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The Kowalewski and Hénon-Heiles Motions as Manakov Geodesic Flows on $SO(4)$ – a Two-Dimensional Family of Lax Pairs*

Mark Adler[1] and Pierre van Moerbeke[1,2]

[1] Department of Mathematics, Brandeis University, Waltham, MA 02254, USA
[2] Department of Mathematics, University of Louvain, Louvain-la-Neuve, Belgium

Abstract. The invariant surfaces for the Kowalewski top, the Hénon-Heiles system and the Manakov geodesic flow on $SO(4)$ complete into Abelian surfaces A, by adjoining, in each case, a divisor D of arithmetic genus 9; these divisors belong to the same linear system on A and they each define a polarization (2,4). Therefore there are rational maps transforming the Kowalewski top and the Hénon-Heiles system into Manakov's geodesic flow on $SO(4)$. This paper deals with the precise geometric relationship between these three problems; it is based on the splitting of the 8-dimensional space of sections of D (theta-functions) into an even and an odd part and also on a normal form for the six quadrics describing A, as embedded in $\mathbb{P}^7$. As a byproduct, we get a 2-dimensional family of Lax pairs for both the Kowalewski top and the Hénon-Heiles system.

1. Introduction

Integrable systems have been integrated classically in terms of quadratures, usually through a sequence of very ingenious algebraic manipulations especially tailored to the problem. More recently, it was realized that whenever a system could be represented as a family of Lax pairs – often arising in the context of coadjoint orbits of Kac-Moody Lie algebras – the system could be linearized on the Jacobian of a spectral curve, defined by the characteristic polynomial of one of the matrices in the Lax pair. However this approach has remained unsatisfactory; indeed (i) finding such families of Lax pairs often requires just as much ingenuity and luck as to actually solve the problem; (ii) it often conceals the actual geometry of the problem. Therefore we have engaged in a systematic approach towards solving integrable systems, based on the Laurent solutions of the differential equations [5]; This is done in the context of *algebraically completely integrable systems*. The latter means: the system has polynomial invariants, in sufficient

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number, their (compact) invariant surfaces are real tori (by the Arnold-Liouville theorem), the invariant surfaces viewed as complex manifolds extend to complex algebraic tori, upon adjoining some divisor, and the phase variables are meromorphic functions on those tori. The Laurent solutions to the differential equations, depending on a sufficient number of parameters, provide the way to complete the affine invariant surfaces to complex algebraic tori; and these solutions, properly decoded, provide all the information about the tori and their periods.

Two integrable Hamiltonian systems may look very different and yet be related by some rational map, involving all the phase variables. It is hopeless to guess this map by mere investigation of the differential equations, but the study of the nature of the tori yields the key to whether the systems are rationally related and it provides the explicit rational map. In this paper we show that three seemingly unrelated problems – the Kowalewski top, the Hénon-Heiles system, and the geodesic flow on $SO(4)$ for the Manakov metric – are rationally related. Moreover we give the precise rational map from one to another; it is closely tied up with the beautiful geometry of line bundles on Abelian surfaces of polarization $(2,4)$. The birational equivalence between those systems enables us to carry properties from one system to another; in particular as a by-product we write down a two-dimensional family of Lax pairs for the Kowalewski top and the Hénon-Heiles problem. This two-dimensional family of Lax pairs leads to a spectral surface, rather than a spectral curve. The nature of this surface will also be discussed in this paper.

We give a brief description of the three problems:

I. Kowalewski's top [21, 22] rotates about a fixed point, its principal moments of inertia $\lambda = \operatorname{diag}(A, B, C)$ (with regard to the fixed point) satisfy the relation $A = B = 2C$ and its center of mass belongs to the equatorial plane (AB plane) through the fixed point. The motion is governed by the equations

$$\dot{m} = m \wedge \lambda m + \gamma \wedge l, \qquad \dot{\gamma} = \gamma \wedge \lambda m, \tag{0}$$

where m, l, and γ denote respectively the angular momentum, the center of mass and the unit vector in the direction of gravity, which after some rescaling and normalization may be taken as $l = (1,0,0)$ and $\lambda m = (m_1/2, m_2/2, m_3)$. Besides the two trivial invariants $\langle m, \gamma\rangle = B$, $\langle \gamma, \gamma\rangle = C$, and the energy $\langle \lambda m, m\rangle/2 + \langle l, \gamma\rangle = A/2$, the system has one other invariant, $y_1 y_2 = D^2$, upon defining the change of variables

$$\begin{aligned}(x_1, x_2, x_3, y_1, y_2, y_3) = \Bigg(\left(\frac{m_1 + i m_2}{2}\right), \left(\frac{m_1 - i m_2}{2}\right), \\ m_3,\ x_1^2 - (\gamma_1 + i\gamma_2),\ x_2^2 - (\gamma_1 - i\gamma_2), \gamma_3\Bigg),\end{aligned} \tag{1}$$

as S. Kowalewski shows in her famous 1889 Acta Mathematica paper. Through a sequence of very clever algebraic manipulations, especially adapted to the problem, she integrates the flow in terms of hyperelliptic quadratures, involving the curve

$$\mathscr{H} : y^2 = T(x)(x - D)(x + D), \tag{2}$$

where the cubic

$$T(x)=\det(M-xI)=x^3-\frac{A}{2}x^2+Cx+\frac{B^2-AC}{2}=(x-a_1)(x-a_2)(x-a_3) \quad (3)$$

is the characteristic polynomial of the matrix

$$M=\tfrac{1}{2}\begin{pmatrix} C-1 & -B & i(C+1) \\ -B & A & -iB \\ i(C+1) & -iB & -(C-1) \end{pmatrix}. \quad (4)$$

For future use, we also introduce the orthogonal matrix U which diagonalizes M (in terms of the spectrum a_1, a_2, a_3 of M) namely

$$UMU^T=\operatorname{diag}(a_1,a_2,a_3). \quad (5)$$

We now provide the geometric background to this problem. It was shown by Lesfari [2] and Adler and van Moerbeke [5] that the affine surface defined by the 4 constants of motion of the Kowalewski top completes into an Abelian surface A by adjoining a divisor D consisting of two isomorphic genus 3 curves $D^{(1)}$ and $D^{(2)}$ intersecting in 4 points. Each D^i is a double cover of an elliptic curve $\mathscr{E}$ ramified at 4 points: it defines a line bundle and a polarization (1, 2) on A. Then $A=\mathbb{C}^2/\Lambda$, where the lattice Λ is generated by the period matrix

$$\begin{pmatrix} 2 & 0 & a & b \\ 0 & 4 & b & c \end{pmatrix}, \qquad \operatorname{Im}\begin{pmatrix} a & b \\ b & c \end{pmatrix}>0.$$

The divisors $2D^1$, $2D^2$ or D^1+D^2 are all very ample and define polarizations (2, 4); the 8-dimensional space of sections (θ-functions) of the corresponding line bundle embeds the abelian surface A into $\mathbb{P}^7$. For instance, setting $D\equiv D^{(2)}$, the space $L(2D)$ is spanned by the following functions (in terms of (1)):

$$L(2D)=\{x_2^2-1,\ -2x_2, i(x_2^2+1), y_2(x_1^2-1),\ -2y_2x_1, iy_2(x_1^2+1),$$
$$-x_3x_2+y_3, y_2(x_1x_3-y_3)\}. \quad (6)$$

Also A is the dual of a Prym variety, namely $A=\operatorname{Prym}(D/\mathscr{E})^{\vee}$.

II. The Hénon-Heiles system

$$\dot{x}_i=\frac{\partial H}{\partial y_i}, \qquad \dot{y}_i=-\frac{\partial H}{\partial x_i}, \qquad i=1,2,$$

with

$$H=Q_1=\tfrac{1}{2}(y_1^2+y_2^2)+x_1^2x_2+2x_2^3=A \quad (7)$$

is algebraically completely integrable with additional integral

$$Q_2=y_1y_2x_1-y_1^2x_2+x_1^2x_2^2+\frac{x_1^4}{4}=B$$

(see Bountis et al. [10]). The affine surface defined by the intersection $\{Q_1=A\}\cap\{Q_2=B\}$ completes into an Abelian surface, by adjoining a smooth genus 3 hyperelliptic curve D; the latter is a double ramified cover of an elliptic curve $\mathscr{E}$, but also a double unramified cover of a genus 2 hyperelliptic curve

$$\mathscr{H}: y^2 = x(-4x^4 + 2Cx + 1), \qquad C = A_1 A_2^{-3/4},$$

on whose Jacobian the flow linearizes. The divisor D defines on A a polarization $(1,2)$. As before, the functions of

$$L(2D) = \left\{1, x_1, x_1^2, ix_2, -\frac{i}{2}(2x_1x_2^2 + y_1y_2), -i(y_1^2 + x_1^2x_2), y_1, y_2x_1 - 2y_1x_2\right\} \tag{8}$$

embed A smoothly into $\mathbb{P}^7$ with a polarization $(2,4)$, and $A = \mathrm{Prym}(D/\mathscr{E})^{\vee}$ is a double unramified cover of $\mathrm{Jac}(\mathscr{H})$.

III. The geodesic flow on $SO(4)$ for the Manakov metric is given by

$$(X + \alpha h)^{\cdot} = \left[X + \alpha h, \frac{\partial Q}{\partial X} + \beta h\right], \tag{9}$$

where

$$X = (X_{ij}) = \begin{pmatrix} 0 & -x_3 & x_2 & -x_4 \\ x_3 & 0 & -x_1 & -x_5 \\ -x_2 & x_1 & 0 & -x_6 \\ x_4 & x_5 & x_6 & 0 \end{pmatrix} \in so(4),$$

$$\alpha = \mathrm{diag}(\alpha_1, \ldots, \alpha_4), \qquad \beta = \mathrm{diag}(\beta_1, \ldots, \beta_4), \tag{10}$$

$$Q = \tfrac{1}{2} \sum_{1 \leq i < j \leq 4} \lambda_{ij} X_{ij}^2, \qquad \lambda_{ij} = \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j}.$$

The system has 4 invariants

$$\begin{aligned} Q_i(X) &= \sum_{j \neq 1} \frac{X_{ij}^2}{\alpha_i - \alpha_j} = A_i, \qquad i = 1, 2, 3, \\ Q_4(X) &= \sqrt{\det X} = X_{23}X_{14} + X_{31}X_{24} + X_{12}X_{34} = A_4, \end{aligned} \tag{11}$$

and linearizes on the Jacobian of the spectral curve of (9), namely

$$P(z, h) = \det(X + \alpha h - zI) = 0.$$

The invariant surface $\bigcap_1^4 \{Q_i = A_i\}$ completes into an Abelian surface A by adjoining a smooth curve $\mathscr{C}$ of genus 9, which is a 4-fold unramified cover of a curve D of genus 3; the latter is a double ramified cover of an elliptic curve $\mathscr{E}$ and therefore $A = \mathrm{Prym}(D/\mathscr{E})$. The functions of $L(\mathscr{C})$

$$L(\mathscr{C}) = \{x_1, \ldots, x_6, 1, (\alpha_3 - \alpha_1)(\alpha_4 - \alpha_2)x_1x_4 + (\alpha_3 - \alpha_2)(\alpha_4 - \alpha_1)x_2x_5\}$$

embed A smoothly into $\mathbb{P}^7$ and the divisor $\mathscr{C}$ defines on A a polarization $(2,4)$ (see Haine [16]).

To conclude, the invariant surfaces for these three problems complete into Abelian surfaces by adjoining divisors; they each define a polarization (2, 4) and they fortunately turn out to belong to the same linear system. Therefore there are birational maps taking the Kowalewski top and the Hénon-Heiles problem to Manakov's geodesic flow on $SO(4)$; to be precise, there is a one-dimensional family of birational maps between the Kowalewski and Hénon-Heiles invariant tori and those of the Manakov problem.

The aim of this paper is to provide an effective method to produce such birational maps. They are given by identifying the three 8-dimensional spaces $L(2D^{(2)})$, $L(2D)$, and $L(\mathscr{C})$ of Kowalewski, Hénon-Heiles, and Manakov; the space of sections of the corresponding line bundles can be given a *canonical basis* respecting the involutions on the Abelian surfaces. The exact map is then given by identifying the bases of $L(2D^{(2)})$, $L(2D)$, and $L(\mathscr{C})$ with the canonical basis.

To elaborate on the above procedure, consider a line bundle $\mathscr{L}$ on the Abelian surface defining a polarization (1, 2); for Kowalewski's problem, pick D_1 or D_2, and for the Hénon-Heiles problem, D itself. Then for some origin on A and for the natural reflection about this origin, the 8-dimensional space of sections of the line bundle $\mathscr{L}^{\otimes 2}$ splits into two subspaces H^+ and H^- of even and odd sections (theta functions)

$$H^0(\mathscr{L}^{\otimes 2})=H^+\oplus H^-=\{\theta_1,\ldots,\theta_6\}\oplus\{\theta_7,\theta_8\}\,. \tag{12}$$

They have the remarkable property that

$$\{H^+,H^-\}\subset(H^+)^{\otimes 2}, \tag{13}$$

where $\{\,,\}$ denotes the Wronskian $\{\theta_i,\theta_j\}\equiv\theta_iX(\theta_j)-\theta_jX(\theta_i)$ between two theta-functions, with respect to an arbitrary holomorphic vector field X on A. Then the Abelian surface A embedded in $\mathbb{P}^7$ can be described by 6 quadratic relations between the θ-functions, 3 of which involve even sections only and another 3 involving even and odd sections. However, the space of the three first quadrics contains 4 collinear rank 3 quadrics. Therefore a canonical basis θ_1 can be picked, up to a finite number of choices – such that the 6 quadrics have the form

$$\begin{gathered}
Q_1=\sum_1^3\theta_i^2=0\,,\qquad Q_2=\sum_1^3\theta_{i+3}=0\,,\\
Q_3=\sum_1^3(\gamma_i\theta_i+\gamma_{i+3}\theta_{i+3})^2=0\,,\qquad Q_4(\theta_1,\ldots,\theta_6)+\theta_7^2=0\,,\\
Q_5(\theta_1,\ldots,\theta_6)+2\theta_7\theta_8=0\,,\qquad Q_6(\theta_1,\ldots,\theta_6)+\theta_8^2=0\,.
\end{gathered}\tag{14}$$

Consider now the linear pencil of curves

$$\mathscr{C}_{\kappa/\lambda}=\{\kappa\theta_7+\lambda\theta_8=0\}$$

on A and the corresponding affine surface $\mathscr{A}_{\kappa/\lambda}=A\backslash\mathscr{C}_{\kappa/\lambda}$; the latter is cut out by the intersection of the 4 quadrics

$$\begin{gathered}
\sum_1^3 u_1^2=0\,,\qquad \sum_1^3 u_{i+3}^2=0\,,\qquad \sum_1^3(\gamma_iu_i+\gamma_{i+3}u_{i+3})^2=0\,,\\
Q_{\kappa/\lambda}(u)\equiv(\kappa^2Q_4+\kappa\lambda Q_5+\lambda^2Q_6)(\kappa\theta_7+\lambda\theta_8)^{-2}=0\,,
\end{gathered}\tag{15}$$

expressed in terms of the affine variables

$$(u_0, u_1, \ldots, u_6) = (\kappa\theta_7 + \lambda\theta_8)^{-1}(\kappa\theta_7 + \lambda\theta_8, \theta_1, \ldots, \theta_6), \qquad u_0 = 1. \tag{16}$$

As a consequence of (13), this surface $\mathscr{A}_{\kappa/\lambda}$ supports a closed system of quadratic differential equations, depending linearly on κ/λ. We show that this system is nothing but Manakov's geodesic flow and that the affine surface $\mathscr{A}_{\kappa/\lambda}$ is cut out by the intersection of the 4 quadrics (11). The linear map u to the Manakov variables x in (10) has the form

$$x_i = \sqrt{a_i^+}(u_i + b_i^+ u_{i+3}), \qquad x_{i+3} = \sqrt{a_i^-}(u_i + b_i^- u_{i+3}), \tag{17}$$

where $a_i^\pm$ and $b_i^\pm$ are algebraic functions of the γ_i and κ/λ, to be spelled out in Sect. 3.

Returning to Kowalewski's problem, the next step is to identify the space $L(2D)$ with the space (12) given by its canonical basis. The affine invariant surface $\mathscr{A}$ defined by the 4 constants of motion, given in I, have the following involution, in terms of the variables defined in (1) (not to be confused with the Manakov x_i-variables):

$$\tau: (x_1, x_2, x_3, y_1, y_2, y_3) \to (x_1, x_2, -x_3, y_1, y_2, -y_3), \tag{18}$$

which amounts to a reflection about some appropriately chosen origin on $\mathscr{A}$. A different choice of origin would lead to a different involution. $\mathscr{A}$ has also a second involution

$$\sigma: (x_1, x_2, x_3, y_1, y_2, y_3) \to (x_2, x_1, -x_3, y_2, y_1, -y_3). \tag{19}$$

Then the map from the functions (6) in $L(2D)$ to the u_i variables is given by

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = U \begin{pmatrix} -(x_2^2-1) \\ 2x_2 \\ -i(x_2^2+1) \end{pmatrix} \frac{1}{2p}, \qquad \begin{pmatrix} u_4 \\ u_5 \\ u_6 \end{pmatrix} = U \begin{pmatrix} -(x_1^2-1) \\ 2x_1 \\ -i(x_1^2+1) \end{pmatrix} \frac{1}{2p^\sigma}, \tag{20}$$

where U is the orthogonal matrix defined in (5) and where

$$p = \kappa(y_3 - x_2x_3) - \lambda y_2(y_3 - x_1x_3), \qquad p^\sigma = \lambda(-y_3 + x_1x_3) - \kappa y_1(-y_3 + x_2x_3).$$

Thus the combination of the maps (17) and (120) yields the one-dimensional family of linear maps (depending on the parameter κ/λ) to the Manakov problem, thus mapping the affine surface $\mathscr{A}$ of Kowalewski to the surface $\mathscr{A}_{\kappa/\lambda}$ of Manakov. For the Hénon-Heiles problem one proceeds in a similar fashion.

