ADVANCES IN MATHEMATICS 108, 140–204 (1994)

Birkhoff Strata, Bäcklund Transformations, and Regularization of Isospectral Operators

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Let differential operators

$$P=\left(\frac{d}{dx}\right)^n+q_2(x,t)\left(\frac{d}{dx}\right)^{n-2}+\cdots+q_n(x,t), \tag{0.1}$$

with holomorphic coefficients in $x\in\mathbb{C}$ and $t=(t_1,t_2,\ldots)\in\mathbb{C}^\infty$, flow according to the isospectral equations

$$\frac{\partial P}{\partial t_k}=[P_+^{k/n},P],\qquad k=1,2,\ldots, \tag{0.2}$$

where $P_+^{k/n}$ denotes the differential part of the k/nth power of P. They form an infinite-dimensional isospectral manifold $\mathcal{M}$ of differential operators. Since the t_1 flow and x translation coincide, it is convenient to replace throughout the variables (x,t) by $t+\bar{x}$, with $\bar{x}=(x,0,0,\ldots)$. In this study we address the following questions:

(i) What is the behavior of P near its blow-up locus $t=t^*$; that is, how do the functions $q_i(x,t)$ blow-up near t^*, and what does it depend on?

(ii) How does one desingularize P near the blow-up locus t^*? In geometrical language, how does one complete the isospectral manifold $\mathcal{M}$?

* The support of National Science Foundation Grant DMS-9203246 is gratefully acknowledged.

† The support of National Science Foundation Grant DMS-9203246, and NATO, FNRS, and Francqui Foundation grants is gratefully acknowledged.

0001-8708/94 $9.00

For the KdV isospectral manifold ($n=2$), we can answer those questions by elementary methods. Indeed it has been observed by many that the solution to the KdV equations ($' \equiv \partial/\partial_x$)

$$\frac{\partial q}{\partial t_k}\frac{\partial P}{\partial t_k}=[P_+^{k/2}, P],\ P=\frac{d^2}{dx^2}+q(x,t)=\frac{d^2}{dx^2}+2(\log\tau)'', \qquad k=1,3,5,\ldots$$

blows up after a finite time t^*, and that there the potential q behaves for small x as

$$q(t^*+\bar{x}) = -\frac{j(j-1)}{x^2}+\text{higher order terms}, \qquad j=2,3,\ldots, \tag{0.3}$$

as one checks by elementary computations.

The operator P can be transformed into a new one, $\tilde{P}$, by means of the well-known Bäcklund transformation. Recall the method: consider first an eigenfunction Ψ_1 going with a large but fixed eigenvalue z_1

$$P\Psi_1 \equiv \left(\frac{d^2}{dx^2}+q\right)\Psi_1 = z_1\Psi_1,$$

and associated linear operators

$$A_1=\Psi_1\frac{d}{dx}\Psi_1^{-1}\equiv\frac{d}{dx}-v_1 \qquad \text{and} \qquad A_1^{\mathrm{T}}=-\frac{d}{dx}-v_1$$

with $v_1\equiv\Psi_1'/\Psi_1$. Then $P-z_1$ admits the decomposition

$$P=\frac{d^2}{dx^2}+q=\frac{d^2}{dx^2}-\frac{\Psi_1''}{\Psi_1}+z_1=-A_1^{\mathrm{T}}A_1+z_1$$

with $q=-\Psi_1''/\Psi_1+z_1=-v_1'-v_1^2+z_1$. The new linear operator $\tilde{P}$ is now defined by conjugation,

$$\tilde{P}=A_1PA_1^{-1}=-A_1A_1^{\mathrm{T}}+z_1=\frac{d^2}{dx^2}+v_1'-v_1^2+z_1=\frac{d^2}{dx^2}+\tilde{q};$$

it induces a transformation

$$q \curvearrowright \tilde{q}=q+2v_1'=2\frac{d^2}{dx^2}\log\tau+2\frac{d^2}{dx^2}\log\Psi_1=2\frac{d^2}{dx^2}\log\tau\Psi_1$$

and, at the τ-level

$$\tau\curvearrowright\tau_1\equiv\tau\Psi_1; \tag{0.4}$$

since $(\tilde{P}-z)A_1\Psi = A_1(P-z)\Psi_1 = 0$, the function Ψ maps into Ψ_1:

$$\Psi \curvearrowright \Psi_1 \equiv A_1\Psi = -\frac{\Psi_1'}{\Psi_1}\Psi + \Psi'. \tag{0.5}$$

That is to say, the new function τ_1 is obtained by multiplying τ with Ψ_1; also, the new eigenfunction Ψ_1 is a linear combination of the old Ψ and Ψ'.

If q behaves as (0.3), then the function v_1, a solution of the Ricatti equation $v_1^2 + v_1' + q - z_1 = 0$, must behave as

$$v_1 = \frac{\alpha}{x} + \cdots, \qquad \text{with} \quad \alpha^2 - \alpha - j(j-1) = 0.$$

Picking the *positive root* $\alpha = j$, we see that the transformation

$$\begin{aligned}\tilde{q} = q + 2v_1' &= -\frac{j(j-1)}{x^2} - 2\frac{j}{x^2} + \text{higher order terms}\\ &= \frac{-j(j+1)}{x^2} + \cdots;\end{aligned}$$

that is, the (integer) leading term $j(j-1)$ of $-q$ is increased to $(j+1)j$.

Picking the *negative root* $\alpha = -j+1$, we see that

$$\begin{aligned}\tilde{q} = q + 2v_1' &= -\frac{j(j-1)}{x^2} - \frac{2(-j+1)}{x^2} + \cdots\\ &= \frac{-(j-1)(j-2)}{x^2} + \cdots\end{aligned}$$

decreases the leading term of $-q$ to $(j-1)(j-2)$. Thus the Bäcklund transformation corresponding to the negative root has the effect of lowering the leading term.

In particular, if $q = -j(j-1)/x^2 + \cdots$, then the function $\tilde{q}$ corresponding to $j-1$ transformations, with negative root is finite for $x \searrow 0$. It is now this simple idea which can be vastly generalized to general pseudo-differential operators. This simple example contains much of the seeds for the results in this paper.

Returning to the general KP case, it is more convenient to pose questions (i) and (ii) for the pseudo-differential operator $L = P^{1/n}$, with

$$L = \frac{d}{dx} + \sum_{j:-1}^{-\infty} a_j(x,t)\left(\frac{d}{dx}\right)^j, \tag{0.6}$$

flowing according to

$$\frac{\partial L}{\partial t_k} = [(L^k)_+, L], \qquad k = 1, 2, \ldots \tag{0.7}$$

Again, since the t_1-flow and x-translation coincide, it is convenient to replace throughout the variables (x, t) by $t + \bar{x}$, with $\bar{x} = (x, 0, 0, \ldots)$. According to Sato's celebrated discovery [S1], L can be conjugated to d/dx by means of the wave operator S:[1]

$$L = S \frac{d}{dx} S^{-1} \qquad \text{with} \quad S = \sum_{n=0}^{\infty} \frac{p_n(-\tilde{\partial})\, \tau(t)}{\tau(t)} \left(\frac{d}{dx}\right)^{-n}; \tag{0.8}$$

that is, the solution L of (0.3) is expressible in terms of a single function $\tau(t)$, a solution of the KP hierarchy. It is also well-known [DJKM] that the wave function $\Psi(t, z)$, a solution of

$$L\Psi = z\Psi \qquad \text{and} \qquad \frac{\partial \Psi}{\partial t_n} = (L^n)_+ \Psi, \tag{0.9}$$

can be represented in terms of τ (for large $z \in \mathbb{C}$) as follows:

$$\Psi(t, z) = S e^{\sum_1^\infty t_i z^i} = e^{\sum_1^\infty t_i z^i} \frac{\tau(t - [z^{-1}])}{\tau(t)} \equiv e^{\sum t_i z^i} \psi(t, z),$$

$$\text{with} \quad [s] \equiv \left(s, \frac{s^2}{2}, \frac{s^3}{3}, \ldots\right). \tag{0.10}$$

To the wave function Ψ one associates a plane, generated by all its partial derivatives with regard to $t_1, t_2, \ldots$ at $t = 0$, viewed as functions of z (see [S, S-W]), which, upon using (0.9), can always be expressed in terms of partial derivatives with regard to t_1:

$$W^0 = \text{span}\left\{ \Psi(t, z)|_{t=0}, \frac{\partial}{\partial t_1} \Psi(t, z)\bigg|_{t=0}, \frac{\partial^2}{\partial t_1^2} \Psi(t, z)\bigg|_{t=0}, \ldots \right\}.$$

Whenever $\tau(t) \neq 0$ (which is so for generic $t \in \mathbb{C}^\infty$), one shows, using the τ-function representation (0.10) of Ψ and the operator $\nabla = \partial/\partial x + z$, that

$$\begin{aligned} W^t \equiv e^{-\sum_1^\infty t_i z^i} W^0 &= \text{span}\{\psi(t, z), \nabla\psi(t, z), \nabla^2\psi(t, z), \ldots\} \\ &= \text{span}\{z^k(1 + O(z^{-1})), k = 0, 1, 2, 3, \ldots\} \end{aligned} \tag{0.11}$$

has a basis of all orders $k = 0, 1, 2, \ldots$, since

$$\nabla^k \psi(t, z) = z^k(1 + O(z^{-1})).$$

[1] $e^{\sum_1^\infty t_i z^i} = \sum_0^\infty p_n(t) z^n$, $p_n(-\tilde{\partial}) = p_n(-\partial/\partial t_1, -\frac{1}{2}(\partial/\partial t_2), -\frac{1}{3}(\partial/\partial t_3), \ldots)$.

Let Gr be the infinite-dimensional Grassmannian consisting of all such planes and their limits.

For the t^*'s such that $\tau(t^*)=0$, the plane W^{t^*} still exists, although the basis (0.11) ceases to exist, it will have some new basis $\varphi_0, \varphi_1, \ldots$ behaving as

$$\varphi_i(z)=z^{s_i}(1+O(z^{-1}))$$

$$s_0<s_1<s_2<\cdots \qquad \text{and} \qquad s_i=i \text{ for large } i.$$

Thus to each plane W^t one associates a finite sequence (partition) (see [S-W] and [P-S])

$$\nu(W^t)\equiv(\nu_0\geqslant\nu_1\geqslant\nu_2\geqslant\cdots\geqslant 0\geqslant 0\geqslant\cdots),$$

where $\nu_i=i-s_i$, which in turn defines a Young diagram; for explanations see Appendix A. Thus the manifold Gr has a cellular decomposition into so-called Birkhoff strata, all parametrized by Young diagrams, with a principal stratum going with $\nu(W^t)=0$. Hence the Young diagram measures how strongly $\tau(t)$ can vanish; it also measures the depth of the singularity of the corresponding operator L, and the eigenfunction Ψ, as $\tau(t)$ appears in the denominator of S and hence in the denominators of L and Ψ (see (0.8)).

This paper deals with the process of desingularizing L and Ψ. An essential ingredient in doing so is provided by the Bäcklund transform and its dual. Given an arbitrary, but fixed, $z_1\in\mathbb{C}$ near ∞, they map a function τ into new τ-functions (see Theorem 4.1):

$$\tau_1(t)\equiv X(t,z_1)\,\tau(t)\equiv e^{\sum t_i z_1^i}\tau(t-[z_1^{-1}])=\Psi(t,z_1)\,\tau(t)$$

and

$$\tilde{\tau}_1(t)\equiv \tilde{X}(t,z_1)\,\tau(t)\equiv e^{-\sum t_i z_1^i}\tau(t+[z_1^{-1}])=\Psi^*(t,z_1)\,\tau(t),$$

with associated wave functions

$$\Psi_1(t,z)=-\frac{z_1}{z}\,\Psi(t-[z_1^{-1}],z)=z^{-1}A_{\Psi(t,z_1)}\Psi(t,z)$$

$$\tilde{\Psi}_1(t,z)=-\frac{z}{z_1}\,\Psi(t+[z_1^{-1}],z)\equiv z\tilde{A}_{\Psi(t,z_1)}\Psi(t,z),$$

expressed in terms of the Bäcklund–Darboux transformation

$$A\equiv A_{\Psi(t,z_1)}=\Psi(t,z_1)\frac{d}{dx}\,\Psi^{-1}(t,z_1)$$

and its dual $\tilde{A} \equiv \tilde{A}_{\Psi(t,z_1)}$, a kind of "inverse." It also induces maps at the level of L,

$$L \mapsto L_1 = ALA^{-1} \quad \text{and} \quad L \mapsto \tilde{L}_1 = \tilde{A}L\tilde{A}^{-1},$$

with

$$L_1 \Psi_1 = z\Psi_1 \quad \text{and} \quad \tilde{L}_1 \tilde{\Psi}_1 = z\tilde{\Psi}_1.$$

The Bäcklund transforms above correspond geometrically to mapping the linear space $W \in \mathrm{Gr}$ into new ones W_1 and $\tilde{W}_1 \in \mathrm{Gr}$ satisfying

$$zW_1' \subset W' \quad \text{and} \quad zW' \subset \tilde{W}_1'.$$

This statement is tantamount to the Fay identity for the τ-function.

Compounding several Bäcklund transformations for fixed but arbitrary $z_1, ..., z_k$ near $z = \infty$, we have the identity

$$\begin{aligned}\Psi_k(t, z) &= z^{-k} A_{\Psi_{k-1}(t,z_k)} \cdots A_{\Psi_1(t,z_2)} A_{\Psi(t,z_1)} \Psi(t, z) \\ &= z^{-k} \frac{\text{Wronskian}[\Psi(t, z_1), ..., \Psi(t, z_k), \Psi(t, z)]}{\text{Wronskian}[\Psi(t, z_1), ..., \Psi(t, z_k)]} \\ &= \prod_{i=1}^{k} (-z_i)\, \Psi\left(t - \sum_{j=1}^{k} [z_j^{-1}], z\right), \quad \text{with} \quad \Psi_0 = \Psi,\end{aligned}$$

which is equivalent to higher Fay identities; see Lemma 5.1. Note that the $\Psi_k(t, z)$ thus obtained is a wave function associated to a plane W_k related to the original plane W by the inclusion

$$z^k W_k' \subset W'.$$

Similarly, compounding dual Bäcklund transformations leads to the wave function

$$\tilde{\Psi}_k(t, z) = z^k \tilde{A}_{\tilde{\Psi}_{k-1}(t,z_k)} \cdots \tilde{A}_{\tilde{\Psi}_0(t,z_1)} \Psi(t, z) \tag{0.12}$$

associated to the plane $\tilde{W}_k$ related to the original plane W by the inclusion

$$z^k W' \subset \tilde{W}_k'.$$

In order to understand and state the main results of the paper, we need to make a small excursion in the theory of symmetric functions. Given a Young diagram $\nu = (\nu_0 \geqslant \nu_1 \geqslant \cdots \geqslant \nu_n \geqslant 0)$, the corresponding Schur polynomial is defined as (see Footnote 1 for a definition of p_k)

$$F_\nu(t) \equiv \det(p_{\nu_i - i + j}(-t))_{0 \leqslant i \leqslant j \leqslant n} \qquad (t = (t_1, t_2, ...)).$$

We need the following facts: a *first ingredient* due to Sato [S] is that the τ-function admits a Fourier expansion in terms of Schur polynomials

$$\tau(t^* + t) = \sum_{\nu} \xi_\nu(W^{t^*})\, F_\nu(t),$$

over all Young diagrams ν where W^{t^*} is the plane associated with t^* and with Fourier coefficients

$$\xi_\nu(W^{t^*}) = \det \operatorname{proj}(W^{t^*} \to H_\nu \equiv \operatorname{span}\{z^{i-\nu_i}, i = 0, 1, 2, \ldots\})$$

satisfying Plücker relations. The latter implies that near a point t^*, where the Young diagram $\nu^* \equiv \nu(W^{t^*}) \neq 0$, we have $\xi_\nu(W^{t^*}) = 0$ for all ν such that $\nu \not\geqslant \nu^*$; therefore, near the point t^*, $\tau(t^* + t)$ has the following Fourier series:

$$\tau(t^* + t) = \xi_{\nu^*}(W^{t^*})\, F_{\nu^*}(t) + \sum_{\substack{\nu \geqslant \nu^* \\ |\nu| > |\nu^*|}} \xi_\nu(W^{t^*})\, F_\nu(t), \qquad \text{with} \quad \xi_{\nu^*}(W^{t^*}) \neq 0.$$

A *second ingredient* is that the Schur polynomial $F_\nu(t - [s])$ (for notation, see (0.10)) admits the following Taylor series in s about $s = 0$,

$$\begin{aligned} F_\nu(t - [s]) &= F_\nu(t) + \cdots + s^k p_k(-\tilde{\partial})\, F_\nu(t) + \cdots + s^{\nu_0} p_{\nu_0}(-\tilde{\partial})\, F_\nu(t) \\ &= F_\nu(t) + \cdots + s^k F_{\nu \backslash (k)}(t) + \cdots + s^{\nu_0} F_{\nu \backslash \text{first row}}(t), \end{aligned}$$

where $F_{\nu\backslash(k)}$ is the skew Schur polynomial associated with the Young diagram $\nu\backslash(k)$; see Appendices A and B.

The main results in the paper can be summarized by the following three theorems, to be found in Section 7.

THEOREM 0.1. *At a point t^* with $\tau(t^*) = 0$ and $\nu(W^{t^*}) = (\nu_0 \geqslant \nu_1 \geqslant \cdots)$, construct planes $W_1, W_2, \ldots$ by means of successive (dual) Bäcklund transforms* (0.12), *thus satisfying*

$$z^{\nu_0} W^t \subset z^{\nu_0 - 1} W_1^t \subset z^{\nu_0 - 2} W_2^t \subset \cdots \subset W_{\nu_0}^t,$$

with τ_k associated with W_k. Then

$$\tau(t^*) = \tau_1(t^*) = \cdots = \tau_{\nu_0 - 1}(t^*) = 0 \qquad \textit{and} \qquad \tau_{\nu_0}(t^*) \neq 0,$$

and

$$\textit{Young diagram } (W_k^{t^*}) = \textit{Young diagram } W^{t^*} \backslash (\textit{first } k \textit{ columns});$$

the associated Schur polynomial determines the leading term of τ_k, *with*[2]

$$\tau_k(t^* + \bar{x}) = a_k x^{|\nu| - \sum_1^k \hat{\nu}_i} + \cdots$$

Successive Bäcklund transforming gives the τ-function a softer and softer zero, according to a well-defined pattern, until it ultimately does not vanish any more. Thus the successive Bäcklund transforms enable one to "climb" out of the singularity by knocking off each time the left-most column of the Young diagram.

In the next theorem we indicate how certain differential polynomials $p_k(-\tilde{\partial})$, applied to τ, behave in the t_1 or x-direction near the point of vanishing t^*.

THEOREM 0.2. *At a point* t^* *with* $\tau(t^*) = 0$ *and* $\nu(W^{t^*}) = (\nu_0 \geqslant \nu_1 \geqslant \cdots)$, *we have the following estimates*

$$\begin{aligned} p_k(-\tilde{\partial})\, \tau(t^* + \bar{x}) &= cc_k^\nu x^{|\nu| - k} + \cdots, && \textit{for} \quad 0 \leqslant k \leqslant \nu_0, \textit{ with } cc_k^\nu \neq 0, \\ &= c_k^\nu x^{|\nu| - \nu_0} + \cdots, && \textit{for} \quad k \geqslant \nu_0, \end{aligned}$$

where the c_k^ν $(0 \leqslant k \leqslant \nu_0)$ *are numbers expressible in terms of the Young diagram* ν *and its dual* $\hat{\nu}$, *by means of the following polynomial identity*:

$$P(z) = \sum_{k=0}^{\nu_0} \frac{c_k^\nu}{c_0^\nu} z(z-1)\cdots(z - \nu_0 + k + 1) = \prod_{i=0}^{\nu_0 - 1} (z - (\nu_0 + \hat{\nu}_i - i - 1)),$$

with

$$c_k^\nu = (-1)^{|\nu| - k} \det\left(\frac{1}{(\nu_i - k\delta_{j,0} - i + j)!}\right)_{0 \leqslant i, j \leqslant \hat{\nu}_0}.$$

The next theorem tells us that $\Psi(t, z)$ multiplied with an appropriate factor (independent of z) tends to a finite limit, when $t \to t^*$ in the t_1-direction and that this limit is—up to a multiplicative factor—a *new wave function* evaluated at $t = t^*$. In fact, this new wave function yields a new frame in Gr with regard to which all the constituents of the limiting plane W^{t^*} can be expressed. Remember the expression (0.10) for $\Psi = \psi e^{\sum t_i z^i}$.

[2] The dual Young diagram $\hat{\nu}$ of ν is the Young diagram obtained by flipping the Young diagram around its diagonal; define

$$|\nu| \equiv \sum_1^{\hat{\nu}_0} \nu_i = \sum_1^{\nu_0} \hat{\nu}_i.$$

THEOREM 0.3. *Consider a point* t^* *where* $\tau(t^*)=0$, *and* $\nu(W^{t^*})=(\nu_0 \geq \nu_1 \geq \cdots)$; *then the following limit exists and equals*

$$\lim_{x\to 0} \psi(t^*+\bar{x}, z)\frac{\tau(t^*+\bar{x})}{p_{\nu_0}(-\tilde{\partial})\,\tau(t^*+\bar{x})}=z^{-\nu_0}\tilde{\psi}_{\nu_0}(t^*, z)\in W^{t^*}$$

where $\tilde{\Psi}_{\nu_0}(t, z)$ *is obtained by compounding* ν_0 *dual Bäcklund transforms, depending on parameters* $z_1, ..., z_{\nu_0}$ *near* $z=\infty$. *Note that the limit is independent of the choice of* z_i.

We also describe in Theorem 7.5 how the whole basis of W^{t^*} can be obtained as a limit of basis elements of W^t for $t \neq t^*$.

The work in this paper was done in 1991. We did a similar study for the Toda lattice (see [AHvM]), where we show how to complete the isospectral set of periodic Jacobi matrices; in the latter, the completion depends heavily on the Birkhoff strata for a space of flags, rather than for the Grassmannian. For lectures on these topics, see [vM].

