A PDE for the joint distributions of the Airy Process

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Abstract

In this paper, we answer a question posed by Kurt Johansson, to find a PDE for the joint distribution of the Airy Process. The latter is a continuous stationary process, describing the motion of the outermost particle of the Dyson Brownian motion, when the number of particles get large, with space and time appropriately rescaled. The question reduces to an asymptotic analysis on the equation governing the joint probability of the eigenvalues of coupled Gaussian Hermitian matrices.

The differential equations lead to the asymptotic behavior of the joint distribution and the correlation for the Airy process at different times t_1 and t_2, when $t_2 - t_1 \to \infty$.

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1 Main result

The Dyson Brownian motion (see [2])

$$\Big(\lambda_1(t),\dots,\lambda_n(t)\Big)\in\mathbb{R}^n,$$

with transition density $p(t,\mu,\lambda)$ satisfies the diffusion equation

$$\begin{aligned}\frac{\partial p}{\partial t} &= \frac{1}{2}\sum_1^n\frac{\partial}{\partial\lambda_i}\Phi(\lambda)\frac{\partial}{\partial\lambda_i}\frac{1}{\Phi(\lambda)}p,\\ &= \sum_1^n\left(\frac{1}{2}\frac{\partial^2}{\partial\lambda_i^2}-\frac{\partial}{\partial\lambda_i}\frac{\partial\log\sqrt{\Phi(\lambda)}}{\partial\lambda_i}\right)p\end{aligned}$$

with

$$\Phi(\lambda)=\Delta^2(\lambda)\prod_1^n e^{-\lambda_i^2/a}.$$

It corresponds to the motion of the eigenvalues $(\lambda_1(t),\dots,\lambda_n(t))$ of an Hermitian matrix B, evolving according to the Ornstein-Uhlenbeck process

$$\frac{\partial P}{\partial t}=\sum_{i,j=1}^{n^2}\left(\frac{1}{4}(1+\delta_{ij})\frac{\partial^2}{\partial B_{ij}^2}+\frac{1}{a^2}\frac{\partial}{\partial B_{ij}}(B_{ij}P)\right)\,, \tag{1.1}$$

with transition density ($c:=e^{-t/a^2}$)

$$P(t,\bar{B},B)=Z^{-1}\frac{1}{(1-c^2)^{n^2/2}}e^{-\frac{1}{a^2(1-c^2)}\operatorname{Tr}(B-c\bar{B})^2},$$

The B_{ij}'s in (1.1) denote the n^2 free parameters in the Hermitian matrix B. In the limit $t\to\infty$, this distribution tends to the stationary distribution

$$Z^{-1}e^{-\frac{1}{a^2}\operatorname{Tr}B^2}dB=Z^{-1}\Delta^2(\lambda)\prod_1^n e^{-\frac{\lambda_i^2}{a^2}}\,d\lambda_i.$$

With this invariant measure as initial condition, one finds for the joint distribution:

$$P(B(0) \in dB_1, B(t) \in dB_2) = Z^{-1} \frac{dB_1 dB_2}{(1-c^2)^{n^2/2}} e^{-\frac{1}{a^2(1-c^2)} \operatorname{Tr}(B_1^2 - 2cB_1B_2 + B_2^2)}. \tag{1.2}$$

Setting $a = 1$, the **Airy process** is defined by an appropriate rescaling of the largest eigenvalue λ_n in the Dyson diffusion,

$$A(t) = \lim_{n\to\infty} \sqrt{2} n^{1/6} \left(\lambda_n(n^{-1/3}t) - \sqrt{2n} \right), \tag{1.3}$$

in the sense of convergence of distributions for a finite number of t's. This process was introduced by Prähofer and Spohn [5] in the context of polynuclear growth models and further investigated by Johansson [3]. Prähofer and Spohn showed the Airy process is a stationary process with continuous sample paths; thus the probability $P(A(t) \leq u)$ is independent of t, and is given by the Tracy-Widom distribution [6],

$$P(A(t) \leq u) = F_2(u) := \exp\left(-\int_u^\infty (\alpha - u) q^2(\alpha) d\alpha \right), \tag{1.4}$$

with $q(\alpha)$ a solution of the **Painlevé II** equation,

$$q'' = \alpha q + 2q^3 \quad \text{with} \quad q(\alpha) \cong \begin{cases} -\frac{e^{-\frac{2}{3}\alpha^{\frac{3}{2}}}}{2\sqrt{\pi}\alpha^{1/4}} \text{ for } \alpha \nearrow \infty \\ \\ \sqrt{-\alpha/2} \text{ for } \alpha \searrow -\infty. \end{cases} \tag{1.5}$$

At MSRI (sept 02), Kurt Johansson posed the question, whether a PDE can be found for the joint probability of the Airy process; see [3]. The present paper answers this question, which enables us to derive the asymptotics of the large time correlation of the Airy Process. We thank Kurt Johansson for introducing us to this process.

