Chapter 5

Spectral statistics of orthogonal and symplectic ensembles

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Abstract

We provide a direct approach to the computing of scalar and matrix kernels, respectively for the unitary ensembles on the one hand and the orthogonal and symplectic ensembles on the other hand. This leads to correlation functions and gap probabilities. In the classical cases (Hermite, Laguerre and Jacobi) we express the matrix kernels for the orthogonal and symplectic ensemble in terms of the scalar kernel for the unitary case, using the relation between the classical orthogonal polynomials going with the unitary ensembles and the skew-orthogonal polynomials going with the orthogonal and symplectic ensembles.

5.1 Introduction

In Chapter 4, it was shown that given a probability distribution function (pdf) of the form

$$P_N^{(\beta)}(x_1,\ldots,x_N) = Z_N^{-1}|\Delta_N(x_1,\ldots,x_N)|^\beta \prod_{j=1}^N w(x_j), \quad \beta = 2, \tag{5.1.1}$$

for $(x_1,\ldots,x_N)=\mathbb{R}^N$, Δ_N the Vandermonde determinant, $w(x)$ a non-negative weight function on $\mathbb{R}$, then it can be rewritten

$$\begin{aligned} P_N^{(2)}(x_1,\ldots,x_N) &= \frac{1}{N!}\det[K_N^{(2)}(x_i,x_j)]_{i\le i,j\le N} \\ K_N^{(2)}(x,y) &= \sum_{i=0}^{N-1}\varphi_i(x)\varphi_i(y),\quad \varphi_i(x)=(w(x))^{\frac{1}{2}}p_i(x), \end{aligned} \tag{5.1.2}$$

with $p_i(x)$ being an i-th degree polynomial, orthonormal with respect to the $\mathbb{R}$-weight $w(x)$, i.e.,

$$\int_{\mathbb{R}} p_i(x)p_j(x)w(x)\,dx=\int_{\mathbb{R}}\varphi_i(x)\varphi_j(x)\,dx=\delta_{ij}. \tag{5.1.3}$$

It was mentioned in Chapter 4, that this example included many interesting examples in random matrix theory and in particular N-by-N random Hermitian matrices, and the set of random matrices handled by a pdf of the form (5.1.1) are known as the unitary ensembles. It was also shown that as a consequence of the reproducing property of the kernels, the n-point correlation function

$$\begin{aligned} R_n(x_1,\ldots,x_n) &:= \frac{N!}{(N-n)!}\int\cdots\int P_N^{(2)}(x_1,\ldots,x_N)\,dx_{n+1}\ldots dx_N \\ &= \det[K_N^{(2)}(x_i,x_j)]_{1\le i,j\le n}, \end{aligned} \tag{5.1.4}$$

which roughly speaking is the probability density that n of the eigenvalues, irrespective of order, lie in infinitesimal neighborhoods of $x_1,\ldots,x_n$. (Since it integrates out to $N!/(N-n)!$, it is not a probability density.)

Now due to the Weyl integration formula [Hel62], [Hel84], when considering the case of conjugation invariant pdf on the ensembles of real symmetric or self-dual Hermitian quaternionic[1] matrices, viewed as the tangent space at the identity of the associated symmetric spaces, one finds formula (5.1.1) with $\beta=1$ or 4, respectively, for the pdf.

In particular they also come up in the so-called chiral models in the physics literature, in which case the weight $w(x)$ contains the factor x^a, see for example [Sen98], [Ake05].

5.2 Direct approach to the kernel

In this section, we give a general method which generalizes the results of Chapter 4 and works for all three cases $\beta=1,2,4$; following the approach of Tracy-Widom [Tra98], [Wid99]. We now state:

[1] The N-by-N matrices H with quaternionic elements q_{ij} are realized: $H=[q_{ij}]_{1\le i,j\le N}$, $q_{ij}\mapsto\begin{bmatrix} z_{ij} & w_{ij} \\ -\bar{w}_{ij} & \bar{z}_{ij}\end{bmatrix}$, $q_{ji}\mapsto\begin{bmatrix} \bar{z}_{ij} & -w_{ij} \\ \bar{w}_{ij} & z_{ij}\end{bmatrix}$, $i\le j$ in which case the eigenvalues are doubly degenerate [Meh04].

Theorem 5.2.1 *Consider the pdf of (5.1.1) for the cases $\beta = 1, 2, 4$. Then we have for the expectation*

$$\begin{aligned}\mathbf{E}\left(\prod_{j=1}^{N}(1+f(x_j))\right) &= \int\ldots\int P_N^{(\beta)}(x_1,\ldots,x_N)\prod_{j=1}^{N}(1+f(x_j))\,dx_j \\ &= \begin{cases}\det(I+K_N^{(\beta)}f) & \beta=2,\\ (\det(I+K_N^{(\beta)}f))^{\frac12} & \beta=1,4,\end{cases}\end{aligned} \tag{5.2.1}$$

where $K_N^{(\beta)}$ is for $\beta = 2$ an operator on $L^2(\mathbb{R})$ with kernel $K_N^{(2)}(x,y)$ and f is the operator, multiplication by f, while for $\beta = 1, 4$, $K_N^{(\beta)}$ is a matrix kernel on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. The kernels are specified below:

$$K_N^{(2)}(x,y) = S_N^{(2)}(x,y), \qquad \underline{\beta=2} \tag{5.2.2}$$

$$K_N^{(\beta)}(x,y) = \begin{pmatrix} S_N^{(\beta)}(x,y) & S_N^{(\beta)}D(x,y)\\ IS_N^{(\beta)}(x,y)-\delta_{\beta,1}\epsilon(x-y) & S_N^{(\beta)}(y,x)\end{pmatrix}, \qquad \underline{\beta=1,4} \tag{5.2.3}$$

with

$$S_N^{(2)}(x,y) = \sum_{i,j=0}^{N-1}\varphi_i(x)\mu_{ij}^{(2)}\varphi_j(y), \tag{5.2.4}$$