CONTENTS

1. Pseudo-differential Operators, Wave Functions, and KP Equations

Consider the (formal) pseudo-differential operator[3]

$$L=\frac{d}{dx}+\sum_{j=-1}^{-\infty} a_j(x;t_1,t_2,...)\left(\frac{d}{dx}\right)^j\in\frac{d}{dx}+\mathscr{D}^-,\qquad (t_1,t_2,...)\in\mathbb{C}^\infty$$

with holomorphic coefficients in x, depending on $t\in\mathbb{C}^\infty$ and the natural set of deformation equations for L given by[4]

$$\frac{\partial L}{\partial t_n}=[(L^n)_+, L],\qquad n=1,2,.... \tag{1.1}$$

[3] $\mathscr{D}^-$ is the space of strictly pseudo-differential operators and $\mathscr{D}^+$ of space of differential operators.

[4] $(L^n)_+$ means the differential part of L^n and $(L^n)_-=L^n-(L^n)_+$.

It defines an infinite number of vector fields on $d/dx + \mathscr{D}^-$, since $[(L^n)_+, L] = [-(L^n)_-, L] \in \mathscr{D}^-$; they commute, since they are equivalent to $(\partial/\partial t_n) L^m_+ - (\partial/\partial t_m) L^n_+ = [L^n_+, L^m_+]$. Note that, setting $\bar{t} = t + \bar{x}$ with $\bar{x} = (x, 0, ...)$,

$$a_j(x; t_1, t_2, ...) \equiv a_j(t + \bar{x}) = a_j(\bar{t}),$$

which is seen by identifying

$$\sum_{j=-1}^{-\infty} \frac{\partial a_j}{\partial t_1}\left(\frac{d}{dx}\right)^j = \frac{\partial L}{\partial t_1} = [L_+, L] = \left[\frac{d}{dx}, L\right] = \sum_{j=-1}^{-\infty} \frac{\partial a_j}{\partial x}\left(\frac{d}{dx}\right)^j.$$

Equations (1.1) define the so-called KP hierarchy. The point of view in this section is chiefly due to Sato [S1] and Date, Jimbo, Kashiwara, and Miwa [DJKM]. Equations (1.1) have a solution, if and only if there exists a *wave function*

$$\begin{aligned} \Psi(t, z) &= \left(\sum_0^\infty w_n(t) z^{-n}\right) e^{\sum_1^\infty t_i z^i}, \qquad z \in \mathbb{C}, \text{ large} \\ &\equiv \psi(t, z) e^{\sum t_i z^i} \end{aligned}$$

satisfying

$$\begin{aligned} L\Psi &= z\Psi \qquad (z = s^{-1}) \\ \frac{\partial \Psi}{\partial t_n} &= (L^n)_+ \Psi. \end{aligned} \tag{1.2}$$

This statement is also equivalent to the existence of two wave functions

$$\Psi = \left(\sum_0^\infty w_n(t) z^{-n}\right) e^{\sum_1^\infty t_i z^i} \qquad \text{and} \qquad \Psi^* = \left(\sum_0^\infty w_n^*(t) z^{-n}\right) e^{-\sum_1^\infty t_i z^i}$$

such that

$$\oint_{z=\infty} \Psi(t, z) \Psi^*(t', z)\, dz = 0 \qquad (\text{all } t \text{ and } t'), \tag{1.3}$$

the integral taken along a small contour around $z = \infty$; see [Che]. Conversely, if (1.3) is satisfied, then Ψ is a wave function and Ψ^* is the wave function of L^T.

Then Sato [S2] showed the existence of a function (called a τ-function) which enabled him to represent Ψ in many different ways (see Appendix A for notations),[5]

[5] Given $s \in \mathbb{C}$, define $[s] = (s, s^2/2, s^3/3, s^4/4, ...) \in \mathbb{C}^\infty$; $p_n(t_1, t_2, ...)$ is the elementary Schur polynomial (A.1) and $p_n(\pm\tilde{\partial}) = p_n(\pm\partial/\partial t_1, \pm\frac{1}{2}(\partial/\partial t_2), \pm\frac{1}{3}(\partial/\partial t_3), ...)$.

$$\begin{aligned}\Psi(t,z)&=\frac{\tau(t-[z^{-1}])}{\tau(t)}\,e^{\sum_1^\infty t_j z^j}\\ &=\sum_{n=0}^{\infty}\frac{p_n(-\tilde{\partial})\,\tau(t)}{\tau(t)}\,z^{-n}e^{\sum t_j z^j}\\ &=\sum_{n=0}^{\infty}\frac{p_n(-\tilde{\partial})\,\tau(t)}{\tau(t)}\left(\frac{d}{dx}\right)^{-n}e^{\sum t_j z^j}\\ &\equiv S(t)\,e^{\sum t_j z^j},\end{aligned}\tag{1.4}$$

where

$$S(t)\equiv\sum_{n=0}^{\infty}\frac{p_n(-\tilde{\partial})\,\tau(t)}{\tau(t)}\left(\frac{d}{dx}\right)^{-n},\tag{1.5}$$

with inverse

$$S(t)^{-1}=\sum_{n=0}^{\infty}\left(\frac{d}{dx}\right)^{-n}\frac{p_n(\tilde{\partial})\,\tau(t)}{\tau(t)}.\tag{1.6}$$

Then also

$$\begin{aligned}\Psi^*(t,z)&=(S^{\mathrm{T}})^{-1}\,e^{-\sum t_j z^j}\\ &=\frac{\tau(t+[z^{-1}])}{\tau(t)}\,e^{-\sum t_j z^j}.\end{aligned}\tag{1.4'}$$

Therefore $L\Psi=z\Psi$ can be expressed as

$$LSe^{\sum_1^\infty t_j z^j}=zSe^{\sum t_j z^j}=S\frac{d}{dx}\,e^{\sum t_j z^j}$$

and thus L admits the representation

$$L=S\frac{d}{dx}S^{-1}\qquad\text{and thus}\qquad L^n=S\left(\frac{d}{dx}\right)^n S^{-1}.\tag{1.7}$$

Moreover $\partial\Psi/\partial t_n=(L^n)_+\,\Psi$ turns into

$$\begin{aligned}\frac{\partial S}{\partial t_n}&=(L^n)_+\,S-S\left(\frac{d}{dx}\right)^n\\ &=((L^n)_+-L^n)\,S,\qquad\text{using (1.7)}\\ &=-(L^n)_-\,S.\end{aligned}\tag{1.8}$$

The representation (1.7) of L^n also implies

$$L^n = S\left(\frac{d}{dx}\right)^n S^{-1}$$
$$= \sum_{i,j}^{\infty} \frac{p_i(-\tilde{\partial})\,\tau}{\tau}\left(\frac{d}{dx}\right)^{n-i-j} \frac{p_j(\tilde{\partial})\,\tau}{\tau}, \qquad \text{using (1.5) and (1.6)}, \tag{1.9}$$

and thus

$$-(L^n)_- = -\sum_{i+j=n+1} \frac{p_i(\tilde{\partial})\,\tau \cdot p_j(-\tilde{\partial})\,\tau}{\tau^2}\left(\frac{d}{dx}\right)^{-1} + \cdots. \tag{1.10}$$

Then, keeping track of the coefficients of $(d/dx)^{-1}$ in $\partial S/\partial t_n = -(L^n)_- S$, we get

$$\left(\tau\frac{\partial^2\tau}{\partial t_n\,\partial t_1} - \frac{\partial\tau}{\partial t_n}\frac{\partial\tau}{\partial t_1}\right) - \sum_{\substack{i+j=n+1\\ i,j\geqslant 0}} p_i(\tilde{\partial})\,\tau\cdot p_j(-\tilde{\partial})\,\tau = 0, \qquad n=1,2,3,..., \tag{1.11}$$

or what is the same, using (A.6), the Hirota bilinear equations[6] for the τ-function

$$(\tfrac{1}{2}D_1 D_n - p_{n+1}(\tilde{\partial}))\,\tau\circ\tau = 0, \qquad n=1,2,3,.... \tag{1.12}$$

The first non-trivial equation (for $n=3$) is the celebrated KP equation

$$(D_1^4 + 5D_2^2 - 4D_1 D_3)\,\tau\circ\tau = 0. \tag{1.13}$$

Remark. It should be pointed out that the operator $S(t)$ is unique up to multiplication by S_0,

$$S(t) \curvearrowright S(t)\,S_0, \qquad S_0 = 1 + \sum_1^{\infty} b_i D^{-i}, \qquad b_i \text{ constants}.$$

It has the following effect on Ψ and τ,

$$\Psi = Se^{\sum t_i z^i} \curvearrowright SS_0 e^{\sum t_i z^i} = S\left(1 + \sum_1^{\infty} b_i z^{-i}\right) e^{\sum t_i z^i}$$
$$= \left(1 + \sum_1^{\infty} b_i z^{-i}\right)\Psi$$
$$= e^{-\sum_1^{\infty}(d_i/i)\,z^{-i}}\Psi$$

[6] Given a polynomial P, define

$$P(\tilde{\partial})\,\tau\circ\tau \equiv P\left(\frac{\partial}{\partial u_1}, \frac{1}{2}\frac{\partial}{\partial u_2}, \frac{1}{3}\frac{\partial}{\partial u_3}, ...\right)\tau(t+u)\,\tau(t-u)\bigg|_{u=0}.$$

and

$$\tau(t) \curvearrowright \tilde{\tau}(t) = \tau(t)\, e^{\sum t_i d_i},$$

with $b_i = p_i(-d_1, -d_2/2, \ldots)$, $i = 1, 2, \ldots$ (for notation see Appendix A). This operation has no effect on L:

$$L \curvearrowright S(t)\, S_0 D S_0^{-1} S(t)^{-1} = S(t)\, D S(t)^{-1} = L.$$

Since for an arbitrary polynomial $P(t)$

$$P\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \ldots\right) \tilde{\tau} \circ \tilde{\tau} = e^{2\sum t_i d_i} P\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \ldots\right) \tau \circ \tau,$$

$\tilde{\tau}$ also satisfies the bilinear relations (1.12).

2. The Fay Identity for the τ-Function

Lemma 2.1. *The τ-function satisfies the Fay identity*

$$\begin{aligned} &+(s_0 - s_1)(s_2 - s_3)\, \tau(t + [s_0] + [s_1])\, \tau(t + [s_2] + [s_3]) \\ &\quad + (s_0 - s_2)(s_3 - s_1)\, \tau(t + [s_0] + [s_2])\, \tau(t + [s_3] + [s_1]) \\ &\quad + (s_0 - s_3)(s_1 - s_2)\, \tau(t + [s_0] + [s_3])\, \tau(t + [s_1] + [s_2]) = 0, \end{aligned} \tag{2.1}$$

and a differential Fay identity[7]

$$\begin{aligned} &\{\tau(t - [s_1]), \tau(t - [s_2])\} \\ &\quad + (s_1^{-1} - s_2^{-1})(\tau(t - [s_1])\, \tau(t - [s_2]) - \tau(t)\, \tau(t - [s_1] - [s_2])) = 0. \end{aligned} \tag{2.2}$$

It also satisfies generalized Fay identities[8]

$$\begin{aligned} &\tau\left(t - \sum_1^n [s_i]\right) \Delta(s_1, \ldots, s_n) \left(\tau\left(t - \sum_1^n [r_i]\right) \Delta(r_1, \ldots, r_n)\right)^{n-1} \\ &\quad = \det\left[\tau\left(t - \sum_1^n [r_l] + [r_i] - [s_j]\right)\right. \\ &\qquad \left. \times \Delta(r_1, \ldots, r_{i-1}, s_j, r_{i+1}, \ldots, r_n)\right]_{1 \leqslant i, j \leqslant n} \end{aligned} \tag{2.3}$$

[7] Wronskian $[f_1, \ldots, f_n] = \det((\partial/\partial x)^{i-1} f_j)_{i,j=1,\ldots,n}$, $\{f, g\} = \text{Wronskian}[g, f] = f'g - fg'$.
[8] $\Delta(s_1, \ldots, s_n) = \prod_{1 \leqslant j < i \leqslant n} (s_i^{-1} - s_j^{-1})$.

and their differential version

$$Wronskian[\Psi(t, s_1^{-1}), ..., \Psi(t, s_n^{-1})]$$

$$= e^{\sum t_i(s_1^{-i} + \cdots + s_n^{-i})} \frac{\tau(t - [s_1] - [s_2] - \cdots - [s_n])}{\tau(t)} \Delta(s_1, ..., s_n). \qquad (2.4)$$

Proof. Identity (2.1) is due to Sato and the proof is due to [DJKM] and [Sh]. Putting (1.4) and (1.4′) into the integral (1.3) and setting $s = z^{-1}$, we find

$$\oint_{s=0} \tau(t - [s])\, \tau(t' + [s])\, e^{\sum (t_i - t_i')/s^i} \frac{ds}{s^2} = 0.$$

Thus, upon changing variables, we have for all t and y

$$\int_C \tau(t - y - [s])\, \tau(t + y + [s])\, e^{-2\sum y_i/s^i} \frac{ds}{s^2} = 0.$$

Then considering a special choice of y and a special shift of t

$$y = ([s_0] - [s_1] - [s_2] - [s_3])/2, \; t \curvearrowright t + ([s_0] + [s_1] + [s_2] + [s_3])/2$$

and using

$$\exp\left(-\sum (a/b)^i/i\right) = 1 - a/b,$$

yield

$$0 = \frac{1}{2\pi i} \int_C \frac{(1 - s_0/s)}{(1 - s_1/s)(1 - s_2/s)(1 - s_3/s)} \tau(t - y - [s])\, \tau(t + y + [s]) \frac{ds}{s^2}$$

$$= \text{“sum of formal residues at } s = s_1, s_2, s_3\text{”}$$

$$= (s_1 - s_2)^{-1} (s_2 - s_3)^{-1} (s_3 - s_1)^{-1} \times \{\text{left hand side of (2.1)}\},$$

which establishes (2.1).

To prove (2.2), differentiate (2.1) with respect to s_0, set $s_0 = s_3 = 0$, divide the relation by $s_1 s_2$, and then shift $t \curvearrowright t - [s_1] - [s_2]$.

Finally, (2.3) and (2.4) are also consequences of (2.1); their proof is postponed until Section 5.

3. Wave Functions, Infinite-Dimensional Grassmannians, and Loops

With Sato [S2], Segal and Wilson [SW], and Pressley and Segal [PS], consider the space Gr of linear spaces W of formal power series in large $z = s^{-1}$ having the property that W possesses an algebraic basis,

$$W = \{w_0(z), w_1(z), w_2(z), \ldots\}; \tag{3.1}$$

it means that the basis elements

$$w_n(z) = \sum_{i:\,-\infty}^{s_n} a_i z^i,$$

have finite orders, which satisfy

$$s_0 < s_1 < s_2 < \cdots \qquad \text{and} \qquad s_n = n \qquad \text{for} \quad \text{large}\, n.$$

Thus to each W we associate a "sequence of virtual genus zero"

$$\mathscr{S}(W) = (s_0, s_1, s_2, \ldots);$$

for more notation, see Appendix A. Of course, for a sequence $\mathscr{S} = (s_0, s_1, \ldots)$, both

$$H_+ = \text{span}\{1, z, z^2, z^3, \ldots\} \text{ and } H_{\mathscr{S}} = \text{span}\{z^{s_0}, z^{s_1}, z^{s_2}, \ldots\} \in Gr \tag{3.2}$$

The principal stratum is the set of linear spaces $W \in Gr$ such that the corresponding $\mathscr{S}_W = \{0, 1, 2, \ldots\}$. The lower strata correspond to other sequences $\mathscr{S} \neq \{0, 1, 2, \ldots\}$.

A wave function Ψ (see (1.2)) leads naturally to a family W^t in Gr, as follows. If $\tau(0) \neq 0$, then define the linear subspaces W^0 of functions of z:

$$W^0 \equiv \text{span}\left\{\Psi(t, z)|_{t=0}, \frac{\partial}{\partial x}\Psi(t, z)\bigg|_{t=0}, \frac{\partial^2}{\partial x^2}\Psi(t, z)\bigg|_{t=0}, \ldots\right\}. \tag{3.4}$$

Remembering that ψ is Ψ without the exponential, we now establish

Lemma 3.1. *If $\tau(t) \neq 0$, then viewing ψ as functions of z, we have*

$$\begin{aligned} W^0 &= \text{span}\{\Psi(t, z), \textit{all } t \in \mathbb{C}^\infty\} \\ &= \text{span}\left\{\left(\frac{\partial}{\partial x}\right)^j \Psi(t, z), j = 0, 1, 2, \ldots, \textit{any } t \in \mathbb{C}^\infty\right\} \end{aligned}$$

and

$$W^t \equiv e^{-\Sigma t_i z^i} W^0 \qquad (W^0 \in \textit{principal stratum})$$
$$= \operatorname{span}\{\psi(t,z), \nabla\psi(t,z), \nabla^2\psi(t,z), \ldots\} \tag{3.5}$$

and[9] $W^t \cap H_- = \phi$, *where*

$$\nabla = \frac{\partial}{\partial x} + z. \tag{3.6}$$

Proof. The differential equations

$$\frac{\partial \Psi}{\partial t_k} = (L^k)_+ \Psi$$

imply

$$W^0 = \operatorname{span}\left\{\begin{matrix} \Psi(t,z)|_{t=0}, \text{ and all } (t_1, t_2, \ldots) \\ -\text{partial derivatives of } \Psi(t,z) \text{ at } t=0 \end{matrix}\right\}$$
$$= \operatorname{span}\{\Psi(t,z), \text{ all } t \in \mathbb{C}^\infty\}, \qquad \text{using Taylor's theorem}$$
$$= \operatorname{span}\left\{\begin{matrix} e^{\Sigma t_i z^i}\psi, (\partial/\partial x)(e^{\Sigma t_i z^i}\psi), (\partial^2/\partial x^2)(e^{\Sigma t_i z^i}\psi), \ldots \\ \text{for fixed } t \in \mathbb{C}^\infty \text{ such that } \tau(t) \neq 0 \end{matrix}\right\}$$
$$= \operatorname{span}\left\{\begin{matrix} e^{\Sigma t_i z^i}\psi, e^{\Sigma t_i z^i}\nabla\psi, e^{\Sigma t_i z^i}\nabla^2\psi, \ldots \\ \text{for fixed } t \in \mathbb{C}^\infty \text{ such that } \tau(t) \neq 0, \end{matrix}\right\}$$
$$= e^{\Sigma t_i z^i} \operatorname{span}\{\psi, \nabla\psi, \nabla^2\psi, \ldots\}$$

since

$$\frac{\partial}{\partial x}(e^{\Sigma t_i z^i}\psi) = e^{\Sigma t_i z^i}\left(z\psi + \frac{\partial\psi}{\partial x}\right) = e^{\Sigma t_i z^i}\nabla\psi$$

and, in general

$$\left(\frac{\partial}{\partial x}\right)^r (e^{\Sigma t_i z^i}\psi) = e^{\Sigma t_i z^i}\nabla^r\psi.$$

Therefore, for all t

$$W^t = e^{-\Sigma t_i z^i} W^0$$
$$= \operatorname{span}\{\psi(t,z), \nabla\psi(t,z), \nabla^2\psi(t,z), \ldots\},$$

[9] $H_- = \operatorname{span}\{z^{-1}, z^{-2}, z^{-3}, \ldots\}$.

establishing (3.5). The expansion (for $k=0, 1, 2, \ldots$)

$$\nabla^k\psi(t,z)$$

$$=\left(z+\frac{\partial}{\partial x}\right)^k \sum_{i=0}^{\infty} \frac{p_i(-\tilde{\partial})\tau}{\tau} z^{-i}$$

$$=\left(z^k+kz^{k-1}\frac{\partial}{\partial x}+\frac{k(k-1)}{2}z^{k-2}\frac{\partial^2}{\partial x^2}+\cdots\right)\left(1-\frac{\tau'}{\tau}z^{-1}+\cdots\right)$$

$$=z^k-\frac{\tau'}{\tau}z^{k-1}+\left(\frac{p_2(-\tilde{\partial})\tau}{\tau}-k\left(\frac{\tau'}{\tau}\right)'\right)z^{k-2}+\cdots \tag{3.7}$$

shows at once that $W^t \cap H_- = \phi$, if $\tau(t)\neq 0$; this establishes Lemma 3.1.

Remark. The statement of Lemma 3.1 is true only when $\tau(t)\neq 0$. At a point t_0, where $\tau(t_0)=0$, the definition $W^t=e^{-\sum t_i z^i}W^0$ makes sense, but the basis elements in (3.5) cease to make sense. This issue is addressed in Section 7.

Now let $Gr^{(n)}\subset Gr$ be the subset of linear spaces W such that $z^nW\subset W$. The corresponding sequences $\mathscr{S}(W)$ then satisfy $\mathscr{S}+n\subset\mathscr{S}$. We now state:

LEMMA 3.2. *If the initial condition $W^0\in Gr^{(n)}$, then if $\tau(t)\neq 0$,*

(i) *$\tau(t)$ is independent of $t_n, t_{2n}, \ldots$, after possible multiplication by $e^{\sum c_r t_{rn}}$;*

(ii) *L^n is a differential operator and $L^n\Psi=z^n\Psi$;*

(iii) $W^t/z^nW^t=\operatorname{span}\{\psi(t,z), \nabla\psi(t,z), \ldots, \nabla^{n-1}\psi(t,z)\}$.

Proof. Since by Lemma 3.1, $\Psi(t,z)$ and $\partial\Psi/\partial t_{rn}\in W^0$, $r=1,2,\ldots$, and $z^{rn}W^0\subset W^0$, we have

$$W^0\ni (L^{rn})_+\Psi-z^{rn}\Psi=\frac{\partial\Psi}{\partial t_{rn}}-z^{rn}\Psi \qquad \text{(using (1.3))}$$

$$=\frac{\partial\psi}{\partial t_{rn}}\in e^{\sum t_i z^i}H_-, \quad \text{since } \psi=1+O(z^{-1}).$$

Since $e^{-\sum t_i z^i}W^0=W^t$ and $W^t\cap H_-=\phi$, (Lemma 3.1), we have

$$\frac{\partial\psi}{\partial t_{rn}}=\frac{\partial}{\partial t_{rn}}\sum_{k=1}^{\infty}\frac{p_k(-\tilde{\partial})\tau(t)}{\tau(t)}=0 \qquad \text{and} \qquad (L^{rn})_+\Psi=z^{rn}\Psi \qquad \text{for all } r.$$

The first relation expresses the fact that $\tau(t-[z^{-1}])/\tau(t)$ is independent of all t_{rn} and thus (i) holds, while (ii) follows from the second relation for $r=1$ and $L^n\Psi=z^n\Psi$. Statement (iii) follows from $z^nW'\subset W'$, (3.5), and the observation that $\nabla^j\psi$ $(j\geqslant 0)$ are elements of order j.

