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Limit matrices for the Toda flow and periodic flags for loop groups

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0 Introduction

Sato's theory of infinite dimensional Grassmannians, has been applied to explain the geometry of the K-P equation ([S; DJKM]), it has been used as a tool to study blow up behaviors and to regularize the solutions near the blow up [A-vM2]. The point is that realizing the K-P flow as a holomorphic flow of planes, enables one to follow what happens to the limiting planes as the equation in the original *bad* coordinates blows up. The blow-up behaviors are characterized by the various strata the orbit of planes visits in the Grassmannian. In this paper such ideas are applied to the N-periodic Toda flow (on periodic Jacobi matrices) which translates into a flow on the space of N-periodic flags of planes in the Grassmannians. Indeed here the N-periodic Toda flow amounts to N coupled KP equations with special interactions between time flows [U-T].

How such matrices blow up has been studied in [Fl; Fl-Ha; A-vM1] for arbitrary Lie algebras and Kac-Moody Lie algebras, whereas this paper focusses on regularizing the flow near the blow up locus; that is, on finding the boundary of isospectral sets.

If N-periodic Jacobi matrices

$$(0.1)\qquad L(z)=\begin{pmatrix} b_1 & a_1 & & & z\\ 1 & b_2 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & a_{N-1}\\ a_N z^{-1} & & & 1 & b_N \end{pmatrix},\quad \sum_1^N b_i=0,\ \prod_1^N a_i\neq 0\,,$$

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flow according to

$$\frac{\partial L}{\partial t_j} = [L, (L^j)^+], \qquad j = 1, 2, \ldots, N-1, \tag{0.2}$$

then the hyperelliptic curve (of genus $g = N - 1$)

(0.3)

$$X: \left\{ \begin{aligned} (z, \lambda) | 0 = \det(\lambda I - L(z)) &\equiv R(\lambda) - (z + Az^{-1}) = 0 \\ &\equiv \lambda^N + I_2 \lambda^{N-2} + \ldots + (-1)^N I_N - (z + Az^{-1}) \end{aligned} \right\}$$

is isospectral, i.e. the I_j and $A \equiv \prod_1^N a_i$ are independent of t_j; the eigenfunctions f of L (see [vM] and [vM-Mu])

$$f(z, \lambda) L(z) = \lambda f(z, \lambda) \qquad f = (f_0 = 1, f_1, \ldots, f_{N-1})$$

are meromorphic functions on the isospectral curve X, having two points P and Q covering ∞, such that for some divisor D of order g

$$(f_k) \geqq -kP + kQ - D \quad \text{and*} \quad (z) = -NP + NQ. \tag{0.4}$$

The set obtained by letting $L(z)$ flow according to the vector fields (0.2) parametrizes an affine part $\mathscr{A}_X$ of the hyperelliptic Jacobian J_X; in precise terms, if $\Theta_0 = \{P + \sum_1^{g-1} x_i, x_i \text{ generic}\} \subset J_X$ denotes the theta-divisor, and $\Theta_r = \Theta_0 + r(Q - P)$ translates of Θ_0 on J_X, then

$$\mathscr{A}_X = J_X \backslash (\Theta_0 \cup \Theta_1 \cup \ldots \cup \Theta_{N-1}),$$

is parametrized by the *isospectral* set** of N-periodic Jacobi matrices. If one approaches the Θ_i's, several entries will blow up, while other will tend to zero. The chief question posed and resolved in this paper is the following: can the whole Jacobian J_X rather than an affine part of it be parametrized by isospectral matrices? That is, when the t_1-trajectories hit the Θ-divisor Θ_1 or any of the translates, can the matrix $L(z, t)$ be conjugated

$$L_{\text{new}}^{\top} = B^{-1} L^{\top} B$$

by means of a matrix B of polynomial entries in the a's and b's such that $\lim_{t \to t^*} L_{\text{new}}$ exists and what is this limit? Thanks to Sato's Grassmannian technology, nicely explained in Pressley and Segal [P-S] and Segal and Wilson [S-W], the answer will turn out to be quite simple.

We sketch the method: a flag of infinite-dimensional planes in Gr, is obtained by viewing L as an infinite N-periodic matrix. Indeed it is natural to consider the (N-periodic) infinite eigenfunction with Floquet multiplier z, i.e.,

$$(\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots) \quad \text{with } f_{k+N} = z f_k; \tag{0.5}$$

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* Follows from the estimates

$$\begin{aligned} z &= R(\lambda) + \cdots \text{ for } \lambda \text{ near } \infty \text{ on the } + \text{sheet} \\ &= \frac{A}{R(\lambda)} + \cdots \text{ for } \lambda \text{ near } \infty \text{ on the } - \text{sheet} \end{aligned}$$

** Maintaining all coefficients of the curve (0.3): A and all the coefficients of $R(\lambda)$

with divisor structure (0.4), where $D \geqq 0$ is a divisor of order g which maps to a point in $\mathscr{A}_X \subset J_X$; such a divisor will be called *regular*. Conversely a point in $\mathscr{A}_X$ maps to a (*regular*) divisor of order g, to a unique set of functions (up to multiplicative constants) on X satisfying $(f_k) \geqq -kP + kQ - D$ and to a unique N-periodic matrix of the form (0.1). Consider now the N-periodic flag

$$\ldots \subset W_{k+1} \subset W_k \subset W_{k-1} \subset \ldots \quad \text{with } zW_k = W_{k+N}$$

defined by

$$W_k(t) = \psi_0^-(t) \cdot \operatorname{span}\{f_k(t), f_{k+1}(t), f_{k+2}(t), \ldots\} \tag{0.6}$$

viewed as functions of $\zeta^{-1} = z^{-1/N}$ defined on a circle $|z| = 1$ around P; the functions $\psi_0^-(t, \zeta) f_k(t, \zeta)$ are the wave functions associated with each W_k. The point of the latter is that then each W_k evolves in a simple way

$$W_k(t) = e^{\sum_{j \geqq 1} t_j \lambda(\zeta)^j}\, W_k(0)\,,$$

where $\lambda(\zeta)$ is the meromorphic function λ expressed in the local coordinate ζ:

$$\lambda(\zeta) = \zeta\left(1 - \frac{I_2}{N}\zeta^{-2} + \ldots\right), \quad \text{near } P\,. \tag{0.7}$$