$\varphi_i(x) = (w(x))^{1/2}p_i(x)$, the $p_i(x)$ any polynomials of degree i, with the symmetric matrix $\mu^{(2)}$ given by

$$[(\mu^{(2)})^{-1}]_{ij} = \int_{\mathbb{R}}\varphi_i(x)\varphi_j(x)\,dx =: \langle\varphi_i,\varphi_j\rangle_{(2)}, \tag{5.2.5}$$

and (for N even)

$$S_N^{(1)}(x,y) = -\sum_{i,j=0}^{N-1}\varphi_i(x)\mu_{ij}^{(1)}\epsilon\varphi_j(y), \tag{5.2.6}$$

$$IS_N^{(1)}(x,y) = -\sum_{i,j=0}^{N-1}\epsilon\varphi_i(x)\mu_{ij}^{(1)}\epsilon\varphi_j(y) = \epsilon S_N^{(1)}(x,y),$$

$$S_N^{(1)}D(x,y) = \sum_{i,j=0}^{N-1}\varphi_i(x)\mu_{ij}^{(1)}\varphi_j(y) = -\frac{\partial}{\partial y}S_N^{(1)}(x,y),$$

with $\varphi_i(x) = w(x)p_i(x)$, the $p_i(x)$ being any polynomials of degree i, $\epsilon(x) = \frac12 sgn(x)$, ϵ = the integral operator with kernel $\epsilon(x-y)$ and the skew-symmetric matrix $\mu^{(1)}$ is given by

$$[(\mu^{(1)})^{-1}]_{ij} = \iint\epsilon(x-y)\varphi_i(x)\varphi_j(y)\,dxdy =: \langle\varphi_i\varphi_j\rangle_{(1)}, \tag{5.2.7}$$

and finally ($f'(x) = \frac{d}{dx} f(x)$)

$$2S_N^{(4)}(x,y) = \sum_{i,j=0}^{2N-1} \varphi_i'(x)\mu_i^{(4)}\varphi_j(y), \tag{5.2.8}$$

$$2IS_N^{(4)}(x,y) = \sum_{i,j=0}^{2N-1} \varphi_i(x)\mu_{i,j}^{(4)}\varphi_j(y) = \int_y^x S_N^{(4)}(v,y)\,dv,$$

$$2S_N^{(4)}D(x,y) = -\sum_{i,j=0}^{2N-1} \varphi_i'(x)\mu_{i,j}^{(4)}\varphi_j'(y) = -\frac{\partial}{\partial y} S_N^{(4)}(x,y),$$

with $\varphi_i(x) = (w(x))^{1/2} p_i(x)$, *where the* $p_i(x)$ *are arbitrary polynomials of degree* i *and* $\mu^{(4)}$ *is the skew-symmetric matrix given by*

$$[(\mu^{(4)})^{-1}] = \frac{1}{2}\int (\varphi_i(x)\varphi_j'(x) - \varphi_i'(x)\varphi_j(x))\,dx =: \langle \varphi_i \varphi_j \rangle_{(4)}. \tag{5.2.9}$$

Remark 1. Note by the definition of $\mu^{(2)}$ (5.2.5), we have the following reproducing property:

$$\langle S_N^{(2)}(x,\cdot), \varphi_k \rangle \equiv \varphi_k(x), \qquad 0 \le k \le N-1,$$

which, given its degree modulo $(w(x)w(y))^{1/2}$, uniquely characterizes $S_N^{(2)}(x,y)$ as the Christelhoff-Darboux kernel of $\langle\ ,\rangle_2$, i.e.,

$$S_N^{(2)}(x,y) = \sum_{i=0}^{N-1} \varphi_i(x)\varphi_j(y), \qquad \langle \varphi_i, \varphi_j \rangle_{(2)} = \delta_{ij}.$$

In other words, the kernel is insensitive to the choice of $\mu^{(2)}$, so we may as well take $\mu^{(2)} = I_N$.

Similarly, we have for $\beta = 1$, the reproducing property

$$\langle S_N^{(1)}D(x,\cdot), \varphi_k \rangle_{(1)} = \varphi_k(x), \qquad 0 \le k \le N-1,$$

which now forces $S_N^{(1)}D(x,y)$ to be the Christelhoff-Darboux kernel for the skew-symmetric inner product $\langle\ ,\ \rangle_{(1)}$ namely[2]

$$S_N^{(1)}D(x,y) = \sum_{i=0}^{N-1} (\varphi_{2i}(y)\varphi_{2i+1}(x) - \varphi_{2i}(x)\varphi_{2i+1}(y)), \quad \langle \varphi_i, \varphi_j \rangle_{(1)} = J_{ij},$$

[2] $J = J_N = I_N \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ = the $2N$-by-N symplectic matrix, while the φ_{2i+1} is well defined module $\varphi_{2i+1} \mapsto \frac{\varphi_{2i+1} + \gamma_i \varphi_{2i}}{\delta_i}$, $\varphi_{2i} \mapsto \delta_i \varphi_{2i}$.

and again the kernel is insensitive to the choice of $\mu^{(1)}$, so we may as well set $\mu^{(1)} = J_N$.

Finally, we also have for $\beta = 4$ the reproducing property

$$\langle IS_N^{(4)}(x,\cdot), \varphi_k\rangle_{(4)} = \varphi_k(x), \quad 0 \le k \le 2N-1,$$

which as before forces $IS_N^{(4)}(x,y)$ to be the Christelhoff-Darboux kernel for the skew-symmetric inner product $\langle\ ,\ \rangle_{(4)}$, i.e.,

$$IS_N^{(4)}(x,y) = \sum_{i=0}^{2N-1} (\varphi_{2i}(y)\varphi_{2i+1}(x) - \varphi_{2i}(x)\varphi_{2i+1}(y)), \quad \langle\varphi_i, \varphi_j\rangle_{(4)} = J_{ij}.$$

Thus in all three cases, $\beta = 1, 2, 4$, the Christelhoff-Darboux kernel of the inner product matrix $\mu^{(\beta)}$, completely determines the kernel $K_N^{(\beta)}(x,y)$, and in fact for the "classical" cases, it is easily determined. In particular the $K_N^{(\beta)}(x,y)$ are insensitive to the choice of the polynomials $p_i(x)$ and for the classical cases we shall see they are all closely related to the $\beta = 2$ kernel.