Substituting the combined maps (17) and (20) in the Lax pair (9) leads to a two-dimensional family of Lax pairs,

$$\dot{A}(k,h) = [A(k,h), B(k,h)], \qquad A, B \in so(4), \qquad k = \kappa/\lambda,$$

depending linearly on h and algebraically on k. The *spectral surface* defined by

$$P(k,h,z) \equiv \det(A(k,h) - zI) = 0$$

is some appropriate projection of the invariant tori A. We *conjecture* it is an expression for the dual $A^\vee$ of the tori A. For six special values of $k = \kappa/\lambda$, the spectral curve $P(k,h,z) = 0$ (k fixed) is hyperelliptic and, in particular for $k = 0$ or ∞, the Lax pair takes on the particularly nice form

$$(M+I)((v-v^{\sigma})\otimes(v-v^{\sigma})+h(v+v^{\sigma})^{\wedge}+Ih^2)^{\cdot}$$
$$=[(M+I)((v-v^{\sigma})\otimes(v-v^{\sigma})+h(v+v^{\sigma})^{\wedge}+Ih^2),\,(M+I)((v+v^{\sigma})^{\wedge}+Ih)], \quad (21)$$

where M is the matrix (4), σ the involution (19),

$$v=-((x_2^2-1),\ -2x_2,\ i(x_2^2+1))/2p, \qquad p=-y_2y_3-x_2x_3+x_1x_3y_2+y_3,$$

and where $^{\wedge}$ denotes the customary map

$$^{\wedge}:\mathbb{R}^3\to so(3):(a,b,c)\curvearrowright\begin{pmatrix}0 & -c & b\\ c & 0 & -a\\ -b & a & 0\end{pmatrix}. \quad (22)$$

Another Lax pair expressed in $sl(6)$ coordinates, reads as follows:

$$\begin{pmatrix}\hat{v} & (\beta-I)h\\ (N-I)h & \hat{v}^{\sigma}\end{pmatrix}^{\cdot}=\left[\begin{pmatrix}\hat{v} & (N-I)h\\ (N-I)h & \hat{v}^{\sigma}\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}(M+I)(v-v^{\sigma})^{\wedge} & 2T(-1)Ih\\ 2T(-1)Ih & -(M+I)(v-v^{\sigma})^{\wedge}\end{pmatrix}\right], \quad (23)$$

where M is the matrix (4), where $T(x)$ is the cubic (3) and where

$$N=\begin{pmatrix}T(-1)+A-C+2 & 2B & iT(-1)\\ 2B & -A+C & 0\\ iT(-1) & 0 & -T(-1)-A-C-2\end{pmatrix}; \quad (24)$$

observe that

$$U(N-I)U^T=2T(-1)\left[\sum_1^3\frac{1}{a_i+1}I-2\,\mathrm{diag}\left(\frac{1}{a_1+1},\frac{1}{a_2+1},\frac{1}{a_3+1}\right)\right]$$

in terms of the spectrum a_i and the diagonalizing map U of M.

During the last ten years, there have been several attempts in constructing meaningful Lax pairs for the Kowalewski top, notably among them the constructions of Perelomov [31] and another one by Buys [11]. In each of the cases there failed to be families of Lax pairs. Applying the methods (Theorems 1 and 2) presented in this paper to the divisor $D^{(1)}+D^{(2)}$ (see description I. above of Kowalewski's problem) with a reflection about a different origin, Haine and Horozov [17] have obtained for the Kowalewski top a different Lax pair from the ones in this paper. Also Fairbanks [13] has shown that every integrable system which can be solved by hyperelliptic quadratures admits a 2×2 Lax pair representation; this result is implicitly contained in the work of Adams et al. [1] and based on the Moser [26] and Mumford [27] description of hyperelliptic Jacobians. Meanwhile, R. Donagi has announced the result that every algebraically completely integrable system can – in principle – be represented as a g-dimensional family of Lax pairs, where g is the dimension of the invariant tori. Also, recently we have received a provocative preprint by Newel et al., [28], who have obtained a Lax pair for the Hénon-Heiles system.

2. Line Bundles on Abelian Surfaces Defining a Polarization (2.4)

Consider a line bundle $\mathscr{L}$ on an Abelian surface A defining a polarization $(1,2)$ and its square $\mathscr{L}^{\otimes 2}$ defining a polarization $(2,4)$. The 8-dimensional space of sections (theta functions) of $\mathscr{L}^{\otimes 2}$ splits into subspaces of even and odd sections for some reflection about the origin. The Wronskian (with regard to any holomorphic vector field on A) of even and odd sections can be expressed quadratically in terms of even sections. Between the 8 sections forming a basis of $H^0(\mathscr{L}^{\otimes 2})$, there are 6 quadratic relations, three of which depend on the even sections only. It is particularly convenient to use the set of quadrics in (3) introduced by Kötter [19, 20] and studied by us in [4]. They have the remarkable property that both the affine surfaces $\bigcap_1^3 \{\Phi_i=0\} \cap \{\Phi_4=0,\ \theta_7 \neq 0\}$ and $\bigcap_1^3 \{\Phi_i=0\} \cap \{\Phi_6=0,\ \theta_8 \neq 0\}$ complete into Abelian surfaces by adjoining 8-fold unramified covers of the hyperelliptic genus 2 curve $y^2 = x\prod_1^4 (x-b_i)$, where the quantities b_i appear in the quadrics below; these curves will play a crucial role in the sequel. Throughout this paper, $V(Q_1, \ldots, Q_n) = \mathbb{P}^{n-1}$ denotes the projective linear span of the quadrics $Q_1, \ldots, Q_n$. Some of the ideas in this section have been inspired by Barth's beautiful paper [7] on Abelian surfaces of type $(1,2)$, by Haine's[16] realization of these surfaces as the intersection of 6 quadrics and by our study of quadrics containing curves of rank 4 quadrics [4].

Theorem 1. *Consider an Abelian surface*[1] *A, and a line bundle $\mathscr{L}$, defining a polarization $(1,2)$ on A. For some origin on A, the 8-dimensional space of sections of the line bundle $\mathscr{L}^{\otimes 2}$ splits into an even and odd subspace for the reflection τ about that origin*

$$H^0(\mathscr{L}^{\otimes 2}) = H^+ \oplus H^- = \{\theta_1, \ldots, \theta_6\} \oplus \{\theta_7, \theta_8\}. \tag{1}$$

Letting X denote any holomorphic vector field on A, the sections θ satisfy the following relationship in terms of the Wronkians $\{\theta_i, \theta_j\} \equiv \theta_j X\theta_i - \theta_i X\theta_j$ of two sections:

$$\{H^+, H^-\} \subset (H^+)^{\otimes 2}. \tag{2}$$

Moreover A, as embedded in $\mathbb{P}^7$ by the sections θ_i, is described by the following 6 quadrics[2]:

$$\begin{aligned}
&\Phi_1(\theta) = \sum_1^3 \theta_i^2, \qquad \Phi_2(\theta) = \sum_1^3 \theta_{i+3}^2,\\
&\Phi_3^{\pm}(\theta) = \tfrac{1}{4}\sum_1^3 \left((b_i^{1/2} \pm b_i^{-1/2})\theta_i + (b_i^{1/2} \mp b_i^{-1/2})\theta_{i+3}\right)^2\\
&\Phi_4(\theta) = \tfrac{1}{4}\sum_1^3 \frac{(\theta_i + \theta_{i+3})^2}{b_4^{-1} - b_i^{-1}} + \theta_7^2\\
&\Phi_5(\theta) = \tfrac{1}{4}\sum_1^3 \frac{\theta_i^2 - \theta_{i+3}^2}{b_i - b_4} + \frac{\theta_7\theta_8}{b_4}\\
&\Phi_6(\theta) = \tfrac{1}{4}\sum_1^3 \frac{(\theta_i - \theta_{i+3})^2}{b_i - b_4} + \theta_8^2, \qquad b_1, b_2, b_3 \in \mathbb{C}^*, \qquad b_4 = b_1 b_2 b_3,
\end{aligned} \tag{3}$$

[1] Not containing an elliptic curve
[2] Note $\Phi_1 - \Phi_2 = \Phi_3^+ - \Phi_3^-$

or alternatively by the following quadrics:

$$\begin{aligned}
\Phi'_1 &= \sum_1^3 \theta_i^2 ,\\
\Phi'_2 &= \sum_1^3 \theta_{i+3}^2 ,\\
\Phi'_3 &= 2\sum_1^3 (-a_i(\theta_i^2+\theta_{i+3}^2)+(d_i+1)\theta_i\theta_{i+3}) ,\\
\Phi'_4 &= 2\sum_1^3 (-a_i\theta_i^2+\theta_i\theta_{i+3})+\theta_7'^2 ,\\
\Phi'_5 &= 2\sum_1^3 a_i\theta_i\theta_{i+3}+\theta'_7\theta'_8 ,\\
\Phi'_6 &= 2\sum_1^3 (-a_i\theta_{i+3}^2+\theta_i\theta_{i+3})+\theta_8'^2 ,
\end{aligned} \tag{3'}$$

where θ'_7 and θ'_8 relate to θ_7 and θ_8 as follows:

$$\theta_7 = \frac{\theta'_7-\theta'_8}{4} b_4^{1/2} , \qquad \theta_8 = \frac{\theta'_7+\theta'_8}{4} b_4^{-1/2} . \tag{4}$$

The parameters a_i, b_i, and d_i are related as follows:

$$\begin{gathered}
d_1 \equiv a_2a_3 - a_3a_1 - a_1a_2 \text{ and cyclic permutation} ,\\
\frac{b_i}{b_4} \equiv \frac{a_i-1}{a_i+1} , \qquad b_4 = \sqrt{\frac{T(-1)}{T(1)}} , \qquad T(x) \equiv (x-a_1)(x-a_2)(x-a_3) .
\end{gathered} \tag{5}$$

The quadrics Φ'_1, Φ'_2, and Φ'_3 are in the linear span of Φ_1, Φ_2, and Φ_3. The set of quadrics Φ has the following involutions:

$$\begin{aligned}
&\tau\colon (\theta_1,\dots,\theta_8) \curvearrowright (\theta_1,\dots,\theta_6, -\theta_7, -\theta_8)\colon \text{ all } \Phi_i \text{ stay} .\\
&\sigma\colon (\theta_1,\dots,\theta_8) \curvearrowright (\theta_4,\theta_5,\theta_6,\theta_1,\theta_2,\theta_3,\theta_7,-\theta_8)\colon \ \Phi_1 \leftrightarrow \Phi_2,\ \Phi_3^+ \leftrightarrow \Phi_3^- ,\\
&\qquad\qquad \Phi_4, \Phi_5, \Phi_6 \text{ stays} .\\
&\varrho_1\colon (\theta_i+\theta_{i+3}, \theta_i-\theta_{i+3}, \theta_7, \theta_8)\\
&\quad \curvearrowright \left(\frac{(\theta_i-\theta_{i+3})}{\sqrt{b_i}}, (\theta_i+\theta_{i+3})\sqrt{b_i}, \theta_8\sqrt{b_4}, \frac{\theta_7}{\sqrt{b_4}}\right) \begin{cases} \Phi_1 \leftrightarrow \Phi_3^+ ,\ \Phi_2 \leftrightarrow \Phi_3^- \\ \Phi_4 \leftrightarrow \Phi_6 ,\ \Phi_5 \text{ stays} . \end{cases}
\end{aligned} \tag{6}$$

Notice ϱ_1 is based on replacing Φ_1, Φ_2 by Φ_3^+, Φ_3^-, suggesting two additional involutions: namely ϱ_2 and ϱ_3, which are based on the interchanges $\Phi_2 \leftrightarrow \Phi_3^+$ and $\Phi_2 \leftrightarrow \Phi_3^-$.

The projective linear span $V(\Phi_1, \dots, \Phi_6)$ of the quadrics (3) *contains a surface of rank* 4 *quadrics, itself given by the intersection of* 4 *dependent cones K_i (rank* 3 *quadrics) in $\mathbb{P}^5$. This surface is precisely the Kummer surface $KmA^\vee$ associated with the dual $A^\vee$ of A. The linear span of the first three quadrics $\Phi_1, \Phi_2, \Phi_3^\pm$ depends on even sections $\theta_1, \dots, \theta_6$ only and it contains four rank* 3 *quadrics Φ_1, Φ_2, Φ_3^+, and Φ_3^-,* *which can thus be viewed as points on* $A^\vee$*; their intersection defines the Kummer surface* KmA.

Proof. First observe that $\dim H^0(\mathscr{L})=2$ and thus $|\mathscr{L}|$ is a one-dimensional pencil of curves; let D_0 be a generic curve among them. Barth [7] proves that the base locus of $|\mathscr{L}|$ consists of four distinct points $e_1,\ldots,e_4$, such that $D_0+e_i\cong D_0$. Moreover the general $D_0\in|\mathscr{L}|$ is smooth and has genus 3. Then there exists an origin on A and a reflection τ having as fixed points (half-periods) the 4 points $e_1,\ldots,e_4$ and 12 additional points $e_5,\ldots,e_{16}$. Then according to Barth, for all $D\in|\mathscr{L}|$, $\tau D=D$ and (for D smooth) D/τ is an elliptic curve $\mathscr{E}$ on the Kummer surface A/τ, showing that D is a double cover of an elliptic curve. Then the line bundle $\mathscr{L}^{\otimes 2}$ is very ample and there is a basis of even sections $\theta_1,\ldots,\theta_6$ and odd sections θ_7,θ_8 such that A (as embedded into $\mathbb{P}^7$ by these sections) is given by the 6 quadrics

$$\begin{aligned}
&Q_i=(v_i+v_i^{-1})(\theta_1^2\pm\theta_4^2)-2(\theta_2^2\pm\theta_5^2)+(v_i-v_i^{-1})(\theta_3^2\pm\theta_6^2)=0\,,\\
&\qquad i=1,2 \text{ associated with } \pm\\
&Q_3=2((v_3+v_3^{-1})\theta_1\theta_4-2\theta_2\theta_5+(v_3-v_3^{-1})\theta_3\theta_6)=0\,,\\
&Q_4==(v_1+v_2)(\theta_1^2+\theta_3^2)+(v_1-v_2)(\theta_4^2+\theta_6^2)-2\theta_2^2+2\theta_7^2=0\,,\\
&\qquad\qquad Q_5=2v_3(\theta_1\theta_4+\theta_3\theta_6)-2\theta_2\theta_5+2\theta_7\theta_8\,,\\
&Q_6=(v_1-v_2)(\theta_1^2+\theta_3^2)+(v_1+v_2)(\theta_4^2+\theta_6^2)-2\theta_5^2+2\theta_8^2=0\,;
\end{aligned}\tag{7}$$

the first three depend on even sections only and the three remaining ones depend also on the odd sections θ_7 and θ_8.

Any linear combination of the Q_i's has the block form

$$\begin{aligned}
XQ_1+\ldots+WQ_6\equiv\sum_1^3(\delta_i\theta_i^2+2\delta_{i,i+3}\theta_i\theta_{i+3}+\delta_{i+3}\theta_{i+3}^2)&\\
+(\delta_7\theta_7^2+2\delta_{7,8}\theta_7\theta_8+\delta_8\theta_8^2)\,,&
\end{aligned}\tag{8}$$

revealing the existence of 4 involutions. The locus of points $p=(X,\ldots,W)\in\mathbb{P}^5$, such that (8) is a rank 4 quadric is given by the intersection of the four quadratic cones

$$K_i=\{p|\delta_i\delta_{i+3}-\delta_{i,i+3}^2=0\}\,,\qquad K_4=\{p|\delta_7\delta_8-\delta_{7,8=0}^2\}\quad i=1,2,3\,,$$

each having rank 3; the explicit expressions for the cones are the following

$$\begin{aligned}
&\left.\begin{matrix}K_1\\K_3\end{matrix}\right\}[(v_1\pm v_1^{-1})X+v_1(U+W)]^2-[(v_2\pm v_2^{-1})Y+v_2(U-W)]^2\\
&\quad-[(v_3\pm v_3^{-1})Z+v_3V]^2=0\,,\\
&K_2\colon 4(X+Y+U)(X-Y+U)-(2Z+V)^2=0\,,\\
&K_4\colon 4UW-V^2=0\,.
\end{aligned}$$

By straightforward computation we have $K_1-K_3=K_2-K_4$ and therefore $\bigcap\limits_1^4 K_i$

defines a surface, which Barth identifies with the Kummer surface $KmA^\vee$ of the dual $A^\vee$ of the Abelian surface A:

$$V(Q_1, \ldots, Q_6) \simeq \mathbb{P}^5 \supset KmA^\vee \equiv \bigcap_1^4 K_i = \text{surface of rank 4 quadrics}.$$

Taking into account the linear relation between the cones K_i, the hyperplane section

$$KmA^\vee \cap \{L^2U - K^2W = 0\} = K_1 \cap K_2 \cap \{U = K^2,\ V = 2KL,\ W = L^2\}$$

is an elliptic curve, which can be viewed as the curve of rank 4 quadrics in the linear span of the quadrics.

$$Q_1, Q_2, Q_3, Q_{K/L} = K^2Q_4 + 2KLQ_5 + L^2Q_6 .$$

According to Theorem 6 of [3], if the span of 4 quadrics of the block form (8) contains a (nondegenerate) elliptic curve of rank 4 quadrics, then it contains a new basis, which after a block-preserving change of variables, has the following form

$$\Phi_1(\theta), \Phi_2(\theta), \Phi_3(\theta), \Phi_{\kappa/\lambda} \equiv \kappa^2\Phi_4 + 2\kappa\lambda b_4\Phi_5 + \lambda^2\Phi_6 \tag{9}$$

in terms of the basis (4), for an appropriate choice of κ and λ. To show that the spaces spanned by the two sets of 6 quadrics (7) and (3) match, we observe that Φ_1, Φ_2, and Φ_3 are in the span of Q_1, Q_2, Q_3 and the rest of the argument proceeds by picking three distinct values of K/L. This shows the basis (7) can be replaced by the Φ's of (3).