The so-called loop $\gamma^-(z) \in LGL_N(\mathbb{C})$ encodes the entire information about the periodic flag*

$$\gamma^-(z) = [(\psi_0^- f_0)^\wedge, (\psi_0^- f_1)^\wedge, \ldots, (\psi_0^- f_{N-1})^\wedge] \in N^- \subset LGL_N(\mathbb{C})\,;$$

N^- is the subspace of matrices γ^- of holomorphic series in z^{-1} with $\gamma^-(\infty)$ upper triangular (with 1's on the diagonal) and B^+ matrices of holomorphic series γ^+ in z with $\gamma^+(0)$ lower triangular. The loop $\gamma^-(z)$ moves according to the j^{th} flow as

$$\gamma^-(t_j) = \gamma^-(0) \cdot \exp(t_j L^j(0, z)) (\operatorname{mod} B^+) \tag{0.8}$$

and the matrix

$$b^-(t_j) \equiv \gamma^-(0)^{-1} \gamma^-(t_j) = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} + O(z^{-1}) \in N^-$$

provides the concomitant factorization (Theorem 2.1)

$$\exp(t_j L^j(0, z)) = b^-(t_j) b^+(t_j)\,, \quad \text{with } b^+(t_j) \in B^+\,. \tag{0.9}$$

The matrix L blows up precisely when the divisor $D(t)$ tends to

$$D(t^*) \in \bigcap\nolimits_{j \in J} \Theta_j \bigcap\nolimits_{j \notin J} \Theta_j^c,\ J \subset \{0, 1, \ldots, N-1\}\,;$$

then instead of the flag W composed of subspaces W_k with basis given by functions with poles at P of order $\{k, k+1, k+2, \ldots\}$, the bases of the W_k now have pole behavior at P very simply specified by J (see Lemmas 3.2 and 3.3 in §3) and we say

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* Given a holomorphic series $w(\zeta)$ near P, define $\hat{w}(z) \equiv (w^{(0)}(z), w^{(1)}(z), \ldots, w^{(N-1)}(z))^\top$, where $w(\zeta) \equiv \sum_0^{N-1} \zeta^i w^{(i)}(z)$, $z = \zeta^N$

the flag W has left the main stratum. In addition (0.8) and (0.9) fail in a delightful way: the same holds, after throwing in an affine Weyl group element w_J (Theorem 3.1)

$$\gamma^-(0)\exp(t_J^* L^J(0)) = \gamma_*^- w_J (\bmod B^+),$$

and

$$\exp(t_J^* L^J(0)) = b_*^- w_J b_*^+, \quad b_*^+ \in B^+,$$

where

$$w_J = \operatorname{diag}(z^{\ell_0}, \ldots, z^{\ell_{N-1}}) \times \text{a permutation matrix}$$

with both the $\ell_i \in \mathbb{Z}(\Sigma \ell_i = 0)$ and the permutation matrix constructed from the sequence of leading ζ-exponents of the algebraic basis for the flag $\ldots \subset W_{k+1} \subset W_k \subset W_{k-1} \subset \ldots$ (Theorem 3.1).

To construct the matrix mentioned earlier, we notice the following: one set of constituents of $W_0(t)$ is given by (0.6), whereas another set is obtained by acting with $\nabla = \partial/\partial t_1 - \lambda$ on $\psi_0^-(t)$, yielding two descriptions of $W_0(t)$:

$$\begin{aligned} W_0(t) &= \operatorname{span}\{\psi_0^-(t), \psi_0^-(t) f_1(t), \psi_0^-(t) f_2(t), \ldots\} \\ &= \operatorname{span}\{\psi_0^-(t), \nabla\psi_0^-(t), \nabla^2\psi_0^-(t), \ldots\}, \quad \nabla = \partial/\partial t_1 - \lambda. \end{aligned} \tag{0.10}$$

A well-known fact in Sato's theory is that the wave functions $\psi_k^-(t) = \psi_0^-(t) f_k(t)$ are ratios, whose denominator is the τ-function, which vanishes simply along Θ_k. Therefore the first basis (0.10) of $W_0(t)$ ceases to make sense when $t \to t^* \in \Theta_0^c \cap \Theta_1 \cap \ldots \cap \Theta_s \cap \Theta_{s+1}^c$ because the f_k's blow up, while ψ_0^- remains finite and so the second basis remains finite. Therefore the trick is to find the map B from one basis to another which here turns out to be polynomial in the a's and b's; then as $t \to t^*$ the matrix $B^{-1} L^\top B$ tends to a finite limit, where the upper $s+1$ by $s+1$ block (which blew up in the matrix $L^\top$) gets replaced by its associated "companion matrix", the rest of the matrix being almost unchanged.

A $(s+1) \times (s+1)$ companion matrix has the form

$$\begin{pmatrix} I_1^{(s)} & -I_2^{(s)} & \cdots & & (-1)^s I_{s+1}^{(s)} \\ 1 & 0 & & & \\ & & & O & \\ & \ddots & \ddots & & \vdots \\ & O & & & \\ & & & 1 & 0 \end{pmatrix};$$

it can be interpreted as the sum of the negative simple roots of $s\ell(s+1, \mathbb{C})$, with the invariant polynomials of $s\ell(s+1, \mathbb{C})$ along the first row.