We shall now sketch a proof of this theorem, following [Tra98], first providing a necessary lemma, of de Bruijn, found in [deB55].

Lemma 5.2.1 *We have the following three identities involving N-fold integrals with determinental entries*

$$\begin{aligned}\int\cdots\int \quad & \det[\varphi_i(x_j)]_{1\le i,j\le N} \det[\psi_i(x_j)]_{1\le i,j\le N}\, dx_1 \ldots dx_N \\ & = N! \det[\textstyle\int \varphi_i(x)\varphi_j(x)\, dx]_{1\le i,j\le N}\end{aligned}$$

$$\begin{aligned}\underset{x_1\le x_2\le\ldots\le x_N}{\int\cdots\int} \quad & \det[\varphi_i(x_j)]_{1\le i,j\le N}\, dx_1 \ldots dx_N \\ & = Pf[\textstyle\int\int \operatorname{sgn}(y-x)\varphi_i(x)\varphi_j(y)\, dydx]_{1\le i,j\le N}\end{aligned}$$

$$\begin{aligned}\int\cdots\int \quad & \det[(\varphi_i(x_j), \psi_i(x_j))]_{\substack{1\le i\le 2N\\ 1\le j\le N}}\, dx_1 \ldots dx_N \\ & = (2N)! Pf[\textstyle\int (\varphi_i(x)\psi_j(x) - \varphi_j(x)\psi_i(x))\, dx]_{i\le i,j\le 2N}\end{aligned}$$

where $PfA = (\det A)^{1/2}$, for A a skew-symmetric matrix, and the second identity requires N to be even.

Sketch of Proof of Theorem 5.2.1: We first do the $\beta = 2$ case, the other cases being technically more complicated but conceptually no different. From

(5.1.1) and Lemma 5.2.1,

$$\begin{aligned}
&\mathbf{E}\left(\prod_{j=1}^{N}(1+f(x_j))\right)\\
&\quad=\int\cdots\int \det[x_j^{i-1}]_{i\leq i,j\leq N}\det[x_j^{i-1}w(x_j)(1+f(x_j))]_{1\leq i,j\leq N}\,dx_1\ldots dx_N\\
&\quad=\det\left[\int x^{i+j}w(x)(1+f(x))\,dx\right]_{0\leq i,j\leq N-1}
\end{aligned}$$

(after replacing $x^i(w(x))^{1/2}\mapsto\varphi_i(x)=p_i(x)(w(x))^{1/2}$, the $p_i(x)$ being arbitrary polynomials of degree i)

$$\begin{aligned}
&= C_N\det[\textstyle\int\varphi_i(x)\varphi_j(x)(1+f(x))\,dx]_{0\leq i,j\leq N-1}\\
&= C_N'\det\left[\delta_{ij}+\int\sum_{k=0}^{N-1}\mu_{ik}^{(2)}\varphi_k(x)\varphi_j(x)f(x)\,dx\right]_{0\leq i,j\leq N-1}\\
&\qquad (C_N'=1,\ \text{since the L.H.S.} = 1 \text{ when } f=0)\\
&= \det(I+K_N^{(2)}f).
\end{aligned}$$

In the last step, we applied the fundamental identity $\det(I+AB)=\det(I+BA)$ for arbitrary Hilbert-Schmidt operators, true as long as the products make sense. Indeed, set $A: L^2(\mathbb{R})\mapsto\mathbb{R}^N$, $B:\mathbb{R}^N\mapsto L^2(\mathbb{R})$, with

$$A(i,x)=\sum_{k=0}^{N-1}\mu_{ik}^{(2)}\varphi_k(x),\qquad B(x,j)=\varphi_j(x)f(x)$$

i.e.,

$$Ah(x)=\left(\int\sum_{k=0}^{N-1}\mu_{ik}^{(2)}\varphi_k(x)h(x)\,dx\right)_{i=0}^{N-1},\quad B(v)=\sum_{j=0}^{N-1}v_j\varphi_j(x)f(x)$$

so

$$AB(i,j)=\int\sum_{k=0}^{N-1}\mu_{ik}^{(2)}\varphi_k(x)\varphi_j(x)\,dx$$

$$BA(x,y)=\sum_{i,j=0}^{N-1}\varphi_i(x)\mu_{ij}^{(2)}\varphi_j(y)=K_N^{(2)}(x,y),$$

yielding the $\beta=2$ case.

Now consider the $\beta=4$ case, and observe the crucial identity

$$\Delta_N^4(x)=\det[(x_j^i,(x_j^i)')]_{\substack{0\leq i\leq 2N-1\\ 1\leq j\leq N}}$$

(a consequence of L'Hôpital's rule), but replacing in the above $x^i \mapsto p_i(x)$, the $p_i(x)$ being arbitrary polynomials of degree i, in which case $\Delta_N(x) \mapsto$ constant $\Delta_N(x)$. Then find using (5.1.1) for $\beta = 4$ and Lemma 5.2.1, upon setting $\varphi_i(x) = (w(x))^{1/2} p_i(x)$, that

$$
\begin{aligned}
&\left(E\left(\prod_{i=1}^{N}(1+f(x_i))\right)\right)^2 \\
&\qquad = C_N \det\left[\int \frac{1}{2}(\varphi_i(x)\varphi_j'(x) - \varphi_i'(x)\varphi_j(x))(1+f(x))\,dx\right]_{0\le i,j\le 2N-1} \\
&\qquad = C_N' \det\left[\delta_{ij} + \int (\tilde\varphi_i(x)\varphi_j'(x) - \tilde\varphi_i'(x)\varphi_j(x))\frac{f(x)}{2}\,dx\right]_{0\le i,j\le 2N-1} \\
&\qquad \left(\tilde\varphi_i(x) = \sum_{k=0}^{2N-1}\mu_{ik}^{(4)}\varphi_k(x) \quad C_N' = 1 \text{ by setting } f=0\right) \\
&\qquad = \det(I + K_N^{(4)} f),
\end{aligned}
$$

once again using $\det(I+AB) = \det(I+BA)$. Indeed, set

$$A : L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \mapsto \mathbb{R}^{2N}, \quad B : \mathbb{R}^{2N} \mapsto L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$$