To see that the spans of the quadrics $\Phi_1, \ldots, \Phi_6$ and $\Phi'_1, \ldots, \Phi'_6$ are the same, we check

$$\begin{aligned}
\Phi_1 &= \Phi'_1 ,\\
\Phi_2 &= \Phi'_2 ,\\
\Phi_3^{\pm} &= \frac{1}{2\Delta}\{(r \pm \Delta)\Phi'_1 + (r \mp \Delta)\Phi'_2 + 2\Phi'_3\} ,\\
\Phi_4 &= \Phi'_1 + \Phi'_2 + \frac{b_4}{16}(\Phi'_4 - 2\Phi'_5 + \Phi'_6) ,\\
\Phi_5 &= \Phi'_1 - \Phi'_2 + \frac{1}{16b_4}(\Phi'_4 - \Phi'_6) ,\\
\Phi_6 &= \Phi'_1 + \Phi'_2 + \frac{1}{16b_4}(\Phi'_4 + 2\Phi'_5 + \Phi'_6) ,
\end{aligned}$$

where

$$r = 2(a_1 + a_2 + a_3 - a_1a_2a_3) , \qquad \Delta^2 = 4T(1)T(-1) .$$

To do this identification, the b_i must be related to the a_i by the fractional linear map

$$z \curvearrowright \frac{z-1}{z+1} b_4 . \tag{10}$$

It is trivial to check from the equations $\Phi=0$ that the maps τ, σ, ϱ_1 are involutions on A. The involution ϱ_2 is obtained by changing variables $\theta_i \curvearrowright \theta_i'$ such that Φ_1, Φ_3^+ get mapped to Φ_2, Φ_3^-. This is achieved by letting Φ_1, Φ_3^+ play the role of Φ_1 and Φ_2 and Φ_2, Φ_3^- play the role of Φ_3^+, Φ_3^-; this leads to a new set of quadrics $\Phi_1, \ldots, \Phi_6$, having the same form, but with new b_i' expressed in terms of the old b_is as follows:

$$(b_i')^{1/2} = \frac{1-b_i^{1/2}}{1+b_i^{1/2}}, \qquad i=1,2,3.$$

The involution ϱ_3 is similar to ϱ_2.

In order to establish the Wronskian relationship, observe that the affine surface $\mathscr{A}_{\kappa/\lambda}$ can be viewed as the intersection of 4 quadrics:

$$\mathscr{A}_{\kappa/\lambda} \equiv A \backslash \{\kappa\theta_7 + \lambda\theta_8 = 0\} = \bigcap_1^3 \{\Phi_i(u)=0\} \cap \{\Phi_{\kappa/\lambda}(u)=0\}, \tag{11}$$

expressed in the affine coordinates

$$(u_0, u_1, \ldots, u_6) = (\kappa\theta_7 + \lambda\theta_8)^{-1}(\kappa\theta_7 + \lambda\theta_8, \theta_1, \ldots, \theta_6), \qquad u_0 \neq 0, \tag{12}$$

where $\Phi_{\kappa/\lambda}$ has been defined in (9). These four quadrics are those obtained in [4, Sect. 5]; there we exhibit two quadratic commuting vector fields $u_i^{\cdot} = f_i(u, \kappa/\lambda)$ defined on $\mathscr{A}_{\kappa/\lambda}$:

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \end{pmatrix} = \kappa \begin{pmatrix} \lambda_{65}u_6u_5 \\ \lambda_{46}u_4u_6 \\ \lambda_{54}u_5u_4 \\ \lambda_6u_2u_6 - \lambda_5u_3u_5 \\ \lambda_4u_3u_4 - \lambda_6u_1u_6 \\ \lambda_5u_1u_5 - \lambda_4u_2u_4 \end{pmatrix} - (\lambda/b_4) \begin{pmatrix} \lambda_3u_3u_5 - \lambda_2u_2u_6 \\ \lambda_1u_1u_6 - \lambda_3u_3u_4 \\ \lambda_2u_2u_4 - \lambda_1u_1u_5 \\ \lambda_{32}u_3u_2 \\ \lambda_{13}u_1u_3 \\ \lambda_{21}u_2u_1 \end{pmatrix}$$

and (13)

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \end{pmatrix} = \kappa \begin{pmatrix} \lambda'_{65}u_3u_2 \\ \lambda'_5u_3u_1 - \lambda'_4u_4u_6 \\ \lambda'_4u_5u_4 - \lambda'_6u_2u_1 \\ \lambda'_6u_2u_6 - \lambda'_5u_3u_5 \\ \lambda'_{54}u_4u_3 \\ \lambda'_{46}u_2u_4 \end{pmatrix} - (\lambda/b_4) \begin{pmatrix} \lambda'_3u_3u_5 - \lambda'_2u_2u_6 \\ \lambda'_{21}u_1u_6 \\ \lambda'_{13}u_5u_1 \\ \lambda'_{32}u_6u_5 \\ \lambda'_2u_4u_6 - \lambda'_1u_3u_1 \\ \lambda'_1u_2u_1 - \lambda'_3u_4u_5 \end{pmatrix},$$

where $\lambda_{ij} \equiv \lambda_i - \lambda_j$ and $\lambda'_{ij} \equiv \lambda'_i - \lambda'_j$, with

$$\lambda_i = \frac{1}{b_4^{-1} - b_i^{-1}}, \quad \lambda'_i = \frac{1}{b_4'^{-1} - b_i'^{-1}}, \quad \lambda_{i+3} = \frac{1}{b_4 - b_i}, \quad \lambda'_{i+3} = \frac{1}{b'_4 - b'_i},$$

and $(b'_1, b'_2, b'_3, b'_4) = (b_4, b_2, b_3, b_1)$. These vector fields on $\mathscr{A}_{\kappa/\lambda}$ extend to holomorphic vector fields X on T^2. Therefore substituting (12) for u_i in $\dot{u}_i = f_i(u, \kappa/\lambda)$, we find

$$\{\theta_i, \kappa\theta_7 + \lambda\theta_8\}(\kappa\theta_7 + \lambda\theta_8)^{-2} = f_i(\theta_1, \ldots, \theta_6, \kappa, \lambda)(\kappa\theta_7 + \lambda\theta_8)^{-2},$$

where the f_i denote quadratic polynomials of $\theta_1, \ldots, \theta_6$ depending on κ/λ. Picking $(\kappa, \lambda) = (1, 0)$ and $(0, 1)$, we find

$$\{\theta_i, \theta_7\} = f_i(\theta_1, \ldots, \theta_6, 1, 0) \quad \text{and} \quad \{\theta_i, \theta_8\} = f_i(\theta_1, \ldots, \theta_6, 0, 1),$$

which leads to the inclusion (2).

The part of Theorem 1 concerning the surface of rank 4 quadrics is straightforward by using the fact that the Q_i and the Φ_i span the same space of quadrics, and that $V(Q_1, \ldots, Q_6)$ has been shown to contain a surface of rank 4 quadrics. Finally, the statements concerning the Kummer surfaces are proven in Barth [7].

3. Removing from A the Zero Locus of Odd Sections and Manakov's Geodesic Flow

Whatever the $k = k/\lambda$, the affine surface $\mathscr{A}_k$ defined in (2.11) supports commuting vector fields having a striking form and, in some new coordinates, $\mathscr{A}_k$ can be viewed as the intersection of the four quadratic invariants of Manakov's geodesic flow on $SO(4)$. Therefore any holomorphic vector field on A restricted to $\mathscr{A}_k$ can be realized as a Manakov geodesic flow and thus it can be represented as a Lax pair, which becomes particularly simple when $k = 0$ and ∞. The techniques and arguments in this section rely heavily on our work about the intersection of quadrics [4]. The Kötter quadrics, introduced in Sect. 2 play an important role in the sense that $\mathscr{A}_0$ and $\mathscr{A}_\infty$ both complete to an Abelian surface by adjoining an 8 fold unramified cover of a hyperelliptic curve, whereas the general affine part $\mathscr{A}_k$ is obtained by some kind of "interpolation" process. Throughout this paper given any vector $u \in \mathbb{R}^6$, set $u' = (u_1, u_2, u_3)$ and $u'' = (u_4, u_5, u_6)$; also given a and $x \in \mathbb{R}^n$, we define $a \cdot x \equiv (a_1 x_1, \ldots, a_n x_n) \in \mathbb{R}^n$.

Theorem 2. I. *The family of divisors $\mathscr{C}_k$ on A,*

$$\mathscr{C}_k = \{\kappa\theta_7 + \lambda\theta_8 = 0\} \cap A, \qquad k = \kappa/\lambda,$$

forms a linear pencil of (generically) smooth curves having as base points the 16 *half-periods $e_1, \ldots, e_{16}$ and having genus* 9. *They are* 4–1 *unramified covers of genus* 3 *curves $D_k^\vee \subset A^\vee$, via the isogeny $\phi: A \to A^\vee$. Moreover through the projection $A^\vee \to KmA^\vee$, the curves $D_k^\vee$ are* 2–1 *ramified covers of the elliptic curves $\mathscr{E}_k^\vee$, ramified at* 4 *half-periods $\phi(e_i) = e_i^\vee$ $(i = 1, \ldots, 4)$ on $A^\vee$. The spaces*

$$L(\mathscr{C}_k) = L^- \oplus L^+ = \{u_1, \ldots, u_6\} \oplus \left\{1, \frac{\theta_7}{\kappa\theta_7 + \lambda\theta_8}\right\}, \tag{1}$$

where

$$(u_0, u_1, \ldots, u_6) = (\kappa\theta_7 + \lambda\theta_8)^{-1}(\kappa\theta_7 + \lambda\theta_8, \theta_1, \ldots, \theta_6), \qquad u_0 \neq 0.$$

For convenience, define

$$\kappa' = \tfrac{1}{4}(\kappa b_4^{1/2} + \lambda b_4^{-1/2}) \quad \text{and} \quad \lambda' = \tfrac{1}{4}(-\kappa b_4^{1/2} + \lambda b_4^{-1/2}). \tag{2}$$

such that $\kappa\theta_7 + \lambda\theta_8 = \kappa'\theta_7' + \lambda'\theta_8'$ in the notation of (2.4).

II. *For every $k = \kappa/\lambda$, define the affine surface $\mathscr{A}_k$,*

$$\mathscr{A}_k = A \backslash \mathscr{C}_k = \bigcap_1^3 \{\Phi_i(u) = 0\}$$
$$\cap \{\Phi_k(u) \equiv (\kappa^2\Phi_4 + 2\kappa\lambda b_4 \Phi_5 + \lambda^2 \Phi_6)(\theta)(\kappa\theta_7 + \lambda\theta_8)^{-2} = 0\}. \tag{3}$$

The holomorphic vector fields on A, restricted to $\mathscr{A}_k$, take on the form

$$\begin{aligned}\dot{u}' &= u' \wedge \left((\kappa'+\lambda')\frac{\partial H^+}{\partial u'} + (\kappa'-\lambda')\frac{\partial H^-}{\partial u'}\right),\\ \dot{u}'' &= u'' \wedge \left((\kappa'+\lambda')\frac{\partial H^+}{\partial u''} - (\kappa'-\lambda')\frac{\partial H^-}{\partial u''}\right).\end{aligned} \tag{4}$$

Picking

$$H^{\pm} = \mp b_4^{\pm 1} \sum_{i=1}^{3} \frac{(u_i \mp u_{i+3})^2}{b_i^{\pm 1} - b_4^{\pm 1}},$$

(essentially Φ_6 and Φ_4) in (4) *yields the vector field*[3] *(particularly distinguished in view of Theorem 3):*

$$\begin{aligned}\dot{u}' &= u' \wedge (a \cdot (\kappa' u' - \lambda' u'') - \kappa' u''),\\ \dot{u}'' &= u'' \wedge (a \cdot (\lambda' u'' - \kappa' u') - \lambda' u').\end{aligned} \tag{5}$$

whereas another vector field is given by picking

$$H^{\pm} = \mp b_4^{\pm 1} \left(\frac{(u_2 \pm u_5)^2}{b_1^{\pm 1} - b_3^{\pm 1}} + \frac{(u_3 \pm u_6)^2}{b_1^{\pm 1} - b_2^{\pm 1}} + \frac{(u_1 \mp u_4)^2}{b_1^{\pm 1} - b_4^{\pm 1}} \right), \tag{6}$$

remembering the relationship (2.5) *between the parameters a_i and b_i appearing in the quadrics Φ and Φ'. For $\kappa = 0$ or $\lambda = 0$, the flow* (5) *takes on the following simple form*

$$\begin{array}{c|c} \kappa = 0 \ (\kappa' = \lambda') & \lambda = 0 \ (\kappa' = -\lambda') \\ (u'+u'')^{\cdot} = (u'-u'') \wedge ((a+1)\cdot(u'-u'')) & (u'-u'')^{\cdot} = (u'+u'') \wedge ((a-1)\cdot(u'+u'')) \\ (u'-u'')^{\cdot} = (u'+u'') \wedge ((a+1)\cdot(u'-u'')) & (u'+u'')^{\cdot} = (u'-u'') \wedge ((a-1)\cdot(u'+u'')). \end{array} \tag{7}$$

III. *Another description for $\mathscr{A}_k$ is given by the Manakov quadrics, namely*

$$\mathscr{A}_k = \bigcap_1^4 \{Q_i = 0,\ x_0 = 1\}, \tag{8}$$

where

$$\begin{aligned} Q_1 &= \frac{x_4^2}{\alpha_1 - \alpha_4} + \frac{x_2^2}{\alpha_1 - \alpha_3} + \frac{x_3^2}{\alpha_1 - \alpha_2} - A_1 x_0^2,\\ Q_2 &= \frac{x_1^2}{\alpha_2 - \alpha_3} + \frac{x_5^2}{\alpha_2 - \alpha_4} + \frac{x_3^2}{\alpha_2 - \alpha_1} - A_2 x_0^2,\\ Q_3 &= \frac{x_1^2}{\alpha_3 - \alpha_2} + \frac{x_2^2}{\alpha_3 - \alpha_1} + \frac{x_6^2}{\alpha_3 - \alpha_4} - A_3 x_0^2,\\ Q_4 &= x_1 x_4 + x_2 x_5 + x_3 x_6 - A_4 x_0^2 \end{aligned}$$

for some appropriate values of α_i and A_i (given below) depending on the b_i and k^2. The surface $\mathscr{A}_k$ supports commuting vector fields having the Lax form

$$(X + \alpha h)^{\cdot} = [X + \alpha h, \lambda \cdot X + \beta h], \qquad \lambda_{ij} = \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j}, \tag{9}$$

with $X \in so(4)$ parametrized by (1.10). *The linear map $u \curvearrowright x$ connecting the two descriptions* (3) *and* (8) *is given by*

$$x_i = \sqrt{a_i^+}\,(u_i + b_i^+ u_{i+3}), \qquad x_{i+3} = \sqrt{a_i^-}\,(u_i + b_i^- u_{i+3}). \tag{10}$$

[3] For $a, b \in \mathbb{R}^n$, $a \cdot b = (a_1 b_1, \ldots, a_n b_n) \in \mathbb{R}^n$

In (8), (9), *and* (10), *the* α_i, A_i, $a_i^{\pm}$, $b_i^{\pm}$ *are functions of the* b_i *and* $k=\kappa/\lambda$, *to wit:*

$$\alpha_i=(b_4k-b_ik^{-1})(b_4k-b_i^{-1}k^{-1}),\quad \alpha_4=0,\quad i=1,2,3,$$

$$A_i=\sum_1^3\tau_j-2\tau_i,\qquad A_4=(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)(\alpha_1-\alpha_2)(k^{-2}-b_4^2k^2),$$

$$B_1=\sum_1^3\alpha_iA_i,\qquad B_2=\sum_1^3\alpha_i(\alpha_{i+1}+\alpha_{i-1})A_i,$$

$$B_3=\alpha_1\alpha_2\alpha_3(A_1+A_2+A_3),\qquad B_4=A_4,$$

$$\alpha_i-\alpha_j=(b_i-b_j)(b_k-b_4),\qquad \tau_i=b_4(b_i^{-1}-b_i)(\alpha_j-\alpha_k)(\alpha_j\alpha_k)^{1/2},$$

$$\alpha_1^+=-\frac{1}{H_2(b_1,b_2,b_3)}\left(\frac{1}{\alpha_3^{1/2}H_2(b_2,b_3,b_1)}\pm\frac{1}{\alpha_2^{1/2}H_2(b_3,b_1,b_2)}\right),$$

$$\alpha_1^-=\frac{\alpha_1}{\alpha_2-\alpha_3}\alpha_1^+,\tag{11}$$

$$b_1^{\pm}=H^{-1}(H_1(b_1,b_2,b_3)\pm\sqrt{\alpha_2\alpha_3}),$$

$$H_1(b_1,b_2,b_3)=(b_4+b_1-b_2-b_3)k^{-1}+(b_4^{-1}+b_1^{-1}-b_2^{-1}-b_3^{-1})b_4^2k,$$

$$H_2(b_1,b_2,b_3)=(b_4+b_1-b_2-b_3)k^{-1}$$
$$-(b_4^{-1}+b_1^{-1}-b_2^{-1}-b_3^{-1})b_4^2k+2b_4(b_1-b_1^{-1}),$$

$$H(b_1,b_2,b_3)=2(k^{-2}-b_4^2k^2)+k^{-1}\sum_1^4 b_i-kb_4^2\sum_1^4 b_i^{-1},$$

and cyclic permutations. The indices i,j,k *in* $\alpha_i-\alpha_j$ *and* τ_i *denote cyclic permutations of* 1, 2, 3.

