus, $\Gamma_{n-N+1}(0)$ and $\tilde{\Gamma}_{n-N+1}(0)$ depend on their leading terms only, to wit $c_{n-2N+1}, \ldots, c_{n-1}, a_{n+1}$ and $a, a_\pm$. It follows that the only *new* parameters that appear at step (5) are a_+ and a. Let us show that they appear in such a way that we can solve for them (linearly) in terms of the other parameters. We do this as in the self-dual case by isolating the leading term in Γ_{n-N+1} as given in proposition 8.2, namely we write Γ_{n-N+1} as

$$\Gamma_{n-N+1} = -u_N(x_nw_1 - x_{n+1})(1-x_{n-1}y_{n-1}) \prod_{i=n-N+1}^{n-2} (1-x_iy_i) + \mathcal{F}(z_{n-2N+2}, \ldots, z_n), \tag{70}$$

The relation (70) was obtained by writing the leading term

$$x_{n+1}(1-x_ny_n) = x_{n+1} - (x_nw_1 - x_n^2y_{n-1}),$$

and throwing the $x_n^2y_{n-1}$ term into $\mathcal{F}$. Since $\Gamma_{n-N+1}(t) = O(1)$ and since the first two terms in (70) belong to $\mathcal{A}'_n$, the last term in (70) is also $O(1)$ in t; since in addition this term does not contain z_{n+1}, by proposition 6.2 and (59) x_n and y_n can only appear in it multiplied by $1-x_{n-1}y_{n-1}$, and so by proposition 3.2 we may conclude that the contribution from this term in $\Gamma_{n-N+1}(0)$ will not involve a_+ or a. Also, the second term in (70), $u_Nx_{n+1}(1-x_{n-1}y_{n-1})\prod_{i=n-N+1}^{n-2}(1-x_iy_i)$ does not contribute to $\Gamma_{n-N+1}(0)$ since $1-x_{n-1}(t)y_{n-1}(t) = O(t)$ while all other factors are $O(1)$. Thus, the dependence on a_+ and a in $\Gamma_{n-N+1}(0)$ comes entirely from the first term in (70), which in view of proposition 3.2 and (61) is given by

$$\Gamma_{n-N+1}(0) = -u_Na_-\Omega \prod_{i=n-N+1}^{n-2} (1-a_ib_i) + \text{previous parameters.}$$

By duality,

$$\tilde{\Gamma}_{n-N+1}(0) = u_{-N} \frac{a_-a_{n-1}}{a_{n+1}} \sigma(\Omega) \prod_{i=n-N+1}^{n-2} (1-a_ib_i) + \text{previous parameters,}$$

where $\sigma(\Omega)$ was given in (62). Since Ω and $\sigma(\Omega)$ are linearly independent, as linear functions of a_+ and a, we can indeed solve $\tilde{\Gamma}_{n-N+1}(0) = 0$ and $\tilde{\Gamma}_{n-N+1}(0) = 0$ linearly for a_+ and a in terms of the other parameters.

Steps[12] (6) – (8) are easy, the point being that by proposition 8.2, for $2 \le k \le N-1$

$$\Delta_{n-N+k}(0) = \begin{pmatrix} u_N & \\ & u_{-N} \end{pmatrix} c_{n+k}((1-x_{n-1}y_{n-1})(1-x_ny_n)(1-x_{n+1}y_{n+1}))^{(0)} \prod_{\substack{i=n-N+k\\ i\neq n-1,n,n+1}}^{n+k-1} (1-a_ib_i) + \text{known}.$$