with

$$A(i,x) = \frac{f(x)}{2}(\tilde\varphi_i(x), -\tilde\varphi_i'(x)), \quad B(x,i) = \begin{pmatrix}\varphi_i'(x)\\ \varphi_i(x)\end{pmatrix}, \quad 0 \le i \le 2N-1,$$

and so

$$A(h_1,h_2) = \frac{1}{2}\left(\int f(x)\tilde\varphi_i(x)h_1(x)\,dx, \quad -\int f(x)\tilde\varphi_i'(x)h_2(x)\,dx\right)_{i=0}^{2N-1},$$

$$B(v_0,\dots,v_{2N-1})^T = \sum_{i=0}^{2N-1} v_i\begin{pmatrix}\varphi_i'(x)\\ \varphi_i(x)\end{pmatrix},$$

hence

$$
\begin{aligned}
AB(i,j) &= \int \frac{f(x)}{2}(\tilde\varphi_i(x), -\tilde\varphi_i'(x))\begin{pmatrix}\varphi_j'(x)\\ \varphi_j(x)\end{pmatrix}dx \\
&= \int \frac{f(x)}{2}(\tilde\varphi_i(x)\varphi_j'(x) - \tilde\varphi_i'(x)\varphi_j(x))\,dx,
\end{aligned}
$$

while

$$
\begin{aligned}
BA(x,y) &= \sum_{i=0}^{2N-1} B(x,i)A(i,y) \\
&= \frac{1}{2}\begin{pmatrix} \sum_{i=0}^{2N-1}\varphi_i'(x)\tilde\varphi_i(y), & -\sum_{i=0}^{2N-1}\varphi_i'(x)\tilde\varphi_i'(y) \\ \sum_{i=0}^{2N-1}\varphi_i(x)\tilde\varphi_i(y), & -\sum_{i=0}^{2N-1}\varphi_i(x)\tilde\varphi_i'(y)\end{pmatrix} f(y),
\end{aligned}
$$

yielding the $\beta = 4$ case.

Finally consider the $\beta = 1$ case, with N even, and so from (5.1.1) and Lemma 5.2.1, find

$$\left(\mathbf{E}\left(\prod_{j=1}^{N}(1+f(x_j))\right)\right)^2$$

$$= (N!)^2 \int_{x_1\le\ldots\le x_N}\cdots\int \prod_{i<j}(x_j - x_i)\prod_{j=1}^{N}(w(x_j)(1+f(x_j))\; dx_1\ldots dx_N$$

$$= C_N \int\cdots\int \det[p_{i-1}(x_j)w(x_j)(1+f(x_j)]_{1\le i,j\le N}\; dx_1\ldots dx_N$$

$$= C_N' \det\left[\iint \epsilon(x-y)\varphi_i(x)\varphi_j(y)(1+f(x))(1+f(y))\; dxdy\right]_{0\le i,j\le N-1}$$

(setting $\varphi_i(x) = w(x)p_i(x)$, $p_i(x)$ an arbitrary polynomial of degree i, $\tilde\varphi_i(x) = \sum_{j=0}^{N-1}\mu_{ij}^{(1)}\varphi_j(x)$)

$$= \det\left[\delta_{ij} + \int (f\tilde\varphi_i\epsilon\varphi_j - f\varphi_j\epsilon\tilde\varphi_i - f\varphi_j\epsilon(f\tilde\varphi_i))\; dx\right]_{0\le i,j\le N-1}$$

(remember $\epsilon f = \int\epsilon(x-y)f(y)\; dy = \frac{1}{2}\int \mathrm{sgn}(x-y)f(y)\; dy$)

$$= \det(I + K_N^{(1)}f),$$

once again using $\det(I+AB) = \det(I+BA)$. Indeed set

$$A : L^2(\mathbb{R})\oplus L^2(\mathbb{R}) \mapsto \mathbb{R}^N, \quad B : \mathbb{R}^N \mapsto L^2(\mathbb{R})\oplus L^2(\mathbb{R})$$

with

$$A(i,x) = f(-\epsilon\tilde\varphi_i - \epsilon(f\tilde\varphi_i), \tilde\varphi_i), \quad B(x,i) = \begin{pmatrix}\varphi_i\\ \epsilon\varphi_i\end{pmatrix}$$

and so

$$\begin{aligned}
\det(I+BA) &= \det\left(I + \sum_{i=0}^{N-1} B(x,i)A(i,y)\right)\\
&= \det\begin{pmatrix} I-\sum\varphi_i\otimes(f\epsilon\tilde\varphi_i + f\epsilon(f\tilde\varphi_i)), & \sum\varphi_i\otimes f\tilde\varphi_i\\ -\sum\epsilon\varphi_i\otimes(f\epsilon\tilde\varphi_i+f\epsilon(f\tilde\varphi_i)), & I+\sum\epsilon\varphi_i\otimes f\tilde\varphi_i\end{pmatrix}\\
&= \det\left(\begin{pmatrix} I-\sum\varphi_i\otimes f\epsilon\tilde\varphi_i, & \sum\varphi_i\otimes f\tilde\varphi_i\\ -\sum\epsilon\varphi_i\otimes f\epsilon\tilde\varphi_i - \epsilon f, & I+\sum\epsilon\varphi_i\otimes f\tilde\varphi_i\end{pmatrix}\begin{pmatrix} I & 0\\ \epsilon f & I\end{pmatrix}\right)\\
&= \det\begin{pmatrix} I-\sum\varphi_i\otimes f\epsilon\tilde\varphi_i, & \sum\varphi_i\otimes f\tilde\varphi_i\\ -\sum\epsilon\varphi_i\otimes f\epsilon\tilde\varphi_i - \epsilon f, & I+\sum\epsilon\varphi_i\otimes f\tilde\varphi_i\end{pmatrix}
\end{aligned}$$

(using $\epsilon^T = -\epsilon$, $\det XY = \det X \cdot \det Y$, $\det\left(\begin{smallmatrix} I & 0 \\ \epsilon f & I \end{smallmatrix}\right) = 1$)

$$\begin{aligned} &= \det\left(I + \begin{pmatrix} -\sum \varphi_i \otimes \epsilon\tilde{\varphi}_i, & \sum \varphi_i \otimes \tilde{\varphi}_i \\ -\sum \epsilon\varphi_i \otimes \epsilon\tilde{\varphi}_i - \epsilon, & \sum \epsilon\varphi_i \otimes \tilde{\varphi}_i \end{pmatrix} f\right) \\ &= \det(I + K_N^{(1)} f), \end{aligned}$$

concluding the case $\beta = 1$ and the proof of Theorem 5.1.1.