COROLLARY 3.2. *When* $z^2W^0\subset W^0$, *then* $\tau(t+[-s])=\tau(t-[s])$, *possibly after multiplication* $\tau\to\tau e^{\sum c_it_i}$.

Proof. For $n=2$, τ depends only on the odd variables $t_1, t_3, t_5, \ldots$ and thus the result follows.

4. The Bäcklund–Darboux Transformation and the Inclusion $zW_1\subset W$

For a function Φ, consider the operator

$$A_\Phi=\Phi\frac{d}{dx}\Phi^{-1}=\frac{d}{dx}-v \qquad \text{with} \quad v=\frac{\Phi'}{\Phi},\ {}'=\frac{d}{dx}$$

and

$$A_\Phi^{\mathrm{T}}=-\frac{d}{dx}-v.$$

The Bäcklund transformation maps the pseudo-differential operator L into a new pseudo-differential operator by conjugation,

$$L\curvearrowright\tilde{L}=A_\Phi LA_\Phi^{-1}, \tag{4.0}$$

and the function Ψ into a new function $\tilde{\Psi}$

$$\Psi\curvearrowright\tilde{\Psi}=A_\Phi\Psi=\Phi\frac{d}{dx}\Phi^{-1}\Psi=\frac{\{\Psi,\Phi\}}{\Phi}, \tag{4.1}$$

satisfying

$$\tilde{L}\tilde{\Psi}=A_\Phi LA_\Phi^{-1}A_\Phi\Psi=A_\Phi L\Psi=A_\Phi z\Psi=zA_\Phi\Psi=z\tilde{\Psi}.$$

But only when Φ is itself a wave function do we have that $A_\Phi\Psi$ satisfies the KP hierarchy, as we show later. Therefore we consider the Bäcklund transformation

$$A_{\Psi(t,z_1)}\Psi(t,z)=\frac{\{\Psi(t,z),\Psi(t,z_1)\}}{\Psi(t,z_1)}. \tag{4.1$'$}$$

We now state:

THEOREM 4.1. *For W and $W_1 \in Gr$, let Ψ and Ψ_1 be the corresponding wave functions and τ and τ_1 the corresponding τ-functions. Then if $\tau(t)$, $\tau_1(t) \neq 0$, the following three statements are equivalent*[10]

(i) $zW_1^t \subset W^t$.

(ii) $z\Psi_1(t, z) = (\partial/\partial x)\, \Psi(t, z) - \alpha\Psi(t, z)$ *for some function* $\alpha = \alpha(t)$.
$$\tag{4.2}$$

(iii) $\{\tau(t-[s]), \tau_1(t)\} + s^{-1}(\tau(t-[s])\, \tau_1(t) - \tau_1(t-[s])\, \tau(t)) = 0.$
$$\tag{4.3}$$

If any of these statements hold, then $\alpha = (\log(\tau_1/\tau))' = \tau_1'/\tau_1 - \tau'/\tau$.

Given a generic W (and thus Ψ, τ, and L), a solution to these equivalent problems is given by a new τ-function, defined for arbitrary z_1 near $z = \infty$, by

$$\tau_1(t) \equiv X(t, z_1)\, \tau = \Psi(t, z_1)\, \tau(t) = e^{\sum t_i z_1^i} \tau(t - [z_1^{-1}]), \tag{4.4}$$

the corresponding wave function

$$\Psi_1(t, z) = e^{\sum t_i z^i} \frac{\tau_1(t - [z^{-1}])}{\tau_1(t)} = z^{-1} A_{\Psi(t, z_1)} \Psi(t, z) = -\left(\frac{z_1}{z}\right) \Psi(t - [z_1^{-1}], z), \tag{4.5}$$

and the pseudo-differential operator L_1

$$\begin{aligned} L_1(t) &\equiv A_{\psi(t, z_1)} L(t)\, A^{-1}_{\Psi(t, z_1)} \\ &= S_1 \frac{d}{dx} S_1^{-1} \end{aligned} \tag{4.6}$$

with

$$S_1 = \sum_{n=0}^{\infty} \frac{p_n(-\tilde{\partial})\, \tau_1(t)}{\tau_1(t)} \left(\frac{d}{dx}\right)^{-n} = A_{\Psi(t, z_1)} S \left(\frac{d}{dx}\right)^{-1}.$$

Conversely, given W_1 and an associated τ_1, then a space $\tilde{W} \in Gr$ such that $zW_1 \subset \tilde{W}$ is constructed by means of the new $\tilde{\tau}$-function

[10] $\{f, g\} = \text{Wronskian}[g, f] = f'g - fg'$.

$$\tilde{\tau}(t) \equiv \tilde{X}(t, z_0)\, \tau_1 = e^{-\sum t_j z_0^j} \tau_1(t + [z_0^{-1}]), \qquad z_0 \in \mathbb{C} \textit{ arbitrary, near } z = \infty$$

and the corresponding wave function

$$\tilde{\Psi}(t, z) = e^{\sum t_j z^j} \frac{\tilde{\tau}(t - [z^{-1}])}{\tilde{\tau}(t)} = z\tilde{A}_{\Psi_1(t, z)} \Psi_1(t, z)$$

$$= -\frac{z}{z_0} \Psi_1(t + [z_0^{-1}], z).$$

Remark 1. It must be pointed out that the pseudo-differential operator L_1, defined in (4.6), is insensitive to the exponential appearing in τ_1, according to the remark at the end of Section 1.

Remark 2. Only by analogy we consider the inverse Bäcklund–Darboux transformation $\tilde{X}(t, z_1)$ of $X(t, z_1)$. Strictly speaking the inverse does not exist, as is seen from Corollary 4.2; indeed $\tilde{X}(t, z_0)\, X(t, z_1)$ blows up when $z_0 \to z_1$.

COROLLARY 4.2. *It holds that*

$$\tilde{\tau} = \tilde{X}(t, s_0^{-1})\, X(t, s_1^{-1})\tau$$

$$= \frac{s_1}{s_1 - s_0} e^{\sum t_i(s_1^{-i} - s_0^{-i})} \tau(t - [s_1] + [s_0])$$

$$= \frac{s_1}{s_1 - s_0} X(t, s_0^{-1}, s_1^{-1})\tau$$

$$= \frac{s_1}{s_1 - s_0} \sum_0^{\infty} \frac{(s_1^{-1} - s_0^{-1})^{\nu}}{\nu!} \sum s_0^{n+\nu} W_n^{(\nu)},$$

where

$$X(t, s_0^{-1}, s_1^{-1}) = e^{\sum_1^{\infty} t_i(s_1^{-i} - s_0^{-i})} e^{\sum_1^{\infty} (s_0^i - s_1^i)(1/i)(\partial/\partial t_i)} \qquad (\textit{vertex operator})$$

and the $W_n^{(\nu)}$ *form the generators of a W-algebra (see* [AvM]).

Proof of Theorem 4.1. First we show (i) $\Rightarrow$ (ii). Indeed $zW_1^t \subset W^t$ and Lemma 3.1 imply $z\psi_1(t, z) \in W^t$; but order $z\psi_1(t, z) = 1$, and again according to Lemma 3.1, the only functions of order $\leqslant 1$ are linear combinations of $\psi(t, z)$ and $\nabla\psi(t, z)$. Therefore

$$z\psi_1(t, z) = \nabla\psi(t, z) - \alpha(t)\, \psi(t, z) \tag{4.7}$$

and multiplying with $e^{\sum t_i z^i}$, we get

$$z\Psi_1(t, z) = \frac{\partial}{\partial x} \Psi(t, z) - \alpha(t)\, \Psi(t, z). \tag{4.8}$$

Compute α by equating the z^0 coefficient of both sides of (4.8).

To show that (ii) $\Rightarrow$ (i), observe that (Lemma 3.1)

$$W^0 = \text{span}\{\Psi, \Psi', \Psi'', \ldots\}; \left({}' = \frac{\partial}{\partial x}\right)$$

and so (ii) yields

$$z\Psi_1 = \Psi' - \alpha\Psi \in W^0.$$

Then taking j derivatives of this relation with respect to x, one finds

$$z\Psi_1^{(j)} = \Psi^{(j+1)} + \beta_1 \Psi^{(j)} + \cdots + \beta_{j+1}\Psi$$

for some $\beta_1, \ldots, \beta_{j+1}$ depending on t only. Since all $\Psi^{(k)} \in W^0$, we also have

$$z\Psi_1^{(j)} \in W^0,$$

implying (i).

Finally, the equivalence (ii) $\Leftrightarrow$ (iii) follows from a straightforward computation: multiplying (ii) by $e^{-\Sigma t_i z^i}$ yields

$$z\psi_1(t, z) = \nabla\psi(t, z) + \left(\frac{\tau'}{\tau} - \frac{\tau_1'}{\tau_1}\right)\psi(t, z)$$

or using the τ-function representation of $\Psi(t, z)$ ($z \equiv s^{-1}$)

$$s^{-1}\frac{\tau_1(t-[s])}{\tau_1(t)} = \left(\frac{\tau'(t-[s])}{\tau(t)} - \frac{\tau'(t)\,\tau(t-[s])}{\tau^2(t)} + s^{-1}\frac{\tau(t-[s])}{\tau(t)}\right)$$
$$+\left(\frac{\tau'(t)}{\tau(t)} - \frac{\tau_1'(t)}{\tau_1(t)}\right)\frac{\tau(t-[s])}{\tau(t)},$$

yielding (iii).

We now turn to the solution to this problem: define for an arbitrary point $z_1 = s_1^{-1}$ in a neighborhood of ∞ the function

$$\tau_1(t) \equiv e^{\Sigma_1^\infty t_i z_1^i}\tau(t - [z_1^{-1}]) = \Psi(t, z)\,\tau(t). \tag{4.9}$$

We must show several facts

(a) $\tau_1(t)$ is a τ-function, using the remark at the end of Section 1.

(b) Given $\tau(t)$, we must show that $\tau_1(t)$ is a solution to (iii) for any choice of small $s_1 \in \mathbb{C}$. Indeed using

$$e^{-\Sigma (s/s_1)^i/i} = 1 - \frac{s}{s_1} \tag{4.11}$$

in the second equality, we compute

$$\begin{aligned}
&\{\tau(t-[s]), \tau_1(t)\} + s^{-1}(\tau(t-[s])\,\tau_1(t) - \tau_1(t-[s])\,\tau(t)) \\
&\quad = \tau'(t-[s])\, e^{\sum t_i s_1^i} \tau(t-[s_1]) \\
&\qquad - \tau(t-[s])(s_1^{-1}\tau(t-[s_1]) + \tau'(t-[s_1]))\, e^{\sum t_i s_1^i} \\
&\qquad + s^{-1}\tau(t-[s])\,\tau(t-[s_1])\, e^{\sum t_i s_1^i} \\
&\qquad - s^{-1}\tau(t)\,\tau(t-[s]-[s_1])\, e^{\sum (t_i - s^i/i) s_1^{-i}} \\
&\quad = e^{\sum t_i s_1^i}(\tau'(t-[s])\,\tau(t-[s_1]) - \tau(t-[s])\,\tau'(t-[s_1]) \\
&\qquad + (s^{-1} - s_1^{-1})(\tau(t-[s])\,\tau(t-[s_1]) - \tau(t)\,\tau(t-[s]-[s_1]))) \\
&\quad = 0,
\end{aligned}$$

using in the last line the differential Fay identity (2.2), with (s_1, s_2) replaced by (s, s_1). This establishes (4.4) as a solution to (iii).

(c) Defining

$$\Psi_1(t, z) \equiv e^{\sum t_i z^i} \frac{\tau_1(t-[z^{-1}])}{\tau_1(t)},$$

we check that

$$A_{\Psi(t, z_1)} \Psi(t, z) = z\Psi_1(t, z). \tag{4.12}$$

Indeed

$$\begin{aligned}
A_{\Psi(t, z_1)} \Psi(t, z) &= \frac{\{\Psi(t, z), \Psi(t, z_1)\}}{\Psi(t, z_1)} \\
&= \Psi'(t, z) - \beta\Psi(t, z), \\
&= \Psi'(t, z) - \left(\log \frac{\tau_1(t)}{\tau(t)}\right)' \Psi(t, z) \\
&= z\Psi_1(t, z).
\end{aligned}$$

The third equality follows from the simple computation

$$\begin{aligned}
\beta &= \frac{\Psi'(t, z_1)}{\Psi(t, z_1)} = (\log \Psi(t, z_1))' \\
&= \left(\log e^{\sum t_i z_1^i} \frac{\tau(t-[z_1^{-1}])}{\tau(t)}\right)' \\
&= \left(\log \frac{\tau_1(t)}{\tau(t)}\right)' = \alpha,
\end{aligned}$$

and the fourth equality in (4.12) follows at once from the fact that τ and τ_1 satisfy (iii), combined with (iii) $\Leftrightarrow$ (ii).

(d) Finally, using (4.11), we compute (setting $s = z^{-1}$ and $s_1 = z_1^{-1}$)

$$\begin{aligned}
\Psi_1(t, s^{-1}) &= e^{\sum t_i/s^i} \frac{\tau_1(t-[s])}{\tau_1(t)}, \qquad \text{with} \quad \tau_1(t) = e^{\sum t_i/s_1^i} \tau(t-[s_1]) \\
&= e^{\sum_1^\infty t_i/s^i} e^{\sum (t_i - s^i/i) s_1^{-i}} e^{-\sum t_i/s_1^i} \frac{\tau(t-[s_1]-[s])}{\tau(t-[s_1])} \\
&= e^{-\sum_1^\infty (s/s_1)^i/i} e^{\sum (s_1/s)^i/i} e^{\sum (t_i - s_1^i/i) s^{-i}} \frac{\tau(t-[s_1]-[s])}{\tau(t-[s_1])} \\
&= \left(1 - \frac{s}{s_1}\right)\left(1 - \frac{s_1}{s}\right)^{-1} e^{\sum (t_i - s_1^i/i) s^{-i}} \frac{\tau(t-[s_1]-[s])}{\tau(t-[s_1])} \\
&= -\frac{s}{s_1} \Psi(t-[s_1], s),
\end{aligned}$$

establishing (4.5). Relations (4.6) then follow from (4.1), (4.2), (4.3) and (4.5), ending the first part of Theorem 4.1.

We now turn to the converse: given W_1 and τ_1, construct $\tilde{W} \supset zW_1$ and $\tilde{\tau}$. From the above we know that $zW_1 \subset \tilde{W}$ is equivalent to the identity (iii) between τ-functions. Therefore it suffices to prove ($s = z^{-1}$, $s_0 = z_0^{-1}$)

$$\{\tilde{\tau}(t-[s]), \tau_1(t)\} + s^{-1}(\tilde{\tau}(t-[s])\, \tau_1(t) - \tau_1(t-[s])\, \tilde{\tau}(t)) = 0$$

for

$$\tilde{\tau}(t) \equiv e^{-\sum t_i/s_0^i} \tau_1(t+[s_0]).$$

Indeed

$$\begin{aligned}
\tilde{\tau}(t-[s]) &= e^{-\sum (t_i - s^i/i) s_0^{-i}} \tau_1(t-[s]+[s_0]) \\
&= \frac{s_0}{s_0 - s} e^{-\sum t_i/s_0^i} \tau_1(t-[s]+[s_0])
\end{aligned}$$

and thus

$$\begin{aligned}
&\{\tilde{\tau}(t-[s]), \tau_1(t)\} + s^{-1}(\tilde{\tau}(t-[s])\, \tau_1(t) - \tau_1(t-[s])\, \tilde{\tau}(t)) \\
&\quad = \frac{s_0}{s_0 - s} e^{-\sum t_i/s_0^i} [(\tau_1'(t-[s]+[s_0])\, \tau_1(t) - \tau_1(t-[s]+[s_0])\, \tau_1'(t)) \\
&\qquad + (s^{-1} - s_0^{-1})(\tau_1(t-[s]+[s_0])\, \tau_1(t) - \tau_1(t+[s_0])\, \tau_1(t-[s]))] \\
&\quad = 0,
\end{aligned}$$

using the differential Fay identity (2.2) with $s_1 \curvearrowright s$, $s_2 \curvearrowright s_1$ and $t \curvearrowright t + s_0$.

Also,

$$\begin{aligned}
\tilde{\Psi}(t, s^{-1}) &= e^{\sum t_i/s^i}\frac{\tilde{\tau}(t-[s])}{\tilde{\tau}(t)} \\
&= e^{\sum t_i/s^i} e^{-\sum (t_i - s^i/i)\, s_0^{-i}} e^{\sum t_i/s_0^i}\frac{\tau_1(t-[s]+[s_0])}{\tau_1(t+[s_0])} \\
&= \frac{s_0}{s_0-s}\, e^{\sum t_i/s^i}\frac{\tau_1(t-[s]+[s_0])}{\tau_1(t+[s_0])} \\
&= \left(-\frac{s_0}{s}\right)\frac{s}{s-s_0}\, e^{\sum t_i/s^i}\frac{\tau_1(t-[s]+[s_0])}{\tau_1(t+[s_0])} \\
&= \left(-\frac{s_0}{s}\right) e^{\sum (t_i + s_0^i/i)\, s^{-i}}\frac{\tau_1(t+[s_0]-[s])}{\tau_1(t+[s_0])} \\
&= -\frac{s_0}{s}\,\Psi_1(t+[s_0], s^{-1}),
\end{aligned} \tag{4.13}$$

ending the proof of Theorem 4.1.

The proof of Corollary 4.2 is a straightforward computation, analogous to (4.13).

COROLLARY 4.3. *If both spaces W_0^t and W_1^t satisfy $z^2 W_i \subset W_i$, then the inclusions*

$$zW_1^t \subset W_0^t \Leftrightarrow zW_0^t \subset W_1^t$$

are equivalent.

Proof. $zW_1 \subset W_0$ is equivalent to the bilinear relation (iii) in Theorem 4.1, i.e.,

$$\begin{aligned}
&\{\tau_0(t-[s]), \tau_1(t)\} + s^{-1}(\tau_0(t-[s])\,\tau_1(t) - \tau_1(t-[s])\,\tau_0(t)) \\
&\quad \equiv B(\tau_0, \tau_1) = 0.
\end{aligned}$$

By Corollary 3.2, $\tau(t-[s]) = \tau(t+[-s])$; then setting first $s \quad -s$ and next $t \quad t-[s]$, one finds

$$\begin{aligned}
&\{\tau_0(t), \tau_1(t-[s])\} - s^{-1}(\tau_0(t)\,\tau_1(t-[s]) - \tau_1(t)\,\tau_0(t-[s])) \\
&\quad \equiv -B(\tau_1, \tau_0) = 0,
\end{aligned}$$

implying upon using (iii) again that $zW_0^t \subset W_1^t$.

5. Bäcklund Transformations and the Flag $z^k W_k \subset \cdots \subset z W_1 \subset W$

Applying the results of the previous section to each of the inclusions in the flag $z^k W_k' \subset \cdots \subset z W_1' \subset W'$, we find the wave functions

$$zW_1 \subset W: \Psi_1(t, z) = z^{-1} A_{\Psi(t, z_1)} \Psi(t, z)$$

$$zW_2 \subset W_1: \Psi_2(t, z) = z^{-1} A_{\Psi_1(t, z_2)} \Psi_1(t, z)$$

$$\vdots$$

$$zW_k \subset W_{k-1}: \Psi_k(t, z) = z^{-1} A_{\Psi_{k-1}(t, z_k)} \Psi_{k-1}(t, z).$$

Then compounding these Bäcklund transformations, we find that for an arbitrary choice of $z_1, ..., z_k$ near $z = \infty$, the function Ψ_k given by

$$\Psi_k(t, z) \equiv z^{-k} A_{\Psi_{k-1}(t, z_k)} \cdots A_{\Psi_1(t, z_2)} A_{\Psi(t, z_1)} \Psi(t, z)$$

is the wave function of a plane W_k such that $z^k W_k \subset W$. We first state

Lemma 5.1.

$$\begin{aligned} z^k \Psi_k(t, z) &\equiv A_{\Psi_{k-1}(t, z_k)} \cdots A_{\Psi_1(t, z_2)} A_{\Psi(t, z_1)} \Psi(t, z) \\ &\overset{*}{=} \frac{\textit{Wronskian}[\Psi(t, z_1), ..., \Psi(t, z_k), \Psi(t, z)]}{\textit{Wronskian}[\Psi(t, z_1), ..., \Psi(t, z_k)]} \\ &= \prod_{i=1}^{k} (-z_i)\, \Psi(t - [z_1^{-1}] - [z_2^{-1}] - \cdots - [z_k^{-1}], z). \end{aligned} \tag{5.1}$$

This lemma, a consequence of Theorem 4.1, then leads to a proof of the higher Fay identities and to a generalization of Theorem 4.1:

Theorem 5.1. *For generic t, the following two statements are equivalent*:

(i) $z^k W_k' \subset W'$

(ii) $z^k \psi_k(t, z) = \alpha_k(t)\, \psi(t, z) + \alpha_{k-1}(t)\, \nabla\psi(t, z) + \cdots + \alpha_0(t)\, \nabla^k \psi(t, z)$, *for some functions $\alpha_i(t)$, with $\psi(t, z)$ and $\psi_k(t, z)$ the wave functions associated with W' and W_k'.*

If these statements hold, then

$$\alpha_j = \frac{p_j(-\tilde{\partial})\tau_k}{\tau_k} + \sum_{i=1}^{j} \frac{p_{j-i}(-\tilde{\partial})\tau_k}{\tau_k} h_i^j, \qquad j = 0, 1, ..., k \tag{5.2}$$

with the h_i^j defined as

$$h_0^j = 1, \qquad h_1^j = -f_1^{(j)},$$

$$h_2^j = -f_2^{(j)} + f_1^{(j-1)} f_1^{(j)}, \ldots, h_r^j = -\sum_1^r h_{r-i}^{j-i} f_i^{(j)}, \ldots \qquad (r = 1, 2, \ldots, j)$$

with

$$f_i^{(m)} = \sum_{\substack{0 \leqslant j, l \\ j + l = i \\ 0 \leqslant j \leqslant i + k - m}} \binom{k - m + i}{j} \left(\frac{\partial}{\partial x}\right)^j \frac{p_l(-\tilde{\partial})\,\tau}{\tau}.$$

Given W, the wave function of the generic W_k such that $z^k W_k' \subset W'$ is given by

$$\begin{aligned} \Psi_k(t, z) &\equiv z^{-k} A_{\Psi_{k-1}(t, z_k)} \cdots A_{\Psi_1(t, z_2)} A_{\Psi(t, z_1)} \Psi(t, z) \\ &= z^{-k} \frac{\text{Wronskian}[\Psi(t, z_1), \ldots, \Psi(t, z_k), \Psi(t, z)]}{\text{Wronskian}[\Psi(t, z_1), \ldots, \Psi(t, z_k)]}. \end{aligned} \tag{5.3}$$

COROLLARY 5.1. *The τ-functions τ and τ_k associated with $z^k W_k \subset W$ satisfy for $k = 1, 2$ the differential equation $(s = z^{-1})$*

$$\begin{aligned} &\sum_{l=0}^{k-1} s^{-l-1} p_{k-l}(\tilde{\partial})\,\tau(t - [s]) \circ \tau_k(t) \\ &\qquad + s^{-k}(\tau(t - [s])\,\tau_k(t) - \tau(t)\,\tau_k(t - [s])) = 0, \end{aligned} \tag{5.4}$$

or equivalently (using the notation $p_k^{\pm}(\tau) = p_k(\pm\tilde{\partial})\,\tau$)

$$\begin{aligned} &k = 1: (p_{r+1}^-(\tau)\tau_1 - \tau p_{r+1}^-(\tau_1)) + (p_r^-(\tau')\tau_1 - p_r^-(\tau)\,\tau_1') = 0, \textit{ all } r \geqslant 0 \\ &k = 2: (p_{r+2}^-(\tau)\tau_2 - \tau p_{r+2}^-(\tau_2)) + (p_{r+1}^-(\tau')\tau_2 - p_{r+1}^-(\tau)\tau_2') \\ &\qquad\qquad + p_r^-(p_2^+(\tau))\tau_2 - p_r^-(\tau')\tau_2' + p_r^-(\tau)\,p_2^-(\tau_2) = 0, \textit{ all } r \geqslant 0. \end{aligned} \tag{5.5}$$

Remark. For $k \geqslant 3$, the equations relating τ and τ_k are more complicated.