Let us concentrate on the next steps, which are more exciting. In step (9a) we need to compute the linear term in $\Gamma_{n-1}(t)$, where we recall from propositions 4.2 and 6.2 that $\Gamma_{n-1} \in \mathcal{A}'_n$, hence that this linear term only depends on the constant and linear terms of the elements of $\Gamma_{n-1} \in \mathcal{A}'_n$. Since $\Gamma_{n-1} \in \mathbf{R}[x_{n-N-1},\dots,x_{n+N-1},y_{n-N},\dots,y_{n+N-2}]$, with leading term

$$\Gamma_{n-1} = u_N x_{n+N-1} \prod_{i=0}^{N-1}(1-x_{n+i-1}y_{n+i-1}) + \cdots,$$

we have from proposition 3.2 that

$$\begin{aligned}\Gamma_{n-1}(t) = \Gamma_{n-1}^{(0)} &+ \\ &\Big(u_N a_{n+N}((1-x_{n-1}y_{n-1})(1-x_ny_n)(1-x_{n+1}y_{n+1}))^{(0)} \textstyle\prod_{i=3}^{N}(1-a_{n+i-1}b_{n+i-1}) + \cdots\Big)t \\ &+O(t^2),\end{aligned}$$

where the dots only involve previous parameters. Therefore we may solve $\Gamma_{n-1}^{(1)} = 0$ (linearly) for a_{n+N}. Step (9b) is similar to step (9) in the self-dual case; notice that we postpone again Δ_n to the next step. First of all $\Gamma_{n+1}(t) = O(1)$ and so $\Gamma_{n+1} \in \mathcal{A}'_n$. The leading term in Γ_{n+1}, namely the term $u_N x_{n+N+1}(1-x_{n+1}y_{n+1})\prod_{i=1}^{N-1}(1-x_{n+1+i}y_{n+1+i})$ cannot contribute to $\Gamma_{n+1}(0)$ because it is $O(t)$, which explains the absence of a_{n+N+1} in $\Gamma_{n+1}(0)$. By proposition 8.2, b_{n+N} can come only from y_{n+N}, which appears only once, namely in

$$-u_{-N}y_{n+N}x_nx_{n+1}\prod_{i=0}^{N-2}(1-x_{n+1+i}y_{n+1+i}) = -u_{-N}y_{n+N}x_{n+1}(x_n(1-x_{n+1}y_{n+1}))\prod_{i=n+2}^{n+N-1}(1-x_iy_i),$$

yielding at $t=0$ a non-zero linear term in b_{n+N}, as $x_n(t)(1-x_{n+1}(t)y_{n+1}(t)) = O(1)$.

Step (10) is the hardest one, but we dealt with it in lemma 6.4. Notice that after this step we have that $\Delta_n(t) = O(t)$ since the nullity of the previous $\Delta_k(0)$ already implies that $\Delta_n(t) = O(1)$ (proposition 4.2). Starting from step (11) everything goes smoothly, as $\Delta_k(t) = O(1)$ for $k > n+1$ and the leading term of $\Gamma_k(0)$, resp. $\tilde{\Gamma}_k(0)$ will produce precisely the new parameter a_{k+N}, resp. b_{k+N} (linearly).

7 Singularity confinement

We have constructed in the previous sections formal Laurent series for the Toeplitz lattice (self-dual and general case) solving the recursion relations $\Gamma_k(x(t);u(t)) = 0$ ($\Delta_k(x(t),y(t);u(t)) = 0$ in the general case). We will now transform these into solutions of the recursion relations $\Gamma_k(x;u) = 0$ (resp. $\Delta_k(x,y;u) = 0$), depending on a certain number of free parameters, and blowing up for only one (resp. two) variables. We will mainly concentrate on the self-dual case, as the general case is dealt with in precisely the same way.

[12] Skip these steps if $N = 2$.

The main tool to do this transformation is a formal version of the implicit function theorem, which we explain in the case of one variable, the scalar case. Suppose that we have a formal series in t,

$$x(t;a) = a + f_1(a)t + f_2(a)t^2 + \cdots; \tag{71}$$

one may think for example of $x(t;a)$ as a formal solution of a vector field (differential equation $\frac{dx}{dt} = F(x)$) on the real line, with initial condition $x(0;a) = a$. In our case the functions f_i will be rational. We wish solve the equation $x(t;a) = \alpha$ formally, namely we wish to construct the formal series in t

$$a(t;\alpha) = \alpha + g_1(\alpha)t + g_2(\alpha)t^2 + \cdots$$

with the property that $x(t;a(t;\alpha)) = \alpha$, as a formal t-series identity. Precisely, we claim that there exist for any $s \in \mathbf{N}$ unique (rational) functions $g_1(\alpha), \ldots, g_s(\alpha)$, such that