Remark 2. The above methods also work for the circular ensembles, see [Tra98].

Remark 3. For $\beta = 2$, we have shown

$$\mathbf{E}\left(\prod_{j=1}^{N}(1 + f(x_j))\right) = \det(I + K_N^{(2)} f). \tag{5.2.10}$$

Setting $f(x) = \sum_{r=1}^{n} z_r \delta(x - y_r)$, we find

$$\det(I + K_N^{(2)} f) = \det[\delta_{ij} + K_N^{(2)}(y_i, y_j) z_j]_{1 \le i,j \le n}$$

and so it is easy to see from the definition (5.1.4) that

$$\begin{aligned} R_n(y_1, \ldots, y_n) &= \mathrm{coeff}_{z_1 \cdots z_n} \mathbf{E}\left(\prod_{j=1}^{N}\left(1 + \sum_{r=1}^{n} z_r \delta(x_j - y_r)\right)\right) \\ &= \mathrm{coeff}_{z_1 \cdots z_n} \det[\delta_{ij} + K_N^{(2)}(y_i, y_j) z_j]_{1 \le i,j \le n} \\ &= \det[K_N^{(2)}(y_i, y_j)]_{1 \le i,j \le n}, \end{aligned} \tag{5.2.11}$$

which we saw in Chapter 4. The probability that no eigenvalues lie in $J \in \mathbb{R}$, $\mathbf{E}(0; J)$ is clearly:[3]

$$\mathbf{E}(0; J) = \mathbf{E}\left(\prod_{j-i}^{N}(1 - \chi_J(x_j))\right) = \det(I - K_N^{(2)} \chi_J), \tag{5.2.12}$$

[3] $\chi_J(x)$ is the indicator function for the set J.

and more generally the probability of n_i eigenvalues in J_i, $1 \le i \le m$ is given by

$$\begin{aligned}
E(n_1, \ldots, n_m; J_1, \ldots, J_m) &= \underset{\begin{pmatrix} n_i \text{ of } x_j \in J_i,\ 1\le i\le m \\ \text{all other } x_j \in (\cup_{i=1}^m J_i)^c \end{pmatrix}}{\int \cdots \int} P_N^{(2)}(x_1, \ldots, x_N)\, dx_1 \ldots dx_N \\
&= \underset{\prod_{i=1}^m (z_i+1)^{n_i}}{\text{coeff}} \mathbf{E}\left(\prod_{j=1}^N \left(\left(1 - \sum_{i=1}^m \chi_{J_i}(x_j) \right) + \sum_{i=1}^m (z_i+1)\chi_{J_i}(x_j) \right) \right) \\
&= \frac{1}{n_1! \ldots n_m!} \frac{\partial^{\sum n_i}}{\partial z_1^{n_1} \ldots \partial z_m^{n_m}} \det\left(I + K_N^{(2)} \sum_{i=1}^m z_i \chi_{J_i} \right) |_{z_1 = \ldots = z_m = -1}.
\end{aligned} \tag{5.2.13}$$

For $\beta = 1$ and 4, we have shown

$$\mathbf{E}\left(\prod_{j=1}^N (1 + f(x_j)) \right) = (\det(I + K_N^{(\beta)} f))^{1/2}, \tag{5.2.14}$$

and so as before we have

$$\begin{aligned}
R_n(y_1, \ldots, y_n) &= \text{coeff}_{z_1 \cdots z_n} \mathbf{E}\left(\prod_{j=1}^N \left(1 + \sum_{r=1}^n z_r \delta(x_j - y_r) \right) \right) \\
&= \text{coeff}_{z_1 \cdots z_n} (\det[\delta_{ij} + K_N^{(\beta)}(y_i, y_j) z_j]_{1 \le i,j \le n})^{\frac{1}{2}}
\end{aligned} \tag{5.2.15}$$

and that

$$E(0, J) = \mathbf{E}\left(\prod_{j=1}^N (1 - \chi_J(x_j)) \right) = (\det(I - K_N^{(\beta)} \chi_J))^{\frac{1}{2}},$$

while

$$\begin{aligned}
E(n_1, \ldots, n_m; J_1, \ldots, J_m) &= \frac{1}{n_1! \ldots n_m!} \frac{\partial^{\sum n_i}}{\partial z_1^{n_1} \ldots \partial z_m^{n_m}} \\
&\quad \det\left(I + K_N^{(\beta)} \sum_{i=1}^m z_i \chi_{J_i} \right)^{\frac{1}{2}} |_{z_1 = \ldots z_n = -1}.
\end{aligned}$$

While $R_n(y_1, \ldots, y_n)$ is much more complicated in the $\beta = 1, 4$ case, than the $\beta = 2$ case, that is not true for the so-called cluster functions, see [Tra98].

5.3 Relations between $K_N^{(2)}$ and $K_N^{(1)}$, $K_N^{(4)}$ via skew-orthogonal polynomials

Theorem 5.2.1 describes the kernels in terms of

$$\hat{S}_N^{(\beta)}(x,y) := \sum_{i,j=0}^{\sigma N-1} \varphi_i(x)\mu_{ij}^{(\beta)}\varphi_j(y),$$

with $\varphi_i(x) = w(x)p_i(x)$, $\beta = 1$, $\varphi_i(x) = (w(x))^{\frac{1}{2}}p_i(x)$, $\beta = 2,4$ while $\sigma = 1$, $\beta = 1,2$, and $\sigma = 2$, $\beta = 4$, with $p_i(x)$ arbitrary polynomials of degree i and

$$\mu_{ij}^{(\beta)} \text{ given by } [(\mu^{(\beta)})^{-1}]_{ij} = \langle \varphi_i, \varphi_j \rangle_\beta,$$