IV. *The curves* $D_k^\vee$ *underlying* $\mathscr{C}_k$, *form a linear pencil* $|D_k^\vee|$ *on* $A^\vee$, *which projects down to a system of elliptic curves on* $KmA^\vee$ *given by*

$$\mathscr{E}_k^\vee=KmA^\vee\cap\{\lambda^2U-\kappa^2W=0\}\subset V(\Phi_1,\Phi_2,\Phi_3^+,\Phi_k)\equiv\mathbb{P}^3,\qquad k=\kappa/\lambda.\tag{12}$$

Each $\mathscr{E}_k^\vee$ *is the locus of rank* 4 *quadrics in the linear span* $V(\Phi_1,\dots,\Phi_k)$; *in the* Q_i*-coordinates,* $\mathscr{E}_k^\vee$ *can be represented as follows:*

$$\begin{aligned}\mathscr{E}_k^\vee=\Big\{&t^2\sum_1^3\alpha_iQ_i-t\sum_1^3\alpha_i(\alpha_{i-1}+\alpha_{i+1})Q_i\\&+\alpha_1\alpha_2\alpha_3\sum_1^3Q_i+2\prod_1^4(t-\alpha_i)^{1/2}Q_4\\&=\sum_1^3\left(\sqrt{(t-\alpha_i)t}\,x_i+\sqrt{(t-\alpha_j)(t-\alpha_k)}\,x_{i+3}\right)^2\\&+x_0^2\Big(B_1t^2-B_2t+B_3+2B_4\prod_1^4(t-\alpha_i)^{1/2}\Big),\ t\in\mathbb{C}\Big\}\end{aligned}$$

$=$double cover of $\mathbb{P}^1$ ramified at the 4 points $\alpha_1,\alpha_2,\alpha_3,\alpha_4$

$=$double cover of $\mathbb{P}^1$ ramified at the 4 points $b_ik+\dfrac{1}{b_ik}$,

$(i=1,2,3,4)$, (13)

with B_i defined in (11). *The curve $D_k^\vee$ is a* 2–1 *cover of $\mathscr{E}_k^\vee$ ramified at the points where the ranks drops to* 3; *thus we have*

$$D_k^\vee : (w^2-(B_1t^2-B_2t+B_3))^2-4B_4^2\prod_1^4(t-\alpha_i)=0; \tag{14}$$

the base points for this linear system $D_k^\vee$ are precisely these 4-*rank* 3 *quadrics. Moreover the linear pencil $|D_k^\vee|$ contains* 6 *smooth hyperelliptic curves, isomorphic two by two; the first pair corresponding to $k=0$ and ∞, together with the two other pairs, are given by the following equations:*

$$y^2=\prod_1^4(x^2-b_i), \quad y^2=\prod_1^4(x^2-b_i'), \quad y^2=\prod_1^4(x^2-b_i''), \tag{15}$$

each covering the elliptic curves

$$y^2=\prod_1^4(x-b_i), \quad y^2=\prod_1^4(x-b'), \quad y^2=\prod_1^4(x-b_i'').$$

The corresponding curves in the pencil $|\mathscr{C}_k|$ on A can also be viewed as 8–1 *unramified covers of the genus* 2 *hyperelliptic curves*[4]

$$y^2=x\prod_1^4(x-b_i), \quad y^2=x\prod_1^4(x-b_i'), \quad y^2=x\prod_1^4(x-b_i''), \tag{16}$$

where

$$(b_l')^{1/2}=\frac{1-b_l^{1/2}}{1+b_l^{1/2}}, \quad (b_l'')^{1/2}=\frac{i-b_l^{1/2}}{i+b_l^{1/2}}, \quad l=1,2,3.$$

Then $A^\vee$ is a double unramified cover of 3 *different hyperelliptic Jacobians, corresponding to the three curves* (16). *For further use, also consider the curve $D_{b_4^{-1}}^\vee$, which is a double cover of*

$$\mathscr{E}_{b_4^{-1}}^\vee : y^2=\prod_1^3(x-a_i^2).$$

V. *Finally upon setting $v=1/h$ and $u\equiv z/h$, the spectral curve going with the Lax pair* (9) *reads as follows:*

$$\begin{aligned}\Sigma_k&=\{\det(X+\alpha h-zI)=0\}\\ &=\{B_4^2v^4+(B_1u^2-B_2u+B_3)v^2+\prod_1^4(u-\alpha_i)=0\}\subset A\end{aligned} \tag{17}$$

with B_i given by (11). *The curves Σ_k sweep out the linear pencil of curves going with the original line bundle $\mathscr{L}$ of Theorem 1 and the following linear equivalence holds:*

$$2\Sigma_k \simeq \mathscr{C}. \tag{18}$$

This induces a map from the pencil $|D_k^\vee|$ to the pencil $|\Sigma_k|$, which maps the smooth hyperelliptic curves to the singular curves and the singular to the smooth hyperelliptic curves. The map $|\check{D}_k| \curvearrowright |\Sigma_k|$ is obtained by flipping around the covers, for some appropriately chosen projection, as illustrated in Fig. 2.

[4] Notice $y^2=x\prod_1^4(x-b_i)$ is conformal to the curve $z^2=(x^2-1)\prod_1^3(x-a_i)$, using (2.5)

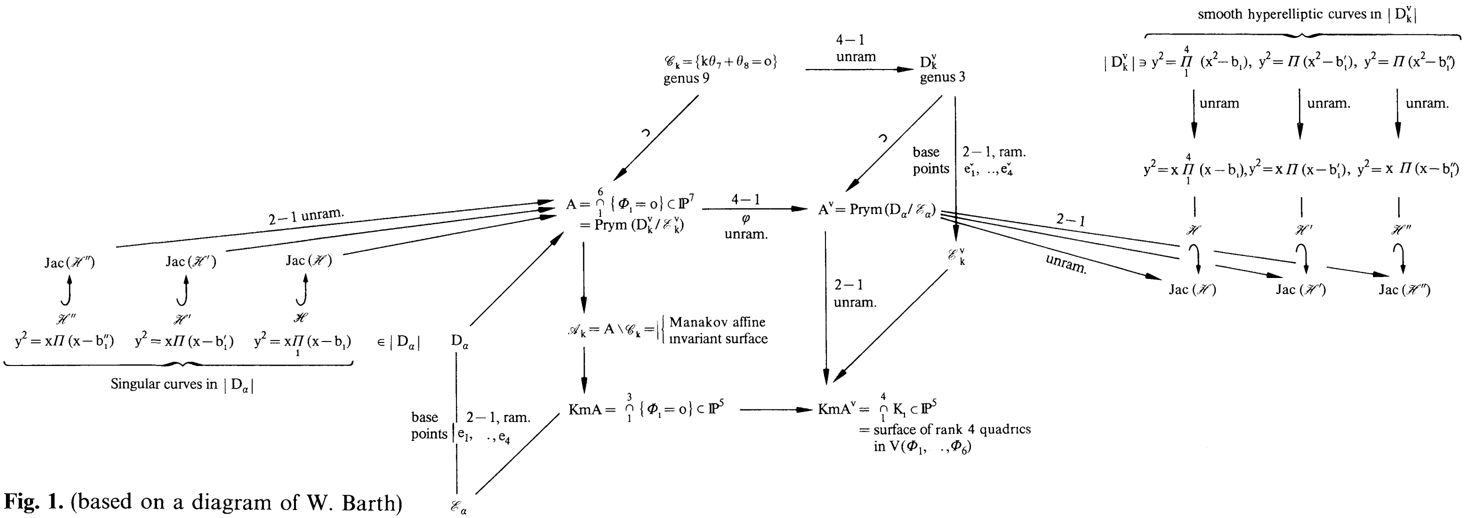

Fig. 1. (based on a diagram of W. Barth)

$$
\begin{tikzcd}
D_\alpha \arrow[d, "\substack{\text{ramified at}\\ \alpha_1,\dots,\alpha_4}"] & D_k^\vee \arrow[d, "\substack{\text{ramified at 4 roots}\\ \text{of } (B_1u^2-B_2u+B_3)^2-4B_4^2\prod_1^4(u-\alpha_i)}"] \\
\mathscr{E}_\alpha \arrow[d, "\substack{\text{ramified at 4 roots of}\\ (B_1u^2-B_2u+B_3)^2-4B_4^2\prod_1^4(u-\alpha_i)}"] & \mathscr{E}_k^\vee \arrow[d, "\substack{\text{ramified at}\\ \alpha_1,\dots,\alpha_4}"] \\
\mathbb{P}^1 & \mathbb{P}^1
\end{tikzcd}
$$

Fig. 2

Corollary 1. *For $\kappa=0$ (and similarly for $\lambda=0$) the flow* (5) *admits the following simple Lax pairs (remembering the map* $\hat{}:\ \mathbb{R}^3\to so(3)$, *defined in* (1.22)*):*

(i) $$(X+\alpha h)^{\cdot}=[X+\alpha h,\ Y+\beta h],\qquad X,Y\in so(4)$$

with

$$X=\begin{pmatrix} 0 & -u_3-u_6 & u_2+u_5 & -u_1+u_4 \\ u_3+u_6 & 0 & -u_1-u_4 & -u_2+u_5 \\ -u_2-u_5 & u_1+u_4 & 0 & -u_3+u_6 \\ u_1-u_4 & u_2-u_5 & u_3-u_6 & 0 \end{pmatrix}$$

$$Y=\begin{pmatrix} 0 & 0 & 0 & \dfrac{-u_1+u_4}{b_1-b_4} \\ 0 & 0 & 0 & \dfrac{-u_2+u_5}{b_2-b_4} \\ 0 & 0 & 0 & \dfrac{-u_3+u_6}{b_3-b_4} \\ \dfrac{u_1-u_4}{b_1-b_4} & \dfrac{u_2-u_5}{b_2-b_4} & \dfrac{u_3-u_6}{b_3-b_4} & 0 \end{pmatrix}$$

$$\alpha=\operatorname{diag}(b_1,b_2,b_3,b_4),\qquad \beta=\operatorname{diag}(0,0,0,-1).$$

The corresponding spectral curves Σ_0 and Σ_∞ are hyperelliptic of genus 2, namely $y^2=x\prod_1^4(x-b_i)$; the latter is isomorphic to $y^2=(x^2-1)T(x)$.

(ii) $$\begin{pmatrix} \hat{u}' & \mathscr{D}h \\ \mathscr{D}h & \hat{u}'' \end{pmatrix}^{\cdot}=\left[\begin{pmatrix} \hat{u}' & \mathscr{D}h \\ \mathscr{D}h & \hat{u}'' \end{pmatrix},\begin{pmatrix} \hat{w} & Ih \\ Ih & -\hat{w} \end{pmatrix}\right]$$

with

$$\begin{gathered} w=\left(\frac{u_1-u_4}{b_1-b_4},\frac{u_2-u_5}{b_2-b_4},\frac{u_3-u_6}{b_3-b_4}\right), \\ \mathscr{D}=\tfrac{1}{2}(b_1+b_2+b_3-b_4)I-\operatorname{diag}(b_1,b_2,b_3). \end{gathered}\tag{19}$$

The spectral curve going with this Lax pair is a cover of the spectral curve obtained in (i).

(iii)

$$(a+1)((u'-u'')\otimes(u'-u'')+(\hat{u}'+\hat{u}'')h+Ih^2)^{\cdot}$$
$$=[(a+1)((u'-u'')\otimes(u'-u'')+(\hat{u}'+\hat{u}'')h+Ih^2),\ (a+1)(\hat{u}'+\hat{u}''+Ih)]\,.$$

where the 3×3 *matrix* $a+1$ *stands for* $\operatorname{diag}(a_1+1, a_2+1, a_3+1)$. *Here also the spectral curve is given by the hyperelliptic curve* $y^2=(x^2-1)T(x)$.

Proof of Theorem 3. The affine surface $\mathscr{A}_k$, $k=\kappa/\lambda$, is defined by the intersection of the quadrics Φ_1, Φ_2, Φ_3, and Φ_k. In (2.13), we gave a set of two commuting vector fields on $\mathscr{A}_{\kappa/\lambda}$; the first vector field has the form (5), whereas the second has the form (4) with $H^{\pm}$ as in (6). For $\kappa=0$ or $\lambda=0$, the vector field (5) transforms into (7) by making sums and differences of the Eqs. (5). Having shown Part II we now proceed to Part III.

Since the linear span $V(\Phi_1, \Phi_2, \Phi_3^+, \Phi_k)=\mathbb{P}^3$ contains a non-degenerate curve of rank 4 quadrics

$$\mathscr{E}_k^{\vee}=A^{\vee}\cap\{\lambda^2 U-\kappa^2 W=0\}\,,\qquad k=\kappa/\lambda\,,$$

the space V can instead be spanned by Φ_1, Φ_2, Φ_3^+ and one other rank 4 quadric. Thus in the variables u_i they have the general form

$$Q_1=\sum_1^3 u_i^2\,,\qquad Q_2=\sum_1^3 u_{i+3}^2\,,$$
$$Q_3=\sum_1^3(\gamma_i u_i+\gamma_{i+3}u_{i+3})^2+\gamma_0 u_0^2\,,\tag{20}$$
$$Q_4=\sum_1^3(\delta_i u_i+\delta_{i+3}u_{i+3})^2+\delta_0 u_0^2$$

with $\gamma_0=0$. By taking appropriate linear combinations, Q_3 and Q_4 can be replaced by new quadrics Q_3 and Q_4 of the same form, but with $\gamma_1=\delta_2=0$, $\gamma_2=\delta_1=0$. This new set $Q_1,\ldots,Q_4$ corresponds to 4 points on the curve $\mathscr{E}_{\kappa/\lambda}^{\vee}$; this curve can be viewed as the intersection of the three quadratic cones in $\mathbb{P}^3$:

$$K_i:\mu_i\mu_{i+3}-\mu_{i,i+3}^2\equiv(X+\gamma_i^2 Y+\delta_i^2 U)(Y+\gamma_{i+3}^2 Z+\delta_{i+3}^2 U)$$
$$-(\gamma_{i,i+3}Z+\delta_{i,i+3}U)^2=0$$

expressed in the coordinates X, Y, Z, U for the basis $Q_1,\ldots,Q_4$ of (20). The functions $\mu_i=\mu_i(X, Y, Z, U)$ can be viewed as meromorphic functions on $\mathscr{E}_k$.

As explained in [4], the curve $\mathscr{E}_k^{\vee}$ contains 3 points $p_i=(X_i, Y_i, Z_i, U_i=1)$ leading to three simultaneously diagonalized quadrics $Q(p_i)=X_iQ_1+Y_iQ_2+Z_iQ_3+Q_4$:

$$Q(p_i)=\mu_1(p_i)\left(u_1+\frac{\mu_{14}}{\mu_1}(p_i)u_4\right)^2+\mu_2(p_i)\left(u_2+\frac{\mu_{25}}{\mu_2}(p_i)u_5\right)^2$$
$$+\mu_3(p_i)\left(u_3+\frac{\mu_{36}}{\mu_3}(p_i)u_6\right)^2+d_i\,,\tag{21}$$

with

$$\frac{\mu_{14}}{\mu_1}(p_2)=\frac{\mu_{14}}{\mu_1}(p_3), \quad \frac{\mu_{25}}{\mu_2}(p_3)=\frac{\mu_{25}}{\mu_2}(p_1), \quad \frac{\mu_{36}}{\mu_3}(p_1)=\frac{\mu_{36}}{\mu_3}(p_2). \tag{22}$$

These three quadrics along with a fourth one

$$\sum_1^3 e_i\left(u_i+\frac{\mu_{i,i+3}}{\mu_i}(p_{i+1})u_{i+3}\right)\left(u_i+\frac{\mu_{i,i+3}}{\mu_i}(p_i)u_{i+3}\right)+d_4 \tag{23}$$

spans the linear space $V(Q_1, \ldots, Q_4)$. The fact that the points $p \in \mathscr{E}^{\vee}_{\kappa/\lambda}$ implies that

$$\frac{\mu_{i,i+3}}{\mu_i}(p)=\frac{\mu_{i+3}}{\mu_{i,i+3}}(p). \tag{24}$$

Therefore, using the relations (22) and (24), the quadrics (21) and (23) have the following "simultaneously diagonalized" form:

$$\begin{gathered}
Y_4^2+Y_2^2-Y_3^2-c(-U, V)Y_0^2, \\
Y_5^2+Y_3^2-Y_1^2-c(U, -V)Y_0^2, \\
Y_6^2+Y_1^2-Y_2^2+c(U, V)Y_0^2, \\
aY_1Y_4+bY_2Y_5+cY_3Y_6+dY_0^2.
\end{gathered} \tag{25}$$

A minor rescaling of the Y_i's yields the 4 Manakov quadrics (8).

This program is carried out explicitly in [4]; namely the γ_i and δ_i have the following form in terms of the b_i, $k=\kappa/\lambda$ and the quantities defined in (12):

$$\begin{gathered}
\gamma_1=\delta_2=0, \quad \gamma_2=\delta_1=1, \\
\gamma_4^2=\frac{4(\alpha_1-\alpha_3)\alpha_1}{H_2^2(b_1,b_2,b_3)}, \quad \gamma_5=\frac{H_1(b_3,b_1,b_2)}{H_2(b_1,b_2,b_3)}, \\
\gamma_0=\frac{(b_4-b_1)(b_4-b_2)(b_1-b_1^{-1})}{(b_1-b_2)H_2(b_1,b_2,b_3)}, \\
(\gamma_3^2, \gamma_{36}, \gamma_6^2)=\frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_2}\left(1, \frac{H_1(b_2,b_3,b_1)}{H_2(b_1,b_2,b_3)}, \frac{H_1(b_2,b_3,b_1)^2}{H_2(b_1,b_2,b_3)^2}\right),
\end{gathered}$$

whereas the δ_i are obtained from the γ_i by performing the following involution:

$$(\gamma_0, \gamma_4, \gamma_5, \gamma_3, \gamma_{36}, \gamma_6, b_1, b_2, b_3) \curvearrowright (\delta_0, \delta_5, \delta_4, \delta_3, \delta_{36}, \delta_6, b_2, b_1, b_3).$$

The linear change of variables $u \curvearrowright Y$ [in terms of (20)] is given by (see [4])

$$Y_i=\sqrt{a_i^+}(u_i+b_i^+u_{i+3}), \quad Y_{i+3}=\sqrt{a_i^-}(u_i+b_i^-u_{i+3}),$$

where the coefficients $a_i^{\pm}$, $b_i^{\pm}$, a, b, c, d, U, V are expressed in terms of the γ and δ's appearing in the quadrics (20); namely

$$a_1^+ = -\gamma_3^2(\gamma_{36}+\delta_{36})+U\,, \quad a_2^+ = -\delta_3^2(\gamma_{36}+\delta_{36})+V, \quad a_3^+ = -\delta_3^2 U - \gamma_3^2 V,$$

$$a_1^- = -a_1^+(-U), \quad a_2^- = a_2^+(-V), \quad a_3^- = a_3^+(-U, V),$$

$$b_1^+ a_1^+ = \delta_4(-\gamma_{36}+U), \quad b_2^+ a_2^+ = \gamma_5(-\delta_{36}+V), \quad b_3^+ a_3^+ = -\gamma_{36}V - \delta_{36}U\,,$$

$$b_1^- = b_1^+(-U), \quad b_2^- = b_2^+(-V), \quad b_3^- = b_3^+(-U/V),$$

$$V = (i\gamma_3/\delta_5)W, \quad V = (\delta_3/\gamma_4)W, \quad W^2 = -\gamma_5^2\delta_5^2\gamma_3^4 + \gamma_4^2\delta_4^2\delta_3^4\,,$$

$$a = i\gamma_4\delta_4\delta_3^2\,, \quad b = -\gamma_5\delta_5\gamma_3^2\,, \quad c = W,$$

$$c(U,V) = (\delta_{36}-V)\gamma_0 - (\gamma_{36}-U)\delta_0\,,$$

$$d = \gamma_3^2\delta_3^2\gamma_4^{-1}\delta_5^{-1}(\gamma_3^2\delta_{36}-\delta_3^2\gamma_{36})^{-1}(\gamma_4^2\delta_{36}+\delta_5^2\gamma_{36})(\gamma_4^2\delta_{36}\delta_0 - \delta_5^2\gamma_{36}\gamma_0)\,.$$

Combining these two sets of formulas, together with a minor rescaling yields the formulas (11) in the statement of this theorem. This shows that the affine surface $\mathscr{A}_k$ can also be viewed as the intersection of the four quadrics (8). These quadrics support commuting vector fields of the Lax type (9).