The results above seem to have a natural generalization to the Toda flows associated with the extended Dynkin diagrams. As explained in [A-vM1], these flows are linear on a complex torus (Abelian variety with a certain polarization) and the isospectral "Jacobi" matrices parametrize an affine part, obtained by removing from the torus a number of divisors, one for each dot in the Dynkin diagram (only for $s\ell(N)$ are they all translates of each other). The intersection pattern between several divisors is governed by the form of the corresponding sub-Dynkin diagram. Therefore it seems natural that an appropriate conjugation of the matrix $L^\top$ tends to a limit, where the submatrix which blew up gets replaced by the "companion" matrix associated with that sub-Dynkin diagram.

When $t^* \in \Theta_1, t^* \notin \Theta_i (i \neq 1)$, then guessing the matrix B for which the limit exists is quite easy, in view of the Painlevé analysis for the Toda lattice. Indeed in [Fl-Ha] and [A-vM1] it is shown that

$$
(0.11) \qquad a_1 = -\frac{1}{(t_1 - t_1^*)^2} + \dots \quad b_1 = -\frac{1}{t_1 - t_1^*} + \dots \quad b_2 = \frac{1}{t_1 - t_1^*} + \dots
$$

$$
a_0 = \alpha(t_1 - t_1^*) + \dots \qquad a_2 = \beta(t_1 - t_1^*) + \dots, \quad \alpha\beta \neq 0
$$

with all the other entries a_i and b_i bounded near t^*. Then conjugating $L^\top$ by the matrix

$$
B = \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & -b_1 & O \\ \hline \multicolumn{2}{c|}{O} & I \end{array}\right)
$$

leads to

$$
B^{-1} L^\top B = \left(\begin{array}{cc|cccccc}
I_1^{(1)} & -I_2^{(1)} & 1 & 0 & & \cdots & 0 & a_N b_1 z^{-1} \\
1 & 0 & 0 & 0 & & \cdots & 0 & a_N z^{-1} \\
\hline
a_2 & -a_2 b_1 & b_3 & 1 & & \cdots & & \\
0 & 0 & a_3 & b_4 & 1 & & O & \\
0 & 0 & & & & \ddots & & \\
\vdots & \vdots & & & & \ddots & & \\
 & & & O & & \ddots & & 1 \\
0 & z & & & & & a_{N-1} & b_N
\end{array}\right)
$$

where $I_1^{(1)}$ and $I_2^{(1)}$, defined by

$$
\det\left(\lambda I - \begin{pmatrix} b_1 & a_1 \\ 1 & b_2 \end{pmatrix}\right) = \lambda^2 - (b_1 + b_2)\lambda + b_1 b_2 - a_1 = \lambda^2 - I_1^{(1)}\lambda + I_2^{(1)}
$$

are invariant polynomials of $s\ell(2, \mathbb{C})$.

Then letting $t_1 \to t_1^*$, the matrix above tends to a finite limit, as is seen from the leading behaviors (0.11)

$$\lim_{t_1 \to t_1^*} B^{-1} L^{\top} B = \left(\begin{array}{cc|cccccc} I_1^{(1)} & -I_2^{(1)} & 1 & 0 & & \cdots & 0 & -\alpha z^{-1} \\ 1 & 0 & 0 & 0 & & \cdots & 0 & 0 \\ \hline 0 & \beta & b_3 & 1 & & & & \\ 0 & 0 & a_3 & b_4 & 1 & & O & \\ 0 & 0 & & & & & & \\ & & & & & \ddots & & \\ \vdots & \vdots & & & & \ddots & & \\ & & & & & \ddots & & \\ 0 & 0 & & O & & & & 1 \\ 0 & z & & & & & a_{N-1} & b_N \end{array} \right).$$

When t^* belongs to the intersection of Θ with more and more translates, the form of B gets considerably more complicated; but thanks to the Grassmannian point of view, the answer will be quite simple.

1 The periodic Toda flag

The object of this section is to associate with $L(z)$ a N-periodic flag of subspaces belonging to some infinite dimensional Grassmannian. The construction is based on the (unique) existence of functions f_k such that

$$(f_k) = -D + D_k - kP + kQ ,$$

as long as $D = \sum_1^g x_i \in \mathscr{A}$ (regular divisor); regularity is equivalent to the conditions

$$\dim L(D - kQ + (k-1)P) = 0, \quad \text{all } k \in \mathbb{Z} ;$$

in fact, N consecutive conditions will suffice, since $-NP + NQ = (z)$.

In particular, a regular divisor is non special* and does not contain P. As already pointed out in the introduction, we pick the local parameter $\zeta^{-1} = z^{-1/N}$ around P leading to the expansion (0.7) of λ in terms of ζ, which converges for $|\zeta|$ big enough, so that ζ defines an isomorphism from some closed neighborhood X_P of P in X to some disk $\{|\zeta| \geqq R\}$ on the Riemann sphere. We denote by X_Q, the complement of the interior of X_P: thus the closed sets X_P and X_Q cover X, and we call S^1 their intersection. Let

$$H = L^2(S^1, \mathbb{C}) = H_+ \oplus H_-$$

with $H_+ = \overline{\{1, \zeta, \zeta^2, \dots\}}$ and $H_- = \overline{\{\zeta^{-1}, \zeta^{-2}, \dots\}}$, and let

$$\mathrm{Gr}(H) = \{\text{closed } W \subset H \,|\, W \text{ is "comparable in size with } H_+\text{"}\}$$

---

* D is non special if and only if $\dim L(D) = 1$. For hyperelliptic X, this is equivalent to $x + ix \notin D$ for $x \in X$.

be the Grassmannian of H (see [P-S, Chap. 7] for the precise definition). Let L be a holomorphic line bundle on X and let φ be a trivialization of L over X_P. We use φ to identify sections of L over X_P with complex valued functions. To the quintuple $(X, L, P, \zeta, \varphi)$ Segal and Wilson [S-W] associate the following space $W \in \mathrm{Gr}(H)$:

$$W(X, L, P, \zeta, \varphi) = \text{the closure of the space of analytic functions on } S^1$$
$$\text{that extend to holomorphic sections of } L \text{ over } X_Q\,.$$

In the sequel, since X, P and ζ will be fixed as above, we shall shorten this notation by $W(L, \varphi)$.