Proof of Lemma 5.1. In view of the expressions in Lemma 5.1, we introduce the notation

$$\mathscr{W}_k \equiv \text{Wronskian}[\Psi(t, z_1), \ldots, \Psi(t, z_k)],$$

$$\mathscr{W}_k(\chi) \equiv \text{Wronskian}[\Psi(t, z_1), \ldots, \Psi(t, z_k), \chi],$$

with the latter a linear differential operator of order k; define inductively

$$\begin{aligned} \Psi_k(t, z) &\equiv z^{-k} A_{\Psi_{k-1}(t, z_k)} \cdots A_{\Psi(t, z_1)} \Psi(t, z) = z^{-1} A_{\Psi_{k-1}(t, z_k)} \Psi_{k-1}(t, z) \\ \Psi_0(t, z) &\equiv \Psi(t, z) \end{aligned} \tag{5.6}$$

and

$$\tilde{\Psi}_k(t, z) \equiv z^{-k} \frac{\mathcal{W}_k(\Psi(t, z))}{\mathcal{W}_k}, \qquad \tilde{\Psi}_0(t, z) = \Psi(t, z); \tag{5.7}$$

observe that

$$\tilde{\Psi}_k(t, z_{k+1}) = z_{k+1}^{-k} \frac{\mathcal{W}_{k+1}}{\mathcal{W}_k}. \tag{5.8}$$

The proof proceeds in two steps (see Adler and Moser [AM]):

(a) An identity of Jacobi

$$\{\mathcal{W}_k(\chi), \mathcal{W}_{k+1}\} = \mathcal{W}_{k+1}(\chi)\mathcal{W}_k.$$

To check this fact, observe that the left hand side is a linear differential operator of order $k+1$, acting on χ, which vanishes for $\chi = \Psi(t, z_1)$, $\Psi(t, z_2), \ldots, \Psi(t, z_k)$ and also for $\chi = \Psi(t, z_{k+1})$. Since the functions $\Psi(t, z_1), \ldots, \Psi(t, z_{k+1})$ are typically linearly independent, the left hand side must be a multiple of $\mathcal{W}_{k+1}(\chi)$; then one checks that the highest coefficients of both sides agree.

(b) The Crum identity

$$A_k A_{k-1} \cdots A_1(\chi) = \frac{\mathcal{W}_k(\chi)}{\mathcal{W}_k},$$

where

$$A_j \equiv A_{\tilde{\Psi}_{j-1}(t, z_j)} = \tilde{\Psi}_{j-1}(t, z_j) \frac{d}{dx} \tilde{\Psi}_{j-1}^{-1}(t, z_j) = \frac{\mathcal{W}_j}{\mathcal{W}_{j-1}} \frac{d}{dx} \frac{\mathcal{W}_{j-1}}{\mathcal{W}_j}.$$

We first check (b) for $k = 1$,

$$\begin{aligned} A_1 \chi &= A_{\tilde{\Psi}_0(t, z_1)} \chi \\ &= A_{\Psi(t, z_1)} \chi \\ &= \frac{\{\chi, \Psi(t, z_1)\}}{\Psi(t, z_1)} \\ &= \frac{\mathcal{W}_1(\chi)}{\mathcal{W}_1}, \end{aligned}$$

and then inductively (using the identity $\{af, ag\} = a^2\{f, g\}$), assuming Crum's identity for k,

$$\begin{aligned}
A_{k+1}A_k \cdots A_1(\chi) &= A_{k+1}(A_k \cdots A_1)\,\chi \\
&= A_{k+1}\frac{\mathscr{W}_k(\chi)}{\mathscr{W}_k} \qquad \text{using Crum's identity for } k \\
&= \frac{\mathscr{W}_{k+1}}{\mathscr{W}_k}\frac{d}{dx}\frac{\mathscr{W}_k}{\mathscr{W}_{k+1}}\frac{\mathscr{W}_k(\chi)}{\mathscr{W}_k} \qquad \text{by definition of } A_{k+1} \\
&= \{\mathscr{W}_k(\chi), \mathscr{W}_{k+1}\}\frac{1}{\mathscr{W}_{k+1}\mathscr{W}_k} \\
&= \frac{\mathscr{W}_{k+1}(\chi)}{\mathscr{W}_{k+1}} \qquad \text{using Jacobi's identity,}
\end{aligned}$$

completing the induction and thus the proof of (b).

(c) $$\tilde{\Psi}_k(t, z) = \Psi_k(t, z).$$

Indeed, since

$$\tilde{\Psi}_0(t, z) = \Psi(t, z),$$

we have

$$\begin{aligned}
\tilde{\Psi}_1(t, z) &= z^{-1}\frac{\mathscr{W}_1(\Psi(t, z))}{\mathscr{W}_1}, && \text{using (5.7)} \\
&= z^{-1}A_1\Psi(t, z), && \text{using (b)} \\
&= z^{-1}A_{\tilde{\Psi}_0(t, z_1)}\Psi(t, z), && \text{by definition} \\
&= z^{-1}A_{\Psi(t, z_1)}\Psi(t, z) = \Psi_1(t, z)
\end{aligned}$$

and so, by induction,

$$\tilde{\Psi}_k(t, z) = \Psi_k(t, z),$$

which leads at once to equality $\stackrel{*}{=}$ in Lemma 5.1. The second equality, follows at once by induction using relation (4.5) of Theorem 4.1, thus ending the proof of Lemma 5.1.

Proof of Lemma 2.1 (higher Fay identities). From Lemma 5.1, the definition of Ψ_k, and relation (4.5), it follows that $(s_i = z_i^{-1})$

$$s_{k+1}^{k}\frac{W_{k+1}}{W_k}=\Psi_k(t,s_{k+1}^{-1}), \qquad \text{using (5.8)}$$

$$=s_{k+1}A_{\Psi_{k-1}(t,s_k^{-1})}\Psi_{k-1}(t,s_{k+1}^{-1}), \qquad \text{using (5.6)}$$

$$=-\frac{s_{k+1}}{s_k}\Psi_{k-1}(t-[s_k],s_{k+1}^{-1}), \qquad \text{using (4.5)}$$

$$=-\frac{s_{k+1}}{s_k}e^{\sum(t_i-s_k^i/i)s_{k+1}^{-i}}\frac{\tau_{k-1}(t-[s_k]-[s_{k+1}])}{\tau_{k-1}(t-[s_k])}$$

$$=-\frac{s_{k+1}}{s_k}\left(1-\frac{s_k}{s_{k+1}}\right)e^{\sum t_is_{k+1}^{-i}}\frac{\tau_{k-1}(t-[s_k]-[s_{k+1}])}{\tau_{k-1}(t-[s_k])}, \tag{5.9}$$

using (4.11). From (4.4) in Theorem 4.1, it follows that

$$\tau_j(t)=e^{\sum t_i/s_j^i}\tau_{j-1}(t-[s_j]), \qquad j=1,2,\ldots \text{ with } \tau_0(t)=\tau(t),$$

and by induction

$$\tau_l(t)=e^{\sum t_i(s_1^{-i}+s_2^{-i}+\cdots+s_l^{-i})}\tau(t-[s_1]-[s_2]-\cdots-[s_l])\prod_{i\leqslant i<j\leqslant l}\left(1-\frac{s_j}{s_i}\right);$$

therefore

$$\frac{\tau_{k-1}(t-[s_k]-[s_{k+1}])}{\tau_{k-1}(t-[s_k])}$$

$$=e^{-\sum(s_{k+1}^i/i)(s_1^{-i}+\cdots+s_{k-1}^{-i})}\frac{\tau(t-[s_1]-[s_2]-\cdots-[s_{k+1}])}{\tau(t-[s_1]-[s_2]-\cdots-[s_k])}$$

$$=\left(1-\frac{s_{k+1}}{s_1}\right)\left(1-\frac{s_{k+1}}{s_2}\right)\cdots\left(1-\frac{s_{k+1}}{s_{k-1}}\right)\frac{\tau(t-[s_1]-\cdots-[s_{k+1}])}{\tau(t-[s_1]-\cdots-[s_k])}. \tag{5.10}$$

Hence (5.9) and (5.10) combined yield

$$s_{k+1}^{k}\frac{\mathscr{W}_{k+1}}{\mathscr{W}_k}=s_{k+1}^{k}\prod_{j=1}^{k}(s_{k+1}^{-1}-s_j^{-1})\,e^{\sum t_is_{k+1}^{-i}}\frac{\tau(t-[s_1]-\cdots-[s_{k+1}])}{\tau(t-[s_1]-\cdots-[s_k])}.$$

We conclude that

$$\text{Wronskian}[\Psi(t,s_1^{-1}),\ldots,\Psi(t,s_n^{-1})]$$

$$=\mathscr{W}_n$$

$$=\left(\frac{\mathscr{W}_n}{\mathscr{W}_{n-1}}\right)\cdots\left(\frac{\mathscr{W}_2}{\mathscr{W}_1}\right)\mathscr{W}_1$$

$$=\prod_{1\leqslant j<i\leqslant n}(s_i^{-1}-s_j^{-1})\,e^{\sum t_i(s_1^{-1}+\cdots+s_n^{-1})}\frac{\tau(t-[s_1]\cdots-[s_n])}{\tau(t)}, \tag{5.11}$$

yielding the result (2.4) of Lemma 2.1. Now using the rule

$$\text{Wronskian}[a_1, ..., a_n] \cdot (\text{Wronskian}[b_1, ..., b_n])^{n+1}$$
$$= \det(\text{Wronskian}[b_1, ..., b_{i-1}, a_j, b_{i+1}, ..., b_n])_{1 \leqslant i,j \leqslant n},$$

and setting

$$a_i = \Psi(t, s_i^{-1}), \qquad b_i = \Psi(t, r_i^{-1}),$$

lead to the result (2.3) of Lemma 2.1.

Proof of Theorem 5.1. We first show (i) $\Rightarrow$ (ii). Indeed, $z^k W'_k \subset W'$ together with Lemma 3.1 implies $z^k \psi_k(t, s) \in W'$; since $z^k \psi_k(t, z) = z^k + O(z^{k-1})$, this function must be, for generic t, a linear combination of functions of order $\leqslant k$ in W', leading to (ii). To verify (ii) $\Rightarrow$ (i), it suffices to prove $z^k W_k^0 \subset W^0$, with

$$W^0 = \text{span}\{\Psi^{(j)}(t, z), j = 0, 1, ..., \text{for all } t \in \mathbb{C}^\infty\}$$

and

$$W_k^0 = \text{span}\{\Psi_k^{(j)}(t, z), j = 0, 1, ..., \text{for all } t \in \mathbb{C}^\infty\},$$

by virtue of (3.5). First,

$$\begin{aligned} z^k \Psi_k(t, z) &= e^{\sum t_i z^i} z^k \psi_k(t, z) \\ &= e^{\sum t_i z^i}(\alpha_k(t)\, \psi(t, z) + \cdots + \alpha_0 \nabla^k \psi(t, z)) \qquad \text{using (ii)} \\ &= \alpha_k(t)\, \Psi(t, z) + \cdots + \alpha_0(t)\, \Psi^{(k)}(t, z) \in W^0 \end{aligned}$$

and subsequently taking j derivatives in x of the equality above leads to

$$z^k \Psi_k^{(j)}(t, z) = \beta_{k+j}(t)\, \Psi(t, z) + \cdots + \beta_0(t)\, \Psi^{(k+j)}(t, z) \in W^0,$$

yielding (ii) $\Rightarrow$ (i).

That $\Psi_k(t, z)$ obtained by Bäcklund transforming Ψ exactly k times provides a solution to (i) and (ii) follows immediately from the introduction to this section, which together with Lemma 5.1 ends the proof of Theorem 5.1, except for expression (5.2), which we now prove.

The form of the α_k's is given as follows: on the one hand ($s \equiv z^{-1}$)

$$\psi_k = \frac{\tau_k(t - [s])}{\tau_k(t)} = \sum_{m=0}^{\infty} \frac{p_m(-\tilde{\partial})\, \tau_k(t)}{\tau_k(t)} s^m \tag{5.12}$$

and on the other hand, in view of the formula

$$\nabla^r = \left(\frac{\partial}{\partial x} + s^{-1}\right)^r = \sum_{j=0}^{r} \binom{r}{j} s^{j-r} \left(\frac{\partial}{\partial x}\right)^j$$

and upon multiplying with $s^k = z^{-k}$, (ii) becomes

$$\begin{aligned}
\psi_k &= s^k \sum_{r=0}^{k} \alpha_{k-r}(t)\, \nabla^r \psi(t, s^{-1}) \\
&= s^k \sum_{r=0}^{k} \alpha_{k-r}(t) \sum_{j=0}^{r} \binom{r}{j} s^{j-r} \left(\frac{\partial}{\partial x}\right)^j \psi(t, s^{-1}) \\
&= s^k \sum_{r=0}^{k} \alpha_{k-r}(t) \sum_{j=0}^{r} \binom{r}{j} s^{j-r} \left(\frac{\partial}{\partial x}\right)^j \sum_{l=0}^{\infty} \frac{p_l(-\tilde{\partial})\,\tau}{\tau} s^l \\
&= \sum_{r=0}^{k} \alpha_{k-r}(t) \sum_{\substack{0 \leqslant j \leqslant r \\ 0 \leqslant l \leqslant \infty}} \binom{r}{j} s^{k+j-r+l} \left(\frac{\partial}{\partial x}\right)^j \left(\frac{p_l(-\tilde{\partial})\,\tau}{\tau}\right).
\end{aligned} \tag{5.13}$$

Comparing both expressions (5.12) and (5.13) for ψ_k, we find

$$\begin{aligned}
g_m \equiv \frac{p_m(-\tilde{\partial})\,\tau_k(t)}{\tau_k(t)} &= \sum_{\substack{k+j-r+l=m \\ 0 \leqslant j \leqslant r \leqslant k \\ 0 \leqslant l}} \alpha_{k-r}(t) \binom{r}{j} \left(\frac{\partial}{\partial x}\right)^j \left(\frac{p_l(-\tilde{\partial})\,\tau}{\tau}\right) \\
&= \sum_{i=0}^{m} \alpha_{m-i}(t) \sum_{\substack{0 \leqslant j,\, l \\ j+l=i \\ 0 \leqslant j \leqslant i+k-m}} \binom{k-m+i}{j} \left(\frac{\partial}{\partial x}\right)^j \left(\frac{p_l(-\tilde{\partial})\,\tau}{\tau}\right) \\
&\equiv \alpha_m(t) + \alpha_{m-1}(t) f_1^{(m)} + \alpha_{m-2}(t) f_2^{(m)} + \cdots + \alpha_0(t) f_m^{(m)},
\end{aligned}$$

yielding a linear system of equations in $\alpha_0, \ldots, \alpha_m$

$$\begin{aligned}
g_m &= \alpha_m + \alpha_{m-1} f_1^{(m)} + \alpha_{m-2} f_2^{(m)} + \cdots + \alpha_0 f_m^{(m)} \\
g_{m-1} &= \alpha_{m-1} + \alpha_{m-2} f_1^{(m)} + \cdots + \alpha_0 f_{m-1}^{(m-1)} \\
&\vdots
\end{aligned}$$

which can be solved inductively to yield (5.2)

$$\begin{aligned}
\alpha_m = g_m &+ g_{m-1}(-f_1^{(m)}) + g_{m-2}(-f_2^{(m)} + f_1^{(m-1)} f_1^{(m)}) \\
&+ g_{m-3}(-f_3^m + f_1^{m-2} f_2^m + f_2^{m-1} f_1^m - f_1^m f_1^m f_1^{m-2}) + \cdots,
\end{aligned}$$

ending the proof of Theorem 5.1.

Proof of Corollary 5.1. For $k=1$, the proof of (5.4) was given in Theorem 4.1; for $k=2$, we compute the coefficients $\alpha_j(t)$ in (ii) by expanding relation (ii) in powers of z^{-1}:

$$z^2\psi_2(t,z)=\left(\frac{p_2^-(\tau_2)}{\tau_2}-\frac{\tau'\tau_2'}{\tau\tau_2}-\frac{p_2^-(\tau)}{\tau}+\frac{2\tau''}{\tau}-\left(\frac{\tau'}{\tau}\right)^2\right)\psi(t,z)$$
$$+\left(-\frac{\tau_2'}{\tau_2}+\frac{\tau'}{\tau}\right)\nabla\psi(t,z)+\nabla^2\psi(t,z).$$

First expressing ψ and ψ_2 in terms of τ and τ_2 and expanding out ∇ and ∇^2, the terms containing τ^{-3} all cancel and those containing τ^{-2} can be eliminated by using the equation

$$(-zp_1(-\tilde{\partial})+p_2(-\tilde{\partial}))\,\tau(t-[z^{-1}])\circ\tau(t)=0,$$

itself a consequence of

$$\frac{\partial\Psi}{\partial t_2}(t,z)=(L^2)_+\,\Psi(t,z)$$
$$=\left(\frac{d^2}{dx^2}+2(\log\tau(t))''\right)\Psi(t,z)$$

with

$$\Psi(t,z)=e^{\sum t_iz^i}\frac{\tau(t-[z^{-1}])}{\tau(t)}.$$

Equations (5.5) are obtained by expanding Eq. (5.4) in powers of z^{-1}, by setting the coefficients of z^{-r} equal to zero and using the Taylor series (A.4).

6. Young Diagrams, Schur Polynomials, and Vanishing Properties

This section of a combinatorial nature deals first with Taylor expanding Schur polynomials $F_\nu(t\pm[s])$, associated with a Young diagram ν, in s around $s=0$; for notations see Appendices A, B, and C. The point is that their Taylor expansion terminates earlier than expected (at first sight) and the Taylor coefficients are themselves Schur polynomials for skew-Young diagrams. At the same time we give the precise leading behavior of $p_k(-\tilde{\partial})\,F_\nu$ near $t=0$ in that direction. These results are instrumental for Section 7.

THEOREM 6.1. *Given a partition* $\nu \equiv (\nu_0 \geqslant \nu_1 \geqslant \cdots \nu_n \geqslant 0)$ *and the corresponding Schur polynomial* $F = F_\nu$, *the following Taylor expansions terminate at* s^{ν_0} *and* $s^{\hat{\nu}_0}$, *respectively,*

$$F(t-[s]) = F(t) + sp_1(-\tilde{\partial})\, F(t) + \cdots + s^{\nu_0} p_{\nu_0}(-\tilde{\partial})\, F(t) = \sum_{i=0}^{\nu_0} s^i F_{\nu \backslash (i)}(t) \tag{6.1}$$

and

$$F(t+[s]) = F(t) + sp_1(\tilde{\partial})\, F(t) + \cdots + s^{\hat{\nu}_0} p_{\hat{\nu}_0}(\tilde{\partial})\, F(t) = \sum_{i=0}^{\hat{\nu}_0} (-s)^i F_{\nu \backslash \underbrace{(1, \ldots, 1)}_{i}}(t), \tag{6.2}$$

with the highest degree terms

$$F_{\nu \backslash (\nu_0)} = F_{(\nu \backslash \text{first row})} \qquad \text{and} \qquad F_{\nu \backslash \underbrace{(1, \ldots, 1)}_{\hat{\nu}_0}} = F_{(\nu \backslash \text{first column})}.$$

Moreover

$$\begin{aligned} p_k(-\tilde{\partial})\, F_\nu(x, 0, 0, \ldots) &= c_k^{(\nu)} x^{|\nu| - k}, \qquad && k \leqslant \nu_0 \\ &= 0, && k > \nu_0, \end{aligned} \tag{6.3}$$

where the c_k^ν *are numbers determined by the partition* ν *via the polynomial identity*[11]

$$P(z) = \sum_{k=0}^{\nu_0} (z)_{\nu_0 - k} \frac{c_k^\nu}{c_0^\nu} = \prod_{i=0}^{\nu_0 - 1} (z - (\nu_0 + \hat{\nu}_i - i - 1)); \tag{6.4}$$

if $f^{\nu/\mu}$ *denotes the number of standard tableaux of the skew Young diagram* $\nu - \mu$ *(see Appendix B), then*

$$\begin{aligned} c_k^\nu &= (-1)^{|\nu| - k} \frac{f^{\nu/(k)}}{(|\nu| - k)!} \neq 0 \\ &= (-1)^{|\nu| - k} \det \left[\frac{1}{(\nu_i - k\delta_{j0} - i + j)!} \right]_{0 \leqslant i, j \leqslant l}. \end{aligned} \tag{6.5}$$

Also,[12]

$$\frac{c_{\nu_0 - r}^{(\nu)}}{c_0^{(\nu)}} = \frac{1}{r!} \sum_{j=0}^{r} (-1)^j \binom{r}{j} P(r-j) \equiv \Delta^r P(z)\big|_{z=0} \quad \text{for } r = 0, \ldots, \nu_0 - 1. \tag{6.6}$$

[11] $(z)_a \equiv z(z-1) \cdots (z-a+1)$.