Theorem 1.1 *Given $t_1 < t_2$, the joint probability for the Airy process*

$$h(t_2 - t_1; \frac{y+x}{2}, \frac{y-x}{2}) := \log P\left(A(t_1) < \frac{y+x}{2}, A(t_2) < \frac{y-x}{2} \right),$$

satisfies a non-linear PDE[1] *in* x, y *and* $t = t_2 - t_1$,

$$2t\frac{\partial^3 h}{\partial t \partial x \partial y} = \left(t^2\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 h}{\partial x^2} - \frac{\partial^2 h}{\partial y^2}\right) + 8\left\{\frac{\partial^2 h}{\partial x \partial y}, \frac{\partial^2 h}{\partial y^2}\right\}_y, \qquad (1.6)$$

with initial condition

$$\lim_{t \searrow 0} h\left(t;\ \frac{y+x}{2}, \frac{y-x}{2}\right) = \log F_2\left(\min(\frac{y+x}{2}, \frac{y-x}{2})\right).$$

Conjecture *For any fixed* $t > 0$, $x \in \mathbb{R}$, *the conditional probability satisfies:*

$$\lim_{z \to \infty} P(A(t) \geq x + z \mid A(0) \leq -z) = 0. \qquad (1.7)$$

Accepting this conjecture, we prove:

Theorem 1.2 *For large* t, *the joint probability admits the asymptotic series*

$$P(A(0) \leq u, A(t) \leq v) = F_2(u)F_2(v) + \frac{F_2'(u)F_2'(v)}{t^2} + \frac{\Phi(u,v) + \Phi(v,u)}{t^4} + O\left(\frac{1}{t^6}\right), \qquad (1.8)$$

with ($q(u)$ *is the function (1.5))*

$$\Phi(u,v) := F_2(u)F_2(v)\begin{pmatrix} \frac{1}{4}\left(\int_u^\infty q^2 d\alpha\right)^2 \left(\int_v^\infty q^2 d\alpha\right)^2 \\ \\ + q^2(u)\left(\frac{1}{4}q^2(v) - \frac{1}{2}(\int_v^\infty q^2 d\alpha)^2\right) \\ \\ + \int_v^\infty d\alpha \big(2(v-\alpha)q^2 + q'^2 - q^4\big) \int_u^\infty q^2 d\alpha \end{pmatrix}.$$

Moreover, the covariance for large t *behaves as*

$$E(A(t)A(0)) - E(A(t))E(A(0)) = \frac{1}{t^2} + \frac{c}{t^4} + \ldots\ , \qquad (1.9)$$

where

$$c := 2\iint_{\mathbb{R}^2} \Phi(u,v)\, du\, dv.$$

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[1] in terms of the Wronskian $\{f(y), g(y)\}_y := f'(y)g(y) - f(y)g'(y)$.

Remark: The equation (1.6) for the probability

$$h(t_2-t_1;u,v):=\log P(A(t_1)\leq u, A(t_2)\leq v),\quad t=t_2-t_1,$$

takes on the alternative form in the variables u and v,

$$\begin{aligned} t\frac{\partial}{\partial t}\Big(\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}\Big)h &= \frac{\partial^3 h}{\partial u^2\partial v}\Big(2\frac{\partial^2 h}{\partial v^2}+\frac{\partial^2 h}{\partial u\partial v}-\frac{\partial^2 h}{\partial u^2}+u-v-t^2\Big)\\ &\quad -\frac{\partial^3 h}{\partial v^2\partial u}\Big(2\frac{\partial^2 h}{\partial u^2}+\frac{\partial^2 h}{\partial u\partial v}-\frac{\partial^2 h}{\partial v^2}-u+v-t^2\Big)\\ &\quad +\Big(\frac{\partial^3 h}{\partial u^3}\frac{\partial}{\partial v}-\frac{\partial^3 h}{\partial v^3}\frac{\partial}{\partial u}\Big)\Big(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\Big)h\ , \end{aligned}\qquad (1.10)$$

with initial condition

$$\lim_{t\searrow 0} h(t;u,v)=\log F_2(\min(u,v)).$$

This equation enjoys an obvious $u\leftrightarrow v$ duality.

The proof of this theorem is based on a PDE, which was obtained in [1] for the spectrum of coupled random matrices.

In [5], Spohn and Prähofer pose the question about the asymptotics of the covariant functions of $A(t)$ and $A(0)$ for large t. Moreover, in a very recent paper, Tracy and Widom [7] expressed the joint distribution, for several times $t_1,\dots,t_m$, as the exponential of a certain integral; its integrand involves traces of matrices, which satisfy a coupled system of non-linear ODE's. The quantities involved are entirely different and their methods are functional-theoretical; it remains unclear what the connection is between the two results.

2 The spectrum of coupled random matrices

Consider a product ensemble $(M_1,M_2)\in\mathcal{H}_n^2:=\mathcal{H}_n\times\mathcal{H}_n$ of $n\times n$ Hermitian matrices, equipped with a Gaussian probability measure,

$$c_n dM_1 dM_2\, e^{-\frac{1}{2}\mathrm{Tr}(M_1^2+M_2^2-2cM_1M_2)},\qquad (2.1)$$

where dM_1dM_2 is Haar measure on the product $\mathcal{H}_n^2$, with each dM_i,

$$dM_1=\Delta_n^2(x)\prod_1^n dx_i dU_1 \text{ and } dM_2=\Delta_n^2(y)\prod_1^n dy_i dU_2\qquad (2.2)$$

decomposed into radial and angular parts. In [1], we define differential operators $\mathcal{A}_k$, $\mathcal{B}_k$ of "weight" k, which form a closed Lie algebra, in terms of the coupling constant c, appearing in (2.1), and the boundary of the set

$$E = E_1 \times E_2 := \cup_{i=1}^{r}[a_{2i-1}, a_{2i}] \times \cup_{i=1}^{s}[b_{2i-1}, b_{2i}] \subset \mathbb{R}^2. \tag{2.3}$$