with

$$\langle f,g\rangle_1 = \iint \epsilon(x-y)f(x)g(y)\,dxdy, \quad \langle f,g\rangle_4 = \int (f(x)g'(x) - f'(x)g(x))\,dx,$$

$$\langle f,g\rangle_2 = \int f(x)g(x)\,dx.$$

Note $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_4$ are skew-symmetric inner products, while $\langle\ ,\ \rangle_2$ is a symmetric inner product. In Remark 1, it was mentioned that the $\hat{S}_N^{(\beta)}(x,y)$ were insensitive to the choice of polynomials $p_i(x)$ and in all three cases the $\hat{S}_N^{\beta}(x,y)$ is the Christelhoff-Darboux kernel corresponding to the inner product $\langle\ ,\ \rangle_\beta$. Thus the case $\beta = 2$ seems related to orthonormal polynomials, while the cases $\beta = 1$ and 4, seem related to skew-orthonormal polynomials, due to the canonical form of the Christelhoff-Darboux kernels in these two cases, i.e., upon picking $\mu^{(2)} = I_N$, $\mu^{(1)} = \mu^{(4)} = J_N$. If we do this, it turns out, at least for the classical weights, the $S_N^{(\beta)}(x,y)$ for $\beta = 1,4$ can be described using the $S_{N'}^{(2)}(x,y)$ for appropriate N', plus a rank 1 perturbation.

Before stating the fundamental theorem relating the orthonormal and skew-orthonormal polynomials that enter into the Christelhoff-Darboux kernels $\hat{S}_N^{(\beta)}(x,y)$, we need some preliminary observations. Indeed, given a weight $w_2(x)$, perhaps with support on an interval I, it can be represented as $\widetilde{w}_2(x)\chi_I(x)$; however, we shall suppress the $\chi_I(x)$ and the $\sim$, while still integrating over $\mathbb{R}$ and making the assumption that $w_2'/w_2 = -g/f$, with g and f polynomials with no common factor such that $f(x)w_2(x)$ vanishes at the endpoints of the support interval I (in the limiting sense for endpoints at $\pm\infty$) and $f > 0$ in the interior of I.

Then given the inner product:

$$(\varphi,\psi)_2 = \int_{\mathbb{R}} \varphi(x)\psi(x)w_2(x)\,dx = \int_I \varphi(x)\psi(x)w_2(x)\,dx,$$

we have two natural operators (on the space of polynomials in x) going with $w_2(x)$. The first being the operator multiplication by x, and the second the first-order operator (see [Adl02])

$$n := f\frac{d}{dx} + f' - g = \left(\frac{f}{w_2}\right)^{\frac{1}{2}} \frac{d}{dx}(fw_2)^{\frac{1}{2}} \tag{5.3.1}$$

and we have

$$(x\varphi, \psi)_2 = (\varphi, xw)_2, \quad (n\varphi, \psi)_2 = (-\varphi, n\psi)_2,$$

i.e., x is a symmetric operator and n a skew-symmetric operator with respect to $(\ ,\ )_2$. The operator n is unique up to a constant, but we can rquire $\pm f$ to be monic, making n unique. The existence of x forces a 3-term recursion relation involving x on the orthonormal polynomials with respect to $(\ ,\ )_2$, and in the case of the classical weights, it forces a 3-term recursion relation involving n. This follows from the fact that x and n, in the basis of orthonormal polynomials, are represented respectively by a 3-band symmetric matrix L and a 3-band skew-symmetric matrix N. In general N has $2d+1$ bands, with $d = \max(\text{degree } f - 1, \text{degree}(f' - g))$, giving rise to a $2d+1$ skew-symmetric recursion relation involving n.

Let us now define ($\chi_I(x)$ suppressed as usual):

$$w_1(x) := \left(\frac{w_2(x)}{f(x)}\right)^{\frac{1}{2}}, \qquad w_4(x) := w_2(x)f(x), \tag{5.3.2}$$

and the associated inner products:

$$\begin{aligned}
(\varphi,\psi)_1 &:= \iint\limits_{\mathbb{R}^2} \varphi(x)\psi(y)\epsilon(x-y)w_1(x)w_1(y)\,dxdy, \qquad (5.3.3)\\
(\varphi,\psi)_4 &:= \int_{\mathbb{R}} \frac{1}{2}(\varphi(x)\psi'(x) - \varphi'(x)\psi(x))w_4(x)\,dx,\\
(\varphi,\psi)_2 &= \int_{\mathbb{R}} \varphi(x)\psi(x)w_2(x)\,dx.
\end{aligned}$$

This brings us to following theorem, whose proof[4] is found in [Adl02]; relating the above inner products, and in the classical cases, relating the skew-symmetric orthonormal polynomials going with $(\ ,\ )_1$ and $(\ ,\ )_4$ with the orthonormal polynomials going with $(\ ,\ )_2$. This theorem generalizes work of [Bre91].

Theorem 5.3.1 *Given the three weights $w_\beta(x)$ of (5.3.2) and inner products $(\ ,\ )_\beta$ of (5.3.3), and the operator n of (5.3.1), then all the inner products are determined by $w_2(x)$ and n as follows:*

[4] In fact in [Adl02], in sections 6 and 7, (5.3.5) is proven along the way in getting a different choice of skew-orthonormal polynomials. There is a small error in the $\beta = 4$ case.

$$(\varphi, n^{-1}\psi)_2 = (\varphi,\psi)_1, \quad (\varphi, n\psi)_2 = (\varphi,\psi)_4. \tag{5.3.4}$$

The mapping of orthonormal polynomials $p_i(x)$ with respect to $(\ ,\ )_2$ into a specific set of skew-orthonormal polynomials $q_i(x)$ with respect to $(\ ,\ )_\beta$, $\beta = 1, 4$ is given by, in the three classical cases:

$$q_{2n} = p_{2n}, \qquad q_{2n+1} = c_{2n}p_{2n+1} - c_{2n-1}p_{2n-1}, \qquad \beta = 1, \tag{5.3.5}$$

$$q_{2n} = p_{2n} + \sum_{\ell=0}^{n-1}\prod_{k=\ell+1}^{n}\left(\frac{c_{2k-1}}{c_{2k-2}}\right)p_{2\ell}, \qquad q_{2n+1} = \frac{p_{2n+1}}{c_{2n}}, \quad \beta = 4,$$

with the c_k's defined by the operator n as follows

$$np_k = c_{k-1}p_{k-1} - c_kp_{k+1}, \quad n = f\frac{d}{dx} + \frac{f'-g}{2}, \quad \frac{w_2'}{w_2} = -\frac{f'}{g}, \tag{5.3.6}$$

with f and g polynomials having no common root.