[12] $\Delta^r P(z)$ denotes the rth difference of P (on the integers) at $z = 0$.

Remark. In particular

$$\frac{c^{\nu}_{\nu_0}}{c^{\nu}_0} = P(0)$$

$$\frac{c^{\nu}_{\nu_0-1}}{c^{\nu}_0} = P(1) - P(0) \tag{6.7}$$

$$\frac{c^{\nu}_{\nu_0-2}}{c^{\nu}_0} = \frac{1}{2}(P(2) - 2P(1) + P(0)), \ldots .$$

COROLLARY 6.1. *The ν_0 roots $\nu_0 + \hat{\nu}_i - i - 1$ $(0 \leqslant i \leqslant \nu_0 - 1)$ of the polynomial $P(z)$ coincide with the ν_0 gaps $(\geqslant 1)$ in the sequence*

$$0,\ \nu_0 - \nu_1 + 1,\ \nu_0 - \nu_2 + 2,\ \nu_0 - \nu_3 + 3, \ldots$$

and thus $P(\nu_0 - \nu_i + i) \neq 0$ $(i = 0, 1, 2, \ldots)$. Moreover

$$\det(\gamma_{ij})_{\substack{i=0,1,\ldots,r \\ j=0,\nu_0-\nu_1+1,\ldots,\nu_0-\nu_r+r}} = \prod_{i=0}^{r} \frac{P(\nu_0 - \nu_i + i)}{(\nu_0 - \nu_i + i - 1)!} \neq 0, \tag{6.8}$$

with

$$\gamma_{ij} \equiv \sum_{l} (-1)^{i-l} \binom{i}{l} (\nu_0 + i - j - 1)_{i-l} \frac{c^{\nu}_{\nu_0-j+l}}{c^{\nu}_0}$$

$$= \sum_{l} \sum_{k=0}^{j-l} (-1)^{i-l+k} P(j-l-k) \frac{(\nu_0 + i - j - 1)_{i-l}}{k!\,(j-l-k)!} \binom{i}{l},$$

where the summation ranges over $\max(0, j-\nu_0) \leqslant l \leqslant \min(i, j)$.

Proof of Theorem 6.1. For facts and notations concerning symmetric polynomials, see Appendices *A* and *B*. To prove (6.1) and (6.2), set $x_0 = s$ and $\lambda = \nu$ in (B.13) and (B.14); note that the Taylor expansions of $F_\nu(t - [s])$ and $F_\nu(t + [s])$, as given by (A.4), terminate at $n = \nu_0$ and $n = \hat{\nu}_0$, respectively.

Consider now the ring homomorphism θ, which acts on the ring of symmetric functions on $x_1, x_2, \ldots$ as follows $(u_r = \sum_{i \geqslant 1} x_i^r)$:

$$\theta: \{\text{Ring of symmetric functions of } x_1, x_2, \ldots\} \to \mathbb{C}[u_1]$$

with

$$\theta(u_1) = u_1 \qquad \text{and} \qquad \theta(u_i) = 0, \qquad i > 0$$

$$\text{fixing } x_0 = s \tag{6.9}$$

Then on the one hand,

$$
\begin{aligned}
F_\nu&\left(-s-u_1,-\frac{s^2}{2},-\frac{s^3}{3},\ldots\right)\\
&=\theta\left(F_\nu\left(-\left(u_1,\frac{u_2}{2},\frac{u_3}{3},\ldots\right)-[s]\right)\right)\\
&=\sum_{k=0}^{\nu_0}\theta(S_{\nu\backslash k}(x_1,\ldots))s^k \qquad \text{using (B.13)}\\
&=\sum_{k=0}^{\nu_0} f^{\nu\backslash k}\frac{u_1^{|\nu|-k}}{(|\nu|-k)!}s^k \qquad \text{using (B.23),}
\end{aligned}
$$

and on the other hand,

$$
\begin{aligned}
F_\nu&\left(-s-u_1,-\frac{s^2}{2},-\frac{s^3}{3},\ldots\right)\\
&=F_\nu(-u-[s])|_{u_2=u_3=\cdots=0}\\
&=\sum_{k=0}^{\nu_0} s^k p_k(-\tilde{\partial})\,F_\nu(-u_1,0,0,\ldots), \qquad \text{by (6.1).}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
p_k&(-\tilde{\partial})\,F_\nu(-u_1,0,0,\ldots)\\
&=f^{\nu\backslash(k)}\frac{u_1^{|\nu|-k}}{(|\nu|-k)!}\\
&=u_1^{|\nu|-k}\det\left(\frac{1}{(\nu_i-k\delta_{j0-i}+j)!}\right)_{0,j\leqslant l}, \qquad \text{by (B.19).}
\end{aligned}
$$

leading to (6.3) with c_k^ν given by the first or the second expression (6.5). Note that $f^{\nu/(k)}\neq 0$ for $k\leqslant\nu_0$, because $f^{\nu/(k)}$ is the number of standard tableaux of shape $\nu/(k)$.

We now provide a first proof of (6.4), which is elementary and which was supplied by Bruce Sagan. First, note that the right hand side of the identity can be written in terms of *the hook length*. Thus setting $b_k^\lambda=(-1)^{\nu-k}c_k^\lambda$, we must prove, in terms of a partition $\lambda=(\lambda_1,\lambda_2,\ldots)$, that

$$
b_0^\lambda\prod_{j=1}^{\lambda_1}(z-h_{(1,j)}^\lambda)=\sum_0^{\lambda_1}(-1)^k\,b_k^\lambda(z)_{\lambda_1-k} \tag{6.10}
$$

with

$$
b_0^\lambda=\frac{1}{\prod_{\text{all } i,j}h^\lambda(i,j)}.
$$

We prove the identity *by induction on the number of columns*, noting that both sides are 1, if λ has no columns. Assume the relation (6.10) valid for the partition λ and then add a column of height m to the left of the Young diagram of λ, forming a new partition μ with $\mu_i = \lambda_i + 1$.
Then

$$\begin{aligned}&\{\text{hook lengths of } \mu\}\\ &\qquad = \{\text{hook lengths of } \lambda\} \cup \{h^{\mu}_{(1,1)} = \lambda_1 + m,\ h^{\mu}_{(2,1)} = \lambda_2 + m - 1, ...\}.\end{aligned}$$

Now multiply identity (6.10) (which is assumed true) by

$$(z - h^{\mu}_{(1,1)}) \frac{b_0^{\mu}}{b_0^{\lambda}} = \frac{(z - m - \lambda_1)}{(\lambda_1 + m)(\lambda_2 + m - 1)\cdots}$$

to yield the identity

$$\begin{aligned}
b_0^{\mu} \prod_{j=1}^{\mu_1} (z - h^{\mu}_{(1,j)})
&= \sum_{k=0}^{\lambda_1} (-1)^k b_k^{\lambda}(z)_{\lambda_1 - k} \frac{z - (\lambda_1 - k) - (m + k)}{(\lambda_1 + m)(\lambda_2 + m - 1)\cdots} \\
&= \frac{1}{(\lambda_1 + m)(\lambda_2 + m - 1)\cdots} \sum_{k=0}^{\lambda_1} (-1)^k b_k^{\lambda}((z)_{\lambda_1 - k + 1} - (m + k)(z)_{\lambda_1 - k}) \\
&= \sum_{k=-1}^{\lambda_1} (-1)^{k+1} \frac{(b_{k+1}^{\lambda} + (m + k)\, b_k^{\lambda})}{(\lambda_1 + m)(\lambda_2 + m - 1)\cdots} (z)_{(\lambda_1 + 1) - (k+1)}, \\
&\qquad \text{upon relabeling (with } b_{-1}^{\lambda} = 0) \\
&= \sum_{k=0}^{\mu_1} (-1)^k \frac{b_k^{\lambda} + (m + k - 1)\, b_{k-1}^{\lambda}}{(\mu_1 + m - 1)(\mu_2 + m - 2)\cdots} (z)_{\mu_1 - k} \\
&\qquad \text{upon setting } k + 1 \curvearrowright k \\
&= \sum_{k=0}^{\mu_1} (-1)^k b_k^{\mu}(z)_{\mu_1 - k}.
\end{aligned} \tag{6.11}$$

The last equality is shown as follows: the determinants b_{k+1}^{λ} and b_k^{λ} can be written as $m \times m$ determinants

$$b_k^{\lambda} = \det \left[\frac{1}{(\lambda_i - k\delta_{j1} + i + j)!} \right]_{1 \leq i,j \leq m}$$

$$b_{k+1}^{\lambda} = \det \left[\frac{1}{(\lambda_i - (k+1)\, \delta_{j1} - i + j)!} \right]_{1 \leq i,j \leq m}.$$

Since these two matrices agree in all but the first column, $b^{\lambda}_{k+1}+(m+k)\,b^{\lambda}_{k}$ can be combined into one matrix which has the same last $m-1$ columns as the above two determinants. Therefore defining the columns

$$c_1=\left[\frac{m+\lambda_i-i+1}{(\lambda_i-k-i+1)!}\right]_{i=1}^{m}$$

and

$$c_{m-l}=\left[\frac{1}{(\lambda_i+(m-i)-l)!}\right]_{i=1}^{m},$$

we have

$$b^{\lambda}_{k+1}+(m+k)\,b^{\lambda}_{k}=\det[c_1, c_2, ..., c_{m-l}, ..., c_m].$$

Replacing the columns as follows,

$$\begin{aligned}
&c_1 \curvearrowright \tilde{c}_1=c_1\\
&c_{m-l}\curvearrowright \tilde{c}_{m-l}=c_{m-l}+lc_{m-l+1}, \qquad l=0, 1, ..., m-2\\
&\qquad=\left[\frac{1}{(\lambda_i+(m-i)-l)!}+\frac{l}{(\lambda_i+(m-i)-l+1)!}\right]_{i=1}^{m}\\
&\qquad=\left[\frac{\lambda_i+m-i+1}{(\lambda_i+(m-i)-l+1)!}\right]_{i=1}^{m}.
\end{aligned}$$

Since the numerator is independent of l, we divide the ith row of the new matrix obtained by $\lambda_i+m-i+1$, yielding

$$\begin{aligned}
\frac{b^{\lambda}_{k+1}+(m+k)\,b^{\lambda}_{k}}{\prod_{i=1}^{m}(\lambda_i+m-i+1)}&=\det\left[\left[\frac{1}{(\lambda_i-k-i+1)!}\right]_{i=1}^{m}, c_2, c_3, c_4, ..., c_{m+1}\right]\\
&=\det\left[\frac{1}{(\lambda_i+1-(k+1)\,\delta_{j1}-i+j)!}\right]_{1\leqslant i,j\leqslant m+1}\\
&=b^{\mu}_{k+1},
\end{aligned}$$

establishing the last equality in (6.11) and thus proving (6.4).

We now give a second, less elementary proof of (6.4), which is due to Ian Goulden and based on the work of Andrews, Goulden, and Jackson [AGJ]. The starting point is, in the notation of Appendix C, the identity (C.9),

$$L_x u^n_1=\sum_{j=0}^{n}(-1)^j(x\text{-})_{n-j}(n)_j u^{n-j}_1 e_j.$$

Then on the one hand, if the inner product is taken with S_λ, $|\lambda| = n$,

$$\begin{aligned}\langle L_x u_1^n, S_\lambda\rangle &= \sum_{j=0}^{n} (-1)^j (x)_{n-j}(n)_j \langle u_1^{n-j} e_j, S_\lambda\rangle \\ &= \sum_{j=0}^{n} (-1)^j (x)_{n-j}(n)_j f^{\lambda\setminus(1^j)};\end{aligned} \tag{6.12}$$

the last equality follows from

$$\begin{aligned}\langle u_1^{|\lambda|-j} e_j, S_\lambda\rangle &= \langle u_1^{|\lambda|-j} S_{(1^j)}, S_\lambda\rangle \qquad \text{by (B.11)} \\ &= \langle u_1^{|\lambda|-j}, S_{\lambda\setminus(1^j)}\rangle = f^{\lambda\setminus(1^j)} \qquad \text{using (C.3) and (C.4).}\end{aligned}$$

On the other hand, compute

$$\begin{aligned}\langle L_x u_1^n, S_\lambda\rangle &= \left\langle L_x \sum_{|\mu|=n} f^\mu S_\mu, S_\lambda\right\rangle \qquad \text{using (C.5)} \\ &= \left\langle \sum_{|\mu|=n} f^\mu L_x S_\mu, S_\lambda\right\rangle \qquad \text{since } L_x \text{ is a linear operator} \\ &= \left\langle \sum_{|\mu|=n} f^\mu q_\mu(x)\, S_\mu, S_\lambda\right\rangle \qquad \text{by (C.8)} \\ &= \sum f^\mu q_\mu(x) \langle S_\mu, S_\lambda\rangle = \sum f^\mu q_\mu(x)\, \delta_{\mu\lambda} = f^\lambda q_\lambda(x) \qquad \text{by (C.2)}\end{aligned} \tag{6.13}$$

Comparing (6.12) and (6.13), we now get an identity, upon setting $\lambda \curvearrowright \hat{\lambda}$:

$$\sum_{k=0}^{|\lambda|} (-1)^k (x)_{|\lambda|-k} (|\lambda|)_k f^{\hat{\lambda}\setminus(1^k)} = f^{\hat{\lambda}} \prod_{i=1}^{|\lambda|} (x - (\hat{\lambda}_i + |\lambda| - i)). \tag{6.14}$$

If we set $x = z + |\lambda| - \lambda_1$ the left hand side equals

$$\begin{aligned}&\sum_{k=0}^{|\lambda|} (-1)^k (x)_{|\lambda|-k} (|\lambda|)_k f^{\hat{\lambda}\setminus(1^k)} \\ &\quad = \sum_{k=0}^{\lambda_1} (-1)^k (x)_{|\lambda|-k} (|\lambda|)_k f^{\hat{\lambda}\setminus(1^k)} \qquad \text{since } f^{\hat{\lambda}\setminus(1^k)} = 0 \text{ for } k > \lambda_1 \\ &\quad = \sum_{k=0}^{\lambda_1} (-1)^k (z + |\lambda| - \lambda_1)_{|\lambda|-k} (|\lambda|)_k f^{\hat{\lambda}\setminus(1^k)} \\ &\quad = \sum_{k=0}^{\lambda_1} (-1)^k (z)_{\lambda_1-k} (|\lambda|)_k f^{\lambda\setminus(k)} (z + |\lambda| - \lambda_1)_{|\lambda|-\lambda_1}, \\ &\qquad (\text{since } f^{\lambda/\mu} = f^{\hat{\lambda}\setminus\hat{\mu}} \text{ by (B.16)})\end{aligned} \tag{6.15}$$

while the right hand side equals

$$
\begin{aligned}
&f^{\hat\lambda} \prod_{j=1}^{|\lambda|} (x-(\hat\lambda_j+|\lambda|-j))\\
&\quad = f^{\hat\lambda} \prod_{j=1}^{|\lambda|} (z-(\lambda_1+\hat\lambda_j-j))\\
&\quad = f^{\hat\lambda} \prod_{j=1}^{\lambda_1} (z-(\lambda_1+\hat\lambda_j-j)) \prod_{j=\lambda_1+1}^{|\lambda|} (z-(\lambda_1+\hat\lambda_j-j))\\
&\quad = f^{\lambda} \prod_{j=1}^{\lambda_1} (z-(\lambda_1+\hat\lambda_j-j))(z+|\lambda|-\lambda_1)_{|\lambda|-\lambda_1},\\
&\qquad\qquad \text{since } \hat\lambda_j=0 \text{ for } j>\lambda_1 \text{ and } f^{\lambda}=f^{\hat\lambda}.
\end{aligned}
\tag{6.16}
$$

Substituting (6.15) and (6.16) into (6.14) and dividing both sides by

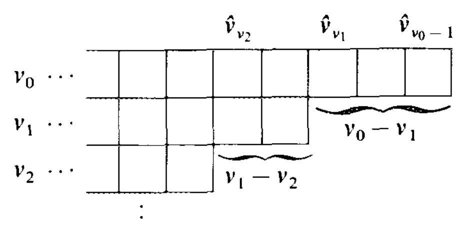
$$f^{\lambda}(z+|\lambda|-\lambda_1)_{|\lambda|-\lambda_1}$$

lead to

$$\sum_{k=0}^{\lambda_1} (-1)^k (z)_{\lambda_1-k} (|\lambda|)_k \frac{f^{\lambda\setminus(k)}}{f^{\lambda}} = \sum_{j=1}^{\lambda_1} (z-(\lambda_1+\hat\lambda_j-j)),$$

thus providing a second proof of (6.4) and ending the proof of Theorem 6.1.

Proof of Corollary 6.1. Typically the Young diagram $(\nu_0 \geqslant \nu_1 \geqslant \cdots)$ looks like



and thus

$$
\begin{aligned}
\hat\nu_i &= 1 \quad \text{for} \quad i=\nu_0-1, ..., \nu_1\\
&= 2 \quad \text{for} \quad i=\nu_1-1, ..., \nu_2\\
&= 3 \quad \text{for} \quad i=\nu_2-1, ..., \nu_3, \text{ etc.}
\end{aligned}
$$

Therefore starting at $i = \nu_0 - 1$ and going down, the numbers $\nu_0 + \hat{\nu}_i - i - 1$ have the values 1, 2, 3, ... up to the point where $i = \nu_1$; at $i = \nu_1 - 1$ it skips one, taking on the value $\nu_0 - \nu_1 + 2$ and then grows linearly until $i = \nu_2$, where again $\nu_0 + \hat{\nu}_i - i - 1$ skips one, taking on the value $\nu_0 - \nu_2 + 3$, proving the first part of the corollary.

As to proving (6.8), we proceed by induction: first observe that for $r = 0$, the determinant (6.8) equals $\gamma_{00} = P(0) = c_{\nu_0}^{(\nu)}/c_0^{(\nu)} \neq 0$ (see (6.7)). Then one first computes

$$\begin{aligned}
\gamma_{00}^{-1} \det(\gamma_{ij})_{\substack{0 \leqslant i \leqslant 1 \\ j \in \{0, \mu\}}} &= \gamma_{00}^{-1} \det \begin{pmatrix} \gamma_{00} & \gamma_{0\mu} \\ \gamma_{10} & \gamma_{1\mu} \end{pmatrix} \\
&= \frac{1}{(\mu-1)!} \left(\sum_{j=0}^{\mu} P(\mu - j) \binom{\mu}{j} (-1)^j \right. \\
&\quad \left. + \sum_{j=0}^{\mu-1} P(\mu - 1 - j) \binom{\mu-1}{j} (-1)^j \right) \\
&= \frac{1}{(\mu-1)!} \left(\sum_{j=1}^{\mu} P(\mu - j)(-1)^j \left(\binom{\mu}{j} - \binom{\mu-1}{j-1} \right) + P(\mu) \right) \\
&= \frac{1}{(\mu-1)!} \sum_{j=0}^{\mu-1} P(\mu - j)(-1)^j \binom{\mu-1}{j},
\end{aligned} \tag{6.17}$$

and thus for $\mu = \nu_0 - \nu_1 + 1$, we have

$$\gamma_{00}^{-1} \det(\gamma_{ij})_{\substack{0 \leqslant i \leqslant 1 \\ j = 0,\, \nu_0 - \nu_1 + 1}} = \frac{P(\nu_0 - \nu_1 + 1)}{(\nu_0 - \nu_1)!}$$

since $P(k) = 0$, as long as $1 \leqslant k < \nu_0 - \nu_1 + 1$, leading to the conclusion

$$\det(\gamma_{ij})_{\substack{i = 0, 1 \\ j = 0,\, \nu_0 - \nu_1 + 1}} = \frac{P(0)\, P(\nu_0 - \nu_1 + 1)}{(\nu_0 - \nu_1)!}.$$

The proof for general r then proceeds by induction, establishing Corollary 6.1.

7. Vanishing of τ-Function, Regularization, and Birkhoff Strata

In this section, the actual core of the paper, we examine the behavior of the τ-function at a point t^* of vanishing, in terms of the Young diagram associated with the plane W^{t^*}. For instance, we give precise estimates of τ and $p_k(-\tilde{\partial})\, \tau$ for all $k \geqslant 1$.

The plane W^{t^*} is said to have a Young diagram $\nu(W^{t^*}) \equiv \nu$, when

$$W^{t^*} = \text{span}\{z^{i-\nu_i}(1+O(z^{-1})), \qquad i=0, 1, 2, \ldots\}.$$

For future use, we define

$$H_\nu = \text{span}\{z^{i-\nu_i}, i=0, 1, 2, \ldots\}.$$

Successive Bäcklund transforms form a "ladder" enabling one to "climb out" of the singularity by each time knocking out the leftmost column of the Young diagram, until exhaustion. The final wave function thus obtained is—roughly speaking—the limit of the old Ψ, after multiplication by an appropriate z-independent function. Although the plane W^t tends to W^{t^*}, the basis of W^t "goes to hell"; only certain definite linear combinations in W^t tend to functions in W^{t^*}. A precise result is given in the final theorem of this section.