Here we only need the first few ones:

$$\begin{aligned} \mathcal{A}_1 &= \frac{1}{c^2-1}\left(\sum_1^r \frac{\partial}{\partial a_j} + c\sum_1^s \frac{\partial}{\partial b_j}\right) & \mathcal{B}_1 &= \frac{1}{1-c^2}\left(c\sum_1^r \frac{\partial}{\partial a_j} + \sum_1^s \frac{\partial}{\partial b_j}\right) \\ \mathcal{A}_2 &= \sum_{j=1}^{r} a_j \frac{\partial}{\partial a_j} - c\frac{\partial}{\partial c} & \mathcal{B}_2 &= \sum_{j=1}^{s} b_j \frac{\partial}{\partial b_j} - c\frac{\partial}{\partial c}. \end{aligned} \tag{2.4}$$

In [1], we prove the following theorem:

Theorem 2.1 *Given the joint distribution*

$$P_n(E) := P(\mathit{all}(M_1\text{-}\mathit{eigenvalues}) \in E_1,\ \mathit{all}(M_2\text{-}\mathit{eigenvalues}) \in E_2), \tag{2.5}$$

the function $F_n(c; a_1, \ldots, a_{2r}, b_1, \ldots, b_{2s}) := \log P_n(E)$, *satisfies the non-linear third-order partial differential equation*[2]*:*

$$\left\{\mathcal{B}_2\mathcal{A}_1 F_n\ ,\ \mathcal{B}_1\mathcal{A}_1 F_n + \frac{nc}{c^2-1}\right\}_{\mathcal{A}_1} - \left\{\mathcal{A}_2\mathcal{B}_1 F_n\ ,\ \mathcal{A}_1\mathcal{B}_1 F_n + \frac{nc}{c^2-1}\right\}_{\mathcal{B}_1} = 0. \tag{2.6}$$

Corollary 2.2 *For* $E = E_1 \times E_2 := (-\infty, a] \times (-\infty, b]$, *it is convenient to use the new variables* $x := -a + cb$, $y := -ac + b$. *In these variables, the equation (2.6) for*

$$f_n(c; x, y) := \log P_n(E) = F_n\left(c; \frac{x-cy}{c^2-1}, \frac{cx-y}{c^2-1}\right) \tag{2.7}$$

takes on the following form:

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{(c^2-1)^2\frac{\partial^2 f_n}{\partial x \partial c} + 2ncx - n(1+c^2)y}{(c^2-1)\frac{\partial^2 f_n}{\partial x \partial y} + nc}\right) \\ &= \frac{\partial}{\partial y}\left(\frac{(c^2-1)^2\frac{\partial^2 f_n}{\partial y \partial c} + 2ncy - n(1+c^2)x}{(c^2-1)\frac{\partial^2 f_n}{\partial y \partial x} + nc}\right). \end{aligned} \tag{2.8}$$

[2]in terms of the Wronskian $\{f, g\}_X = Xf.g - f.Xg$, with regard to a first order differential operator X.

Proof: It is an immediate consequence of Theorem 2.1, upon observing the simple form of the differential operators $\mathcal{A}_1 = \partial/\partial x$ and $\mathcal{B}_1 = \partial/\partial y$, when expressed in terms of x and y. ∎

3 Proof of Theorem 1.1

Taking into account the scaling in the definition of the Airy process (1.3), and using the Ornstein-Uhlenbeck transition probability (1.2), we compute the probability (setting $c = e^{-n^{-1/3}t}$, with $t = t_2 - t_1$)

$$P\left(\sqrt{2}n^{1/6}\left(\lambda_n(n^{-1/3}t_1) - \sqrt{2n}\right) \leq u, \sqrt{2}n^{1/6}\left(\lambda_n(n^{-1/3}t_2) - \sqrt{2n}\right) \leq v\right)$$

$$= \iint_{\substack{\text{all } B_1\text{-eigenvalues} \leq \frac{1}{\sqrt{2}}(2n^{1/2}+n^{-1/6}u) \\ \text{all } B_2\text{-eigenvalues} \leq \frac{1}{\sqrt{2}}(2n^{1/2}+n^{-1/6}v)}} Z^{-1}\frac{dB_1 dB_2}{(1-c^2)^{n^2/2}} e^{-\frac{1}{1-c^2}\operatorname{Tr}(B_1^2+B_2^2-2cB_1B_2)}$$

$$= \iint_{\substack{\text{all } M_1\text{-eigenvalues} \leq \frac{2n^{1/2}+n^{-1/6}u}{\sqrt{1-c^2}} \\ \text{all } M_2\text{-eigenvalues} \leq \frac{2n^{1/2}+n^{-1/6}v}{\sqrt{1-c^2}}}} Z'^{-1} dM_1 dM_2\, e^{-\frac{1}{2}\operatorname{Tr}(M_1^2+M_2^2-2cM_1M_2)}, \tag{3.1}$$

using the change of variables

$$M_i = \frac{B_i}{\sqrt{(1-c^2)/2}}.$$

The integral (3.1) turns out to coincide with the statistics (2.5) of the coupled matrix model with the Gaussian distribution (2.1). Therefore, we set

$$a = \frac{2n^{1/2}+n^{-1/6}u}{\sqrt{1-c^2}} \quad \text{and} \quad b = \frac{2n^{1/2}+n^{-1/6}v}{\sqrt{1-c^2}}; \tag{3.2}$$

in formula (2.8), via x and y. Then, setting $k = n^{1/6}$, we now express x and y in terms of u, v and $c = e^{-n^{-1/3}t} = e^{-t/k^2}$, using the change of variables in