$$x(t;\alpha + g_1(\alpha)t + g_2(\alpha)t^2 + \cdots + g_s(\alpha)t^s) - \alpha = O(t^{s+1}),$$

where $x(t;\cdot)$ is given by (71). This is a trivial consequence of a formal version of Taylor's Theorem. For example, for $s=1$ we neglect all terms in t^2 and the condition on g_1 becomes

$$\begin{aligned} x(t;\alpha + g_1(\alpha)t) - \alpha + O(t^2) &= g_1(\alpha)t + f_1(\alpha + g_1(\alpha)t)t + O(t^2) \\ &= (g_1(\alpha) + f_1(\alpha))t + O(t^2), \end{aligned}$$

so that $g_1(\alpha) = -f_1(\alpha)$. For $s=2$ we neglect the terms in t^3, giving

$$\begin{aligned} &x(t;\alpha - f_1(\alpha)t + g_2(\alpha)t^2) - \alpha + O(t^3) \\ &\quad = -f_1(\alpha)t + g_2(\alpha)t^2 + f_1(\alpha - f_1(\alpha)t)t + f_2(\alpha)t^2 + O(t^3) \\ &\quad = g_2(\alpha)t^2 + f_1'(\alpha)(-f_1(\alpha)t)t + f_2(\alpha)t^2 + O(t^3) \\ &\quad = (g_2(\alpha) - f_1(\alpha)f_1'(\alpha) + f_2(\alpha))t^2 + O(t^3), \end{aligned}$$

which has $g_2(\alpha) := f_1(\alpha)f_1'(\alpha) - f_2(\alpha)$ as a unique solution. Continuing in this way it is clear that $g_i(\alpha)$ equals $-f_i(\alpha)$, up to a differential polynomial in the $f_j(\alpha)$, with $j < i$. Notice that when all $f_i(a)$ are rational function the same will be true for all $g_j(\alpha)$.

Let us apply this to the formal Laurent series that we have constructed for the self-dual Toeplitz lattice, and that yield formal solutions to the recursion relations $\Gamma_k(t) := \Gamma_k(x(t);u(t)) = 0$, where $k \in \mathbf{Z}$. Recall from proposition 5.4 that these formal Laurent solutions $x_k(t)$ depend on $2N-1$ parameters $a_{n-2N}, \ldots, a_{n-2}$, which are the leading coefficients of $x_{n-2N}, \ldots, x_{n-2}$, namely

$$x_k(t) = a_k + O(t), \qquad k = n-2N, \ldots, n-2, \tag{72}$$

where the higher order terms are rational functions of the parameters $a_{n-2N}, \ldots, a_{n-2}$. Besides the parameters a_k these functions also depend (polynomially) on the parameters $u = (u_1, \ldots, u_N)$ that define the recursion relations, namely $x_k(t) = x_k(t;a_{n-2N}, \ldots, a_{n-2};u)$, for $n-2N \le k \le n-2$. The formal implicit function theorem then leads to the following proposition.

Proposition 7.1 *There exist for $k = n-2N, \ldots, k = n-2$ rational functions*

$$a_k^{(i)} = a_k^{(i)}(\alpha_{n-2N}, \ldots, \alpha_{n-2}; u_1, \ldots, u_N)$$

such that $a_k := \sum_{i=0}^{\infty} a_k^{(i)} t^i$, $k = n-2N, \ldots, n-2$ formally inverts (72), i.e.,

$$x_k\left(t; \sum_{i=0}^{\infty} a_{n-2N}^{(i)} t^i, \ldots, \sum_{i=0}^{\infty} a_{n-2}^{(i)} t^i; u\right) = \alpha_k,$$

for $k = n-2N, \ldots, n-2$, with $a_k^{(0)} = \alpha_k$.