Remark 4. In the three classical cases of Hermite, Laguerre and Jacobi one finds for the orthonormal polynomials $p_k(x)$ that[5]

$$xp_k = a_{k-1}p_{k-1} + b_kp_k + a_kp_{k+1}, \quad np_k = c_{k-1}p_{k-1} - c_kp_{k+1}, \tag{5.3.7}$$

with

Hermite: $w_2(x) = e^{-x^2}, \quad f = 1, \quad a_{n-1} = \sqrt{n/2}, \quad b_n = 0, \quad c_n = a_n,$ (5.3.8)

Laguerre: $w_2(x) = e^{-x}x^\alpha\chi_{[0,\infty)}(x)$, $f = x$,

$$a_{n+1} = \sqrt{n(n+\alpha)}, \quad b_n = 2n+\alpha+1, \quad c_n = \frac{a_n}{2},$$

Jacobi: $w_2(x) = (1-x)^\alpha(1+x)^\beta\chi_{[-1,1]}(x)$, $f = 1-x^2$,

$$a_{n-1} = \left(\frac{4n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)(2n+\alpha+\beta-1)}\right)^{\frac{1}{2}},$$

$$b_n = \frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \quad c_n = a_n\left(\frac{\alpha+\beta}{2}+n+1\right).$$

Sketch of Proof of Theorem 5.3.1: Formula (5.3.4) is a consequence of

$$\left(\frac{d}{dx}\right)^{-1}\varphi(x) = \int_{\mathbb{R}}\epsilon(x-y)\varphi(y)\,dy, \quad \frac{d}{dx}\varphi(x) = \int_{\mathbb{R}}\delta'(x-y)\varphi(y)\,dy.$$

[5] Here we include χ rather than suppressing it.

The map O which takes $p := (p_i(x))_{i\geq 0}$ into $q := (q_i(x))_{i\geq 0}$, i.e., $q = Op$, with O a lower triangular semi-infinite matrix is given respectively for the cases $\beta = 1, 4$ by performing the skew-Borel decomposition

$$-N^{-1} = O^{-1}J_\infty(O^{-1})^T, \beta = 1, \quad -N = O^{-1}J_\infty(O^{-1})^T, \beta = 4, \tag{5.3.9}$$

with N the skew-symmetric 3-band semi-infinite matrix, which expresses the operator n in the orthonormal basis $\{p_k\}$ (given in (5.3.6) and (5.3.7) in terms of the c_k, $k \geq 0$) and J_∞ is the semi-infinite symplectic matrix, $I_\infty \otimes \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. One makes use of the non-uniqueness of the skew-orthonormal polynomials

$$(q_{2n}, q_{2n+1}) \mapsto \left(\delta_n q_{2n}, \frac{1}{\delta_n}(q_{2n+1} + \gamma_n q_{2n})\right),$$

to maximize the simplicity of the transformation $p \mapsto q$.

For general m semi-infinite and skew-symmetric, one performs the "skew-Borel decomposition", $m = O^{-1}J_\infty(O^{-1})^T$, for O lower triangular, by forming the skew-orthogonormal polynomials $(h_i(z))_{i\geq 0}$ going with the skew-symmetric inner product defined by $\langle z^i, z^j\rangle = m_{ij}$, $i, j \geq 0$ and setting

$$O\colon\ O(1, z, z^2 \ldots, \ )^T = (h_0(z), h_1(z), \ldots, \ )^T.$$

This is fully explained in [Adl99] and is an immediate generalization of m symmetric case (see [Adl97]). In [Adl99] the recipe for the $h_i(z)$ is given, to wit:

$$h_{2n}(z) = \frac{1}{(Pf(m_{2n})Pf(m_{2n+2}))^{\frac{1}{2}}} Pf \left(\begin{array}{c|c} & 1 \\ & z \\ m_{2n+1} & \vdots \\ & \\ & z^{2n} \\ \hline -1, -z, \ldots, \ , -z^{2n} & 0 \end{array}\right)$$

$$\begin{aligned} h_{2n+1}(z) \quad &= \quad \frac{1}{(Pf(m_{2n})Pf(m_{2n+2}))^{\frac{1}{2}}} \\ & Pf \left(\begin{array}{c|cc} & 1 & m_{0,2n+1} \\ & z & m_{1,2n+1} \\ m_{2n} & \vdots & \vdots \\ & z^{2n-1} & m_{2n-1,2n+1} \\ \hline -1, -z, \ldots, \ -z^{2n-1} & 0 & -z^{2n+1} \\ -m_{0,2n+1}, \ldots, -m_{2n-1,2n+1} & z^{2n+1} & 0 \end{array}\right) \end{aligned}$$

with $m_k := (m_{ij})_{0\le i,j\le k-1}$ and $Pf(A) = (\det A)^{\frac{1}{2}}$ for A a skew-symmetric matrix.

Remark 5. In the nonclassical case, one still has the same recipe for O, (5.3.9), but in general N will have $2d+1$ bands, $d>1$, and so O will increase in complexity with increasing $d>1$.

We can apply Theorem 5.3.1 to compute $\hat{S}_N^{(\beta)}(x,y)$ and hence $S_N^{(\beta)}(x,y)$ for $\beta = 1,4$ by setting $\mu_{ij}^{(\beta)} = J_{ij}$, so that (up to the weight factor) the $\varphi_i(x)$ are skew-orthogonal polynomials. This leads to the following theorem, found in [Adl00].