Corollary 2.2,

$$
\begin{aligned}
x(c(t),u,v) &= -a+bc = -\frac{1}{\sqrt{1-c^2}}\left(\left(2k^3+\frac{u}{k}\right)-\left(2k^3+\frac{v}{k}\right)c\right)\\
y(c(t),u,v) &= -ac+b = -\frac{1}{\sqrt{1-c^2}}\left(\left(2k^3+\frac{u}{k}\right)c-\left(2k^3+\frac{v}{k}\right)\right)
\end{aligned}
\tag{3.3}
$$

with inverse given by

$$
\begin{aligned}
u &= \frac{1}{\sqrt{1-c^2}}(k(cy-x)-2k^4\sqrt{1-c^2})\\
v &= \frac{1}{\sqrt{1-c^2}}(k(y-cx)-2k^4\sqrt{1-c^2}), \quad \text{with } c=e^{-t/k^2}.
\end{aligned}
\tag{3.4}
$$

So, $f_n(c;x,y)$ satisfies equations (2.8), which written out, has the form

$$
\begin{aligned}
&\frac{\partial^3 f}{\partial x^2\partial y}\left((c^2-1)^2\frac{\partial^2 f}{\partial x\partial c}+2cnx-(c^2+1)ny\right)\\
&-\frac{\partial^3 f}{\partial x\partial y^2}\left((c^2-1)^2\frac{\partial^2 f}{\partial y\partial c}+2cny-(c^2+1)nx\right)\\
&+(c^2-1)\left((c^2-1)\frac{\partial^2 f}{\partial x\partial y}+nc\right)\left(\frac{\partial^3 f}{\partial y^2\partial c}-\frac{\partial^3 f}{\partial x^2\partial c}\right)=0.
\end{aligned}
\tag{3.5}
$$

Using (3.3) and definition (2.7), one defines, at a first stage, a new function

$$
\begin{aligned}
g_n(c(t);u,v) &:= f_n(c(t);x(c(t);u,v),y(c(t);u,v))\\
&= F_n\left(c(t);\frac{x(c;u,v)-cy(c;u,v)}{c^2-1},\frac{cx(c;u,v)-y(c;u,v)}{c^2-1}\right),
\end{aligned}
$$

one expresses the (x,y,c)-partials of f in terms of (u,v,c)-partials of g, using the inverse map (3.4), e.g.,

$$
\frac{\partial^3 f}{\partial x^2\partial y}=\left(\frac{k}{\sqrt{1-c^2}}\right)^3\left(c(\frac{\partial^3 g}{\partial u^3}+c\frac{\partial^3 g}{\partial v^3})+(2c^2+1)\frac{\partial^3 g}{\partial u^2\partial v}+c(c^2+2)\frac{\partial^3 g}{\partial u\partial v^2}\right),
$$

etc... .One substitutes these expressions in the differential equation (3.5) for $f(c;x,y)$, yielding a differential equation in $g(c;u,v)$, with coefficients depending on u, v, c and k. After division by k^4, this differential equation is a quadratic polynomial in k^4,

$$
\begin{aligned}
& k^8(c-1)^2\ 2\left(\frac{\partial^3 g}{\partial u^2 \partial v}-\frac{\partial^3 g}{\partial u \partial v^2}\right) \\
& +k^4 \left\{\begin{array}{l}
+\dfrac{\partial^3 g}{\partial u^2 \partial v}\left(2(c^2-c+1)\dfrac{\partial^2 g}{\partial u^2}-4c\dfrac{\partial^2 g}{\partial v^2}+v(c^2+1)-2cu\right) \\
-\dfrac{\partial^3 g}{\partial u \partial v^2}\left(2(c^2-c+1)\dfrac{\partial^2 g}{\partial v^2}-4c\dfrac{\partial^2 g}{\partial u^2}+u(c^2+1)-2cv\right) \\
-2c\left(\dfrac{\partial^3 g}{\partial u^3}\dfrac{\partial^2 g}{\partial v^2}-\dfrac{\partial^3 g}{\partial v^3}\dfrac{\partial^2 g}{\partial u^2}+\left(\dfrac{\partial^3 g}{\partial u^2 \partial v}-\dfrac{\partial^3 g}{\partial u \partial v^2}\right)\dfrac{\partial^2 g}{\partial u \partial v}\right) \\
-2(c^2-c+1)\dfrac{\partial^2 g}{\partial u \partial v}\left(\dfrac{\partial^3 g}{\partial u^3}-\dfrac{\partial^3 g}{\partial v^3}\right) \\
+c(c^2-1)\left(\dfrac{\partial^3 g}{\partial u^2 \partial c}-\dfrac{\partial^3 g}{\partial v^2 \partial c}\right)
\end{array}\right\}+\ldots.
\end{aligned}
\tag{3.6}
$$