We can use these series to replace the free parameters $a_{n-2N},\dots,a_{n-2}$ in the series $x_k(t)$, $k\in\mathbf{Z}$, by $\alpha:=(\alpha_{n-2N},\dots,\alpha_{n-2})$, where we think of the latter as (partial) initial conditions to the recursion relation. To do this, one simply substitutes $a_k=\sum_{i=0}^\infty a_k^{(i)}t^i$ for $k=n-2N,\dots,n-2$ in each of the series $x_k(t)=x_k(t;a_{n-2N},\dots,a_{n-2};u)$, and rewrites this as a series in t; by construction, this simply gives $x_k(t)=\alpha_k$ for $k=n-2N,\dots,k=n-2$. For $k=n-1$, this yields

$$x_{n-1}(t)=\varepsilon+\sum_{i=1}^\infty x_{n-1}^{(i)}(a;u)t^i=\varepsilon+\sum_{i=1}^\infty \xi_{n-1}^{(i)}(\alpha;u)t^i,$$

where we recall that $\varepsilon^2=1$. The functions $\xi_{n-1}^{(i)}$ are rational in α and u. We will now use the formal implicit[13] function theorem again, but in a form which is different from the one explained above: putting $x_{n-1}(t)=\varepsilon+\lambda(t)$, i.e., we put

$$\lambda:=\sum_{i=1}^\infty \xi^{(i)}(\alpha;u)t^i,$$

which we solve for t as a formal series in λ,

$$t(\lambda)=\sum_{i=1}^\infty \tau^{(i)}(\alpha;u)\lambda^i, \tag{73}$$

where it is important to note that the constant term in this series is absent. Indeed, let us first substitute (73) in the series for a_k that was obtained in proposition (7.1), to get $a_k=a_k(\alpha;\lambda;u)$. Then, the latter and $t(\lambda)$ are substituted in all $x_k(t)$, to yield series in λ whose coefficients are rational functions of $\alpha=(\alpha_{n-2N},\dots,\alpha_{n-2})$ (and of $u=(u_1,\dots,u_N)$), which take the following form.

$$\begin{array}{rclrcl}
x_k(\lambda,\alpha;u) &=& \sum_{i=0}^\infty \chi_k^{(i)}(\alpha;u)\lambda^i, & k &<& n-2N,\\
x_k(\lambda,\alpha;u) &=& \alpha_k, & n-2N &\le& k<n-1,\\
x_{n-1}(\lambda,\alpha;u) &=& \varepsilon+\lambda, &&&\\
x_n(\lambda,\alpha;u) &=& \frac{1}{\lambda}\sum_{i=0}^\infty \chi_n^{(i)}(\alpha;u)\lambda^i, &&&\\
x_{n+1}(\lambda,\alpha;u) &=& -\varepsilon+\sum_{i=1}^\infty \chi_{n+1}^{(i)}(\alpha;u)\lambda^i, &&&\\
x_k(\lambda,\alpha;u) &=& \sum_{i=0}^\infty \chi_k^{(i)}(\alpha;u)\lambda^i, & n+1 &<& k.
\end{array}$$

It may seem that we have reached the final result, but we should not forget that these series are constructed from solutions $x=x(t)$ to the recursion relations $\Gamma_k(x;u(t))$, where $u(t)=(u_1+t,u_2,\dots,u_N)$. However, letting $U=(U_1,\dots,U_n):=u(t)$, and using (73) to get rid of t, we have that

$$x_k(\lambda,\alpha;(U_1-t(\lambda),U_2,\dots,U_N)),\ k\in\mathbf{Z}\ \text{ solves }\ \Gamma_k(x;U)=0,\ k\in\mathbf{Z}.$$

Notice that, when it is all worked out, the x_k are formal power series in λ (except x_n which has a simple pole in λ), and their coefficients are rational functions of the initial conditions $\alpha_{n-2N},\dots,\alpha_{n-2}$ and of the parameters $U_1,\dots,U_n$. Writing

$$\begin{aligned}
x_k(\lambda,\alpha;(U_1-t(\lambda),U_2,\dots,U_N)) &= \sum_{i=0}^\infty x_k^{(i)}(\alpha;U)\lambda^i,\ k\in\mathbf{Z}\setminus\{n\}\\
x_n(\lambda,\alpha;(U_1-t(\lambda),U_2,\dots,U_N)) &= \sum_{i=-1}^\infty x_n^{(i)}(\alpha;U)\lambda^i,
\end{aligned}$$

[13] Call this the formal inverse function theorem, if you wish.

leads to our final result.