The computation presented above becomes invalid whenever the quantities in (12) vanish or become infinity; i.e., this happens for the values of κ and λ for which $A_4 = B_4 = 0$. Then the curve Σ_k in (17) becomes hyperelliptic. This proves Part III.

We now sketch the verification of Parts I and IV. From the asymptotic analysis of these differential equations in [5] the intersection of the four quadrics (8) completes into an Abelian surface upon adding a curve $\mathscr{C}_k$ of genus 9, which is a 4-fold unramified cover of a curve $D_k^\vee \subset A^\vee$ of genus 3; the latter is a double cover of the elliptic curve of rank 4 quadrics $\mathscr{E}_k^\vee$, ramified at the 4 points corresponding to the quadrics where the rank drops to 3.; an explicit representation of $\mathscr{E}_k^\vee$ (due to Haine [16]) in terms of the Manakov quadrics $Q_1, \ldots, Q_4$, is given in (13). The linear system $|D_k^\vee|$ on $A^\vee$ contains 3 pairs of smooth hyperelliptic curves of genus 3, and 12 singular curves, as shown by Horozov-van Moerbeke [18]. There it was shown that given a line bundle $\mathscr{L}$ defining on A a polarization (1, 2) and given the linear system $|D|$ going with $\mathscr{L}$, the Abelian surface A is a double unramified cover of the Jacobian of the smooth hyperelliptic sections in D, whereas the Jacobians of the singular sections (upon normalization) are double unramified covers of A. The representation (13) of $\mathscr{E}_k^\vee$ as a curve of rank 4 quadrics in terms of the Manakov quadrics follows from a straightforward computation. The curves $\mathscr{E}_\infty^\vee$ and $\mathscr{E}_{b_4^{-1}}^\vee$ have the form announced in (15) because for those specific values of $k = \kappa/\lambda$, the points α_i, defined in (12), can be moved to $b_1, \ldots, b_4$ and 0, a_1^2, a_2^2, a_3^2 respectively. The form of the spectral curve Σ_k is due to Haine [16] and comparing the formulas for $D_k^\vee$ and Σ_k confirms the diagram in Fig. 2. Except for Part V of the theorem (to be shown in Sect. 5), this ends the proof of Theorem 2.

Proof of Corollary 1. The equations (7) for $k=0$ are readily reformulated in terms of the Lax pair (up to some time rescaling)

$$(X+\alpha h)^{\cdot} = \left[X+\alpha h, \frac{\partial Q}{\partial X} + \beta h\right]$$

with

$$x_i = u_i + u_{i+3}, \qquad x_{i+3} = u_i - u_{i+3}, \qquad i = 1, 2, 3,$$
$$\alpha = \operatorname{diag}(b_1, \ldots, b_4), \qquad \beta = \operatorname{diag}(0, 0, 0, -1),$$
$$Q = \tfrac{1}{2}\left\{\frac{x_4^2}{b_1 - b_4} + \frac{x_5^2}{b_2 - b_4} + \frac{x_5^2}{b_3 - b_4}\right\}$$
$$= \frac{1}{4b_4}((a_1+1)x_4^2 + (a_2+1)x_5^2 + (a_3+1)x_6^2),$$

using the fractional linear relation (2.5) between a_i and b_i. The statement concerning the spectral curve Σ will be given in Sect. 5. The second Lax pair is obtained by considering the representation of $sl(4)$ as acting on $\Lambda^2\mathbb{C}$. In particular skew-symmetric and diagonal matrices transform as follows:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ -A_{12} & A_{22} & A_{23} & A_{24} \\ -A_{13} & -A_{23} & A_{33} & A_{34} \\ -A_{14} & -A_{24} & -A_{34} & A_{44} \end{pmatrix}$$

$$\curvearrowright \left(\begin{array}{ccc|ccc} 0 & A_{12}+A_{34} & A_{13}-A_{24} & A_{22}+A_{33} & 0 & 0 \\ -A_{12}-A_{34} & 0 & A_{23}+A_{14} & 0 & A_{33}+A_{11} & 0 \\ -A_{13}+A_{24} & -A_{23}-A_{14} & 0 & 0 & 0 & A_{11}+A_{22} \\ \hline A_{22}+A_{33} & 0 & 0 & 0 & A_{12}-A_{34} & A_{13}+A_{24} \\ 0 & A_{33}+A_{11} & 0 & -A_{12}+A_{34} & 0 & A_{23}-A_{14} \\ 0 & 0 & A_{11}+A_{22} & -A_{13}-A_{24} & -A_{23}+A_{14} & 0 \end{array}\right).$$

Using this representation we immediately get the second Lax pair, after a slight modification of α and β, so as to make α and β traceless. The third Lax pair is a straightforward consequence of (7) for $k=0$, which is a variation on a Lax pair due to Perelomov [30].

4. The Six Quadrics Associated with Kowalewski's Top

In this section we show how to apply the theory developed in Sect. 2 to the specific situation of Kowalewski's top. Given a line bundle $\mathscr{L}^{\otimes 2}$ and an origin, the splitting of the space of sections into even and odd subspaces, with regard to the reflection τ, can be found by picking a divisor D_0 in the linear system $|\mathscr{L}^{\otimes 2}|$ which is defined by an even or an odd section θ_0. Then the space $L(D_0)$ is spanned by the functions $\theta_1/\theta_0, \ldots, \theta_8/\theta_0$, and thus $L(D)$ splits into a 6-dimensional and a 2-dimensional space of even and odd functions, when θ_0 is even and the other way around when θ_0 is odd. By picking an appropriate basis of the function space $L(D_0)$, there must be 6 quadratic relations Φ between the functions, exactly of the type (2.3) discussed in Sect. 2. The salient features of the quadrics Φ are the following: they all have the block form (2.8) and their linear span has 4 rank 3 quadrics depending only on even functions when θ_0 is even and only on odd functions when θ_0 is odd. The program is thus as follows:

(i) find a divisor D_0 defined by either an even or an odd section.

(ii) write the 6 quadratic relations between the functions of $L(D_0)$,

(iii) find a new basis of $L(D_0)$ and thus a change of variables transforming the 6 quadratic relations into relations of the type Φ or Φ' in Sect. 2.

Besides $T(x) = \det(M - xI)$ already defined in (1.3), we introduce the following polynomials

$$\begin{aligned} R(x,y) &= -x^2y^2 + Axy - B(x+y) + C\,, \\ R_1(x,y) &= -Ax^2y^2 + 2Bxy(x+y) - C(x+y)^2 + AC - B^2\,, \\ P(x) &= R(x,x)\,, \qquad Q(x) = R_1(x,y)\,. \end{aligned} \tag{1}$$

The polynomials $T(x)$ and $P(x)$ have the same invariants g_2 and g_3.

As pointed out in Sect. 1, the invariant surface for the Kowalewski top completes into an Abelian surface, by adjoining two isomorphic genus 3 curves D^1 and D^2, intersecting in four points, each given by

$$D^i : (U^2+1)^2 Z^4 - (U^2+1)(Z^4 + P(Z) + 1) + 1 = 0\,. \tag{2}$$

This is a double cover of the elliptic curve

$$\mathscr{E} : W^2 = ((A-2)Z^2 - 2BZ + C + 1)((A+2)Z^2 - 2BZ + C + 1) \tag{3}$$

ramified at the 4 points where $U = 0$ covering the four roots of $P(Z) = 0$. The Abelian surface can be viewed as the dual of the Prym variety: $A = \mathrm{Prym}(D^i/\mathscr{E})^\vee$. The divisors $2D^1$, $2D^2$ or $D^1 + D^2$ are all very ample and they all define polarizations $(2,4)$. The line bundle $\mathscr{L}$ going with D^i has only even sections: the Riemann theta-function and a theta function with characteristic. Therefore D^i and thus $2D^i$ are both cut out by even theta-functions. In the same way $D^1 + D^2$ is defined by an odd theta function, but for a different reflection. As a consequence the program spelled out above can be carried out for any of these three divisors. For the sake of this exposition, take $D \equiv D^2$.

We first observe that the Kowalewski vector field in I. of Sect. 1, with the change of coordinates (1.1), takes the form

$$\begin{aligned} X_1 : \dot{x}_1 &= x_3x_1 - y_3\,, & \dot{y}_1 &= 2x_3y_1\,, \\ \dot{x}_2 &= -x_3x_2 + y_3\,, & \dot{y}_2 &= -2x_3y_2\,, \\ \dot{x}_3 &= x_2^2 - x_1^2 + y_1 - y_2\,, & \dot{y}_3 &= x_1(x_2^2 - y_2) - x_2(x_1^2 - y_1)\,. \end{aligned} \tag{4}$$

A second flow commuting with the first is regulated by the equations

$$\begin{aligned} X_2 : x_1' &= y_1(-x_2x_3 + y_3)\,, & y_1' &= 2y_1(y_3(x_1+x_2) - x_1x_2x_3)\,, \\ x_2' &= y_2(x_1x_3 - y_3)\,, & y_2' &= -2y_2(y_3(x_1+x_2) - x_1x_2x_3)\,, \\ x_3' &= x_2^2y_1 - x_1^2y_2\,, & y_3' &= x_2y_1(x_2^2 - y_2) - x_1y_2(x_1^2 - y_1)\,. \end{aligned}$$

The Kowalewski constants of motion map into

$$\begin{aligned}
Q_1 &= (x_1+x_2)^2+x_3^2-y_1-y_2=A\,,\\
Q_2 &= x_1x_2(x_1+x_2)-y_1x_2-y_2x_1+x_3y_3=B\,,\\
Q_3 &= x_1^2x_2^2+y_3^2-y_1x_2^2-y_2x_1^2=C\,,\\
Q_4 &= y_1y_2=D^2=1\,.
\end{aligned} \tag{5}$$

Define the vector field

$$X_{\kappa/\lambda}=(\kappa b_4^{1/2}/4)(X_1-X_2)+(\lambda b_4^{-1/2}/4)(X_1+X_2)=\kappa' X_1+\lambda' X_2 \tag{6}$$

in terms of κ, λ or κ', λ' related by (3.2). The affine invariant surface $\mathscr{A}$ defined by the four Kowalewski constants of motion (2) has the involution

$$\tau:(x_1,x_2,x_3,y_1,y_2,y_3)\to(x_1,x_2,-x_3,y_1,y_2,-y_3)\,, \tag{7}$$

which maps the vector fields X_i into $-X_i$, which thus amounts to a reflection about some appropriately chosen origin on A. A different choice of origin would lead to a different involution. $\mathscr{A}$ also has a second involution

$$\sigma:(x_1,x_2,x_3,y_1,y_2,y_3)\to(x_2,x_1,-x_3,y_2,y_1,-y_3)\,, \tag{8}$$

which preserves the vector fields. We now state Theorem 3.

Theorem 3. *The space $L(2D)$ splits into two subspaces L^+ and L^- of even and odd functions for the τ-involution*

$$\begin{aligned}
L(2D) &= L^+\oplus L^-\\
&= \{\zeta_1,\dots,\zeta_6\}\oplus\{\zeta_7,\zeta_8\}\\
&= \{1,x_2,x_2^2,y_2,y_2x_1,y_2x_1^2\}\oplus\{X_1(x_2),X_2(x_2)\}\,;
\end{aligned} \tag{9}$$

the involution σ acts on the ζ_i as follows:

$$\sigma:\zeta_1\leftrightarrow\zeta_4,\zeta_2\leftrightarrow\zeta_5,\zeta_3\leftrightarrow\zeta_6,\zeta_7\leftrightarrow\zeta_8\,. \tag{10}$$

Moreover we have the following Wronskian relations, analogous to (2.2)*:*

$$\{L^+,L^-\}\subset(L^+)^{\otimes 2}\,. \tag{11}$$

Between these 8 variables $\zeta_1,\dots,\zeta_8$ there are 6 quadratic relations $\Phi_i''(\zeta)$, the first three involving the even functions $\zeta_1,\dots,\zeta_6$ only:

$$\begin{aligned}
\Phi_1'' &= \zeta_2^2-\zeta_1\zeta_3\,,\\
\Phi_2'' &= \zeta_5^2-\zeta_4\zeta_6=\Phi_1^\sigma\,,\\
\Phi_3'' &= -C(\zeta_1^2+\zeta_4^2)+(B^2-AC)\zeta_1\zeta_4-A(\zeta_2^2+\zeta_5^2)+2(C+1)\zeta_2\zeta_5\\
&\quad+\zeta_3^2+\zeta_6^2+A\zeta_3\zeta_6+2B(\zeta_1\zeta_2+\zeta_4\zeta_5)+(C-1)(\zeta_1\zeta_6+\zeta_3\zeta_4)\\
&\quad-2B(\zeta_2\zeta_6+\zeta_3\zeta_5)=\Phi_3''^\sigma\,,\\
\Phi_4'' &= -C\zeta_1^2-A\zeta_2^2+\zeta_3^2+2B\zeta_1\zeta_2+2\zeta_2\zeta_5-\zeta_3\zeta_4-\zeta_1\zeta_6+\zeta_7^2\,,\\
\Phi_5'' &= C\zeta_1\zeta_4+A\zeta_2\zeta_5-\zeta_3\zeta_6-B(\zeta_2\zeta_4+\zeta_1\zeta_5)+\zeta_7\zeta_8=\Phi_5''^\sigma\,,\\
\Phi_6'' &= -C\zeta_4^2-A\zeta_5^2+\zeta_6^2+2B\zeta_4\zeta_5+2\zeta_2\zeta_5-\zeta_3\zeta_4-\zeta_1\zeta_6+\zeta_8^2=\Phi_4''^\sigma\,.
\end{aligned} \tag{12}$$

Define the map

$$\theta_j=\alpha_j\frac{\zeta_1-\zeta_3}{2}+\beta_j\zeta_2-i\gamma_j\frac{\zeta_1+\zeta_3}{2}, \qquad \theta'_7=\zeta_7,$$
$$\theta_{j+3}=\alpha_j\frac{\zeta_4-\zeta_6}{2}+\beta_j\zeta_5-i\gamma_j\frac{\zeta_4+\zeta_6}{2}, \qquad \theta'_8=\zeta_8, \tag{13}$$

given by the (complex) orthogonal matrix $U=(\alpha_j,\beta_j,\gamma_j)_{j=1,2,3}$, *which diagonalizes the matrix* M, *defined in* (1.4):

$$UMU^T=\operatorname{diag}(a_1,a_2,a_3),$$

the characteristic polynomial of α *being*

$$T(x)=x^3-\frac{A}{2}x^2+Cx+\frac{B^2-AC}{2}.$$

The map (13) *transforms the space of quadrics* Φ'' *into the quadrics* Φ' *(of Theorem 1) and the Kowalewski vector field* (4) *into the vector field (discussed in Theorem 2):*

$$\dot u'=u'\wedge(a\cdot(\kappa'u'-\lambda'u'')-\kappa'u''),$$
$$\dot u''=u''\wedge(a\cdot(\lambda'u''-\kappa'u')-\lambda'u'),$$

expressed in the variables $u_i=(\kappa\theta_7+\lambda\theta_8)^{-1}\theta_i\,(1\leqq i\leqq 6)$, *with the* a_i *being the roots of* $T(x)$.

Conversely, there is a specific rotation $U\in SO(3)$,

$$\begin{pmatrix}\dfrac{\zeta_1-\zeta_3}{2}\\ \zeta_2\\ -i\,\dfrac{\zeta_1+\zeta_3}{2}\end{pmatrix}=U^T\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix}, \quad \begin{pmatrix}\dfrac{\zeta_4-\zeta_6}{2}\\ \zeta_5\\ -i\,\dfrac{\zeta_4+\zeta_6}{2}\end{pmatrix}=U^T\begin{pmatrix}\theta_4\\ \theta_5\\ \theta_6\end{pmatrix} \tag{13'}$$

and a map

$$\zeta_7=a\theta_7+b\theta_8, \qquad \zeta_8=-a\theta_7+b\theta_8,$$

in terms of the theta functions θ *in* [2.3], *such that the variables*

$$(x_1,x_2,x_3)\equiv\left(\frac{\zeta_5}{\zeta_4},\frac{\zeta_2}{\zeta_1},\frac{\zeta_7\zeta_4+\zeta_8\zeta_1}{\zeta_1\zeta_5-\zeta_2\zeta_4}\right)$$
$$(y_1,y_2,y_3)\equiv\left(\frac{\zeta_1}{\zeta_4},\frac{\zeta_4}{\zeta_1},\frac{\zeta_7\zeta_5+\zeta_8\zeta_2}{\zeta_1\zeta_5-\zeta_2\zeta_4}\right)$$

are precisely the Kowalewski variables $(x_1,x_2,x_3,y_1,y_2,y_3)$.