If we denote by pr: $W \to H_+$ the orthogonal projection of W on H_+ along H_-, then

(1.1) $$\text{virtual dimension of } W \equiv \dim\ker \mathrm{pr} - \dim\operatorname{coker}\mathrm{pr}$$
$$= \chi(L) - 1\,,$$

where $\chi(L) = \dim H^0(X, L) - \dim H^1(X, L)$ denotes the Euler characteristic. In our case, for each $k \in \mathbb{Z}$, we pick

(1.2) $$L_k = [D - kQ]\,,$$

to be the line bundle associated with the divisor $D - kQ$. Since $D = \sum_{i=1}^{g} x_i$ is regular, it is easy to check that the f_k's (0.4) are unique and, what is the same, that L_k has a unique meromorphic section s (up to a constant) such that $(s) + kQ \geqq 0$. The divisor of this section is $(s) = D - kQ$, and it therefore defines a trivialization φ_k of L_k over the complement of $\{x_i, Q\}$, and in particular over X_P since $P \notin D$. We define

(1.3) $$W_k = W(L_k, \varphi_k)\,.$$

From (1.1) we have that virtual dimension of W is $-k$.

For $W \in \mathrm{Gr}(H)$, an element $w \in W$ which can be written as

$$w = a_s\zeta^s + a_{s-1}\zeta^{s-1} + \ldots,\ a_s \neq 0\,,$$

is called an *element of order* s. The elements of finite order form a dense subspace of W which is denoted by W^{alg}. In our case,

(1.4) $$W_k^{\mathrm{alg}} = \left\{\begin{array}{l}\text{analytic functions on } S^1 \text{ which extend to meromorphic}\\ \text{sections of } [D - kQ] \text{ which are holomorphic on } X\backslash\{P\}\end{array}\right\}$$
$$= \left\{\begin{array}{l}\text{meromorphic functions } f \text{ on } X \text{ such that}\\ (f) + D - kQ \geqq 0 \text{ on } X\backslash\{P\}\end{array}\right\}$$
$$= \bigcup\nolimits_{j\in\mathbb{Z}} L(D - kQ + jP)$$
$$= \text{span of } \{f_k, f_{k+1}, f_{k+2}, \ldots\}\,.$$

In (1.4), the second equality comes from the fact that the choice of the trivialization φ_k of $[D - kQ]$ described above amounts to write any meromorphic section u of $[D - kQ]$ which is holomorphic on $X\backslash\{P\}$ as

$$u(x) = f(x)s(x), \quad \text{near } P\,,$$

with s the unique meromorphic section of $[D - kQ]$ such that $(s) + kQ \geqq 0$, so that in fact

$$(f) + D - kQ = (u) - (s) + D - kQ = (u) \geqq 0 \quad \text{on} \quad X \backslash \{P\} .$$

It follows from (1.4) and (0.5) that the sequence of subspaces of H

$$\{W_k\}_{k \in \mathbf{Z}} = \ldots \subset W_{k+1} \subset W_k \subset W_{k-1} \subset \ldots$$

satisfies

$$\dim W_k / W_{k+1} = 1 \quad \text{and} \quad z W_k = W_{k+N} .$$

Such a sequence is naturally called a *periodic flag*. The set of those is a complex manifold, called the periodic flag manifold and is denoted by $F\ell^{(N)}$. The periodic flag $\{W_k\}_{k \in \mathbf{Z}}$ which we have associated to a periodic Jacobi matrix $L(z)$, will be called *the Toda flag* associated with $L(z)$.

By formulas (1.2) and (1.3), we can associate a periodic flag $\{W_k\}_{k \in \mathbf{Z}}$ to any divisor $D \in \mathrm{Pic}^g(X)$, not necessarily regular. However, when D is not regular – in particular when D is special or contains P – the trivialization φ_k we described above will cease to make sense. Thus, to define $\{W_k\}_{k \in \mathbf{Z}}$ in this case, we will have to pick another trivialization of $[D - kQ]$ about P, and there seems to be no canonical choice for such a trivialization. In other words, there seems to be no way of getting a well defined map from Jac X to the periodic flag manifold.

For given $W \in \mathrm{Gr}$, we define

$$S_W \equiv \{s \in \mathbb{Z}: W \text{ contains an element of order } s\} .$$

S_W is bounded from below, and contains all sufficiently large integers. Call the set

$$\Sigma_S = \{W \in \mathrm{Gr}(H): S_W = S\} ,$$

the stratum of $\mathrm{Gr}(H)$ corresponding to S. The virtual cardinal of S_W is defined as the virtual dimension of W. An indexing set S of virtual cardinal $-k$ can be written as

$$S = \{s_k, s_{k+1}, s_{k+2}, \ldots\} ,$$

with $s_k < s_{k+1} < s_{k+2} < \ldots$ and $s_j = j$ for large j. The Σ_S form a stratification of $\mathrm{Gr}(H)$.