Then taking into account the fact that

$$
c(t)=e^{-t/k^2}=1-\frac{t}{k^2}+O\left(\frac{1}{k^4}\right), \quad \frac{\partial}{\partial c}=-\frac{k^2}{c}\frac{\partial}{\partial t}, \tag{3.7}
$$

the leading term has order k^4 for large k (noting that $k^4(c-1)^2 = t^2 + O(1/k^2)$), At a second stage, defining

$$
h(t; u, v) := \lim_{k\to\infty} g(e^{-t/k^2}; u, v),
$$

and using the expansion (3.7) for $c(t)$ and the partial $\partial/\partial c = -(k^2/c)\ \partial/\partial t$, the leading term in the expression (3.6) has order k^4; no contribution comes from the k^0-term. This leading term (multiplied with $-1/2$) must therefore vanish, leading first to equation (1.10) and then setting $x = u - v,\ y = u + v$ in that equation, to equation (1.6) given in Theorem 1. ■

4 Proof of Theorem 1.2

Before giving the proof of Theorem 1.2, we remind the reader of the conjecture stated in section 1: for any fixed $t > 0$, $x \in \mathbb{R}$, the conditional probability satisfies:

$$\lim_{z \to \infty} P(A(t) \geq x + z \mid A(0) \leq -z) = 0. \tag{4.1}$$

Under this assumption, we prove the following:

Lemma 4.1 *Considering the series for the probability, for large t,*

$$P(A(0) \leq u, A(t) \leq v) = F_2(u)F_2(v)\left(1 + \sum_{i \geq 1} \frac{f_i(u,v)}{t^i}\right), \tag{4.2}$$

the coefficients $f_i(u,v)$ have the property

$$\lim_{u \to \infty} f_i(u,v) = \lim_{u \to \infty} f_i(v,u) = 0, \text{ for fixed } v \in \mathbb{R} \tag{4.3}$$

and

$$\lim_{z \to \infty} f_i(-z, z + x) = 0, \qquad \text{for fixed } x \in \mathbb{R}. \tag{4.4}$$

Proof: First observe by (5.1) in section 5 that the Airy kernel becomes diagonal when $t \to \infty$. Then the Airy process decouples at ∞, and, using the stationarity, one is lead to

$$\lim_{t \to \infty} P(A(0) \leq u, A(t) \leq v) = P(A(0) \leq u)P(A(0) \leq v) = F_2(u)F_2(v).$$

The next terms follow from the PDE (1.6), although it is more convenient here to use the form (1.10) of the equation. Considering the following conditional probability,

$$\begin{aligned} & P(A(t) \leq v \mid A(0) \leq u) \\ = & \frac{P(A(0) \leq u, A(t) \leq v)}{P(A(0) \leq u)} \\ = & F_2(v)\left(1 + \sum_{i \geq 1} \frac{f_i(u,v)}{t^i}\right), \end{aligned}$$

and setting

$$v = z + x, \qquad u = -z,$$

we have for all t, since $\lim_{z\to\infty} F_2(z+x) = 1$, and by (4.1) that

$$\begin{aligned} 1 &= \lim_{z\to\infty} P(A(t) \le z + x \mid A(0) \le -z) \\ &= 1 + \sum_{i\ge 1} \frac{\lim_{z\to\infty} f_i(-z, z+x)}{t^i}, \end{aligned}$$

implying that

$$\lim_{z\to\infty} f_i(-z, z+x) = 0, \qquad \text{for all } i \ge 1.$$

Similarly, letting $v \to \infty$, we have

$$\begin{aligned} 1 = \lim_{v\to\infty} P(A(t) \le v \mid A(0) \le u) &= \lim_{v\to\infty} \left[F_2(v) \left(1 + \sum_{i\ge 1} \frac{f_i(u,v)}{t^i} \right) \right] \\ &= 1 + \sum_{i\ge 1} \frac{\lim_{v\to\infty} f_i(u,v)}{t^i}. \end{aligned}$$

Hence

$$\lim_{v\to\infty} f_i(u,v) = 0 \tag{4.5}$$

and, considering the same argument for the conditional probability $P(A(0) \le u \mid A(t) \le v)$,

$$\lim_{u\to\infty} f_i(u,v) = 0,$$

ending the proof of Lemma 4.1. ■

Proof of Theorem 1.2: Putting the log of the expansion (4.2)

$$\begin{aligned} h(t;u,v) &= \log P(A(0) \le u, A(t) < v) \\ &= \log F_2(u) + \log F_2(v) + \sum_{i\ge 1} \frac{h_i(u,v)}{t^i} \\ &= \log F_2(u) + \log F_2(v) + \frac{f_1(u,v)}{t} + \frac{f_2(u,v) - f_1^2(u,v)/2}{t^2} + \ldots, \end{aligned} \tag{4.6}$$

in the equation (1.10), leads to:

(i) a leading term of order t, given by

$$\mathcal{L}h_1 = 0, \tag{4.7}$$

where

$$\mathcal{L} := \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\frac{\partial^2}{\partial u \partial v}. \tag{4.8}$$