Theorem 7.2 *The recursion relations $\Gamma_k(x;U) = 0$, $k \in \mathbf{Z}$ admit for any $n \in \mathbf{Z}$ two*[14] *formal Laurent solution $x = (x_k(\alpha, \lambda; U))_{k\in\mathbf{Z}}$, depending on $2N$ free parameters $\alpha = (\alpha_{n-2N}, \ldots, \alpha_{n-2})$ and λ with x_n having a (simple) pole for $\lambda \to 0$, and no other singularities. Explicitly, these series with coefficients rational in α are given by*

$$\begin{array}{rclrcl}
x_k(\lambda,\alpha;U) &=& \sum_{i=0}^{\infty} x_k^{(i)}(\alpha;U)\lambda^i, & k &<& n-2N,\\
x_k(\lambda,\alpha;U) &=& \alpha_k, & n-2N &\leq& k < n-1,\\
x_{n-1}(\lambda,\alpha;U) &=& \varepsilon+\lambda, &&&\\
x_n(\lambda,\alpha;U) &=& \frac{1}{\lambda}\sum_{i=0}^{\infty} x_n^{(i)}(\alpha;U)\lambda^i, &&&\\
x_{n+1}(\lambda,\alpha;U) &=& -\varepsilon+\sum_{i=1}^{\infty} x_{n+1}^{(i)}(\alpha;U)\lambda^i, &&&\\
x_k(\lambda,\alpha;U) &=& \sum_{i=0}^{\infty} x_k^{(i)}(\alpha;U)\lambda^i, & n+1 &<& k.
\end{array}$$

The corresponding theorem for the recursion relations $\Delta_k = 0$, which was formulated in the introduction (Theorem 1.1) follows in the same way, using the formal Laurent solutions $z(t)$ that solve the recursion relations.

8 Appendix

In this appendix we obtain the leading terms of the polynomials Γ_k and $\tilde{\Gamma}_k$, which are needed in sections 5 and 6. The notations are as in the body of the paper, namely P_1 and P_2 are polynomials of degree N (see (19)), the matrices L_1 and L_2 are defined by (14) and the polynomials Γ_k and $\tilde{\Gamma}_k$ are defined by (20). Since Γ_k is given by

$$\Gamma_k(x,y;u) := \frac{1-x_ky_k}{y_k}\left(\begin{array}{c} -(L_1P_1'(L_1))_{k+1,k+1} - (L_2P_2'(L_2))_{k,k} \\ +(P_1'(L_1))_{k+1,k} + (P_2'(L_2))_{k,k+1}\end{array}\right) + kx_k, \tag{74}$$

we need, by duality, only to determine the leading terms of $(L_1^s)_{kk}$ and of $(L_1^s)_{k+1,k}$, for $s,k \in \mathbf{Z}$, with $s \geq 2$, which will be done in the following lemma. Notice that the leading terms of $\tilde{\Gamma}_k$ will also follow from it, by duality.

Lemma 8.1 *For $k \in \mathbf{Z}$ and $s \in \mathbf{N}$, with $s \geq 2$, the diagonal and first subdiagonal entries of the Toeplitz matrices L_1 and L_2, defined in (14), are polynomials in the following variables,*

$$\begin{array}{rcl}
(L_1^s)_{kk} &\in& \mathbf{R}[x_{k-s+1},\ldots,x_{k+s-1},y_{k-s},\ldots,y_{k+s-2}],\\
(L_1^s)_{k+1,k} &\in& \mathbf{R}[x_{k-s+1},\ldots,x_{k+s},y_{k-s},\ldots,y_{k+s-1}].
\end{array}$$

More precisely[15]*,*

$$(L_1^s)_{kk} \quad = \quad -x_{k+s-1}y_{k-1}\prod_{i=1}^{s-1}(1-x_{k+i-1}y_{k+i-1}) + x_{k+s-2}^2y_{k+s-3}y_{k-1}\prod_{i=1}^{s-2}(1-x_{k+i-1}y_{k+i-1})$$

[14] parametrized by $\epsilon = \pm 1$.