Assume that D is regular; then it follows from (1.4) that

$$W_k \in \Sigma_{S_k} \quad \text{and} \quad S_k = \{k, k+1, k+2, \ldots\} .$$

When D is not regular, as noticed earlier, the definition of the planes W_k depends on a choice of a trivialization of $[D - kQ]$ about P. However, since a change of trivialization will amount to multiplying W_k by a non-zero holomorphic function $c_0 + c_1 \zeta^{-1} + \ldots$ $(c_0 \neq 0)$, the sequence S_k will remain unchanged and is therefore intrinsically defined. So, for example, when D is regular, it follows from (1.4) that the sequence S_k is defined by the places where the function

$$h(j) = \dim L(D - kQ + jP)$$

experiences a jump (by one). In Sect. 3, we will reduce the general case (D non regular), to a similar Riemann-Roch type computation (see formula (3.20), Sect. 3).

2 The Birkhoff factorization and the Toda flow

In this section, we show how a Birkhoff factorization leads to the linearization of the Toda flows.

We consider the group $LGL_N(\mathbb{C})$ of loops associated with $GL_N(\mathbb{C})$ and the subgroups

$$B^+ = \left\{ \begin{array}{l} \text{boundary values of } \gamma^+ \colon \{|z| < 1\} \to GL_N(\mathbb{C}) \text{ holomorphic} \\ \text{with } \gamma^+(0) \text{ lower triangular} \end{array} \right\}$$

and

$$N^- = \left\{ \begin{array}{l} \text{boundary values of } \gamma^- \colon \{|z| > 1\} \to GL_N(\mathbb{C}) \text{ holomorphic} \\ \text{with } \gamma^-(\infty) \text{ upper triangular, with 1's on the diagonal} \end{array} \right\}.$$

As in the finite-dimensional situation, there is a natural isomorphism between the periodic flag manifold $F\ell^{(N)}$ and the complex homogeneous space $LGL_N(\mathbb{C})/B^+$; to the N-periodic flag $\{W_k\}_{k\in\mathbb{Z}}$ $(zW_k = W_{k+N})$, we associate the loop of columns

$$\gamma(z) = [\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_{N-1}],$$

where

$$w_k(\zeta) = \zeta^0 w_{0k}(z) + \zeta^1 w_{1k}(z) + \ldots + \zeta^{N-1} w_{N-1,k}(z);\; z = \zeta^N$$

is a function spanning W_k/W_{k+1} and where

$$\hat{w}_k(z) = (w_{0k}, \ldots, w_{N-1,k})^\top \in L^2(S^1, \mathbb{C}^N).$$

Since one can replace $w_k \to \sum_{j\geq k} a_{jk} w_j$, $a_{kk} \neq 0$ $(k = 0, 1, \ldots, N-1)$, changing the coset representative for w_k, has the effect of multiplying $\gamma(z)$ to the right by an element of B^+. So the above map is only defined $\operatorname{mod} B^+$. One shows that it is an isomorphism. The inverse map sends $\gamma \operatorname{mod} B^+$ to the periodic flag $\{W_k\}_{k\in\mathbb{Z}} = \{\gamma\cdot\zeta^k H_+\}_{k\in\mathbb{Z}}$ where $\gamma\cdot W$ means $\{\gamma\hat{w}, w\in W\}$.

In the next section, we will explain how to pick a natural coset representative for $\gamma \operatorname{mod} B^+$. For the purpose of this section, it will be enough to understand the procedure for a "generic" flag. Let

$$\begin{aligned} \Sigma_{\mathrm{id}} &= \{\{W_k\}_{k\in\mathbb{Z}} \in F\ell^{(N)} \colon W_k \cap \zeta^k H_- = \{0\}, \forall k\in\mathbb{Z}\} \\ &= \{\{W_k\}_{k\in\mathbb{Z}} \in F\ell^{(N)} \colon W_k \in \Sigma_{\{k,k+1,k+2,\ldots\}}, \forall k\in\mathbb{Z}\}. \end{aligned}$$

Σ_{id} is an open dense subset of $F\ell^{(N)}$, which is called the *big stratum*. Indeed since W_k is transverse to $\zeta^k H_-$ and virt dim $W_k = -k$, the orthogonal projection $W_k \to \zeta^k H_+$ is an isomorphism. Let w_k be the unique element in W_k, which projects to ζ^k. Since $W_{k+1} \cap \zeta^{k+1} H_- = \{0\}$, $w_k \notin W_{k+1}$. We shall denote by $\gamma^-(z)$ the loop corresponding to this particular choice of w_k's and it is easy to check that $\gamma^-(z) \in N^- \subset LGL_N(\mathbb{C})$. We now formulate Theorem 1.

Theorem 2.1. *Let $\{W_k\}_{k\in\mathbb{Z}}$ be the Toda flag associated with the Jacobi matrix $L(z)$, and for $\lambda = \lambda(\zeta)$ defined in* (0.7) *let*

$$\{W_k(t)\}_{k\in\mathbb{Z}} = \{e^{t\lambda(\zeta)^j} W_k\}_{k\in\mathbb{Z}},\; j \text{ some positive integer}.$$

Then the factorization

$$\exp(tL(z)^j) = b^-(t)b^+(t)^{-1};\ b^-(t) \in N^-,\ b^+(t) \in B^+ , \tag{2.1}$$

can be performed if and only if $\{W_k(t)\}_{k\in\mathbf{Z}} \in \Sigma_{\mathrm{id}}$, *and*

$$b^-(t) = \gamma^-(0)^{-1}\gamma^-(t) , \tag{2.2}$$

*where**

$$\gamma^-(t) = [\psi_0^-(t)^\wedge, \psi_1^-(t)^\wedge, \ldots, \psi_{N-1}^-(t)^\wedge] \in N^- \tag{2.3}$$

is the unique loop in N^- *associated with the flag such that*

$$\{\gamma^-(t)\cdot\zeta^k H_+\}_{k\in\mathbf{Z}} = \{W_k(t)\}_{k\in\mathbf{Z}} .$$

Then the loop $\gamma^-(t)$ *flows according to*

$$\gamma^-(t) = \gamma^-(0)\exp(tL^j(z)) \quad (\mathrm{mod}\, B^+)$$

and

$$L(t,z) \equiv b^\pm(t)^{-1}L(z)b^\pm(t)$$

solves the j^{th} *Toda flow*

$$L(t)^\cdot = [L(t), (L(t)^j)^\pm]$$

and

$$\psi^\pm(t)^\cdot = \psi^\pm(t)(L^j)^\pm(t) \tag{2.4}$$

where

$$\psi^\pm(t) = (\psi_0^\pm(t), \ldots, \psi_{N-1}^\pm(t)) \quad \text{and} \quad \psi_k^+(t) = e^{-t\lambda^j}\psi_k^-(t) .$$

Proof of Theorem 2.1. It will be broken into several steps.