THEOREM 7.1. *For the family $W^t \in Gr_0$ of linear spaces associated with the KP flow, consider a Bäcklund generated*[13] *$W_1^t \in Gr_0$ such that $zW^t \subset W_1^t$. Let t^* be a point where $\tau(t^*)=0$ such that W^{t^*} has Young diagram ν. Then the following two statements hold*:

(i) *If the Young diagram about W^{t^*} is ν, then for small s the Young diagram about $W^{t^*+[s]}$ is ($\nu\backslash$first column).*

(ii) *If the Young diagram about W^{t^*} is ν, then the Young diagram about $W_1^{t^*}$ is ($\nu\backslash$first column).*

Proof. In general, from (A.11), it follows that

$$\tau(t^*+t) = \sum_\mu \xi_\mu(W^{t^*})\, F_\mu(t), \tag{7.1}$$

where W^{t^*} is the plane associated with t^* and

$$\xi_\mu = \det \text{proj}(W^{t^*} \to H_\mu)$$

are the Plücker coordinates of the plane W^{t^*} (Sato [S1]). But since W^{t^*} has Young diagram ν, we have $\xi_\nu(W^{t^*}) \neq 0$ and $\xi_\mu(W^{t^*})=0$ for all ν such that[14] $\mu \not\geqslant \nu$, therefore

$$\begin{aligned}\tau(t^*+t) &= \sum_{\mu \geqslant \nu} \xi_\mu(W^{t^*})\, F_\mu(t) \\ &= \xi_\nu(W^{t^*})\, F_\nu(t) + \sum_{\substack{|\mu|>|\nu| \\ \mu \geqslant \nu}} \xi_\mu(W^{t^*})\, F_\mu(t) \quad \text{with } \xi_\nu(W^{t^*}) \neq 0,\end{aligned} \tag{7.2}$$

[13] A plane W_1^t such that $zW^t \subset W_1^t$, generated by a dual Bäcklund transform.
[14] $\mu \geqslant \nu$ iff $\mu_i \geqslant \nu_i$.

i.e., the term of minimal degree, where degree $(\prod_i t_i^{k_i}) \equiv \sum_i ik_i$, in the expansion of $\tau(t^* + t)$ is the Schur polynomial going with the stratum of t^*. In other words, the Schur polynomial of minimal degree in the (unique) expansion for $T(t^* + t)$ determines the stratum of W^{t^*}.

Given the neighborhood $t^* + [s] + t$ of $t^* + [s]$, consider the expansion of τ in Schur polynomials about t^* and subsequently the Taylor expansion (6.2) of $F_\mu(t + [s])$ about t:

$$\begin{aligned}
&\tau(t^* + [s] + t)\\
&\quad = \xi_\nu(t^*)\, F_\nu([s] + t) + \sum_{\substack{\mu_i \geqslant \nu_i \\ |\mu| > |\nu|}} \xi_\mu(t^*)\, F_\mu([s] + t)\\
&\quad = \xi_\nu(t^*)\, (F_\nu(t) - sF_{\nu\backslash(1)}(t) + s^2 F_{\nu\backslash(11)}(t) + \cdots + (-s)^{\hat{\nu}_0} F_{(\nu\backslash\text{first column})}(t)\\
&\qquad + \sum_{\substack{\mu_i \geqslant \nu_i \\ |\mu| > |\nu|}} \xi_\mu(t^*)(F_\mu(t) + \cdots + (-s)^{\hat{\mu}_0} F_{(\mu\backslash\text{first column})}(t)).
\end{aligned} \tag{7.3}$$

In this sum, the polynomial of smallest degree is given by $F_{(\nu\backslash\text{first column})}$, except possibly for those μ's such that

$$\mu \geqslant \nu \qquad \text{with} \quad |\mu| > |\nu| \quad \text{and} \quad (\nu\backslash\text{first column}) = (\mu\backslash\text{first column});$$

but then $\hat{\mu}_0 > \hat{\nu}_0$ and the coefficient of $F_{(\nu\backslash\text{first column})}$ is given by

$$\xi_\nu(t^*)(-s)^{\hat{\nu}_0} + \xi_\mu(t^*)(-s)^{\mu_0} \qquad \text{with} \quad \xi_\nu(t^*) \neq 0.$$

This expression is $\neq 0$ for s sufficiently small and thus the polynomial of smallest degree remains $F_{(\nu\backslash\text{first column})}$, establishing (i).

Part (ii) follows at once from (i) and from the result in Theorem 4.1: namely the τ-function going with the Bäcklund generated W_1^t such that $zW^t \subset W_1^t$ is given by

$$\tau_1(t) = e^{-\sum t_i/s^i} \tau(t + [s]) \qquad (s = z^{-1})$$

and, since the exponential does not matter here, the stratum of $W_1^{t^*}$ is totally governed by the behavior of $\tau((t^* + [s]) + t)$ around $t^* + [s]$.

THEOREM 7.2. *Consider the Bäcklund generated flag*

$$z^{\nu_0} W_0 \subset z^{\nu_0 - 1} W_1 \subset \cdots \subset W_{\nu_0}$$

such that at a specific point t^*, the plane $W_0^{t^*}$ lies in the stratum

$$\nu(W_0^{t^*}) = (\nu_0 \geqslant \nu_1 \geqslant \nu_2 \geqslant \cdots); \tag{7.4}$$

then,

$$\textit{Young diagram } (W_j^{t^*}) = \textit{Young diagram } (W^{t^*}) - \textit{first } j \textit{ columns.} \tag{7.5}$$

In particular, if τ_l denotes the τ-function associated with W_l, then

$$\tau(t^*) = \tau_1(t^*) = \cdots = \tau_{\nu_0 - 1}(t^*) = 0 \qquad \textit{and} \qquad \tau_{\nu_0}(t^*) \neq 0. \tag{7.6}$$

Proof. This statement follows readily from Theorem 7.1, which applied to the flag

$$zW_0 \subset W_1$$

leads to the statement (7.5) for $j = 1$, and so on inductively. Since going from W_j to W_{j+1} amounts to removing the leftmost column of the Young diagram and since the Young diagram associated with $W_0^{t^*}$ has exactly ν_0 columns, there will be a non-trivial Young diagram associated with each $W_0^{t^*}, W_1^{t^*}, ..., W_{\nu_0 - 1}^{t^*}$ and at $W_{\nu_0}^{t^*}$ the stratum will have an empty Young diagram, i.e., $W_{\nu_0}^{t^*}$ belongs to the principal cell, which implies (7.6).

The estimate (7.7) in Theorem 7.3 below generalizes the one found by Segal and Wilson [SW] for $k = 0$; see also a footnote by Laumon in [SW].

THEOREM 7.3. *Define $t = t^*$ such that $\tau(t^*) = 0$ and such that the plane W^{t^*} belongs to the stratum $\nu = (\nu_0 \geqslant \nu_1 \geqslant \cdots)$. Then the expressions below behave as follows near t^* in the x direction (consider an increment $\bar{x} = (x, 0, 0, ...)$):*

$$\begin{aligned} p_k(-\tilde{\partial})\, \tau(t^* + \bar{x}) &= c c_k^{\nu} x^{|\nu| - k} + \cdots && \textit{for} \quad 0 \leqslant k \leqslant \nu_0, \\ &= c_k^{\nu} x^{|\nu| - \nu_0} + \cdots && \textit{for} \quad k > \nu_0 \\ p_k(\tilde{\partial})\, \tau(t^* + \bar{x}) &= c c_k^{\hat{\nu}} x^{|\hat{\nu}| - k} + \cdots && \textit{for} \quad 0 \leqslant k \leqslant \hat{\nu}_0, \\ &= c_k^{\hat{\nu}} x^{|\hat{\nu}| - \hat{\nu}_0} + \cdots && \textit{for} \quad k > \hat{\nu}_0, \end{aligned} \tag{7.7}$$

where $c = \xi(W^{t^*}) \neq 0$ and where the c_k^{ν} $(0 \leqslant k \leqslant \nu_0)$ and $c_k^{\hat{\nu}}$ $(0 \leqslant k \leqslant \hat{\nu}_0)$ are given by Theorem 6.1:

$$c_k^{\nu} = (-1)^{|\nu| - k} \frac{f^{\nu/(k)}}{(|\nu| - k)!} \neq 0, \qquad 0 \leqslant k \leqslant \nu_0$$

and

$$c_k^{\hat{\nu}} = (-1)^{\hat{\nu} - k} \frac{f^{\hat{\nu} \setminus (1^k)}}{(|\hat{\nu}| - k)!} \neq 0, \qquad 0 \leqslant k \leqslant \hat{\nu}_0. \tag{7.8}$$

Remark. Observe that the ratios of c_k^{ν} for $0 \leqslant k \leqslant \nu_0$ are independent of t^*, whereas beyond ν_0 the c_k^{ν} will depend on t^*, as appears from the proof.

Proof. As in (7.2) (the summations are over μ with fixed ν),

$$\tau(t^* + t) = \xi_\nu(W'^*)\, F_\nu(t) + \sum_{\substack{\text{all } \mu \geqslant \nu \\ |\mu| > |\nu|}} \xi_\mu(W'^*)\, F_\mu(t) \qquad \text{with} \quad \xi_\nu(W'^*) \neq 0.$$

Applying $p_k(-\tilde{\partial})$ to $\tau(t^* + t)$, and then setting $t = \bar{x} \equiv (x, 0, 0, ...)$, we have

$$p_k(-\tilde{\partial})\, \tau(t^* + \bar{x}) = \xi_\nu\, p_k(-\tilde{\partial})\, F_\nu(\bar{x}) + \sum_{\substack{\mu \geqslant \nu \\ |\mu| > |\nu|}} \xi_\mu\, p_k(-\tilde{\partial})\, F_\mu(\bar{x})$$

(i) for $0 \leqslant k \leqslant \nu_0$

$$\overset{(*)}{=} \xi_\nu c_k^\nu x^{|\nu| - k} + \sum_{\substack{\mu \geqslant \nu \\ |\mu| > |\nu|}} \xi_\mu c_k^\mu x^{|\mu| - k}$$

$$= \xi_\nu c_k^\nu x^{|\nu| - k} + \text{higher order terms, with } \xi_\nu c_k^\nu \neq 0.$$

To prove equality $(*)$, note that in the first and the subsequent terms, we have $\nu_0 \leqslant \mu_0$ and thus $k \leqslant \mu_0$. Hence in all terms, $p_k(-\tilde{\partial})\, F_\mu(\bar{x})$ is estimated by (6.3) of Theorem 6.1 with c_k^ν given by (6.5).

(ii) for $k > \nu_0$ (let $m \equiv k - \nu_0$)

$$= \sum_{i=0}^{m-1} \Bigg(\sum_{\substack{|\mu| = |\nu| + i \\ \mu \geqslant \nu}} \xi_\mu\, p_k(-\tilde{\partial})\, F_\mu(\bar{x}) \Bigg) + \sum_{\substack{|\mu| = |\nu| + m \\ \mu \geqslant \nu}} \xi_\mu\, p_k(-\tilde{\partial})\, F_\mu(\bar{x})$$

$$+ \sum_{i=m+1}^{\infty} \Bigg(\sum_{\substack{|\mu| = |\nu| + i \\ \mu \geqslant \nu}} \xi_\mu\, p_k(-\tilde{\partial})\, F_\mu(\bar{x}) \Bigg)$$

$$\overset{(*)}{=} \sum_{\substack{|\mu| = |\nu| + m \\ \mu \geqslant \nu}} \xi_\mu\, p_k(-\tilde{\partial})\, F_\mu(\bar{x}) + \sum_{i=m+1}^{\infty} \Bigg(\sum_{\substack{|\mu| = |\nu| + i \\ \mu \geqslant \nu}} \xi_\mu\, p_k(-\tilde{\partial})\, F_\mu(\bar{x}) \Bigg)$$

$$\overset{(**)}{=} \text{constant} \times (\partial/\partial x)^k\, x^{|\nu| + m} + \text{higher order terms in } x$$

$$= \text{constant} \times x^{|\nu| + m - \nu_0 - m} + \text{higher order terms (since } k = \nu_0 + m)$$

$$= \text{constant} \times x^{|\nu| - \nu_0} + \text{higher order terms.}$$

Equality $(*)$ holds because $|\mu| \leqslant |\nu| + m - 1$ and $\mu \geqslant \nu$ imply $\mu_0 \leqslant \nu_0 + m - 1 < \nu_0 + m = k$ and thus using (6.3) for such μ:

$$p_k(-\tilde{\partial})\, F_\mu(\bar{x}) = 0.$$

Equality $(**)$ follows from the fact that on the one hand (see (A.2))

$$p_k(t) = \frac{t_1^k}{k!} + \cdots \qquad \text{and thus} \qquad p_k(-\tilde{\partial}) = \frac{1}{k!}\left(-\frac{\partial}{\partial x} \right)^k + \cdots;$$

on the other hand for μ such that $|\mu| = |\nu| + m$

$$F_\mu(\bar{x}) = c_\mu(-x)^{|\mu|} = c_\mu(-x)^{|\nu|+m}, \qquad c_\mu \neq 0.$$

Finally, $p_k(-\tilde{\partial})\, F_\mu(\bar{x})$ with $|\mu| > |\nu| + m$ will produce higher powers of x, thus proving the first half of (7.7). As to the second, it follows in the same way from Theorem 6.1.

COROLLARY 7.3. *At a point t^* where $\nu(W^{t^*}) = \nu$ we have the estimate*

$$\frac{\partial \tau}{\partial t_k}(t^* + \bar{x}) = c b_k^\nu x^{|\nu| - k} + \cdots \qquad \text{for} \quad 1 \leqslant k < \min(\nu_0, \hat{\nu}_0),$$

where $c = \xi_\nu(W^{t^*}) \neq 0$ is independent of k and where

$$b_k^\nu = \text{coefficient of } t_k t_1^{|\nu| - k} \text{ in } F_\nu(t).$$

Proof. Using the KP equation (see (1.11)) and the estimates (7.7), we have

$$\left(\frac{(\partial\tau/\partial t_k)(t^* + \bar{x})}{\tau(t^* + \bar{x})}\right)' = \frac{1}{\tau^2}\left(\sum_{\substack{i+j=k+1\\ i,j \geqslant 0}} p_i(-\tilde{\partial})\,\tau \cdot p_j(\tilde{\partial})\,\tau\right)$$

$$= x^{-k-1} \sum_{\substack{i+j=k+1\\ i,j \geqslant 0}} (-1)^i \frac{c_i^{\hat{\nu}} c_j^\nu}{c_0^{\hat{\nu}} c_0^\nu} + O(x^{-k})$$

$$\text{for} \quad 1 \leqslant k \leqslant \min(\nu_0, \hat{\nu}_0) - 1.$$

Integrating and using the estimate (7.7) for τ yield ($c_0^{\hat{\nu}} = c_0^\nu$)

$$\frac{\partial \tau}{\partial t_k}(t^* + \bar{x}) = c b_k^\nu x^{|\nu| - k} + O(x^{|\nu| - k + 1}),$$

$$\text{with} \quad b_k^\nu = -\frac{(c_0^\nu)^{-1}}{k} \sum_{\substack{i+j=k+1\\ i,j \geqslant 0}} (-1)^i c_i^{\hat{\nu}} c_j^\nu,$$

where the leading coefficient b_k^ν is a number expressible purely in terms of the Young diagram ν, except for the constant c, independent of k.

The Schur polynomial F_ν is also a solution, although very special, of the KP equation, such that at $t^* = 0$, the corresponding W^{t^*} is in the stratum ν. Therefore, it must satisfy the same estimate as above, that is,

$$\frac{\partial F_\nu}{\partial t_k}(\bar{x}) = b_k^\nu x^{|\nu| - k},$$

showing that b_k^ν is the coefficient of $t_k t_1^{|\nu| - k}$ in $F_\nu(t)$.

THEOREM 7.4. *Consider a point t^*, where $\tau(t^*)=0$, and with $W'^* \in Gr$ belonging to the stratum $\nu=(\nu_0 \geqslant \nu_1 \geqslant \cdots)$; then the following limit exists and equals,*

$$\lim_{x\to 0} \psi(t^*+\bar{x}, z)\frac{\tau(t^*+\bar{x})}{p_{\nu_0}(-\tilde{\partial})\,\tau(t^*+\bar{x})} = z^{-\nu_0}\psi_{\nu_0}(t^*, z) \in W'^*, \tag{7.9}$$

where (using the notation $\tilde{A}_{z_k} \equiv \tilde{A}_{\Psi_{k-1}(t,z_k)}$ of Theorem 4.1)

$$\Psi_{\nu_0}(t,z) = z^{\nu_0}\tilde{A}_{z_{\nu_0}}\tilde{A}_{z_{\nu_0-1}}\cdots\tilde{A}_{z_1}\Psi(t,z), \qquad z_1, z_2, ..., z_{\nu_0} \textit{ arbitrary}$$

is the wave function of a plane W'_{ν_0}, generated by ν_0 successive inverse Bäcklund transformations, thus satisfying

$$z^{\nu_0}W' \subset W'_{\nu_0}.$$

Remark. Observe that by Theorem 7.2, $W'^*_{\nu_0}$ belongs to the principal stratum and that $\Psi_{\nu_0}(t,z)$ is finite in a neighborhood of t^*.

Proof of Theorem 7.4. Using the estimates in Theorem 7.3, the following expression tends to a non-zero holomorphic limit when $x \to 0$; indeed if we set $k=\nu_0$ and $s=z^{-1}$,

$$\begin{aligned}
&\psi(t^*+\bar{x}, s^{-1})\frac{\tau(t^*+\bar{x})}{p_k(-\tilde{\partial})\,\tau(t^*+\bar{x})}\\
&\quad=\frac{\tau(t^*+\bar{x}-[s])}{p_k(-\tilde{\partial})\,\tau(t^*+\bar{x})}\\
&\quad=\frac{\sum_{j=0}^{\infty} p_j(-\tilde{\partial})\,\tau(t^*+\bar{x})\,s^j}{p_k(-\tilde{\partial})\,\tau(t^*+\bar{x})}\\
&\quad=\sum_{j=0}^{k-1}\frac{p_j(-\tilde{\partial})\,\tau(t^*+\bar{x})}{p_k(-\tilde{\partial})\,\tau(t^*+\bar{x})}s^j + s^k + \sum_{j=k+1}^{\infty}\frac{p_j(-\tilde{\partial})\,\tau(t^*+\bar{x})}{p_k(-\tilde{\partial})\,\tau(t^*+\bar{x})}s^j\\
&\quad=\sum_{j=0}^{k-1}\frac{x^{|\nu|-j}(c_j+O(x))}{x^{|\nu|-k}(c_k+O(x))}s^j + s^k + \sum_{j=k+1}^{\infty}\frac{x^{|\nu|-k}(c_j+O(x))}{x^{|\nu|-k}(c_k+O(x))}s^j\\
&\qquad\text{with } c_k \neq 0\\
&\quad=\sum_{j=0}^{k-1} x^{k-j}\left(\frac{c_j}{c_k}+O(x)\right)s^j + s^k + \sum_{j>k}\left(\frac{c_j}{c_k}+O(x)\right)s^j\\
&\quad\xrightarrow[x\to 0]{} s^k + \sum_{j>k}\frac{c_j}{c_k}s^j.
\end{aligned}$$

In order to identify this limit, we first construct $W'_k \in Gr$ and the associated wave functions Ψ_k, via $k=\nu_0$ (dual) Bäcklund transforms, such that

$z^k W^t \subset W_k^t$. Then from Theorem 5.1(ii), we have upon division by $\alpha_k(t^* + \bar{x})$

$$z^k \frac{\psi(t^* + \bar{x}, z)}{\alpha_k(t^* + \bar{x})} = \psi_k(t^* + \bar{x}, z) + \frac{\alpha_{k-1}(t^* + \bar{x})}{\alpha_k(t^* + \bar{x})} \nabla\psi_k(t^* + \bar{x}, z)$$
$$+ \cdots + \frac{\alpha_0(t^* + \bar{x})}{\alpha_k(t^* + \bar{x})} \nabla^k \psi_k(t^* + \bar{x}, z). \tag{7.10}$$

By (5.2) and estimate (7.7) in Theorem 7.3, estimate

$$\begin{aligned}\alpha_j(t^* + \bar{x}) &= \frac{p_j(-\tilde{\partial})\,\tau(t^* + \bar{x})}{\tau(t^* + \bar{x})} + \sum_{i=1}^{j} h_i^j(\psi_k(t^* + \bar{x}))\frac{p_{j-i}(-\tilde{\partial})\,\tau(t^* + \bar{x})}{\tau(t^* + \bar{x})} \\ &= \frac{c_j x^{|\nu| - j} + \cdots}{c_0 x^{|\nu|} + \cdots} + \sum_{i=1}^{j} h_i^j(\psi_k(t^* + \bar{x}))\frac{c_{j-i} x^{|\nu| - (j-i)} + \cdots}{c_0 x^{|\nu|} + \cdots} \\ &\overset{(*)}{=} \frac{c_j/c_0 + O(x)}{x^j}, \qquad \text{with} \quad c_j/c_0 \neq 0 \text{ for } j = 0, 1, \ldots, k.\end{aligned} \tag{7.11}$$

The $h_i^{(j)}$ in the identity above are polynomial expressions in $\psi_k(t^* + x)$ and its derivatives, which themselves remain finite, when $x \to 0$, since $\tau_k(t^*) \neq 0$ by Theorem 7.2. Therefore taking the limit $x \searrow 0$ in these expressions does no harm. Therefore, for $j = 0, 1, \ldots, k-1$, the ratios

$$\frac{\alpha_j(t^* + \bar{x})}{\alpha_k(t^* + \bar{x})} = x^{k-j}\left(\frac{c_j}{c_k} + O(x)\right) \to 0 \qquad \text{when} \quad x \to 0,$$

which applied to (7.10) leads to

$$\lim_{x \to 0} z^k \frac{\psi(t^* + \bar{x}, z)}{\alpha_k(t^* + \bar{x})} = \lim_{x \to 0} \psi_k(t^* + \bar{x}, z). \tag{7.12}$$

The limit on the right hand side exists, since $\tau_k(t^*) \neq 0$, according to Theorem 7.2. Dividing (7.11) for $j = k$ by

$$\frac{p_k(-\tilde{\partial})\,\tau(t^* + \bar{x})}{\tau(t^* + \bar{x})}$$

leads to

$$\begin{aligned}&\frac{\alpha_k(t^* + \bar{x})}{(p_k(-\tilde{\partial})\,\tau(t^* + \bar{x})/\tau(t^* + \bar{x}))} \\ &\quad = 1 + \sum_{i=1}^{k} h_i^k(\psi_k(t^* + \bar{x}))\frac{p_{k-i}(-\tilde{\partial})\,\tau(t^* + \bar{x})}{p_k(-\tilde{\partial})\,\tau(t^* + \bar{x})} \\ &\quad = 1 + O(x), \qquad \text{again using the same estimates (7.7).}\end{aligned} \tag{7.13}$$

Then multiplying the two expressions in (7.12) and (7.13) and taking the limit when $x \to 0$ yield

$$\lim_{x \to 0} z^k \psi(t^* + \bar{x}, z) \frac{\tau(t^* + \bar{x})}{p_k(-\tilde{\partial})\, \tau(t^* + \bar{x})} = \lim_{x \to 0} \psi_k(t^* + \bar{x}, z), \tag{7.14}$$

ending the proof of Theorem 7.4.