The most general solution to (4.7) is given by

$$h_1(u,v) = r_1(u) + r_3(v) + r_2(u+v),$$

with arbitrary functions r_1, r_2, r_3. Hence,

$$P(A(0) \leq u, A(t) \leq v) = F_2(u)F_2(v)\left(1 + \frac{h_1(u,v)}{t} + ...\right)$$

with $h_1(u,v) = f_1(u,v)$ as in (4.2). Applying Lemma 4.1,

$$r_1(u) + r_3(\infty) + r_2(\infty) = 0, \quad \text{for all } u \in \mathbb{R},$$

implying

$$r_1(u) = \text{constant} = r_1(\infty),$$

and similarly

$$r_3(u) = \text{constant} = r_3(\infty).$$

Therefore, without loss of generality, we may absorb the constants $r_1(\infty)$ and $r_3(\infty)$ in the definition of $r_2(u+v)$. Hence, from (4.6),

$$f_1(u,v) = h_1(u,v) = r_2(u+v)$$

using (4.5),

$$0 = \lim_{z\to\infty} f_1(-z, z+x) = r_2(x)$$

implying that the $h_1(u,v)$-term in the series (4.6) vanishes.

(ii) One computes that the term $h_2(u,v)$ in the expansion (4.6) of $h(t;u,v)$ satisfies

$$\mathcal{L}h_2 = \frac{\partial^3 g}{\partial u^3}\frac{\partial^2 g}{\partial v^2} - \frac{\partial^3 g}{\partial v^3}\frac{\partial^2 g}{\partial u^2}, \text{ with } g(u) := \log F_2(u). \tag{4.9}$$

This is the term of order t^0, by putting the series (4.6) in the equation (1.10). The most general solution to (4.9) is

$$h_2(u,v) = g'(u)g'(v) + r_1(u) + r_3(v) + r_2(u+v).$$

Then

$$\begin{aligned} P(A(0) \le u, A(t) \le v) &= e^{h(t,u,v)} \\ &= F_2(u)F_2(v)e^{\sum_{i\ge 2} \frac{h_i(u,v)}{t^i}} \\ &= F_2(u)F_2(v)\left(1 + \frac{h_2(u,v)}{t^2} + ...\right). \end{aligned}$$

In view of the explicit formula for the distribution F_2 and the behavior (1.5) of $q(\alpha)$ for $\alpha \nearrow \infty$, we have that

$$\begin{aligned} \lim_{u\to\infty} g'(u) &= \lim_{u\to\infty} (\log F_2(u))' \\ &= \lim_{u\to\infty} \int_u^\infty q^2(\alpha)d\alpha = 0. \end{aligned}$$

Hence

$$0 = \lim_{u\to\infty} f_2(u,v) = \lim_{u\to\infty} h_2(u,v) = r_1(\infty) + r_3(v) + r_2(\infty),$$

showing r_1 and similarly r_3 are constants. Therefore, by absorbing $r_1(\infty)$ and $r_3(\infty)$ into $r_2(u+v)$, we have

$$f_2(u,v) = h_2(u,v) = g'(u)g'(v) + r_2(u+v).$$

Again, by the behavior of $q(x)$ at $+\infty$ and $-\infty$,

$$g'(-z)g'(z+x) = \int_{-z}^\infty q^2(\alpha)d\alpha \int_{z+x}^\infty q^2(\alpha)d\alpha \le cz^{3/2}e^{-2z/3}.$$

Hence

$$0 = \lim_{z\to\infty} f_2(-z, z+x) = r_2(x)$$

and so

$$f_2(u,v) = h_2(u,v) = g'(u)g'(v),$$

yielding the $1/t^2$ term in the series (4.6).

(iii) Next, setting

$$\begin{aligned} h(t;u,v) &= \log P(A(0) \le u, A(t) \le v) \\ &= g(u) + g(v) + \frac{g'(u)g'(v)}{t^2} + \frac{h_3(u,v)}{t^3} + \ldots \end{aligned} \tag{4.10}$$

in the equation (1.10), we find for the t^{-1} term:

$$\mathcal{L}h_3 = 0.$$

As in (4.7), its most general solution is given by

$$h_3(u,v) = r_1(u) + r_3(v) + r_2(u+v).$$

By exponentiation of (4.6), we find

$$\begin{aligned} P(A(0) \le u, A(t) \le v) = F_2(u)F_2(v)\Bigg(1 + \frac{g'(u)g'(v)}{t^2} \\ + \frac{r_1(u) + r_3(v) + r_2(u+v)}{t^3} + \ldots\Bigg). \end{aligned}$$

The precise same arguments lead to $h_3(u,v) = 0$.

(iv) So, at the next stage, we have

$$h(t;u,v) = g(u) + g(v) + \frac{g'(u)g'(v)}{t^2} + \frac{h_4(u,v)}{t^4} + \ldots \tag{4.11}$$

with

$$f_4(u,v) = h_4(u,v) + \frac{1}{2}h_2^2(u,v) = h_4(u,v) + \frac{1}{2}g'(u)^2g'(v)^2. \tag{4.12}$$