Proof. Applying the methods explained by Adler and van Moerbeke [5], one looks for Laurent solutions of the differential equations (4) depending on

$$\dim(\text{phase space})-1=5$$

free parameters. There are two distinct families of Laurent solutions, which upon substituting into the constants of motion Q_i and then upon setting the values of the results obtained equal to A, B, C, and $D=1$, lead to the following:

Laurent solutions D^2	*Laurent solutions* D^1
$x_1=Z-tUZ^2+\ldots,$	$x_1=\frac{U}{t}+Z(U^2+1)+\ldots,$
$x_2=\frac{U}{t}+Z(U^2+1)+\ldots,$	$x_2=Z-tUZ^2+\ldots,$
$x_3=-\frac{1}{t}+UZ+\ldots,$	$x_3=-\frac{1}{t}+UZ+\ldots,$
$y_1=\frac{t^2}{U^2}(U^2Z^2-P(Z))+\ldots,$	$y_1=(U^2+1)\left(\frac{1}{t^2}+\frac{2UZ}{t}+\ldots\right),$
$y_2=(U^2+1)\left(\frac{1}{t^2}+\frac{2UZ}{t}+\ldots\right),$	$y_2=\frac{t^2}{U^2}(U^2Z^2-P(Z))+\ldots,$
$y_3=\frac{Z}{t}-UZ^2+\ldots,$	$y_3=-\frac{Z}{t}+UZ^2+\ldots,$

with $P(Z)$ defined by (1). The table above implies $(x_i)\geqq -D^i$ $(i=1,2)$, $(x_3)\geqq -D^1-D^2$, $(y_1)\geqq 2(D^2-D^1)$, and $(y_2)\geqq 2(D^1-D^2)$, $(y_3)\geqq -D^1-D^2$; therefore all $\zeta_i\in L(D^2)$, as defined in (9). For future use, one checks that along D^2

$$\begin{aligned}(\zeta_1,\zeta_2,\zeta_3,\zeta_7)&=\left(1,\frac{U}{t},\frac{U^2}{t^2},-\frac{U}{t^2}\right)+\text{higher order terms in } t,\\ (\zeta_4,\zeta_5,\zeta_6,\zeta_8)&=\frac{U^2+1}{t^2}(1,Z,Z^2,-UZ)+\text{higher order terms in } t.\end{aligned}\tag{14}$$

To show that σ, defined in (8), acts on the ζ_i as announced in (10), take a point

$$p=(\zeta_1,\ldots,\zeta_6,\zeta_7,\zeta_8)=(1,x_2,x_2^2,\,y_2,y_2x_1,y_2x_1^2,X_1(x_2),X_2(x_2))\in\mathbb{P}^7,$$

and observe that as a vector in $\mathbb{P}^7$, using $y_1y_2=1$ and the form (4) of vector fields X_1 and X_2:

$$\begin{aligned}\sigma(p)&=(1,x_1,x_1^2,y_1,y_1x_2,y_1x_2^2,X_1(x_1),X_2(x_1))\\ &=y_2(1,x_1,x_1^2,y_1,y_1x_2,y_1x_2^2,X_1(x_1),X_2(x_1))\\ &=(\zeta_4,\zeta_5,\zeta_6,\zeta_1,\zeta_2,\zeta_3,\zeta_8,\zeta_7)\in\mathbb{P}^7.\end{aligned}$$

Moreover the functions $\zeta_1,\ldots,\zeta_6$ are even and ζ_7,ζ_8 odd for the involution τ.

An effective way to get quadratic relations between the ζ_i is to use the Laurent series (14) and to match poles. For instance, starting with the seed ζ_6^2 having Laurent series

$$\zeta_6^2=\frac{Z^4(U^2+1)^2}{t^4}+\ldots=\frac{(U^2+1)(AZ^2-2BZ+C+1)-1}{t^4}+\ldots$$

[the latter equality following from the curve relation (2)], and then using appropriate products of the ζ_i, the t^{-4}-term can be peeled off leaving a t^{-3} contribution and so on, leading to the quadratic relation Φ_3''; the same procedure leads to the six quadrics $\Phi_1'',\ldots,\Phi_6''$. This is further simplified using the involution σ.

The Wronskian relations (11) follow at once from the Wronskian relations (2.2) for the sections. In order to identify the Kowalewski vector field with (3.5), we must

– in a first step – compute those Wronskians explicitly, namely

$$\{\zeta_i, \zeta_j\} = \text{quadratic polynomial of } (\zeta_1, \ldots, \zeta_6) \quad \begin{matrix} i=1,\ldots,6, \\ j=7,8, \end{matrix}$$

whatever be the vector fields X_1 or X_2. Again the asymptotics is instrumental in checking this fact. Indeed assuming ζ_i and ζ_j behave as t^{-2}, the Wronskian $\{\zeta_i, \zeta_j\}$ behaves as t^{-4}; this term can be peeled off by subtracting appropriate products of even function $\zeta_1, \ldots, \zeta_6$, leaving a t^{-3} contribution, and so on. We thus obtain a closed system of quadratic differential equations in the variables

$$(w_0, w_1, \ldots, w_6) = (\kappa'\zeta_7 + \lambda'\zeta_8)^{-1}(\kappa'\zeta_7 + \lambda'\zeta_8, \zeta_1, \ldots, \zeta_6);$$

in particular the Kowalewski vector field (4) maps into

$$\begin{aligned}
\dot{w}_1 &= \kappa'[2(w_2w_4 - w_1w_5) - (Bw_1^2 - Aw_1w_2 + 2w_2w_3)] \\
&\quad + \lambda'[Bw_1w_4 - Aw_1w_5 + 2w_2w_6], \\
\dot{w}_2 &= \kappa'[(w_3w_4 - w_1w_6) - (Cw_1^2 + w_3^2 - Bw_1w_2)] \\
&\quad + \lambda'[Cw_1w_4 + w_3w_6 - Bw_1w_5], \\
\dot{w}_3 &= \kappa'[2(w_3w_5 - w_2w_6) - (-2Bw_2^2 + 2Cw_2w_1 - Bw_3w_1 - Aw_3w_2)] \\
&\quad + \lambda'[-2Bw_2w_5 + 2Cw_2w_4 - Bw_3w_4 + Aw_3w_5], \\
\dot{w}_4 &= \lambda'[2(w_1w_5 - w_2w_4) + Aw_4w_5 - 2w_5w_6 - Bw_4^2] \\
&\quad + \kappa'[Bw_1w_4 - Aw_4w_2 + 2w_5w_3], \\
\dot{w}_5 &= \lambda'[-Cw_4^2 + Bw_4w_5 - w_6^2 + (w_1w_6 - w_3w_4)] \\
&\quad + \kappa'[Cw_1w_4 + w_3w_6 - Bw_4w_2], \\
\dot{w}_6 &= \lambda'[2Bw_5^2 - 2Cw_4w_5 + Bw_4w_6 - Aw_5w_6 + 2(w_6w_2 - w_3w_5)] \\
&\quad + \kappa'[-2Bw_2w_5 + 2Cw_5w_1 - Bw_6w_1 + Aw_6w_2].
\end{aligned} \tag{15}$$

We now prove the map (13) transforms the quadrics Φ'' into Φ', at first making some preliminary observations.

Among the quadrics Φ_1'' obtained in (12), the 3 quadrics Φ_1'', Φ_2'', and Φ_3'' are expressed in terms of even functions, two of which already have rank 3; however the Φ_i'' do not have the block form. In order to match Φ_1'' and Φ_2'' to Φ_1' and Φ_2', and in order to identify the involutions σ [defined by (10) and (8)], we require the transformation $\zeta \curvearrowright \theta$ to have the form (13) with a 3×3 (complex) orthogonal matrix

$$U = (\alpha_j, \beta_j, \gamma_j)_{j=1,2,3}.$$

However, rather than work with the map (13), it is more convenient to consider the map

$$\begin{aligned}
\theta_j &= \alpha_j'\zeta_1 + \beta_j'\zeta_2 + \gamma_j'\zeta_3, \quad j=1,2,3 \\
\theta_{j+3} &= \alpha_j'\zeta_4 + \beta_j'\zeta_5 + \gamma_j'\zeta_6, \quad j=1,2,3,
\end{aligned}$$

defined by the matrix

$$U' = (\alpha_j', \beta_j', \gamma_j')_{j=1,2,3} = \left(\frac{\alpha_j - i\gamma_j}{2}, \beta_j, -\frac{\alpha_j + i\gamma_j}{2}\right)_{j=1,2,3}.$$

We will see the quadrics Φ_i'' assume the block form in the ζ-coordinates, provided the α, β, and γ are parametrized according to (13). Prior to proving this claim, we observe that the orthogonality of U implies the following relations for U':

(i) $$\beta'_j\beta'_{j+1}-2\alpha'_j\gamma'_{j+1}-2\gamma'_j\alpha'_{j+1}=0,$$

(i)′ $$\frac{\beta'_j\beta'_{j+1}}{\gamma'_j\gamma'_{j+1}}=2\left(\frac{\alpha'_j}{\gamma'_j}+\frac{\alpha'_{j+1}}{\gamma'_{j+1}}\right),$$

(ii) $$\beta'^2_j-2\alpha'_j\gamma'_j=1,$$

(iii) $$(\alpha'_j,\beta'_j,\gamma'_j)=i(\alpha'_{j-1}\beta'_{j+1}-\alpha'_{j+1}\beta'_{j-1},2(\alpha'_{j-1}\gamma'_{j+1}-\gamma'_{j-1}\alpha'_{j+1}),\gamma'_{j+1}\beta'_{j-1}-\beta'_{j+1}\gamma'_{j-1}),$$

(iv) $$\prod_1^3\frac{\beta'_i}{\gamma'_i}=-\sqrt{2\left(\frac{\alpha'_1}{\gamma'_1}+\frac{\alpha'_2}{\gamma'_2}\right)}\sqrt{2\left(\frac{\alpha'_2}{\gamma'_2}+\frac{\alpha'_3}{\gamma'_3}\right)}\sqrt{2\left(\frac{\alpha'_3}{\gamma'_3}+\frac{\alpha'_1}{\gamma'_1}\right)}.$$

Statements (i) and (ii) follow from the orthonormality of the rows of U, whereas (iii) expresses the fact that $U^{-1}=U^T$ and (iv) follows from (i)′.

In order to prove the exact form of the map (12), we substitute it, using U', into $\Phi''_1,\dots,\Phi''_6$; Φ''_1 and Φ''_2 are as in (2.3′), while the quadrics Φ''_3, Φ''_4, Φ''_5, and Φ''_6 have the block form in the θ-variables if and only if

$$\mathscr{A}\begin{pmatrix}-4\\A\\4C\\2B\end{pmatrix}=0\quad\text{and}\quad\mathscr{B}\begin{pmatrix}A\\C\\E\\B\end{pmatrix}=0,\tag{16}$$

with $E\equiv B^2-AC$; the 3×4 matrices $\mathscr{A}$ and $\mathscr{B}$ read as follows:

$$\mathscr{A}=(\alpha'_j\alpha'_{j+1}\ \beta'_j\beta'_{j+1}\ \gamma'_j\gamma'_{j+1}\ \beta'_j\gamma'_{j+1}+\beta'_{j+1}\gamma'_j)_{j=1,2,3},$$
$$\mathscr{B}=(\alpha'_j\alpha'_{j+1}\ \beta'_j\beta'_{j+1}\ \gamma'_j\gamma'_{j+1}\ \beta'_j\alpha'_{j+1}+\alpha'_j\beta'_{j+1})_{j=1,2,3}.$$

Solving the first linear system of equations for A,B,C, we find (all products and sums are taken cyclically from 1 to 3)

$$\begin{aligned}A&=\frac{4}{\Pi\gamma'_j}\sum_j\alpha'_j\alpha'_{j+1}\gamma'^2_{j+2},\quad\text{upon using formula (iii) for }\gamma'_j\\&=\frac{2i}{\Pi\gamma'_j}(\alpha'_2\beta'_2\gamma'^2_1-\alpha'_1\beta'_1\gamma'^2_2),\quad\text{using (iii) for }\beta'_1,\beta'_2\\&=\frac{2}{\Pi\gamma'_j}\sum\alpha'_j\gamma'_{j+1}\gamma'_{j+2},\quad\text{using (iii) for }\alpha'_3,\gamma'_3\\&=2\sum\frac{\alpha'_j}{\gamma'_j};\end{aligned}\tag{17}$$

$$\begin{aligned}B&=-\frac{2\beta'_3}{\Pi\gamma'_j}(\alpha'_3\gamma'_1-\alpha'_1\gamma'_3)(\alpha'_3\gamma'_2-\alpha'_2\gamma'_3),\\&\qquad\text{using (iii) for }j=3\text{ and (i) for }j=1\\&=-\tfrac{1}{2}\prod\frac{\beta'_j}{\gamma'_j},\quad\text{using (iii)}\\&=\sqrt{2}\left(\frac{\alpha'_1}{\gamma'_1}+\frac{\alpha'_2}{\gamma'_2}\right)^{1/2}\left(\frac{\alpha'_2}{\gamma'_2}+\frac{\alpha'_3}{\gamma'_3}\right)^{1/2}\left(\frac{\alpha'_3}{\gamma'_3}+\frac{\alpha'_1}{\gamma'_1}\right)^{1/2},\quad\text{using (iv)};\end{aligned}\tag{18}$$

$$C=\frac{1}{\Pi\gamma_j'}\sum\alpha_j'\alpha_{j+1}'\gamma_{j+2}'\beta_{j+2}'^2,\quad\text{using (iii) for }\gamma_1',\gamma_2',\gamma_3'$$
$$=\frac{1}{\Pi\gamma_j'}\left(\frac{i}{2}\beta_1'\beta_2'(\alpha_2'\beta_1'-\alpha_1'\beta_2')+\alpha_1'\alpha_2'\gamma_3'\right),$$
$$\text{using (ii) for }\beta_3'^2\text{ and (iii) for }\beta_1',\beta_2'$$
$$=\sum\frac{\alpha_j'}{\gamma_j'}\frac{\alpha_{j+1}'}{\gamma_{j+1}'},\quad\text{using (i) for }\beta_1'\beta_2'\text{ and (iii) for }\alpha_3',\tag{19}$$

yielding

$$E=B^2-AC=-2\prod\frac{\alpha_j'}{\gamma_j'}.$$

Using the above results, the second linear system (16) for A, B, C, E is automatically satisfied. From the above formulas, the A, B, C, and E are symmetric polynomials in the $a_j\equiv\alpha_j'/\gamma_j'$, and hence the a_j are the roots of Kowalewski's polynomial $T(x)$.

Using (iv), (i'), and (ii), the entries α, β, and γ of U can be parametrized as follows:

$$U=(\alpha_j,\beta_j,\gamma_j)_{j=1,2,3}$$
$$=\begin{bmatrix}(a_1-1)\left(\dfrac{a_2+a_3}{2(a_1-a_2)(a_1-a_3)}\right)^{1/2} & -\left(\dfrac{(a_1+a_2)(a_1+a_3)}{(a_1-a_2)(a_1-a_3)}\right)^{1/2}\\ i(a_1+1)\left(\dfrac{a_2+a_3}{2(a_1-a_2)(a_1-a_3)}\right)^{1/2} & \\ \text{and cyclic permutations} & \end{bmatrix}$$

One then checks that $U^T\operatorname{diag}(a_1,a_2,a_3)U$ is a (symmetric) matrix of symmetric polynomials of the a_i, and thus expressible as polynomials of A, B, C, E. Using the exact expressions (17), (18), and (19), one then concludes

$$U^T\operatorname{diag}(a_1,a_2,a_3)U=M,$$

with α defined in (1.4).

Finally using this change of variables $\zeta\to\theta$, and after some effort the quadrics Φ_i'' take on the form (12) and Kowalewski's vector field (4) takes the form (3.5), with a_i being the roots of Kowalewski's polynomial.

To prove the converse, whatever be the rotation U^T in (13'), the new space of sections $(\zeta_1,\ldots,\zeta_8)$ behaves as follows with regard to the involutions σ and τ,

$$\tau:(\zeta_1,\ldots,\zeta_8)\curvearrowright(\zeta_1,\ldots,\zeta_6,-\zeta_7,-\zeta_8)$$
$$\sigma:(\zeta_1,\ldots,\zeta_8)\curvearrowright(\zeta_4,\zeta_5,\zeta_6,\zeta_1,\zeta_2,\zeta_3,-\zeta_8,-\zeta_7).$$

In view of the rotation U, the sections ζ_i satisfy

$$\zeta_2^2-\zeta_1\zeta_3=0\quad\text{and}\quad\zeta_5^2-\zeta_4\zeta_6=0.$$

Since $\zeta_1=0$ implies $\zeta_2^2=0$ and since $\zeta_4=0$ implies $\zeta_5^2=0$, the theta functions ζ_2 and ζ_5 define the following divisors D_i and D_i^- on A, all having genus 3:

$$(\zeta_1)=2D_2,\quad(\zeta_4)=2D_1,\quad(\zeta_2)=D_2+D_2^-,\quad(\zeta_5)=D_1+D_1^-.$$

Therefore the functions

$$x_1 \equiv \zeta_5/\zeta_4\,, \quad x_2 \equiv \zeta_2/\zeta_1\,, \quad y_1 \equiv \zeta_1/\zeta_4\,, \quad y_2 \equiv \zeta_4/\zeta_1\,,$$

satisfy

$$(x_1) = D_1^- - D_1\,, \quad (x_2) = D_2^- - D_2\,, \quad y_1 = 2D_2 - 2D_1\,, \quad y_2 = 2D_1 - 2D_2\,,$$

regardless of the rotation U in (13′). These functions, together with

$$x_3 \equiv \frac{\zeta_7\zeta_4 + \zeta_8\zeta_1}{\zeta_1\zeta_5 - \zeta_2\zeta_4} \quad \text{and} \quad y_3 \equiv \frac{\zeta_7\zeta_5 + \zeta_8\zeta_2}{\zeta_1\zeta_5 - \zeta_2\zeta_4}$$

transform according to (7) and (8), with regard to the involutions τ and σ. It is only after picking U, a, and b as in (13) and (2.4) that x_3 and y_3 satisfy the inequalities

$$(x_3) \geqq -D_1 - D_2 \quad \text{and} \quad (y_3) \geqq -D_1 - D_2$$

and the set $(x_1, x_2, x_3, y_1, y_2, y_3)$ satisfies the Kowalewski system of differential equations, completing the proof of Theorem 3.