Step 1. The Baker vector and the loop.

Introduce Baker functions for the planes $W_k(t)$ provided $\{W_k(t)\}_{k\in\mathbf{Z}} \in \Sigma_{\mathrm{id}}$ (which is always the case if t is small enough, since $\{W_k\}_{k\in\mathbf{Z}} \in \Sigma_{\mathrm{id}}$). Let $\psi_k^-(t,\zeta)$ (in short $\psi_k^-(t)$) be the unique element in $e^{t\lambda^j}W_k$ which projects onto ζ^k under the orthogonal projection $e^{t\lambda^j}W_k \to \zeta^k H_+$. Define

$$\psi_k^+(t) = e^{-t\lambda^j}\psi_k^-(t) \in W_k .$$

By the periodicity of $\{W_k\}_{k\in\mathbf{Z}}$, $\psi_{k+N}^\pm(t) = z\psi_k^\pm(t)$ and all the information is contained in the *Baker vector*

$$\psi^\pm(t) = \psi^\pm(t,\zeta) = (\psi_0^\pm(t,\zeta), \ldots, \psi_{N-1}^\pm(t,\zeta))$$

and thus

$$\gamma^-(t,z) = [\hat{\psi}_0^-(t), \ldots, \hat{\psi}_{N-1}^-(t)] .$$

Notice that, from the uniqueness of $\psi_k^\pm(t)$ and from (1.4) we have

$$\psi^\pm(0) = f = (f_0 = 1, f_1, \ldots, f_{N-1}) ,$$

* $\psi_k^-(t,\zeta)$ is the Baker function of W_k; i.e. the unique function in $e^{t\lambda^j}W_k$ which projects onto ζ^k under the orthogonal projection $e^{t\lambda^j}W_k \to \zeta^k H_+$

where we think of the f_k's as functions of ζ via $f_k(\zeta) = f_k(\lambda(\zeta), \zeta^N)$.

We also define the *Baker matrices*

(2.5) $$\Psi^{\pm}(t)_{ij} = [\psi_j^{\pm}(t, \omega^i \zeta)]_{0 \leq i,j \leq N-1} ,$$

where ω denotes some N^{th} root of unity, $\omega \neq 1$. Thus the i^{th} line of $\Psi(0)$ corresponds to the (normalized) left eigenvector of $L(z)$ with eigenvalue $\lambda_i = \lambda(\omega^i \zeta)$. One easily checks that

(2.6) $$\Psi^{-}(t) = \Delta(\zeta)\gamma^{-}(t) ,$$

where

$$\Delta(\zeta) = [(\omega^i \zeta)^j]_{0 \leq i,j \leq N-1} .$$

In the appendix the Baker functions $\psi_k^{+}(t)$ (resp. $\psi_k^{-}(t)$) are built explicitly from the function theory on the Riemann surface X; their analytic extension to X is meromorphic on $X \setminus \{P\}$ (resp. $X \setminus \{Q\}$), with an essential singularity at P (resp. Q).

Step 2. Let $t \in \mathbb{C}$ be such that the factorization (2.1) can be done. Then

(a) $\{W_k(t)\}_{k \in \mathbb{Z}} \in \Sigma_{\text{id}}$.

(b) $\Psi^{\pm}(t) = \Psi(0) b^{\pm}(t)$.

The Birkhoff factorization theorem implies that a matrix Laurent series $M(z, z^{-1})$ can always be factorized (see [P-S], Theorem 8.7):

(2.7) $$M(z, z^{-1}) = n^{-} w b^{+}, \quad n^{-} \in N^{-}, b^{+} \in B^{+}$$

$$w = \operatorname{diag}(z^{\ell_0}, \ldots, z^{\ell_{N-1}}) \cdot \begin{pmatrix} \text{permutation} \\ \text{matrix} \end{pmatrix},$$

$$\ell_i \in \mathbb{Z} .$$

The generic case is $w = I$ and if $M(z, z^{-1}) = M(z, z^{-1}, t)$ depends on t analytically, so will $n^{-} = n^{-}(t)$ and $b^{+} = b^{+}(t)$.

Since

(2.8) $$\Psi(0) L(z) = \Lambda \Psi(0)$$

with $\Lambda = \operatorname{diag}(\lambda(\zeta), \lambda(\omega\zeta), \ldots, \lambda(\omega^{N-1}\zeta))$ and