[15] We give in each case the terms that will be used, no more, no less. When $s = 2$ only the first two lines survive; the term on fourth line coincides with the first term on the second line and should only be counted once.

$$
\begin{aligned}
&- x_{k+s-2}\left(y_{k-2}(1-x_{k-1}y_{k-1}) - 2y_{k-1}\sum_{j=1}^{s-2} x_{k+j-1}y_{k+j-2}\right)\prod_{i=1}^{s-2}(1-x_{k+i-1}y_{k+i-1})\\
&+ \mathcal{F}_1(x_{k-s+2},\ldots,x_{k+s-3},y_{k-s+1},\ldots,y_{k+s-3})\\
&- x_k y_{k-s}\prod_{i=1}^{s-1}(1-x_{k-i}y_{k-i})
\end{aligned}
$$

and

$$
\begin{aligned}
(L_1^s)_{k+1,k} \quad = \quad & -x_{k+s}y_{k-1}\prod_{i=1}^{s-1}(1-x_{k+i}y_{k+i}) - x_{k+1}y_{k-s}\prod_{i=1}^{s-1}(1-x_{k-i}y_{k-i})\\
&+ \mathcal{F}_2(x_{k-s+2},\ldots,x_{k+s-1},y_{k-s+1},\ldots,y_{k+s-2})
\end{aligned}
$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are polynomials in their arguments.

Proof The following notation is useful for obtaining formulas of this type. To the bi-infinite vector x we associate, for any $k\in\mathbf{Z}$ a bi-infinite diagonal matrices $X^{(k)}$ and $Y^{(k)}$ by putting $X_{ij}^{(k)} := x_{i+k}\delta_{ij}$ and $Y_{ij}^{(k)} := y_{i+k}\delta_{ij}$ and (Kronecker delta). Similarly we introduce the diagonal matrices $V^{(k)}$, by defining $V_{ij}^{(k)} := (1-x_{i+k}y_{i+k})\delta_{ij}$. We denote by Δ the shift operator, which we view as a bi-infinite matrix, with entries $\Delta_{ij} := \delta_{i+1,j}$. It is easy to verify that

$$
\Delta^i X^{(j)} = X^{(i+j)}\Delta^i, \qquad i,j\in\mathbf{Z},
$$

which is the main formula that we will use, as it allows us to push all Δ to the right (or to the left). One obvious consequence is that a monomial in X, Y, V and Δ will only have a non-zero diagonal when it is independent of Δ (i.e., the sum of all powers of Δ is zero). In order to apply this to obtain the above formulas, observe that L_1 and L_2 can be written as

$$
\begin{aligned}
L_1 &= \Delta V^{(-1)} - \sum_{i\geq 0}\Delta^{-i}X^{(i)}Y^{(-1)} = V^{(0)}\Delta - \sum_{i\geq 0}X^{(0)}Y^{(-i-1)}\Delta^{-i},\\
L_2 &= \Delta^{-1}V^{(0)} - \sum_{i\geq 0}\Delta^{i}X^{(-i-1)}Y^{(0)} = V^{(-1)}\Delta^{-1} - \sum_{i\geq 0}X^{(-1)}Y^{(i)}\Delta^{i}.
\end{aligned}
$$

Notice that, in view of what we said, all diagonal entries of $(V^{(0)}\Delta)^{s-1}$ are zero. Therefore, it follows from the second formula for L_1 that the leading term in x of the diagonal terms of L_1^s will be gotten from the product

$$
-(V^{(0)}\Delta)^{s-1}\sum_{i\geq 0}X^{(0)}Y^{(-i-1)}\Delta^{-i}. \tag{75}
$$

The diagonal entries of (75) are obtained by taking $i = s-1$, which yields

$$
\begin{aligned}
\left(-(V^{(0)}\Delta)^{s-1}X^{(0)}Y^{(-s)}\Delta^{-s+1}\right)_{kk} &= -\left(V^{(0)}\ldots V^{(s-2)}X^{(s-1)}Y^{(-1)}\right)_{kk}\\
&= -x_{k+s-1}y_{k-1}\prod_{i=1}^{s-1}(1-x_{k+i-1}y_{k+i-1}).
\end{aligned}
$$

Notice that this leading term already contains x_{k+s-2}, and that it yields, through the factor $1-x_{k+s-2}y_{k+s-2}$, the single term that contains y_{k+s-2}, which is the highest y variable that appears in $(L_1^s)_{kk}$.