5. A Two-Dimensional Family of Lax Pairs and the Spectral Surface

This section deals with the two-dimensional family of Lax pairs associated with the affine surfaces $\mathscr{A}_k = A \backslash \mathscr{C}_k$, $k \in \mathbb{P}^1$, obtained by removing the genus 9 curves $\mathscr{C}_k$ from A. Referring to Fig. 1 in Sect. 3, the curves $\mathscr{C}_k$ are 4–1 covers of genus 3 curves $D_k^\vee$, via the isogeny $\phi: A \to A^\vee$. As shown in Theorem 2, for each value of $k = \kappa/\lambda \in \mathbb{P}^1$, we have a Manakov problem associated to $\mathscr{A}_k$, a one-dimensional family of Lax equations (3.9) and associated spectral curves Σ_k. It turns out the linear system $|D_k^\vee| \subset A^\vee$ and the family Σ_k are intimately related as follows:

Theorem 4. *The spectral curves Σ_k sweep out the linear pencil $|D_\alpha| \subset A$ generated by the curve $D \equiv D^{(2)}$ obtained in Theorem 3. This induces an algebraic map between the linear pencils $|D_k^\vee| \subset A^\vee$ and $|D_\alpha| \subset A$, which takes the smooth hyperelliptic sections in $|D_k^\vee|$ to the singular sections in $|D_\alpha|$ and the singular sections to the smooth hyperelliptic sections. Moreover, the curve $D_{4^{-1}}^\vee$ gets mapped to $D^{(2)} = D_\infty$ in $|D_\alpha|$. When $k = \kappa/\lambda = 0$, the Kowalewski flow has the simple Lax pair representations* (1.21) *and* (1.23) *announced in the introduction.*

Proof. At first we give a description of the linear pencil $|D_\alpha| = |D^{(2)}| \subset A$. In terms of the Kowalewski coordinates x_1, x_2, and $z_1 \equiv x_3 x_1 - y_3$ and the polynomials P, Q, R, R_1 defined in (4.1), the Abelian surface A, suitably projected, is given by the equation

$$\Psi(x_1, x_2, z_1^2) \equiv z_1^4 P(x_2) + z_1^2((x_1 - x_2)^4 - R^2 - P(x_1)P(x_2)) + R^2 P(x_1) = 0\,, \qquad (1)$$

and the Kummer surface KmA, by the equation $\Psi(x_1, x_2, u_1) = 0$, as shown by Horozov and van Moerbeke [18]. Each curve $D_\alpha = \{x_2 = \alpha\}$, obtained by setting $x_2 = \alpha$ in Eq. (1), is a double cover of the elliptic curve defined by the radical of (1):

$$\begin{aligned} \mathscr{E}_\alpha: w^2 = {} & (R_1(x_1, \alpha) - (x_1 - \alpha)^2 - 2R(x_1, \alpha)) \\ & \times (R_1(x_1, \alpha) - (x_1 - \alpha)^2 + 2R(x_1, \alpha)) \end{aligned} \qquad (2)$$

ramified at the 4 points, where $z_1=0$, hence $P(x_1)=0$, corresponding to 4 half periods on A. Moreover there are 12 values of α, given by the 12 zeroes of the polynomial

$$P(\alpha)\left[P^2(\alpha)\left((C+1)^2+A(B^2-AC)\right)+P(\alpha)Q(\alpha)(AC-B^2-A)+Q^2(\alpha)\right]=0,$$

where the curve $\Phi(x_1,\alpha,z_1^2)$ becomes a hyperelliptic curve of genus 2 with one normal crossing; each singular curve passes through one of the 12 remaining half-periods, with the singularity being at the half-period. The hyperelliptic curves corresponding to the 4 roots of $P(\alpha)=0$ are given by

$$\begin{gathered} y^2=(x^2-1)(x-a_1)(x-a_2)(x-a_3) \\ \text{(Kowalewski's hyperelliptic curve)}. \end{gathered} \tag{3}$$

An explicit but tedious computation shows that the spectral curve Σ_k given by Eq. (3.17) with α_i and β_i defined in (3.11) belongs to the linear system $|D_\alpha|$ on A, whatever be $k\in\mathbb{P}^1$. Rather than giving this computation, we shall present two illustrations of this result.

Case 1. $k=\kappa/\lambda=b_4^{-1}$; then $D^{\vee}_{b_4^{-1}}\frown\Sigma_{b_4^{-1}}=D_\infty\subset|D_\alpha|$.

Putting this value of κ/λ into the expressions (3.11), we get the Manakov quadrics (3.8) with

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(a_2+a_3,a_3+a_1,a_1+a_2,0),$$

$$(\beta_1,\beta_2,\beta_3,\beta_4)=\left(r_1^2(r_2^2+r_3^2),r_2^2(r_3^2+r_1^2),r_3^2(r_1^2+r_2^2),\ -r_1r_2r_3\sum_1^3 r_i\right),$$

$$r_i^2=a_{i-1}+a_{i+1},$$

$$A_1=\frac{2(r_1-r_2-r_3)}{(r_2-r_1)(r_3-r_1)(r_2+r_3)}(C+1+r_1r_2r_3(r_1-r_2-r_3)),$$

$$A_4=2(C+1),$$

A_2 and A_3 being defined by cyclic permutations; the linear change of variable $u\frown x$, defined in (3.10), reads as follows:

$$x_i=2\sqrt{r_i(r_{i+1}+r_{i-1})}((r_{i+1}r_{i-1}-a_i)u_i+u_{i+3}),$$

$$x_{i+3}=\frac{2r_i^2}{\sqrt{r_i(r_{i+1}+r_{i-1})}}((-r_{i+1}r_{i-1}-a_i)u_i+u_{i+3}).$$

In the Lax pair

$$(X+\alpha h)^{\cdot}=\left[X+\alpha h,\frac{\partial Q}{\partial x}+\beta h\right],$$

the diagonal matrices α and β are given by (4) and

$$\begin{aligned} Q=\tfrac{1}{2}[&r_1r_2r_3(r_1x_1^2+r_2x_2^2+r_3x_3^2)\\ &+r_2r_3(r_1r_3^2+r_2(r_3+r_1)(r_3+r_2))x_4^2\\ &+r_3r_1(r_2r_1^2+r_3(r_1+r_2)(r_1+r_3))x_5^2\\ &+r_1r_2(r_3r_2^2+r_1(r_2+r_3)(r_2+r_1))x_6^2]. \end{aligned}$$

Given these data, one computes the spectral curve $\Sigma_{b_4^{-1}}$ using the formula (3.17), first in terms of symmetric polynomials of the r_i, and then after a fractional linear transformation in u and a rescaling of v, one finds

$$\begin{aligned} \Sigma_{b_4^{-1}}: v^4+2\left(u^2-\left(A-\frac{2B^2}{C+1}\right)u-C\right)v^2&\\ +(u^4+2Au^3-(2C-A^2)u^2-2(AC-2B^2)u+C^2);& \end{aligned} \tag{5}$$

it is a double cover of the elliptic curve

$$\mathscr{E}_{b_4^{-1}}: w^2=u\left(u^2+\left(A-\frac{B^2}{C+1}\right)u+1\right) \tag{6}$$

ramified at the 4 points where the quartic polynomial

$$u^4+2Au^3-(2C-A^2)u^2-2(AC-2B^2)u+C^2 \tag{7}$$

vanishes. The fractional linear map

$$u=D\frac{x(\varrho+B)-1}{x(2-A)(C+1)(\varrho+B)^{-1}+1}, \qquad \varrho^2=2T(1) \tag{8}$$

maps the cubic (6) to the quartic

$$\mathscr{E}_\infty: w^2=[(1+C)(A-2)x^2-2Bx+1][(1-C)(A+2)x^2-2Bx+1], \tag{9}$$

i.e., the curve $\mathscr{E}_\alpha$ given in (2), with $\alpha\uparrow\infty$.

In realizing a double ramified cover of an elliptic curve branched at 4 points – the elliptic curve given by a definite projection – one still has the freedom to translate the four points on the elliptic curve, without modifying the double cover. We shall show that such a translation transforms (5) to D_∞. Euler (see [18]) has given us an explicit recipe for translating an elliptic curve $u^2=F(x)$ to the same elliptic curve $v^2=F(y)$. The relationship between (x,u) and (y,v) is given by

$$y=\frac{1}{\mathscr{A}(x)}(-\mathscr{B}(x)+8\eta u), \qquad v=-\frac{1}{8y}(\mathscr{B}(y)+\mathscr{A}(y)x),$$

where $\mathscr{A}$ and $\mathscr{B}$ are appropriately chosen polynomials and where

$$\eta^2 = 4\xi^3 - g_2\xi - g_3,$$

g_2 and g_3 being the invariants of the polynomial F.

In view of Euler's method (used here, rather than his exact formulae), consider the symmetric polynomial

$$\begin{aligned}\Phi_\xi(x,y) &\equiv x^2y^2[(4-A^2)(C+1)\xi - 4B^2] + 2B(A\xi+2)xy(x+y)\\ &\quad + 2xy\left(\xi^2 - \frac{2B^2}{C+1}\xi - 1\right) - (\xi^2 + A\xi + 1)(x^2+y^2)\\ &\quad + \frac{2B}{C+1}\xi(x+y) - \frac{\xi}{C+1}\\ &\equiv \mathscr{A}_\xi(y)x^2 + 2\mathscr{B}_\xi(y)x + \mathscr{C}_\xi(y)\\ &= \mathscr{A}_\xi(x)y^2 + 2\mathscr{B}_\xi(x)y + \mathscr{C}_\xi(x) = 0;\end{aligned}$$

with discriminant

$$\begin{aligned}(\mathscr{B}^2 - \mathscr{A}\mathscr{C})(y) = &-\frac{1}{C+1}((A-2)(C+1)y^2 - 2By + 1)\\ &\times((A+2)(C+1)y^2 - 2By + 1)\\ &\times \xi\left(\xi^2 + \left(A - \frac{B^2}{C+1}\right)\xi + 1\right).\end{aligned}$$

The map $(x,u) \curvearrowright (y,v)$ given by

$$x = \frac{-\mathscr{B}(y) \pm \sqrt{\mathscr{B}^2 - \mathscr{A}\mathscr{C}}}{\mathscr{A}} \tag{10}$$

corresponds to a translation on the elliptic curve $\mathscr{E}_\infty$, determined by the point ξ on the isomorphic curve

$$\eta^2 = \xi\left(\xi^2 + \left(A - \frac{B^2}{C+1}\right)\xi + 1\right).$$

Considering the translation corresponding to the point $\xi = -1$, and evaluating (10) at that point leads to the map

$$x = \frac{-\mathscr{B}(y) \pm (C+1)^{-1}\sqrt{2T(1)}\,w(y)}{\mathscr{A}(y)} \tag{11}$$

with w given by (9), with x replaced by y. Combining the two transformations (8) and (11) and rationalizing the denominator in $w\sqrt{2T(1)}$ leads to

$$u = -\frac{A(C+1)y^2 - 2By + 1 + w\sqrt{2T(1)}}{2(C+1)y^2}, \tag{12}$$

to be applied to the expression (5) for $\Sigma_{b_4^{-1}}$. Using this transformation and rationalizing once more, we see the quartic (7) in u maps to the quartic $P(y(C+1))$.

This shows the map (11) transforms the elliptic curve (6) and the 4 roots of the quartic (7), to the curve $\mathscr{E}_\infty$ and the 4 roots of $P(x)$; therefore $\Sigma_{b_4^{-1}} \equiv D_\infty$.

Case 2. $k = \kappa/\lambda = 0$; then

$$D_0^\vee \cap \Sigma_0 = D_\alpha \subset |D_\alpha| \text{ with } P(\alpha) = 0 \text{ (singular sections)},$$

and $\kappa'/\lambda' = 1$; in this instance the entire procedure simplifies considerably. Indeed, the affine surface $\mathscr{A}_0$ is given by the intersection of the four quadrics Φ_1, Φ_2, $\Phi_3^\pm$, and Φ_6 in (2.3). By taking simple linear combinations, this intersection is seen to be defined by the 4 Manakov quadrics, the map (3.10) reduces to

$$x_i = u_i + u_{i+3}, \qquad x_{i+3} = u_i - u_{i+3} \qquad (i = 1, 2, 3);$$

we have $\alpha_i = b_i$ and

$$\begin{aligned} Q_i(x) &= 2 \prod_{\substack{k=1 \\ k \neq i}}^{4} (b_i - b_k)^{-1} \Big(b_i(b_i - b_4)(\Phi_1 + \Phi_2) \\ &\quad - (b_i - b_4)(\Phi_3^+ + \Phi_3^-) + 2 \frac{b_i}{b_4} \prod_1^3 (b_j - b_4)\Phi_6 \Big) \\ &= \frac{x_{i-1}^2}{b_i - b_{i+1}} + \frac{x_{i+1}^2}{b_i - b_{i-1}} + \frac{x_{i+3}^2}{b_i - b_4} + \frac{4b_i \prod_1^3 (b_k - b_4)}{b_4 \prod_{k \neq i} (b_i - b_k)}, \\ Q_4(x) &= \Phi_1 - \Phi_2 = x_1x_4 + x_2x_5 + x_3x_6 . \end{aligned}$$

One then computes the quantities (see 3.11)

$$B_i = 0, \quad i = 1, 3, 4, \quad \text{and} \quad B_2 = -b_4 \prod_1^3 (b_i - b_4),$$

which give at once the hyperelliptic spectral curve

$$\Sigma_0 : v^2 = u \prod_1^4 (u - \alpha_i) = u \prod_1^4 (u - b_i);$$

it is conformal to Kowalewski's hyperelliptic curve $w^2 = (x^2 - 1)T(x)$, by means of the fractional linear map (2.5) between a_i and b_i.

Next we study the associated Lax pairs, using the following property of the isomorphism $\hat{}: \mathbb{R}^3 \to so(3)$ defined in (1.22), namely

$$\text{if} \quad U \in SO(3) \quad \text{and} \quad x \in \mathbb{R}^3, \quad \text{then} \quad (Ux)\hat{} = U\hat{x}U^T .$$

In view of the map of Theorem 3 and the involution σ [see (1.19)]

$$u' = Uv, \qquad u'' = Uv^\sigma,$$

where

$$v = \begin{pmatrix} -(x_2^2 - 1) \\ 2x_2 \\ -i(x_2^2 + 1) \end{pmatrix} \frac{1}{2p}$$

and

$$p = X_1(x_2) + X_2(x_2) = -y_2y_3 - x_2x_3 + x_1x_3y_2 + y_3 ,$$

we conjugate the second Lax pair (ii) in Corollary 1 (Sect. 3),

$$\begin{pmatrix} \hat{u}' & \mathscr{D}h \\ \mathscr{D}h & \hat{u}'' \end{pmatrix}^{\cdot} = \left[\begin{pmatrix} \hat{u}' & \mathscr{D}h \\ \mathscr{D}h & \hat{u}'' \end{pmatrix}, \begin{pmatrix} \hat{w} & Ih \\ Ih & -\hat{w} \end{pmatrix} \right]$$

with the rotation U of Sect. 4, yielding first

$$\begin{aligned} \begin{pmatrix} U^T & 0 \\ 0 & U^T \end{pmatrix} \begin{pmatrix} \hat{u}' & \mathscr{D}h \\ \mathscr{D}h & \hat{u}'' \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} &= \begin{pmatrix} U^T\hat{u}'U & U^T\mathscr{D}Uh \\ U^T\mathscr{D}Uh & U^T\hat{u}''U \end{pmatrix} \\ &= \begin{pmatrix} (U^Tu')^\wedge & U^T\mathscr{D}Uh \\ U^T\mathscr{D}Uh & (U^Tu'')^\wedge \end{pmatrix} \\ &= \begin{pmatrix} \hat{v} & 0 \\ 0 & \hat{v}^\sigma \end{pmatrix} + \frac{2}{\Delta} h \begin{pmatrix} 0 & I-N \\ I-N & 0 \end{pmatrix}, \end{aligned}$$

where N is the matrix (1.24) and $\Delta^2 = 4T(1)T(-1)$. The only difficulty of the computation lies in the last equality, namely in computing that

$$U^T\mathscr{D}U = \frac{2}{\Delta}(I-N). \tag{13}$$

To do this, one first observes that the entries of the diagonal matrix $\mathscr{D}$ equal (cyclically)

$$\begin{aligned} b_2+b_3-b_1-b_4 &= (b_2-b_4)+(b_3-b_4)-(b_1-b_4) \\ &= -2b_4\left(\frac{1}{a_2+1}+\frac{1}{a_3+1}-\frac{1}{a_1+1}\right) \\ &= -2b_4\left(\sum_1^3 \frac{1}{a_i+1}-\frac{2}{a_i+1}\right), \quad b_4^2 = \frac{T(-1)}{T(1)}. \end{aligned}$$

Next one observes that conjugating $\mathscr{D}$ (in terms of a_i) by the matrix U [as given explicitly in (4.20)] yields a matrix of symmetric polynomials in the roots a_i of Kowalewski's cubic, which is thus expressible in terms of A, B, C, modulo a factor Δ defined above. Then using the exact expressions (4.17), (4.18), (4.19) of A, B, C in terms of the a_i leads to the desired result (13).