(2.9) $$\exp(tL(z)^j) = b^{-}(t) b^{+}(t)^{-1}$$

we obtain upon exponentiating (2.8) and substituting (2.9) that

$$\Psi(0) b^{-}(t) = \exp(t\Lambda^j) \Psi(0) b^{+}(t)$$

which is equivalent to the first row

(2.10) $$f(\zeta) b^{-}(t) = e^{t\lambda^j} f(\zeta) b^{+}(t) .$$

The k^{th} component of the left hand side of (2.10) looks like

$$[f(\zeta) b^{-}(t)]_k = \sum_{\ell = -\infty}^{k-1} c_{\ell k} f_\ell(\zeta) + f_k(\zeta) ,$$

$$= \zeta^k (1 + O(\zeta^{-1}))$$

while

$$[f(\zeta)b^{+}(t)]_k = \sum_{\ell=k}^{\infty} d_{\ell k} f_\ell(\zeta) \in W_k \quad \text{(see (1.4))};$$

therefore

$$[f(\zeta)b^{-}(t)]_k \in e^{t\lambda^j} W_k \quad \text{(of order } k)$$

and so it must be $\psi_k^-(t,\zeta)$ if (a) holds. In particular, since $e^{t\lambda^j} W_k \supset e^{t\lambda^j} W_{k+1} \supset \ldots$, the plane $e^{t\lambda^j} W_k$ contains elements of order $k, k+1, \ldots$. If $e^{t\lambda^j} W_k$ would contain some element of order $s < k$, then we would have that virt dim $e^{t\lambda^j} W_k > -k$, which is impossible, since virt dim $W_k = -k$ and $e^{t\lambda^j} W_k$ belongs to the same connected component of $\mathrm{Gr}(H)$ as W_k. Thus $e^{t\lambda^j} W_k$ is transverse to $\zeta^k H_-$, establishing part (a), and so $[f(\zeta)b^-(t)]_k = \psi_k^-(t,\zeta)$, establishing part (b).

Step 3. The loop $\gamma^-(t)$ and the factorization of $\exp(tL(z)^j)$.

Whenever the factorization (2.1) is possible, we have

$$\begin{aligned} (2.11) \qquad b^-(t) &= \Psi(0)^{-1}\Psi^-(t) && \text{by step 2(b)} \\ &= \gamma^-(0)^{-1}\Delta^{-1}\Delta\gamma^-(t) && \text{by (2.6)} \\ &= \gamma^-(0)^{-1}\gamma^-(t)\,. \end{aligned}$$

We can think of formula (2.11) as telling us via (2.1) that, for small t,

$$(2.12) \qquad \gamma^-(0,z)\exp(tL(z)^j) \bmod B^+ = \begin{pmatrix} \text{the loop associated with} \\ \{e^{t\lambda^j} W_k\}_{k\in\mathbb{Z}} \end{pmatrix} \bmod B^+ .$$

By continuity, it must hold for all $t \in \mathbb{C}$. In particular, if $\{e^{t\lambda^j} W_k\}_{k\in\mathbb{Z}} \in \Sigma_{\mathrm{id}}$, we can pick a unique coset representative for the right hand side of (2.12) in N^-, and the factorization can be done and so formula (2.11) holds.

Step 4. The solution of the Toda equations.

Finally, we show that the solution of the Birkhoff factorization problem (2.1) is equivalent to the solution of the Toda flows. Let us denote by $\mathcal{N}^-$ and $\mathcal{B}^+$, the Lie algebras of N^- and B^+. For $L(z) \in Lg\ell_N(\mathbb{C})$ (the loop algebra of $g\ell_N(\mathbb{C})$), we will write

$$(2.13) \qquad L(z) = L(z)^- - L(z)^+ = \pi_{\mathcal{N}^-} L(z) + \pi_{\mathcal{B}^+} L(z)$$

where $\pi_{\mathcal{N}^-}$ (resp. $\pi_{\mathcal{B}^+}$) denote the projections onto $\mathcal{N}^-$ (resp. $\mathcal{B}^+$). Since

$$\begin{aligned} L(t,z) \equiv b^-(t)^{-1}L(z)b^-(t) &= b^+(t)^{-1}\exp(-tL(z)^j)L(z)\exp(tL(z)^j)b^+(t) \\ &= b^+(t)^{-1}L(z)b^+(t), \end{aligned}$$

$L(t)$ must have the form of a periodic Jacobi matrix. Then straightforward differentiation leads to

$$L(t)^{\cdot} = [L(t), b^{\pm}(t)^{-1}b^{\pm}(t)^{\cdot}]\,.$$

Now, by differentiating (2.1) we get

(2.14) $$L(z)^j \exp(tL(z)^j) = b^-(t)^{\cdot} b^+(t)^{-1} - b^-(t) b^+(t)^{-1} b^+(t)^{\cdot} b^+(t)^{-1} .$$

Upon substituting $b^-(t)L(t)^j b^-(t)^{-1}$ for $L(z)^j$ and $b^-(t)b^+(t)^{-1}$ for $\exp(tL(z)^j)$ in the left hand side of (2.14), we find after some simplification

$$L(t)^j = b^-(t)^{-1} b^-(t)^{\cdot} - b^+(t)^{-1} b^+(t)^{\cdot} ,$$

which, using our convention (2.13), shows that

(2.15) $$(L(t)^j)^{\pm} = b^{\pm}(t)^{-1} b^{\pm}(t)^{\cdot}$$

as desired.

Step 5. For the j^{th} Toda flow, we have

(2.16) $$\Psi^{\pm}(t)^{\cdot} = \Psi^{\pm}(t)(L^j)^{\pm}(t) .$$

Indeed, by step 2(b), we have

$$\Psi^{\pm}(t) = \Psi(0) b^{\pm}(t) ,$$

which, by formula (2.15), implies

$$\Psi^{\pm}(t)^{\cdot} = \Psi(0) b^{\pm}(t)(L^j)^{\pm}(t) = \Psi^{\pm}(t)(L^j)^{\pm}(t) ,$$

as desired.