Setting the series (4.11) in the equation (1.10), we find for the t^{-2} term:

$$\begin{aligned}
\mathcal{L}h_4 &= 2\left(\frac{\partial^3 g}{\partial u^3}\left(\frac{\partial^2 g}{\partial v^2}\right)^2 - \frac{\partial^3 g}{\partial v^3}\left(\frac{\partial^2 g}{\partial u^2}\right)^2\right) + \frac{\partial^3 g}{\partial u^3}\frac{\partial^3 g}{\partial v^3}\left(\frac{\partial g}{\partial u} - \frac{\partial g}{\partial v}\right) \\
&\quad + \frac{1}{2}\left(\frac{\partial^4 g}{\partial u^4}\frac{\partial}{\partial v}\left(\frac{\partial g}{\partial v}\right)^2 - \frac{\partial^4 g}{\partial v^4}\frac{\partial}{\partial u}\left(\frac{\partial g}{\partial u}\right)^2\right) \\
&\quad + \left(\frac{\partial^3 g}{\partial u^3}\frac{\partial^2 g}{\partial v^2} + \frac{\partial^3 g}{\partial v^3}\frac{\partial^2 g}{\partial u^2}\right)(u-v) + 2\left(\frac{\partial^3 g}{\partial u^3}\frac{\partial g}{\partial v} - \frac{\partial^3 g}{\partial v^3}\frac{\partial g}{\partial u}\right) \\
&= 2\Big(2q(u)q'(u)(q(v)q'(v)+1) - q(u)q''(u)q^2(v) - (q'(u))^2 q^2(v)\Big)\int_v^\infty q^2 \\
&\quad + 2q(u)\Big(q(u)q'(v)q''(v) + q'(u)q(v)q''(v) - 2q(u)q^3(v)q'(v)\Big) \\
&\quad - \text{same with } u \leftrightarrow v. \qquad (4.13)
\end{aligned}$$

This latter is an expression in $q(u)$, $q(v)$ and its derivatives and in $\int_u^\infty q^2(\alpha)d\alpha$ and $\int_v^\infty q^2(\alpha)d\alpha$. It is obtained by substituting in the previous expression

$$g(u) = \int_u^\infty (u-\alpha)q^2(\alpha)d\alpha$$

and the Painlevé II differential equation for $q(u)$,

$$u\ q(u) = q''(u) - 2q(u)^3,$$

in order to eliminate the explicit appearance of u and v.

Now introducing[3]

$$\begin{aligned}
g(u) &= \int_u^\infty (u-\alpha) q^2(\alpha) d\alpha \\
g_1(u) &= \int_u^\infty (u-\alpha) q'^2(\alpha) d\alpha \\
g_2(u) &= \int_u^\infty (u-\alpha) q^4(\alpha) d\alpha,
\end{aligned}$$

the most general solution to equation (4.13) is given, modulo the null-space of $\mathcal{L}$, by

$$\begin{aligned}
h_4(u,v) &= \frac{1}{2}\Big(g''(u)g'(v)^2 + g''(v)g'(u)^2 + g''(u)g''(v)\Big) \\
&\quad + g'(u)\Big(2g(v) + g_1'(v) - g_2'(v)\Big) \\
&\quad + g'(v)\Big(2g(u) + g_1'(u) - g_2'(u)\Big) \\
&= q^2(u)\left(\frac{q^2(v)}{4} - \frac{1}{2}\left(\int_v^\infty q^2(\alpha)d\alpha\right)^2\right) \\
&\quad + \int_u^\infty q^2(\alpha)d\alpha \int_v^\infty \Big(2(v-\alpha)q^2(\alpha) + q'^2(\alpha) - q^4(\alpha)\Big)d\alpha \\
&\quad + \text{same with } u \leftrightarrow v. \qquad (4.14)
\end{aligned}$$

This form, together with (4.12), implies for the function $f_4(u,v)$:

$$\begin{aligned}
f_4(u,v) &= h_4(u,v) + \frac{1}{2}g'(u)^2 g'(v)^2 + r_1(u) + r_3(v) + r_2(u+v) \\
&= \sum_i a_i(u) b_i(v) + r_1(u) + r_3(v) + r_2(u+v).
\end{aligned}$$

Using the asymptotics of $q(u)$, one finds

[3]Note

$$g'(u) = \int_u^\infty q^2(\alpha)d\alpha, \qquad g''(u) = -q^2(u),$$

and

$$g_1'(u) = \int_u^\infty q'^2(\alpha)d\alpha, \qquad g_2'(u) = \int_u^\infty q^4(\alpha)d\alpha.$$

$$\begin{aligned} a_i(u), b_i(u) &\leq c\, e^{-u} \qquad u \to \infty, \\ &\leq c|u|^3 \qquad u \to -\infty, \end{aligned}$$

and so, by the same argument,

$$r_1(u) = r_2(u) = r_3(u) = 0.$$

Therefore, we have

$$f_4(u,v) = h_4(u,v) + \frac{1}{2} g'(u)^2 g'(v)^2.$$

with $h_4(u,v)$ as in (4.14), thus yielding the formula (1.8).