In order to get the other terms in L_1^s that lead to x_{k+s-2} we need Δ^{s-2} in front of $X^{(0)}$, i.e., we need $s-2$ copies of $V^{(0)}\Delta$ (not necessarily consecutive), on the left of $-\sum_{i\geq 0} X^{(0)}Y^{(-i-1)}\Delta^{-i}$. For the remaining factor we can have another copy of $V^{(0)}\Delta$ or of $-\sum_{i\geq 0} X^{(0)}Y^{(-i-1)}\Delta^{-i}$, inserted at an arbitrary place inside the product $-(V^{(0)}\Delta)^{s-2}\sum_{i\geq 0} X^{(0)}Y^{(-i-1)}\Delta^{-i}$. This leads to three possible types of terms. For the first one, we put another $V^{(0)}\Delta$ at the end

$$-(V^{(0)}\Delta)^{s-2}\sum_{i\geq 0} X^{(0)}Y^{(-i-1)}\Delta^{-i}(V^{(0)}\Delta),$$

and we get the k,k diagonal term by taking $i=s-1$, which gives

$$\left(-(V^{(0)}\Delta)^{s-2}X^{(0)}Y^{(-s)}\Delta^{1-s}V^{(0)}\Delta\right)_{kk} = -x_{k+s-2}y_{k-2}\prod_{i=0}^{s-2}(1-x_{k+i-1}y_{k+i-1}).$$

For the second one we put another $-\sum_{j\geq 0} X^{(0)}Y^{(-j-1)}\Delta^{-j}$ at the end,

$$(V^{(0)}\Delta)^{s-2}\sum_{i\geq 0} X^{(0)}Y^{(-i-1)}\Delta^{-i}\sum_{j\geq 0} X^{(0)}Y^{(-j-1)}\Delta^{-j};$$

its diagonal terms are given by taking $i+j=s-2$, i.e., from

$$(V^{(0)}\Delta)^{s-2}\sum_{j=0}^{s-2} X^{(0)}Y^{(j-s+1)}X^{(j-s+2)}Y^{(-s+1)}\Delta^{2-s},$$

whose k,k term is given by

$$y_{k-1}\left(x_{k+s-2}^2 y_{k+s-3} + x_{k+s-2}\sum_{j=0}^{s-3} x_{k+j}y_{k+j-1}\right)\prod_{i=1}^{s-2}(1-x_{k+i-1}y_{k+i-1}).$$

The third term is obtained by inserting the constant term $-X^{(0)}Y^{(-1)}$ of $-\sum_{j\geq 0} X^{(0)}Y^{(-j-1)}\Delta^{-j}$ at all possible places in the product $(V^{(0)}\Delta)^{s-2}$, namely from

$$\sum_{j=0}^{s-3}(V^{(0)}\Delta)^j(X^{(0)}Y^{(-1)})(V^{(0)}\Delta)^{s-j-2}\sum_{i\geq 0} X^{(0)}Y^{(-i-1)}\Delta^{-i},$$

with $i=s-2$, so that its k,k term is given by

$$\left(y_{k-1}x_{k+s-2}\sum_{j=0}^{s-3} x_{k+j}y_{k+j-1}\right)\prod_{i=1}^{s-2}(1-x_{k+i-1}y_{k+i-1}),$$

which, combined with the first two terms, yields the leading terms of $(L_1^s)_{kk}$. Using the first formula for L_1, the lowest term in y of the diagonal terms of L_1^s is gotten from

$$-\Delta^{-s+1}X^{(s-1)}Y^{(-1)}(\Delta V^{(-1)})^{s-1} = -X^{(0)}Y^{(-s)}V^{(-s+1)}\dots V^{(-1)},$$

whose k,k entry is $-x_k y_{k-s}\prod_{i=1}^{s-1}(1-x_{k-i}y_{k-i})$. It contains the lowest term in x, through the factor $1-x_{k-s+1}y_{k-s+1}$.