Theorem 5.3.2 *In the case of the three classical weights of Remark 4, the $\beta = 1,4$ kernel is given in terms of the $\beta = 2$ kernel as follows:*

$$S_N^{(1)}(x,y) = \left(\frac{f(y)}{f(x)}\right)^{\frac{1}{2}} S_{N-1}^{(2)}(x,y) + c_{N-2}\frac{\varphi_{N-1}^{(2)}(x)}{(f(x))^{\frac{1}{2}}}\epsilon\left(\frac{\varphi_{N-2}^{(2)}}{(f)^{\frac{1}{2}}}\right)(y), \quad N \text{ even} \tag{5.3.10}$$

$$S_N^{(4)}(x,y) = \left(\frac{f(y)}{f(x)}\right)^{\frac{1}{2}} S_{2N}^{(2)}(x,y) - c_{2N-1}\frac{\varphi_{2N}^{(2)}(x)}{(f(x))^{\frac{1}{2}}}\int_y^\infty \frac{\varphi_{2N-1}^{(2)}(t)}{(f(t))^{\frac{1}{2}}}dt, \tag{5.3.11}$$

where $\varphi_k^{(2)} = (w_2)^{\frac{1}{2}}p_k$, with p_k and $S_k^{(2)}(x,y)$ being the usual orthonormal polynomials and Christelhoff-Darboux kernel with respect to the weight w_2. Given the weight $w(x)$ appearing in $P_N^{(\beta)}(x_1,\dots,x_n)$, $\beta = 1,4$, (5.1.1), pick $w_2(x)$ such that

$$w(x) = \left(\frac{w_2(x)}{f(x)}\right)^{\frac{1}{2}}, \quad \beta = 1 \quad \text{and} \quad w(x) = w_2(x)f(x), \quad \beta = 4, \tag{5.3.12}$$

with

$$\frac{w_2'(x)}{w_2(x)} = -\frac{g(x)}{f(x)}, \quad f(x) = 1, x, 1-x^2,$$

respectively for the Hermite, Laguerre and Jacobi cases of Remark 4. The c_k are defined by

$$np_k = c_{k-1}p_{k-1} - c_kp_{k+1}, \quad n = f\frac{d}{dx} + \frac{f'-g}{2},$$

and are given explicitly in Remark 4.

Remark 6. In the special case of the Gaussian potential, $w_2(x) = e^{-x^2}$, [Tra98] showed that $S_N^{(\beta)}(x,y)$ also have the above representation:

$$S_N^{(1)}(x,y) = S_N^{(2)}(x,y) + \left(\frac{N}{2}\right)^{\frac{1}{2}} \varphi_{N-1}(x)\epsilon\varphi_N(y),$$

$$2S_N^{(4)}(x,y) = S_{2N+1}^{(2)}(x,y) + \left(N + \frac{1}{2}\right)^{\frac{1}{2}} \varphi_{2N}(x)\epsilon\varphi_{2N+1}(y),$$

but in order to obtain one formula for all three classical cases we need the above theorem. Indeed, the above formula and different formulas for the Laguerre case due to [For99] are found in [Adl00], section 4, and shown to agree with the theorem at the end of the paper.

Sketch of Proof of Theorem 5.3.2. The proof uses (5.3.5), which is a consequence of (5.3.12). Set $\mu_{ij}^{(\beta)} = J_{ij}$, $\beta = 1, 4$ in (5.2.6) and (5.2.8), and then substitute in $S_N^{(\beta)}(x,y)$, $\beta = 1, 4$ respectively that

$$\epsilon\varphi_k(v) = \frac{1}{w(v)}\langle\delta(x-v), q_k\rangle_1, \quad , \quad \varphi_k = wq_k,$$

$$\epsilon\varphi_k'(v) = \frac{1}{(w(v)^{\frac{1}{2}}}\langle\delta(x-v), q_k\rangle_4, \quad , \quad \varphi_k = w^{\frac{1}{2}}q_k,$$

$$\delta(x-y) = \sum_{n=0}^{\infty} \varphi_n^{(2)}(x)\varphi_n^{(2)}(y).$$

Then using both (5.3.5) and the inverse map, finally yields the theorem.

References

[Adl97] M. Adler and P. van Moerbeke, String-orthogonal polynomials, string equations, and 2-Toda symmetries, Commun. on Pure and Appl. Math. **L** (1997) 241–290.

[Adl99] M. Adler, E. Horozov, and P. van Moerbeke, The Pfaff lattice and skew-orthogonal polynomials, Internat. Math. Res. Notices (1999) 569–588.

[Adl00] M. Adler, P.J. Forrester, T. Nagao, and P. van Moerbeke, Classical skew orthogonal polynomials and random matrices, J. Stat. Phys. **99**, 1/2 (2000) 141–170.

[Adl02] M. Adler and P. van Moerbeke, Toda versus Pfaff lattice and related polynomials, Duke Math. J. **112**, 1 (2002) 1–58.

[Ake05] G. Akemann, *The Complex Laguerre Symplectic Ensemble of Non-Hermitian Matrices* [arXive:hep-th/0507156].

[Bre91] E. Brezin and H. Neuberger, Multicritical points of unoriented random surfaces, Nucl. Phys. **B350** (1991) 513–553.

[deB55] N. G. de Bruijn, On some multiple integrals involving determinants, J. Indian Math.. Soc. **19** (1955) 133–151.

[For99] P. J. Forrester, T. Nagao, and G. Honner, Correlations for the orthogonal-unitary and symplectic-unitary transitions at the hard and soft edges, Nucl. Phys. **B553** (1999) 601–643.

[Hel62] S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, New York 1962.

[Hel84] S. Helgason, Groups and geometric analysis, in: *Integral Geometry, Invariant Differential Operators, and Spherical Functions*, Academic Press, New York 1984.

[Meh04] M.L. Mehta, *Random Matrices*, Academic Press, 3rd Edition, London 2004.

[Sen98] M. K. Sener and J. J. M. Verbaarschot, Universality in Chiral Random Matrix Theory at $\beta = 1$ and $\beta = 4$, Phys. Rev. Lett. **81** (2) (1998) 248–251.

[Tra98] C. A. Tracy and H. Widom, Correlation functions, cluster functions, and spacing distributions for random matrices, J. Stat. Phys. **92** (1998) 809–835.

[Wid99] H. Widom, On the relation between orthogonal, symplectic and unitary random matrices, J. Stat. Phys. **94** (1999) 601–643.