Finally, to prove formula (1.9), we compute, after integration by parts,

$$\begin{aligned} E\Big(A(0)A(t)\Big) &= \iint_{\mathbb{R}^2} uv \frac{\partial^2}{\partial u\, \partial v} P(A(0) \leq u, A(t) \leq v) du\ dv \\ &= \int_{-\infty}^{\infty} u F_2'(u) du \int_{-\infty}^{\infty} v F_2'(v) dv \\ &\quad + \frac{1}{t^2} \int_{-\infty}^{\infty} F_2'(u) du \int_{-\infty}^{\infty} F_2'(v) dv \\ &\quad + \frac{1}{t^4} \iint_{\mathbb{R}^2} \Big(\Phi(u,v) + \Phi(v,u)\Big) du\ dv \\ &\quad + \ \mathrm{O}\left(\frac{1}{t^6}\right) \\ &= \Big(E\Big(A(0)\Big)\Big)^2 + \frac{1}{t^2} + \frac{c}{t^4} + \ \mathrm{O}\left(\frac{1}{t^6}\right), \end{aligned}$$

where

$$c := \iint_{\mathbb{R}^2} \Big(\Phi(u,v) + \Phi(v,u)\Big) du\ dv = 2 \iint_{\mathbb{R}^2} \Phi(u,v) du dv,$$

thus ending the proof of Theorem 1.2. ■

5 The extended Airy kernel

The joint probabilities for the Airy process can also be expressed in terms of the Fredholm determinant of a matrix kernel, the so-called extended Airy kernel (considered in [3], [4] and [5]), namely

$$\begin{aligned} & P(A(t_1) < u_1, \ldots, A(t_m) < u_m) \\ &= \det\left(I - z(\hat{K}_{ij})_{1 \le i,j \le m}\right)\Big|_{z=1} \\ &= 1 + \sum_{k=1}^{\infty} \frac{(-z)^k}{k!} \sum_{1 \le i_1 \le \ldots \le i_k \le m} \int_{\mathbb{R}^k} \det\left(\hat{K}_{i_r i_s}(x_r, x_s)\right)_{1 \le r,s \le k} dx_1 \ldots dx_k \Bigg|_{z=1} \end{aligned}$$

with

$$\hat{K}_{ij}(x,y) := \chi_{[u_i,\infty)}(x) K_{ij}(x,y) \chi_{[u_j,\infty)}(y) \tag{5.1}$$

and

$$K_{ij}(x,y) := \begin{cases} \int_0^{\infty} e^{-z(t_i - t_j)} \mathrm{Ai}(x+z)\mathrm{Ai}(y+z) dz, & \text{if } t_i \ge t_j \\ \\ -\int_{-\infty}^{0} e^{-z(t_i - t_j)} \mathrm{Ai}(x+z)\mathrm{Ai}(y+z) dz, & \text{if } t_i < t_j \ , \end{cases}$$

$\mathrm{Ai}(x)$ being the Airy function. So the Fredholm determinant above is also a solution of the PDE of Theorem 2.1 for $m = 2$.

6 Appendix: remarks about the conjecture

Consider the Dyson Brownian motion $(\lambda_1(t), \ldots, \lambda_n(t))$ and the corresponding Ornstein-Uhlenbeck process on the matrix B. Then, using the change of variables

$$M_i = \frac{B_i}{\sqrt{(1-c^2)/2}},$$

and further $M_2 \mapsto M := M_2 - cM_1$ in the M_2-integrals below and noting that $\max(\mathrm{sp}\ M_1) \le -z$ and $\max(\mathrm{sp}\ M_2) \ge a$ imply $\max(\mathrm{sp}\ (M_2 - cM_1) \ge a + cz$, we have for the conditional probability, the following inequality:

$$
\begin{aligned}
& P\left(\lambda_n(t) \geq a \;\middle|\; \lambda_n(0) \leq -z\right) \\
&= \frac{\displaystyle\int_{\max(\mathrm{sp}\ M_1)\leq -z} dM_1 e^{-\frac{1}{2}(1-c^2)\operatorname{Tr} M_1^2} \int_{\max(\mathrm{sp}\ M_2)\geq a} dM_2 e^{-\frac{1}{2}\operatorname{Tr}(M_2-cM_1)^2}}{\displaystyle\int_{\max(\mathrm{sp}\ M_1)\leq -z} dM_1 e^{-\frac{1}{2}(1-c^2)\operatorname{Tr} M_1^2} \int_{M_2\in\mathcal{H}_n} dM_2 e^{-\frac{1}{2}\operatorname{Tr}(M_2-cM_1)^2}} \\
&\leq \frac{\displaystyle\int_{\max(\mathrm{sp}\ M_1)\leq -z} dM_1\ e^{-\frac{1}{2}(1-c^2)\operatorname{Tr} M_1^2} \int_{\max(\mathrm{sp}\ M)\geq a+cz} dM\ e^{-\frac{1}{2}\operatorname{Tr} M^2}}{\displaystyle\int_{\max(\mathrm{sp}\ M_1)\leq -z} dM_1\ e^{-\frac{1}{2}(1-c^2)\operatorname{Tr} M_1^2} \int_{M\in\mathcal{H}_n} dM\ e^{-\frac{1}{2}\operatorname{Tr} M^2}} \\
&= P(\lambda_n(t) \geq a + cz),
\end{aligned}
$$

implying

$$\lim_{z\to\infty} P\left(\lambda_n(t) \geq a \;\middle|\; \lambda_n(0) \leq -z\right) = 0,$$

and a fortiori,

$$\lim_{z\to\infty} P\left(\lambda_n(t) \geq x + z \;\middle|\; \lambda_n(0) \leq -z\right) = 0.$$

It is unclear why this limit remains valid when $n \to \infty$, using the Airy scaling (1.3). But the extended Airy kernel (5.1) seems to indicate the conjecture is valid.

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