COROLLARY 7.4. *Let W^t belong to the stratum $\nu = (\nu_0 \geqslant \nu_1 \geqslant \cdots)$ for some $t = t^*$; for given $W \in Gr$, consider a space $W_{\nu_0} \in Gr$ such that*

$$z^{\nu_0} W^t \subset W^t_{\nu_0} \tag{7.15}$$

is generated by ν_0 inverse Bäcklund transforms $\tilde{A}_{z_i}$, $1 \leqslant i \leqslant \nu_0$, with corresponding wave function $\Psi_{\nu_0}(t, z)$. Although W_{ν_0} and Ψ_{ν_0} depend on the choice of $z_1, ..., z_{\nu_0}$, the function $\Psi_{\nu_0}(t, z)$, evaluated at $t = t^*$, is independent of that choice.

Proof. This fact follows at once from Theorem 7.4, since according to (7.9), $\Psi_{\nu_0}(t^*, z)$ is the limit of an expression which is independent of the choice of $W^t_{\nu_0}$ and such that (7.15) holds.

It is instructive to give an independent proof for the case $k = 1$. Indeed setting $(s_0, s_1, s_2, s_3) \curvearrowright (0, s_1, s, s_2)$ and $t \curvearrowright t - [s]$, with $s = z^{-1}$ and $s_i = z_i^{-1}$, in the Fay identity (Lemma 2.1), one finds

$$\begin{aligned} s_1(s_2 - s)\, &\tau(t - [s] + [s_1])\, \tau(t + [s_2]) \\ &- s_2(s_1 - s)\, \tau(t - [s] + [s_2])\, \tau(t + [s_1]) \\ &= s(s_2 - s_1)\, \tau(t)\, \tau(t + [s_2] + [s_1] - [s]). \end{aligned}$$

Therefore upon multiplication by an exponential, we have

$$\frac{s_1}{s_1 - s} e^{\sum t_i/s^i} \frac{\tau(t - [s] + [s_1])}{\tau(t + [s_1])} = \frac{s_2}{s_2 - s} e^{\sum t_i/s^i} \frac{\tau(t - [s] + [s_2])}{\tau(t + [s_2])}$$

$$\text{along the locus} \quad \tau(t) = 0. \tag{7.16}$$

Using the third equality in (4.13), both sides of (7.16) are new wave function $\Psi_1(t, z)$ and $\Psi_2(t, z)$ associated with W_1 and W_2 such that $zW \subset W_1$ and $zW \subset W_2$. Now (7.16) claims that both expressions are equal along the locus $\{\tau(t) = 0\}$, establishing the announced independence.

In Theorem 7.4 we saw how to normalize the wave function ψ_0 so that the limit exists when the plane W^t tends to the limit W^{t^*}; the next theorem reveals how the rest of the basis, properly normalized tends to a new basis of W^{t^*}.

THEOREM 7.5. *Consider a family of planes W^t such that W^{t^*} belongs to the stratum $\nu(W^{t^*})=(\nu_0 \geqslant \nu_1 \geqslant \cdots)$; that is,*

$$W^{t^*}=\operatorname{span}\{\varphi_0(t^*, z), \varphi_1(t^*, z), \ldots\}, \tag{7.17}$$

with

$$\varphi_j(t^*, z)=z^{s_j}(a_{s_j}+O(1/z)), \qquad \text{with} \quad s_j=j-\nu_j;$$

the φ_j's are obtained as limits of basis elements of W^t as follows: consider a Bäcklund generated plane W_{ν_0} with $\nu_0=-s_0$, thus satisfying $z^{\nu_0}W^t \subset W^t_{\nu_0}$ with wave function ψ_{ν_0}; then

$$\varphi_0(t^*, z)=\lim_{x\to 0}\frac{\tau(t^*+\bar{x})}{p_{\nu_0}(-\tilde{\partial})\,\tau(t^*+\bar{x})}\psi(t^*+\bar{x}, z)=z^{-\nu_0}\psi_{\nu_0}(t^*, z)$$

$$\varphi_r(t^*, z)=\lim_{x\to 0}\frac{\tau(t^*+\bar{x})}{p_{\nu_r}(-\tilde{\partial})\,\tau(t^*+\bar{x})}\sum_{0\leqslant i\leqslant r}\alpha_{ri}(t^*+\bar{x})\,\nabla^i\psi(t^*+\bar{x}, z)$$

$$\text{with} \quad \alpha_{rr}=1$$

$$=z^{-\nu_0}\sum_{\substack{1\leqslant i\leqslant s_r-s_0\\ i\neq s_1-s_0,\, s_2-s_0,\, \ldots,\, s_{r-1}-s_0}}\beta_{ri}(t^*)\,\nabla^i\psi_{\nu_0}(t^*, z),$$

where the α's are uniquely determined functions of t and the β's functions of t^*, with β_{r, s_r-s_0} a rational number (independent of t^*) determined via the Young diagram ν by

$$\beta_{r, s_r-s_0}=\frac{1}{\Delta^{s_r-s_0-r}P(z)|_{z=0}}\,\frac{P(s_r-s_0)}{(s_r-s_0-1)!}\neq 0 \qquad \text{when} \quad \nu_r\neq 0$$

$$=\frac{P(r-s_0)}{(r-s_0-1)!}.$$

Proof of Theorem 7.5. This proof is broken into several steps:

Step (a). The inclusion $z^{\nu_0}W^t \subset W^t_{\nu_0}$ implies the first relation below by Theorem 5.1(ii) and taking consecutive t_1-derivatives of this relation, multiplied by $e^{\sum t_i z^i}$, yields the subsequent relations (remember $\nu_0=-s_0$)

$$z^{\nu_0}\psi=\sum_{j=0}^{\nu_0}\alpha_{\nu_0-j}\nabla^j\psi_{\nu_0}$$

$$\vdots$$

$$z^{\nu_0}\nabla^i\psi=\sum_{j=0}^{i+\nu_0}\beta_{ij}\nabla^j\psi_{\nu_0},$$

with

$$\beta_{ij} = \sum_{l=\varepsilon_{j\nu_0}}^{\min(i,j)} \binom{i}{l} \partial_x^{i-l} \alpha_{\nu_0 - j + l}, \qquad \varepsilon_{jk} = \max(0, j-k). \tag{7.18}$$

Step (b). We estimate $\beta_{ij}(t^* + \bar{x})$ as a function of x: according to Theorem 5.1 (formula (5.2)) and Theorem 7.3, we have

$$\alpha_j(t) = \frac{p_j(-\tilde{\partial})\,\tau}{\tau} + \sum_{0 \leqslant r < j} \frac{p_r(-\tilde{\partial})\,\tau}{\tau} h^j_{j-r}(\tau_{\nu_0})$$

and

$$\frac{p_r(-\tilde{\partial})\,\tau}{\tau} = \left(\frac{c_r^\nu}{c_0^\nu} + O(x)\right)\frac{1}{x^r}, \qquad 0 \leqslant r \leqslant \nu_0 = -s_0,\ c_r^\nu,\ c_0^\nu \neq 0;$$

but $\tau_{\nu_0} \neq 0$ near t^* by Theorem 7.2 and therefore $h^j_{j-r}(\tau_{\nu_0}) = O(1)$. Combining these facts we find

$$\partial_x^{i-l} \alpha_{\nu_0 - j + l}(t^* + \bar{x}) = \left(\frac{c^\nu_{\nu_0 - j + l}/c_0^\nu + O(x)}{x^{\nu_0 + i - j}}\right)\cdot(\nu_0 + i - j - 1)_{i-l}(-1)^{i-l},$$

which substituted into (7.18) yields

$$\beta_{ij} = \frac{(\gamma_{ij} + O(x))}{x^{\nu_0 + i - j}}$$

with

$$\gamma_{ij} = \sum_{l=\varepsilon_{j\nu_0}}^{\min(i,j)} (-1)^{i-l} \binom{i}{l} (\nu_0 + i - j - 1)_{i-l} \frac{c^\nu_{\nu_0 - j + l}}{c_0^\nu}. \tag{7.19}$$

For later use we make the crucial estimate of this theorem:

$$\begin{aligned} \det[\beta_{ij}]_{\substack{j \in (j_0, j_1, \ldots, j_r) \\ 0 \leqslant i \leqslant r}} &= x^{-\sum_{i=0}^r (\nu_0 + i - j_i)} \left(\det[\gamma_{ij}]_{\substack{0 \leqslant i \leqslant r \\ j \in (j_0, j_1, \ldots, j_r)}} + O(x)\right) \\ &\equiv x^{-\sum_{i=0}^r (\nu_0 + i - j_i)} (\Gamma_{j_0 j_1 \cdots j_r} + O(x)). \end{aligned} \tag{7.20}$$

Step (c). By Kramer's rule, there exists a unique linear combination of the expressions $\{z^{\nu_0}\psi, z^{\nu_0}\nabla\psi, \ldots, z^{\nu_0}\nabla^{r+1}\psi\}$ in step (a) missing $\{\nabla^{j_0}\psi_{\nu_0}, \ldots, \nabla^{j_r}\psi_{\nu_0}\}$, $0 \leqslant j_i \leqslant r + \nu_0 + 1$, which then allows the following estimates in x and s,

$$\begin{aligned}
&\nabla^{r+1}\psi+\alpha_{r+1,r}\nabla^{r}\psi+\cdots+\alpha_{r+1,0}(t^*+\bar{x})\,\psi(t^*+\bar{x},z)\\
&=z^{-\nu_0}\sum_{\substack{\mu\notin\{j_0,\dots,j_r\}\\0\le\mu\le r+\nu_0+1}}\frac{\det[\beta_{ij}]_{\substack{0\le i\le r+1\\ j\in(j_0,j_1,\dots,j_r,\mu)}}}{\det[\beta_{ij}]_{\substack{0\le i\le r\\ j\in(j_0,j_1,\dots,j_r)}}}\nabla^{\mu}\psi_{\nu_0}(t^*+\bar{x},z)\\
&=z^{-\nu_0}\sum_{\substack{\mu\notin\{j_0,\dots,j_r\}\\0\le\mu\le r+\nu_0+1}}\frac{(\Gamma_{j_0j_1\cdots j_r\mu}/\Gamma_{j_0j_1\cdots j_r}+O(x))}{x^{\nu_0+r+1-\mu}}\nabla^{\mu}\psi_{\nu_0}(t^*+\bar{x},z)\\
&\qquad\text{using (7.20)},\qquad(7.21)
\end{aligned}$$

where

$$\nabla^{\mu}\psi_{\nu_0}(t,z)=z^{\mu}(1+O(z^{-1}))\qquad(\text{using (3.7)})\qquad(7.22)$$

is a Laurent series in z^{-1} with holomorphic coefficients in x, in the neighborhood of $t=t^*$. Here one also uses Theorem 7.2, implying that $\tau_{\nu_0}(t^*)\neq 0$.

Step (d). Realizing $\varphi_r(t^*,z)\in W^t$ as a limit of elements in W^t.

The lowest order element ($s_0=-\nu_0$) in W^{t^*} was already obtained as a limit in Theorem 7.5; we now proceed to the next order element $\varphi_1(t^*,z)$ having order s_1: form a linear combination of ($'=\partial/\partial x$)

$$z^{\nu_0}\psi=\alpha_{\nu_0}\psi_{\nu_0}+\alpha_{\nu_0-1}\nabla\psi_{\nu_0}+\cdots+\alpha_{\nu_0-\mu}\nabla^{\mu}\psi_{\nu_0}+\cdots+\alpha_0\nabla^{\nu_0}\psi_{\nu_0}$$

and

$$\begin{aligned}
z^{\nu_0}\nabla\psi&=\alpha'_{\nu_0}\psi_{\nu_0}+(\alpha_{\nu_0}+\alpha'_{\nu_0-1})\nabla\psi_{\nu_0}\\
&\quad+\cdots+(\alpha_{\nu_0-\mu+1}+\alpha'_{\nu_0-\mu})\nabla^{\mu}\psi_{\nu_0}\\
&\quad+\cdots+(\alpha_1+\alpha'_0)\nabla^{\nu_0}\psi_{\nu_0}+\alpha_0\nabla^{\nu_0+1}\psi_{\nu_0}
\end{aligned}$$

with precisely ψ_{ν_0} missing; this is given by Step (c) with $r=0$ and $j_0=0$,

$$\begin{aligned}
&\left(\nabla\psi-\frac{\alpha'_{\nu_0}}{\alpha_{\nu_0}}\psi\right)(t^*+\bar{x},z)\\
&=z^{-\nu_0}\sum_{\mu=1}^{\nu_0+1}\frac{1}{\alpha_{\nu_0}}\det\begin{pmatrix}\alpha_{\nu_0} & \alpha_{\nu_0-\mu}\\ \alpha'_{\nu_0} & \alpha_{\nu_0-\mu+1}+\alpha'_{\nu_0-\mu}\end{pmatrix}\nabla^{\mu}\psi_{\nu_0}(t^*+\bar{x},z)\\
&=\sum_{\mu=1}^{\nu_0+1}\frac{\Gamma_{0\mu}/\Gamma_0+O(x)}{x^{\nu_0-\mu+1}}z^{-\nu_0}\nabla^{\mu}\psi_{\nu_0}(t^*+\bar{x},z)\\
&\equiv\sum_{\mu=1}^{\nu_0+1}\left(\sum_{j=0}^{\infty}a_{\mu j}x^{-\nu_0+\mu-1+j}\right)z^{-\nu_0}\nabla^{\mu}\psi_{\nu_0}\equiv\sum_{\mu=1}^{\nu_0+1}G_{\mu}\equiv G\qquad(a_{\mu j}\in\mathbb{C})
\end{aligned}$$

(7.23)

where

$$\nabla^{\mu}\psi_{\nu_0}(t, z) = z^{\mu}(1 + O(z^{-1})), \tag{7.24}$$

with holomorphic coefficients in x; according to Corollary 6.1 (see also (6.17))

$$\Gamma_0 = P(0) \neq 0, \quad a_{\nu_0 - \nu_1 + 1, 0} = \frac{\Gamma_{0, \nu_0 - \nu_1 + 1}}{\Gamma_0} = \frac{P(\nu_0 - \nu_1 + 1)}{(\nu_0 - \nu_1)!} \neq 0. \tag{7.25}$$

Now if $\lim_{x \to 0} x^{\alpha} G$ exists, it belongs to W'^{*}, since $G \in W'$, even after multiplication with a function of $(x + t_1, t_2, \ldots)$. Now one checks, using (7.22) and (7.23)

$$\begin{aligned}\lim_{x \to 0} x^{\nu_0} G(x, z) &= \lim_{x \to 0} x^{\nu_0} G_1(x, z) \\ &= a_{10} z^{-\nu_0} \lim_{x \to 0} \nabla \psi_{\nu_0} \\ &= a_{10} z^{s_0 + 1}(1 + O(z^{-1})) \in W'^{*} \qquad \text{using (7.24) and } s_0 = \nu_0.\end{aligned}$$

If $s_1 > s_0 + 1$, it must be that $\alpha_{10} = 0$, because otherwise W'^{*} would contain a function of order $s_0 + 1$, which is a contradiction. If $s_1 = s_0 + 1$, i.e., $\nu_0 - \nu_1 = 0$ then $\alpha_{10} \neq 0$ by (7.25) and the proof is finished.

Assume now that we have shown that the coefficients

$$\begin{array}{llll} a_{10} & & & \\ a_{11} & a_{20} & & \\ a_{12} & a_{21} & a_{30} & \\ \vdots & & & \ddots \\ a_{1, k-1} & \cdots & & a_{k0} \end{array} \tag{7.26}$$

vanish; then the following limit exists and equals

$$\begin{aligned}\lim_{x \to 0} x^{\nu_0 - k} G(x, z) &= \lim x^{\nu_0 - k}(G_1 + \cdots + G_{k+1})(x, z) \\ &= a_{1,k} z^{-\nu_0} \lim \nabla \psi_{\nu_0} + \cdots + a_{k+1, 0} z^{-\nu_0} \lim \nabla^{k+1} \psi_{\nu_0} \\ &= a_{1,k} z^{-\nu_0 + 1}(1 + O(z^{-1})) \\ &\quad + \cdots + a_{k+1, 0} z^{-\nu_0 + k + 1}(1 + O(z^{-1})) \in W'^{*}, \\ &\quad 0 \leqslant k \leqslant \nu_0 - \nu_1 - 1,\end{aligned}$$

implying in that order

$$\alpha_{k+1,0}=\alpha_{k,1}=\alpha_{k-1,2}=\cdots=\alpha_{1,k}=0,$$

or else we would have an element of W^{t^*} of order k, $s_0<k<s_1$. So inductively we show that the whole triangle (7.26) vanishes for $0\leqslant k\leqslant \nu_0-\nu_1-1$; finally we set $k=\nu_0-\nu_1$ and so form the well-defined limit

$$\begin{aligned}
\lim x^{\nu_1}G(x,z)&\\
&=\lim x^{\nu_1}(G_1+\cdots+G_{\nu_0-\nu_1+1})(x,z)\\
&=a_{1,\nu_0-\nu_1}z^{-\nu_0}\lim \nabla\psi_{\nu_0}+\cdots+a_{\nu_0-\nu_1+1,0}z^{-\nu_0}\lim\nabla^{\nu_0-\nu_1+1}\Psi_{\nu_0}\\
&=a_{1,\nu_0-\nu_1}z^{-\nu_0+1}(1+O(z^{-1}))\\
&\quad+\cdots+a_{\nu_0-\nu_1+1,0}z^{-\nu_1+1}(1+O(z^{-1}))\in W^{t^*}
\end{aligned}\tag{7.27}$$

with non-zero leading term

$$a_{\nu_0-\nu_1+1,0}z^{-\nu_1+1}=a_{\nu_0-\nu_1+1,0}z^{s_1}$$

by (7.25) and so the remaining coefficients $a_{ij}(t^*)$ need not be zero. Instead of multiplying (7.27) by x^{ν_1} we could as well multiply the expression by

$$\frac{\tau(t^*+\bar{x})}{p_{\nu_1}(-\tilde{\partial})\,\tau(t^*+\bar{x})}=\frac{c_0^\nu}{c_{\nu_1}^\nu}x^{\nu_1}(1+O(x)).$$

So, one sees that

$$\beta_{1j}(t^*)=\frac{c_0^\nu}{c_{\nu_1}^\nu}a_{j,\nu_0-\nu_1-j+1}(t^*)$$

with $\beta_{1,\nu_0-\nu_1+1}$ having the promised value in the statement of the theorem, by Corollary 6.1.

In the next case one considers a linear combination of $\nabla^2\psi$, $\nabla\psi$, and ψ with ψ_{ν_0} and $\nabla^{s_1-s_0}\Psi_{\nu_0}=\nabla^{\nu_0-\nu_1+1}\psi_{\nu_0}$ removed:

$$\begin{aligned}
&(\nabla^2\psi+\alpha_{2,1}\nabla\psi+\alpha_{20}\psi)(t^*+\bar{x},z)\\
&\quad=\sum_{\substack{1\leqslant\mu\leqslant\nu_0+2\\ \mu\neq\nu_0-\nu_1+1}}\frac{\Gamma_{0,\nu_0-\nu_1+1,\mu}/\Gamma_{0,\nu_0-\nu_1+1}+O(x)}{x^{\nu_0-\mu+2}}z^{-\nu_0}\nabla^\mu\psi_{\nu_0}(t^*+\bar{x},z).
\end{aligned}$$

Taking into account that W^{t^*} has no terms of orders $z^{s_0+1}, z^{s_0+2}, \ldots, z^{s_1-1}$, $z^{s_1+1}, \ldots, z^{s_2-1}$ and that $\Gamma_{0,\nu_0-\nu_1+1,\nu_0-\nu_2+2}\neq 0$ (by Corollary 6.1), one proves that $\varphi_2(t^*,z)$ has the promised form (in general we remove $\nabla^{s_j-s_0}\psi_{\nu_0}$, $j=0,1,\ldots,r$, from $\nabla^j\psi_j$, $j=0,1,\ldots,r$) and so it goes, by induction, ending the proof of Theorem 7.5.