One obtains similarly the entries of $(L_1^s)_{k+1,k}$ by selecting the terms in L_1^s that contain precisely Δ^{-1}. Notice in this respect that if M is a bi-infinite diagonal matrix then $(M\Delta^{-1})_{k+1,k} = M_{k+1,k+1}$. It follows that the leading term in x of $(L_1^s)_{k+1,k}$, which contains also the leading term in y, is obtained from the product (75), with $i = s$, yielding

$$-\left(V^{(0)}\dots V^{(s-2)}X^{(s-1)}Y^{(-2)}\right)_{k+1,k} = -x_{k+s}y_{k-1}\prod_{i=1}^{s-1}(1-x_{k+i}y_{k+i}).$$

The lowest term in y, which contains the lowest term in x, is obtained in the same way.

The above lemma and (74) lead by direct substitution to the following proposition.

Proposition 8.2 *For $k \in \mathbf{Z}$, the polynomials Γ_k and $\tilde{\Gamma}_k$ depend on the following variables x_i and y_i:*

$$\begin{aligned}
\Gamma_k(x,y;u) &\in \mathbf{R}[x_{k-N},\dots,x_{k+N},y_{k-N+1},\dots,y_{k+N-1}],\\
\tilde{\Gamma}_k(x,y;u) &\in \mathbf{R}[x_{k-N+1},\dots,x_{k+N-1},y_{k-N},\dots,y_{k+N}].
\end{aligned}$$

More precisely[16],

$$\begin{aligned}
\Gamma_k(x,y;u) \;=\;& u_N x_{k+N}\prod_{i=0}^{N-1}(1-x_{k+i}y_{k+i}) - u_N x_{k+N-1}^2 y_{k+N-2}\prod_{i=0}^{N-2}(1-x_{k+i}y_{k+i})\\
&-u_N x_{k+N-1}\left(x_k y_{k-1} + 2\sum_{j=1}^{N-2} x_{k+j}y_{k+j-1}\right)\prod_{i=0}^{N-2}(1-x_{k+i}y_{k+i})\\
&+(u_{N-1}x_{k+N-1} - u_{-N}y_{k+N-1}x_{k-1}x_k)\prod_{i=0}^{N-2}(1-x_{k+i}y_{k+i})\\
&+(1-x_k y_k)\mathcal{F}(x_{k-N+1},\dots,x_{k+N-2},y_{k-N+2},\dots,y_{k+N-2}) + kx_k\\
&-(u_N x_k x_{k+1} y_{k-N+1} - u_{-N}x_{k-N}(1-x_{k-N+1}y_{k-N+1}))\prod_{i=0}^{N-2}(1-x_{k-i}y_{k-i}),
\end{aligned}$$

where $\mathcal{F}$ is a polynomial in its arguments, with a similar statement for $\tilde{\Gamma}_k$ gotten by duality. In the self-dual case, Γ_k takes the simpler form

$$\begin{aligned}
\Gamma_k(x;u) \;=\;& u_N x_{k+N}\prod_{i=0}^{N-1}(1-x_{k+i}y_{k+i}) + u_{N-1}x_{k+N-1}\prod_{i=0}^{N-2}(1-x_{k+i}y_{k+i})\\
&-u_N x_{k+N-1}\left(x_{k+N-1}x_{k+N-2} + 2\sum_{j=0}^{N-2} x_{k+j}x_{k+j-1}\right)\prod_{i=0}^{N-2}(1-x_{k+i}y_{k+i})\\
&+(1-x_k y_k)\mathcal{F}(x_{k-N+1},\dots,x_{k+N-2}) + kx_k\\
&-u_N(x_k x_{k+1} x_{k-N+1} - x_{k-N}(1-x_{k-N+1}y_{k-N+1}))\prod_{i=0}^{N-2}(1-x_{k-i}y_{k-i}).
\end{aligned}$$

[16] As in the case of lemma 8.1, when $N = 2$ then the term $-u_2 x_k x_{k+1} y_{k-1}(1-x_k y_k)$, which appears twice, should only be taken into account once.

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