The other matrix in the Lax pair (ii) must be conjugated by U as well; namely

$$\begin{pmatrix} U^T & 0 \\ 0 & U^T \end{pmatrix} \begin{pmatrix} \hat{w} & Ih \\ Ih & -\hat{w} \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} (U^Tw)^\wedge & Ih \\ Ih & -(U^Tw)^\wedge \end{pmatrix}.$$

Using the definition (3.19) of w, the matrix

$$M = U^T \operatorname{diag}(a_1, a_2, a_3) U$$

and the fractional linear relation (2.5) between a and b, we find

$$\begin{aligned} U^Tw &= U^T \operatorname{diag}\left(\frac{1}{b_1-b_4}, \frac{1}{b_2-b_4}, \frac{1}{b_3-b_4}\right) \begin{pmatrix} u_1-u_4 \\ u_2-u_5 \\ u_3-u_6 \end{pmatrix} \\ &= -\frac{1}{2b_4} U^T \operatorname{diag}(a_1+1, a_2+1, a_3+1) U(v-v^\sigma) \\ &= -\frac{1}{2b_4}(M+I)(v-v^\sigma). \end{aligned}$$

Appropriately rescaling h and t yields

$$\begin{pmatrix} \hat{v} & (N-I)h \\ (N-I)h & \hat{v}^{\sigma} \end{pmatrix}^{\cdot} = \left[\begin{pmatrix} \hat{v} & (N-I)h \\ (N-I)h & \hat{v}^{\sigma} \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} (M+I)(v-v^{\sigma})^{\wedge} & 2T(-1)h \\ 2T(-1)h & -(M+I)(v-v^{\sigma})^{\wedge} \end{pmatrix} \right].$$

We finally deal with the Lax pair (iii) (in Corollary 1). In view of the transformation $u' = Uv$ and $u'' = Uv^{\sigma}$, we multiply the Lax pair (iii) to the left by U^T and to the right by U, we use the properties $N(x \otimes y)M = (Nx) \otimes (M^T y)$ and $U\hat{v}U^T = (Uv)^{\wedge}$ and we take into account the matrix $M + I = U^T \operatorname{diag}(a_1+1, a_2+1, a_3+1)U$; this leads to

$$\begin{aligned} &(\alpha+I)((v-v^{\sigma})\otimes(v-v^{\sigma}) + h(\hat{v}+\hat{v}^{\sigma}) + Ih^2)^{\cdot} \\ &\quad = [(\alpha+I)((v-v^{\sigma})\otimes(v-v^{\sigma}) + h(\hat{v}+\hat{v}^{\sigma}) + Ih^2), (\alpha+I)((\hat{v}+\hat{v}^{\sigma}) + Ih)], \end{aligned}$$

ending the proof of Theorem 4.

6. Going from Hénon-Heiles to Manakov

In this section we show how a linear map transforms the Hénon-Heiles system to the Manakov problem. As a reminder from Sect. 1, the Hénon-Heiles system

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \dot{y}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1, 2, \ (x, y) \in \mathbb{R}^4, \tag{1}$$

with

$$H = Q_1 = \tfrac{1}{2}(y_1^2 + y_2^2) + x_1^2 x_2 + 2x_2^3 = A_1 = ic$$

has another constant of motion

$$Q_2 = y_1 y_2 x_1 - y_1^2 x_2 + x_1^2 x_2^2 + \frac{x_1^4}{4} = A_2 = -1.$$

The flow (1) has Laurent solutions

$$\begin{aligned} x_1 &= t^{-1}\left(2Z + \frac{2Z^3}{3}t^2 + \frac{U}{3}t^3 - \frac{4Z^5t^4}{9} - \frac{Z^2Ut^5}{9} - Z(27V + 6Z^6)\left(\frac{t}{3}\right)^6 + \dots\right) \\ x_2 &= t^{-2}\left(-1 + \frac{Z^2t^2}{3} + \frac{Z^4t^4}{3} + \frac{2ZUt^5}{9} + \frac{Vt^6}{6} - \frac{Z^3Ut^7}{9}\right. \\ &\quad \left. - \tfrac{3}{2}(126Z^2V + 27U^2 + 10Z^8)\left(\frac{t}{3}\right)^8 + \dots\right). \end{aligned} \tag{2}$$

with $y_i = \dot{x}_i$. Substituting the solutions (4) into $Q_1 = A_1$ and $Q_2 = A_2$, and equating the t^0 terms yields

$$140Z^6 + 63V - 27A_1 = 0,$$

$$172Z^8 + 126VZ^2 - 27U^2 + 27A_2 = 0.$$

Eliminating V from these equations, leads to the following hyperelliptic curve of genus 3,

$$D: U^2 = -4Z^8 + 2A_1Z^2 + A_2. \tag{3}$$

It is a double ramified cover of the elliptic curve

$$\mathscr{E}: V^2 = -4Y^4 + 2A_1Y + A_2, \tag{4}$$

ramified at the four points covering $Y = 0$ and ∞ and a 2–1 unramified cover of the genus 2 hyperelliptic curve

$$\mathscr{H}: w^2 = Y(-4Y^4 + 2A_1Y + A_2). \tag{5}$$

From the theory developed by Adler and van Moerbeke [5] the affine surface $\bigcap_1^2 \{Q_i = A_i\}$ completes into an Abelian surface $A = \mathrm{Prym}(D/\mathscr{E})^{\check{}}$, parametrized by $A_2^{3/4}A_1^{-1}$, after adjoining the divisor D. The latter defines on A a polarization (1, 2). Moreover D is one of the 6 smooth hyperelliptic curves in the linear system $|D|$, and therefore A is a double unramified cover of $\mathrm{Jac}(\mathscr{H})$ (see Horozov and Moerbeke [18]). Also the differentials dt_1 and dt_2, going respectively with the flows generated by $H = Q_1$ and Q_2 become, upon restriction to D, the (odd) Prym differentials, which descend to the differentials on $\mathscr{H}$. Indeed, using the Laurent solutions, one checks:

$$dt_1|_D = \frac{Z^2dZ}{U} = \frac{YdY}{W}, \qquad dt_2|_D = \frac{dZ}{U} = \frac{dY}{W}.$$

The divisor $2D$ defines a (2, 4)-polarization on A and the functions of $L(2D)$ embed T^2 into $\mathbb{P}^7$. For this problem, the τ-involution reads

$$\tau: (x_1, x_2, y_1, y_2) \curvearrowright (x_1, x_2, -y_1, -y_2)$$

and, from the Laurent solutions (2), one checks

$$\begin{aligned} L(2D) &= L^+ \oplus L^- = \{\zeta_1, \ldots, \zeta_6\} \oplus \{\zeta_7, \zeta_8\} \\ &= \left\{1, x_1, x_i^2, ix_2, -\frac{i}{2}(2x_1x_2^2 + y_1y_2), -i(y_1^2 + x_1^2x_2)\right\} \\ &\quad \oplus \{X_1(x_1) = y_1), X_2(x_1) = y_2x_1 - 2y_1x_2\}. \end{aligned}$$

Indeed, by means of (2), compute

$$\{\zeta_1, \zeta_2, \zeta_3, \zeta_7\} = \left(1, \frac{2Z}{t}, \frac{4Z^2}{t^2}, \frac{-2Z}{t^2}\right) + \text{higher order terms in } t,$$

$$\{\zeta_4, \zeta_5, \zeta_6, \zeta_8\} = \left(-\frac{i}{t^2}, -\frac{iU}{t^2}, \frac{4iZ^4}{t^2}, \frac{4Z^3}{t^2}\right) + \text{higher order terms in } t.$$

Using these Laurent solutions, we get 6 quadratic relations in the variables $\zeta_1, \ldots, \zeta_8$, namely, after rescaling A_1 and A_2 into ic and -1:

$$\Phi_1'' = \zeta_2^2 - \zeta_1\zeta_3 = 0,$$

$$\Phi_2'' = -\zeta_1^2 - 2i\zeta_2\zeta_5 - \frac{\zeta_3^2}{4} + \zeta_4\zeta_6 = 0,$$

$$\Phi_3'' = -4\zeta_4^2 - 4\zeta_5^2 - \zeta_6^2 + 2c(\zeta_3\zeta_4 + \zeta_1\zeta_6) = 0,$$

$$\Phi_4'' = -\frac{i}{c}\left(2\zeta_4^2 + 2\zeta_5^2 + \frac{\zeta_6^2}{2}\right) + \zeta_7^2 = 0,$$

$$\Phi_5'' = 2\zeta_1^2 + 2i\zeta_2\zeta_5 + \frac{\zeta_3^2}{2} + \zeta_7\zeta_8 = 0,$$

$$\Phi_6'' = 4i\zeta_1\zeta_4 - 2ic\zeta_2^2 + i\zeta_3\zeta_6 + \zeta_8^2 = 0.$$

Defining the affine variables

$$(w_0, w_1, \ldots, w_6) \equiv (\kappa\zeta_7 + \lambda\zeta_8)^{-1}(\kappa\zeta_7 + \lambda\zeta_8, \zeta_1, \ldots, \zeta_6),$$

leads to the closed system of differential equations

$$\begin{aligned}
\dot{w}_1 &= -2i\kappa w_2 w_4 + \lambda(w_2 w_3 + 2i w_1 w_5),\\
\dot{w}_2 &= i\kappa(w_1 w_6 - w_3 w_4) + \lambda(-2w_1^2 + w_3^2/2),\\
\dot{w}_3 &= 2i\kappa w_1 w_5 + \lambda(2ic w_1 w_2 - i w_2 w_6 - 2w_4 w_6),\\
\dot{w}_4 &= -2\kappa w_2 w_6 + 2\lambda(-2i w_1 w_2 + w_3 w_5),\\
\dot{w}_5 &= \kappa(-2w_1 w_4 + \tfrac{1}{2} w_3 w_6) + \lambda i(-c w_1 w_6 + c w_3 w_4 - 2w_4^2 + w_6^2/2),\\
\dot{w}_6 &= 2\kappa w_3 w_5 + 2\lambda(c w_2 w_3 - 2 w_2 w_4 + i w_5 w_6).
\end{aligned}$$

As in the Kowalewski problem we now perform a rotation so as to get Φ'' into the block form; define for $j = 1, 2, 3$:

$$\eta_j = \alpha_j \frac{\zeta_1 - \zeta_3}{2} + \beta_j\zeta_2 - i\gamma_j \frac{\zeta_1 + \zeta_3}{2}, \qquad \eta_7 = (-ic/2)^{1/2}\zeta_7,$$

$$\eta_{j+3} = \alpha_j \frac{\zeta_4 - \zeta_6}{2} + \beta_j\zeta_5 - i\gamma_j \frac{\zeta_4 + \zeta_6}{2}, \qquad \eta_8 = (-2ic)^{-1/2}\zeta_8,$$

where

$$U = (\alpha_j\ \beta_j\ \gamma_j) = \begin{pmatrix} 0 & 1 & 0 \\ -3/2\sqrt{2} & 0 & -i/2\sqrt{2} \\ -i/2\sqrt{2} & 0 & 3/2\sqrt{2} \end{pmatrix}$$

is a (complex) orthogonal matrix. In these new coordinates, the Φ_i'' have the following form:

$$\begin{aligned}
\Phi_1'' &= \eta_1^2+\eta_2^2+\eta_3^2,\\
\Phi_2'' &= 2i\eta_1\eta_4+\eta_2^2+\eta_5^2-\eta_3^2+\eta_6^2,\\
\Phi_3'' &= \eta_4^2+\eta_5^2+c\eta_2\eta_5-\eta_6^2+c\eta_3\eta_6,\\
\Phi_4'' &= -\eta_4^2-\eta_5^2+\eta_6^2+\eta_7^2,\\
\Phi_5'' &= -i\eta_1\eta_4-\eta_2^2+\eta_3^2+\eta_7\eta_8,\\
\Phi_6'' &= \eta_1^2-\frac{2}{c}\eta_2\eta_5+\frac{2}{c}\eta_3\eta_6+\eta_8^2.
\end{aligned}$$

The final step is to transform these quadrics to the quadrics (2.3′), which is done as follows: the rank 4 quadrics in the projective linear span $V(\Phi_1'',\ldots,\Phi_6'')$ $\{X\Phi_1''+\ldots+W\Phi_6''\}\simeq\mathbb{P}^5$ are given by the intersection of the 4 quadratic cones K_i,

$$\begin{aligned}
&K_1:(X+W)(Z-U)+\left(Y-\frac{V}{2}\right)^2=0,\\
&K_2:(X+Y-V)(Y+Z-U)-\left(\frac{cZ}{2}-\frac{W}{c}\right)^2=0,\\
&K_3:(X-Y+V)(Y-Z+U)-\left(\frac{cZ}{2}+\frac{W}{c}\right)^2=0,\\
&K_4:UW-\frac{V^2}{4}=0.
\end{aligned}\tag{6}$$

Since

$$-2K_1+K_2-K_3-2K_4=0,$$

the locus $\bigcap_1^4 K_i$ defines a surface in P^5. In order to make the identification with the quadrics Φ [see (2.3)], we search for the rank 3 quadrics, besides Φ_1'', in the space of quadrics Φ_1'', Φ_2'', Φ_3'' (depending on even sections only), i.e., the quadrics $\Phi''\in\bigcap_1^4 K_i$ $\cap\{U=V=W=0\}$. From (6), they turn out to have the form

$$\begin{aligned}
\Phi_v'' &\equiv v^2\Phi_1''-v\Phi_2''-\Phi_3''\\
&=(v\eta_1-i\eta_4)^2+(v(v-1)\eta_2^2-(v+1)\eta_5^2-c\eta_2\eta_5)\\
&\quad+(v(v+1)\eta_3^2-(v-1)\eta_6^2-c\eta_3\eta_6);
\end{aligned}$$

these quadrics have rank 3 if and only if

$$T(v)=v^3-v+\frac{c^2}{4}\equiv(v-v_1)(v-v_2)(v-v_3)=0,\tag{7}$$

yielding the 4 rank 3 quadrics Φ_{v_1}'', Φ_{v_2}'', Φ_{v_3}'', $\Phi_\infty''=\Phi_1''$. They are related by the linear relation

$$\sum_1^3(\Phi_{v_i}''/T'(v_i))-\Phi_\infty''=0,$$

by means of the Jacobi trick.

Therefore, comparing the sets of quadrics Φ''_{v_i} and Φ, we make the following identification:

$$\Phi_1(\theta)=\Phi''_{\infty}(\eta), \qquad \Phi_2(\theta)=\Phi''_{v_1}(\eta)/T'(v_1),$$
$$\Phi_3^{+}(\theta)=\Phi''_{v_2}(\eta)/T'(v_2), \qquad \Phi_4(\theta)=-\Phi''_{v_3}(\eta)/T'(v_3),$$

suggesting the change of variables

$$\begin{gathered}\theta_i=\eta_i\ (i=1,2,3), \qquad \theta_4=T(v_1)^{-1/2}(v_1\eta_1-i\eta_4),\\ \theta_{5,6}=T(v_1)^{-1/2}(\sqrt{v_1(v_1\mp 1)}\eta_3+i\sqrt{v_1\pm 1}\eta_6),\\ \theta_7=\sqrt{U_+}\left(\eta_7+\frac{V_+}{2U_+}\eta_8\right), \qquad \theta_8=\sqrt{U_-}\left(\eta_7+\frac{V_-}{2U_-}\eta_8\right),\end{gathered} \tag{8}$$

where

$$b_i=\frac{\mu_i+T'(v_1)^{1/2}}{\mu_i-T'(v_1)^{1/2}}, \qquad b_4=b_1b_2b_3, \qquad i=1,2,3,$$

$$(\mu_1,\mu_2,\mu_3)=(v_2-v_1)\left(1,\frac{2i}{c}\sqrt{v_1+1}(v_3+1),\frac{2i}{c}\sqrt{v_1-1}(v_3-1)\right),$$

$$\begin{pmatrix}U_\pm\\ V_\pm\end{pmatrix}=(\mp 2c^2T'(b_1^{\mp 1}-b_4^{\mp 1})(b_2^{\mp 1}-b_4^{\mp 1}))^{-1}$$
$$\times\begin{pmatrix}-1 & \frac{c^2-2}{2} & -2 & 1 & 1 & \frac{-c^2}{2}\\ 0 & c^2 & c^2 & 0 & -c^2 & 0\end{pmatrix}$$
$$\times\begin{pmatrix}(b_2^{\mp 1}-b_4^{\mp 1})(v_1\pm\sqrt{T'})^2\\ (b_2^{\mp 1}-b_4^{\mp 1})\\ (b_2^{\mp 1}-b_4^{\mp 1})(-v_1\mp\sqrt{T'})^2\\ (b_1^{\mp 1}-b_4^{\mp 1})(\sqrt{v_1^2-v_1}\pm\sqrt{T'})^2\\ (b_1^{\mp 1}-b_4^{\mp 1})(-1-v_1)\\ (b_1^{\mp 1}-b_4^{\mp 1})\left(-1\pm\frac{2}{c}\sqrt{-(v_1+1)T'}\right)\end{pmatrix}$$

the derivative T' of the polynomial (7) being evaluated at $v=v_1$. One then identifies $X_i\Phi''_1+Y_i\Phi''_2+\ldots+W_i\Phi''_6$, expressed in the η-coordinates, with Φ_4,Φ_5,Φ_6 evaluated at $\eta_7=\eta_8=0$; this yields a highly overdetermined linear system in the $X_i,\ldots,W_i$, which is easily solved. In turn, this yields the linear map (8) between η_7, η_8 and θ_7, θ_8.

To summarize, we find the following transformation:

$$\begin{aligned}
\theta_1 &= x_1\,,\\
\theta_2 &= (1/\sqrt{2})(-1+\tfrac{1}{2}x_1^2)\,,\\
\theta_3 &= -(i/\sqrt{2})(1+\tfrac{1}{2}x_1^2)\,,\\
\theta_4 &= T^{-1/2}(v_1x_1-\tfrac{1}{2}(2x_1x_2^2+y_1y_2))\,,\\
\theta_{5,6} &= -i\sqrt{T/2}(\sqrt{v_1(v_1\mp 1)}(1+\tfrac{1}{2}x_1^2)\\
&\quad -\sqrt{v_1\pm 1}(x_2-\tfrac{1}{2}(y_1^2+x_1^2x_2)))\,,\\
\theta_{7,8} &= \sqrt{U_\pm/2}(\sqrt{-ic}\,y_1-(V_\pm/2\sqrt{-ic}\,U_\pm)(y_2x_1-2y_1x_2))\,.
\end{aligned} \tag{9}$$

Define, as before

$$u_i=\frac{\theta_i}{\kappa\theta_7+\lambda\theta_8}$$

with θ_i given by (9). Then this map combined with the transformation $u\curvearrowright x$, given by (3.10) [in terms of the parameters b_i found in (3.11)] provides the linear map from the Hénon-Heiles to the Manakov problem. In particular, setting $k=0$, leads to the Lax pairs of Corollary 1 (Sect. 3), with $u_i=\theta_i/\theta_8$.

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