Appendix A: Schur Polynomials, Young Diagrams, and Differential Operators

The elementary Schur polynomials $p_n(t)$ are defined by

$$\begin{aligned} \exp \sum_{j=1}^{\infty} t_j z^j &= \sum_{n=0}^{\infty} p_n(t_1, t_2, \ldots)\, z^n \\ &= 1 + t_1 z + \left(\frac{1}{2} t_1^2 + t_2\right) z^2 + \left(\frac{1}{6} t_1^3 + t_1 t_2 + t_3\right) z^3 + \cdots \end{aligned} \tag{A.1}$$

with

$$p_n(t) = \frac{t_1^n}{n!} + \cdots + t_n \tag{A.2}$$

The polynomials $p_n(t)$ also appear in the context of symmetric polynomials; this is discussed in Appendix B. The p_n's yield differential operators

$$p_n(\tilde{\partial}) \equiv p_n\left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \ldots\right), \tag{A.3}$$

which arise naturally in the Taylor expansion

$$\begin{aligned} f(t \pm [s]) &\equiv f\left(t_1 \pm s, t_2 \pm \frac{s^2}{2}, t_3 \pm \frac{s^3}{3}, \ldots\right) \\ &= \sum_{n=0}^{\infty} p_n(\pm\tilde{\partial})\, f(t)\, s^n. \end{aligned} \tag{A.4}$$

For an arbitrary polynomial p, define the Hirota operation

$$p(\partial)\, f \circ g(t) \equiv p\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots\right) f(t+y)\, g(t-y)\,|_{y=0}. \tag{A.5}$$

The identity

$$\sum_{\substack{i+j=n+1\\ i,j \geqslant 0}} (p_i(\tilde{\partial}) f)(p_j(-\tilde{\partial})\, g) = p_{n+1}(\tilde{\partial})\, f \circ g \tag{A.6}$$

holds; it follows from expanding $f(t+[s])\, g(t-[s])$ in a Taylor series in two different ways: at first, by expanding each of the terms (using (A.4))

$$\begin{aligned} f(t+[s])\, g(t-[s]) &= \left(\sum_{i \geqslant 0} s^i p_i(\tilde{\partial})\, f(t)\right)\left(\sum_{j \geqslant 0} s^j p_j(-\tilde{\partial})\, g(t)\right) \\ &= \sum_{n \geqslant 0} s^n \left(\sum_{\substack{i+j=n\\ i,j \geqslant 0}} (p_i(\tilde{\partial}) f)(p_j(-\tilde{\partial})\, g(t)\right), \end{aligned}$$

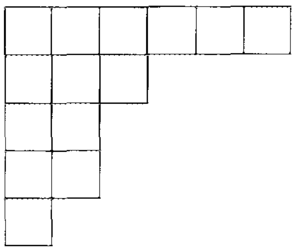
and at second, by expanding the product as a function of s:

$$f(t+[s])\,g(t-[s])=\sum_{n\geqslant 0} s^n p_n(\tilde{\partial})\,f\circ g \qquad \text{using (A.4) and (A.5).}$$

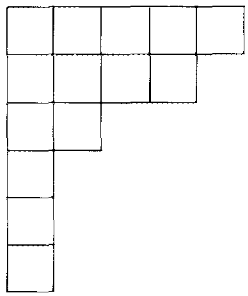
Let $\mathscr{S}=(s_0, s_1, \ldots)$ be a "sequence of virtual genus zero," i.e., a strictly increasing sequence of integers $s_0<s_1<s_2<s_3<\cdots$, such that $s_i=i$ for large enough i. We refer to it as *a sequence*. Such a sequence defines a partition

$$(\nu_0, \nu_1, \nu_2, \ldots, 0, 0, 0, \ldots), \qquad \nu_0\geqslant \nu_1\geqslant \nu_2\geqslant \cdots \geqslant 0 \tag{A.7}$$

of decreasing integers $\nu_i \equiv i-s_i$, all equal to 0 from a certain point on. A partition $(\nu_0, \nu_1, \ldots)$ defines a Young diagram; a dual or transpose Young diagram (obtained by flipping the previous one along its diagonal) defines a transpose partition $(\hat{\nu}_0, \hat{\nu}_1, \ldots)$ and a conjugate sequence $\hat{\mathscr{S}}$,



$\nu=(6, 3, 2, 2, 1)$



$\hat{\nu}=(5, 4, 2, 1, 1, 1)$

which is also a sequence of virtual genus zero. We also define the length of a sequence or partition

$$\begin{aligned} l(\mathscr{S})&=|\nu|=\sum_0^{\infty}\nu_i=\sum_0^{\infty}(i-s_i)\\ l(\hat{\mathscr{S}})&=|\hat{\nu}|=\sum \hat{\nu}_i=\sum (i-\hat{s}_i)=l(\mathscr{S}). \end{aligned} \tag{A.8}$$

To a partition $\nu=(\nu_0, \ldots, \nu_n)$, one associates a polynomial of homogeneous degree $|\nu|$, called the Schur polynomial F, constructed in terms of a matrix of size $n+1$,

$$F_{\nu_0\cdots\nu_n}(t)\equiv \det(p_{\nu_i-i+j}(-t))_{0\leqslant i,j\leqslant n}; \tag{A.9}$$

the p_k $(k\geqslant 0)$ are the elementary Schur polynomials, defined above, and $p_k=0$ for $k<0$, and $p_0=1$. Also (as follows from (B.4) and (B.9) in Appendix B);

$$F_{\hat{\nu}_0\cdots\hat{\nu}_n}(t)=(-1)^{|\nu|}F_{\nu_0\cdots\nu_n}(-t). \tag{A.10}$$

Finally, any $f(t) \in \mathbb{C}[[t_1, t_2, \ldots]]$ admits a Fourier expansion in terms of Schur polynomials (see (C.1) and (C.2))

$$f(t) = \sum c_\nu F_\nu(t) \tag{A.11}$$

with regard to the inner product (C.1) with $\langle F_\nu, F_\mu \rangle = \delta_{\lambda\mu}$, so that

$$c_\nu = F_\nu(-\tilde{\partial}) f|_{t=0} = \langle F_\nu, f \rangle.$$

Appendix B: The Geometry of Symmetric Polynomials

Given a set of finite (N) or infinite ($N = \infty$) variables $x = (x_1, x_2, x_3, \ldots)$, the symmetric functions

$$\begin{aligned} E(z) &= \prod_{i \geqslant 1} (1 + x_i z) = \sum_{r \geqslant 0} e_r(x)\, z^r \\ H(z) &= \prod_{i \geqslant 1} (1 - x_i z)^{-1} = \sum_{r \geqslant 0} h_r(x)\, z^r \\ P(z) &= \sum_{i \geqslant 1} (1 - x_i z)^{-1} = \sum_{r \geqslant 0} u_r(x)\, z^{-r} \end{aligned} \tag{B.1}$$

define symmetric polynomials

$$\begin{aligned} e_r(x) &= \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \\ h_r(x) &= \sum_{\sum i_j = r} x_1^{i_1} x_2^{i_2} \cdots \\ u_r(x) &= \sum_i x_i^r. \end{aligned} \tag{B.2}$$

Each set defines an algebraic basis for the ring of symmetric polynomials in the variables $x_1, x_2, \ldots$. Note that

$$\log H(z) = -\sum_i \log(1 - x_i z) = \sum_i \sum_{l \geqslant 1} \frac{z^l x_i^l}{l} = \sum_{l \geqslant 1} \frac{z^l}{l} u_l(x) \tag{B.3}$$

and thus, taking the exponential and using formula (A.15), we have

$$\sum_{r \geqslant 0} h_r(x) z^r = H(z) = \exp \sum_{l \geqslant 1} \frac{z^l u_l(x)}{l} = \sum_{r \geqslant 0} p_r(-t) z^r, \tag{B.4}$$

which implies

$$p_n(-t) = h_n(x), \tag{B.5}$$

upon setting

$$t_l = -\frac{u_l(x)}{l} \equiv -\frac{\sum_1^\infty x_i^l}{l}.$$

A basis for the symmetric polynomials viewed as a vector space is given by the Schur polynomials S_λ. Indeed given a partition

$$\lambda = (\lambda_1, ..., \lambda_l), \qquad \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l,$$

define

$$\begin{aligned} S_\lambda(x) &\equiv \det(h_{\lambda_i - i + j}(x))_{1 \leqslant i,j \leqslant l} \\ &= \det(p_{\lambda_i - i + j}(-t))_{1 \leqslant i,j \leqslant l}, \qquad \text{using (B.5)} \\ &= F_\lambda(t), \qquad \text{using (A.9).} \end{aligned} \tag{B.6}$$

Moreover, for two partitions λ and μ such that $\lambda \geqslant \mu$ ($\lambda_i \geqslant \mu_i$ for all i), define the skew-Schur polynomial

$$\begin{aligned} S_{\lambda\backslash\mu}(x) &\equiv \det(h_{\lambda_i - \mu_j - i + j}(x))_{1 \leqslant i,j \leqslant l} \\ &= \det(p_{\lambda_i - \mu_j - i + j}(-t))_{1 \leqslant i,j \leqslant l} \\ &\equiv F_{\lambda\backslash\mu}(t). \end{aligned} \tag{B.7}$$

Consider now the involution ω on the ring $R(h)$ of Schur polynomials, defined by

$$\omega(h_n) = e_n \qquad \text{and} \qquad \omega(e_n) = h_n \tag{B.8}$$

and so

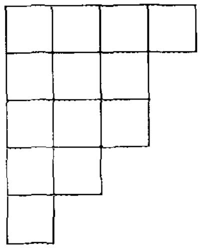
$$\begin{aligned} H(z) = \prod_i (1 - x_i z)^{-1} = \sum_0^\infty h_n(x)\, z^n \overset{\omega}{\curvearrowright} E(z) &= H^{-1}(-z) \\ = \prod_i (1 + x_i z) = \sum_0^\infty e_n(x)\, z^n. \end{aligned} \tag{B.9}$$

Using this involution (see [McD, p. 15]), we have

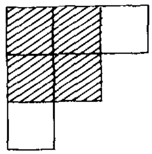
$$\det(h_{\lambda_i - i + j}(x))_{1 \leqslant i,j \leqslant l} = \det(e_{\hat{\lambda}_i - i + j}(x))_{1 \leqslant i,j \leqslant l}$$

from which (A.10) follows.

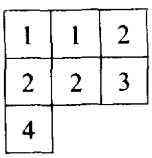
Both S_λ and $S_{\lambda\backslash\mu}$ have a combinatorial interpretation via the so-called Young diagrams associated to the partition. For example,

partition (4, 3, 3, 2, 1) $\leftrightarrow$ 

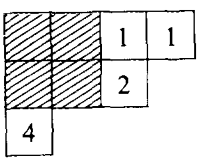
while, given two partitions λ and μ such that $\mu \subset \lambda$, the skew-partition $\lambda\backslash\mu$ has the skew-diagram $\lambda - \mu$ (viewed set-theoretically). For example,

(3, 2, 1)\(2, 2) 

Indeed a *tableau* of shape λ (or $\lambda\backslash\mu$) is an array a_{ij} of positive numbers ($\leqslant N$, if there are N variables) placed in the Young (skew) diagram which are non-decreasing in going to the right along the rows and increasing down the columns. Consider the associated monomials $\prod x_{a_{ij}}$ going with the tableau. For instance,



$\curvearrowright x_1^2 x_2^3 x_3 x_4$



$\curvearrowright x_1^2 x_2 x_4.$

Then we have the important identities

$$S_\lambda = \sum_{\{a_{ij}\} \text{ tableaux of } \lambda} \prod x_{a_{ij}}$$
$$S_{\lambda\backslash\mu} = \sum_{\{a_{ij}\} \text{ tableau of } \lambda\backslash\mu} \prod x_{a_{ij}}. \tag{B.10}$$

In particular, the symmetric polynomials e_j and h_j are themselves Schur polynomials (associated with dual Young diagrams):

$$e_j = S_{\underbrace{(1,1,...,1)}_{j}} \qquad \text{and} \qquad h_j = S_{(j)}. \tag{B.11}$$

As an application of (B.10), consider symmetric polynomials $S(x_0, x_1, ...)$ depending on an additional variable x_0. Then in terms of the t_l defined in (B.5), we have

$$\frac{1}{l}\sum_{i \geqslant 0} x_i^l = \frac{x_0^l}{l} - t_l \tag{B.12}$$

and

$$\begin{aligned} F_\lambda(-[x_0]+t) &= S_\lambda(x_0, x_1, x_2, ...), && \text{using (B.6)},\\ &= \sum_{j=0}^{\lambda_0} x_0^j S_{\lambda\backslash(j)}(x_1, x_2, ...) && \text{(see below)}\\ &= \sum_{j=0}^{\lambda_0} x_0^j F_{\lambda\backslash(j)}(t), && \text{using (B.7)}. \end{aligned} \tag{B.13}$$

The second equality follows from (B.10). Indeed, when x_0^j appears in the Schur polynomial, you will have 0's in the first j blocks in the first row and 1, 2, 3, ...'s in the remaining blocks of the tableau; i.e., for the Young diagram $\lambda\backslash(j)$ you will have an arbitrary tableau. The highest order term in x_0 has degree λ_0, because at most you can have 0's in the first row of the tableau.

$$\begin{aligned} F_\lambda([x_0]+t) &= (-1)^{|\lambda|} F_{\hat\lambda}(-[x_0]-t), && \text{using (A.10)},\\ &= (-1)^{|\lambda|}\sum_{j=0}^{\hat\lambda_0} x_0^j F_{\hat\lambda\backslash(j)}(-t), && \text{using (B.13) with } t - t,\\ &= (-1)^{|\lambda|}\sum_{j=0}^{\hat\lambda_0} x_0^j (-1)^{|\lambda|-j} F_{\lambda\backslash\underbrace{(1,...,1)}_{j}}(t), && \text{using (A.10)},\\ &= \sum_{j=0}^{\hat\lambda_0} (-x_0)^j F_{\lambda\backslash\underbrace{(1,...,1)}_{j}}(t), \end{aligned} \tag{B.14}$$

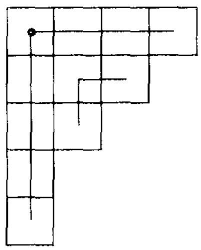
A *standard tableau* of λ is a tableau in which the entries 1, 2, 3 ,..., $n = |\lambda| \equiv \sum \lambda_i$ all appear precisely once. Then

$$\begin{aligned} f^\lambda &\equiv \{\#\text{ of standard tableaux of shape } \lambda\}\\ &= \text{coefficient of } x_1 x_2 \cdots x_n \text{ in } S_\lambda \end{aligned} \tag{B.15}$$

and similarly

$$f^{\lambda\backslash\mu} \equiv \{ \# \text{ of standard tableaux of skew-shape } \lambda\backslash\mu \}$$
$$= \text{coefficient of } x_1 x_2 \cdots x_n \text{ in } S_{\lambda\backslash\mu}, \text{ with } n = |\lambda\backslash\mu| = |\lambda| - |\mu|. \tag{B.16}$$

Given the (i, j)th box in a Young diagram, define the *hook length* $h^\lambda(i, j)$ as the length of the hook formed by drawing a horizontal line emanating from the center of the box to the right and a vertical line emanating from the center of the box to the bottom of the diagram, i.e.,



$$h^\lambda(i, j) = (\lambda_i - (j-1)) + (\hat{\lambda}_j - (i-1)) - 1$$
$$= \lambda_i + \hat{\lambda}_j - i - j + 1. \tag{B.17}$$

We now have the identities.

$$\text{(i)} \quad f^\lambda = \frac{|\lambda|!}{\prod_{\text{all } i,j} h^\lambda_{(i,j)}} = |\lambda|! \det \left[\frac{1}{(\lambda_i - i + j)!} \right] \tag{B.18}$$

and

$$\text{(ii)} \quad f^{\lambda\backslash\mu} = (|\lambda| - |\mu|)! \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]. \tag{B.19}$$

The first equality in (i) follows from [McD, p. 9, Ex. 1], whereas the second equality in (i) and (ii) can be shown as follows:

Consider the same ring homomorphism as in section 6,

$$\theta: \{\text{Ring of symmetric function}\} \to \mathbb{C}[u_1]$$

such that

$$\theta(u_1) = u_1 \qquad \text{and} \qquad \theta(u_i) = 0, \qquad i > 1.$$

Consider now an arbitrary symmetric polynomial f of degree n; it must be expressible as

$$\begin{aligned} f(x) &= cu_1^n + \sum_{\left(\substack{\sum is_i = n \\ s_1 \neq n}\right)} c_s \prod_i u_i^{s_i} \\ &= cn!\, x_1 x_2 \cdots x_n + \text{other terms.} \end{aligned}$$

Therefore

$$\theta(f) = \frac{u_1^n}{n!}\,(\text{coefficient of } x_1 x_2 \cdots x_n) \tag{B.20}$$

and thus, in particular,

$$\theta(e_n) = \theta(h_n) = \frac{u_1^n}{n!}. \tag{B.21}$$

Applying (B.20) to the Schur polynomial $f \equiv S_\lambda(x)$ and using (B.15), we find

$$\begin{aligned} f^\lambda \frac{u_1^{|\lambda|}}{|\lambda|!} &= \theta(S_\lambda) \\ &= \theta(\det[h_{\lambda_i - i + j}]), && \text{using the definition of } S_\lambda \\ &= \det[\theta(h_{\lambda_i - i + j})], && \text{since } \theta \text{ is a homomorphism} \\ &= \det\left[\frac{u_1^{\lambda_i - i + j}}{(\lambda_i - i + j)!}\right], && \text{using (B.21)} \\ &= u_1^{|\lambda|} \det\left[\frac{1}{(\lambda_i - i + j)!}\right] && \text{by homogeneity} \end{aligned} \tag{B.22}$$

and, similarly

$$\begin{aligned} f^{\lambda\setminus\mu} \frac{u_1^{|\lambda| - |\mu|}}{(|\lambda| - |\mu|)!} &= \theta(S_{\lambda\setminus\mu}) \\ &= \theta \det[h_{\lambda_i - \mu_j - i + j}] \\ &= \det[\theta(h_{\lambda_i} - \mu_j - i + j)] \\ &= \det\left[\frac{u_1^{\lambda_i - \mu_j - i + j}}{(\lambda_i - \mu_j - i + j)!}\right. \\ &= u_1^{|\lambda| - |\mu|} \det\left[\frac{1}{(\lambda_i - \mu_j - i + j)!}\right]. \end{aligned} \tag{B.23}$$

APPENDIX C: VANDERMONDE DETERMINANTS AND CAUCHY IDENTITIES

First, define the symmetric inner product [McD, p. 34] between symmetric functions of x, by expressing them in terms of the symmetric functions $u_r(x) = \sum_i x_i^r$ and computing

$$\langle f, g \rangle = \langle f(u), g(u) \rangle = f\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots\right) g(u) \bigg|_{u=0} \tag{C.1}$$

which has the Schur polynomials as an orthonormal basis

$$\langle S_\lambda, S_\mu \rangle = \delta_{\lambda\mu} \tag{C.2}$$

and which satisfies [McD, p. 43, Ex. 3] for any symmetric function φ

$$\langle S_\lambda, S_\mu \varphi \rangle = \langle S_{\lambda \setminus \mu}, \varphi \rangle. \tag{C.3}$$

In particular

$$\begin{aligned} \langle u_1^m, \varphi \rangle &= \left(\frac{\partial}{\partial u_1}\right)^m \varphi \bigg|_{u=0} \\ &= m! \cdot \text{coefficient of } u_1^m \text{ in } \varphi \\ &= \text{coefficient of } x_1 x_2 \cdots x_m \text{ in } \varphi \end{aligned}$$

and so by (B.16) and (C.3) conclude that

$$\langle u_1^{(|\lambda| - |\mu|)}, S_{\lambda \setminus \mu} \rangle = f^{\lambda \setminus \mu}. \tag{C.4}$$

Furthermore, as the Schur polynomials form a basis for the vector space of symmetric functions, we have

$$\varphi = \sum_\lambda \langle \varphi, S_\lambda \rangle S_\lambda$$

and setting $\varphi = u_1^k$ and applying (C.4) with $\mu = 0$, conclude that

$$u_1^k = \sum_{|\lambda| = k} f^\lambda S_\lambda. \tag{C.5}$$

We also introduce the generalized Vandermonde determinants

$$a_\lambda(y) = \sum_{w \in \mathcal{S}_n} \operatorname{sgn}(w) \prod_{i=1}^n y_{w(i)}^{\lambda_i} = \det(y_i^{\lambda_j})|_{1 \leq i, j \leq n}, \tag{C.6}$$

with $y=(y_1, \ldots, y_n)$, and $\mathscr{S}_n$ the permutation group on n letters. Setting $\delta=(n-1, n-2, \ldots, 1, 0)$

$$a_\delta(y)=\prod_{1 \leqslant i<j \leqslant n}(y_i-y_j)=\text{Vandermonde determinant}$$

and clearly $a_\delta(y)$ divides $a_\lambda(y)$, with

$$S_\lambda(y)=a_{\lambda+\delta}/a_\delta \qquad ((\lambda+\delta)_i=\lambda_i+\delta_i). \tag{C.7}$$

As an application of these ideas, consider a differential operator in y, depending on the parameter x:

$$L_x=\frac{1}{a_\delta}\prod_1^n\left(x-y_i\frac{\partial}{\partial y_i}\right)a_\delta$$

and the polynomial

$$q_\lambda(x)\equiv\prod_{i=1}^n(x-(\lambda_i+n-i)). \tag{C.8}$$

We show the $S_\lambda(y)$ form an orthonormal set of eigenvectors of L_x with eigenvalues $q_\lambda(x)$. Indeed, using (C.6) with $\lambda+\delta=(\lambda_1+n-1, \ldots, \lambda_i+n-i, \ldots)$

$$\begin{aligned}
L_x S_\lambda(y) &= L_x\left(\frac{a_{\lambda+\delta}(y)}{a_\delta(y)}\right)\\
&=\frac{1}{a_\delta}\prod_{k:1}^n\left(x-y_k\frac{\partial}{\partial y_k}\right)a_{\lambda+\delta}(y)\\
&=\frac{1}{a_\delta}\prod_{k:1}^n\left(x-y_k\frac{\partial}{\partial y_k}\right)\sum_{w\in\mathscr{S}_n}\operatorname{sgn}(w)\prod_{i=1}^n y_{w(i)}^{(\lambda_i+n-i)}\\
&=\frac{1}{a_\delta}\sum_{w\in\mathscr{S}_n}\operatorname{sgn}(w)\prod_{i=1}^n\left(x-y_{w(i)}\frac{\partial}{\partial y_{w(i)}}\right)y_{w(i)}^{(\lambda_i+n-i)}\\
&=\frac{1}{a_\delta}q_\lambda(x)\sum_{w\in\mathscr{S}_n}\operatorname{sgn}(w)\prod_{i=1}^n y_{w(i)}^{\lambda_i+n-i}\\
&=q_\lambda(x)\frac{a_{\lambda+\delta}(y)}{a_\delta(y)}=q_\lambda(x)\,S_\lambda(y),
\end{aligned}$$

yielding (C.8).

Now there is a classical result of Cauchy,

$$\sum_\lambda S_\lambda(y_1, \ldots, y_n)\,S_\lambda(w_1, \ldots, w_m)=\prod_{j=1}^n\prod_{k=1}^m(1-y_jw_k)^{-1},$$

and in Andrews, Goulden, and Jackson [AGJ], this result is generalized in a straightforward way. A special case of their generalization of Cauchy's identity (namely Corollary 4.5 in [AGJ]) reads

$$L_x u_1^n = \sum_{j=0}^{n} (-1)^j (x)_{n-j} (n)_j u_1^{n-j} e_j. \tag{C.9}$$

Acknowledgments

We thank D. Bennequin, E. Date, J. Fay, M. Gromov, L. Haine, H. Knörrer, G. Lusztig, P. Malliavin, C. L. Terng, R. Palais, P. Vanhaecke, and G. Wilson and especially I. Gessel and T. Shiota for helpful conversations. Lively audiences at Brandeis, IHES, le Collège de France, Luminy, and Oberwolfach have also contributed positively towards the understanding of the subject. We also thank Suzanne D'Addato-Muës for a superb typing performance!

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Printed in Belgium
Uitgever: Academic Press, Inc.
Verantwoordelijke uitgever voor België:
Hubert Van Maele
Altenastraat 20, B-8310 Sint-Kruis