3 Strata and limit loops for non regular divisors

Let*

$$W_k(t) = \exp(t_1 \lambda + \cdots + t_g \lambda^g) W_k, \quad t = (t_1, \ldots, t_g) ,$$

where $\{W_k\}_{k \in \mathbb{Z}}$ denotes the Toda flag associated with some periodic Jacobi matrix $L(z)$, or equivalently associated with a regular divisor D. From the previous section, it follows that

(3.1) $$\{W_k(t)\}_{k \in \mathbb{Z}} \in \Sigma_{\text{id}} \Leftrightarrow \exp(\Sigma_{j \geqq 1} t_j L(z)^j) = b^-(t) b^+(t)^{-1}$$
$$\text{with } b^-(t) \in N^-, b^+(t) \in B^+ ,$$
$$\Leftrightarrow L(t, z) = b^{\pm}(t)^{-1} L(z) b^{\pm}(t)$$

with $L(t, z)$ the Jacobi matrix obtained by following the various Toda flows during times $t_1, \ldots, t_g$.

Let $f_k(t)$ be the left eigenvectors of $L(t, z)$ normalized by $f_0 = 1$, such that

(3.2) $$(f_k(t)) = -D(t) + D_k(t) - kP + kQ ,$$

---

* In Sect. 2, t refers to $t \in \mathbb{C}$, whereas in this section $t \in \mathbb{C}^g$

with $D(t)$ a regular divisor of degree g. Since (by (A.11)) $f_k(t) = \frac{\psi_k^-(t)}{\psi_0^-(t)}$, we have

$$\begin{aligned} W_k^{\text{alg}}(t) &= \{\psi_k^-(t), \psi_{k+1}^-(t), \ldots\}, \quad \text{from (2.3)}\,, \\ &= \psi_0^-(t)\, W_k^{\text{alg}}([D(t) - kQ], \varphi_k(t)), \quad \text{from (1.4)}\,, \end{aligned}$$

with $\varphi_k(t)$ the trivialization of $[D(t) - kQ]$ defined in (1.3). In this section, we will see what happens when $D(t)$ fails to be regular, that is when

$$\begin{aligned} & D(t) \in \Theta_0 \cup \Theta_1 \cup \ldots \cup \Theta_{N-1}, \text{ with} \\ & \Theta_k = \Theta_0 + k(Q - P) \\ & \quad = \{D \mid \deg D = g \text{ and } L(D - kQ + (k-1)P) \neq 0\} \\ & \quad = \left\{ D \mid D \sim \sum_1^{g-1} x_i + kQ - (k-1)P \right\}. \end{aligned} \tag{3.3}$$

Precisely when $D(t) \in \Theta_k$, do we have $\psi_k^-(t) \equiv \infty$ (by (A.6′) since $\int_{gP}^{D(t)} \omega = -(At + \xi + K)$, with K the vector of Riemann constants), and the factorization (3.1) breaks down. The flag $\{W_k(t)\}_{k \in \mathbb{Z}}$ will then fail to be in the big stratum Σ_{id}, or equivalently

$$\exp(\Sigma_{j \geq 1} t_j L(z)^j) = b^-(t) w b^+(t)^{-1}\,, \tag{3.4}$$

with

$$w = \text{diag}(z^{l_0}, \ldots, z^{l_{N-1}}) \cdot (\text{permutation matrix}),\ l_i \in \mathbb{Z},\ \Sigma l_i = 0\,,$$

some affine Weyl group element of $LGL_N(\mathbb{C})$.

We need to characterize the flag when it fails to be in the big stratum [P-S, Chap. 8], in order to state the main result. Let $\{W_k\}_{k \in \mathbb{Z}}$ be a periodic flag. Each W_k belongs to some stratum Σ_{S_k} of the Grassmannian, and $S_k \backslash S_{k+1}$ has exactly one element, say π_k. $\pi = \{\pi_k\}$ is a permutation of $\mathbb{Z}$ with the property $\pi_{k+N} = \pi_k + N$. For each $0 \leq k \leq N-1$, we choose a vector $v_k \in H$ spanning W_k / W_{k+1}, which is of order π_k, that is

$$v_k = \zeta^{\pi_k} + * \zeta^{\pi_k - 1} + \cdots.$$

Write now

$$\begin{aligned} & \pi_0 = N l_0 + \tau_0, \ldots, \pi_{N-1} = N l_{N-1} + \tau_{N-1} \\ & l_k \in \mathbb{Z},\ \Sigma l_k = 0;\ \{\tau_i\} \text{ a permutation of } \{0, 1, \ldots, N-1\}\,. \end{aligned} \tag{3.5}$$

Then the loop

$$\gamma(z) = [\hat{v}_0(z), \ldots, \hat{v}_{N-1}(z)]$$

associated with the flag $\{W_k\}_{k \in \mathbb{Z}}$ is such that

$$\gamma(z) = \gamma^-(z) w\,, \tag{3.6}$$

where $\gamma^-(z) \in N^-$ and w is the affine Weyl group element having the column representation:

$$\begin{aligned} & w = [z^{l_0} e_{\tau_0}, z^{l_1} e_{\tau_1}, \ldots, z^{l_{N-1}} e_{\tau_{N-1}}]\,, \\ & e_i = (0, \ldots, 1, \ldots, 0)^T\,. \end{aligned} \tag{3.7}$$

In particular denote by w_j, $0 \leqq j \leqq N-1$, the Weyl reflections through the affine simple roots of $LGL_N(\mathbb{C})$:

$$w_0 = \begin{pmatrix} 0 & & & z \\ & \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} & & \\ z^{-1} & & & 0 \end{pmatrix},$$

$$\text{(3.8)} \qquad w_j = \begin{pmatrix} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} & & \\ & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \\ & & \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \end{pmatrix} \quad \begin{matrix} 0^{\text{th}} \text{ row} \\ \\ j^{\text{th}} \text{ row} \\ \\ (N-1)^{\text{th}} \text{ row} \end{matrix}$$

Equivalent to the factorization (3.1) breaking down and becoming (3.4) when one leaves the big stratum is that the formula of Theorem 2.1 (see (2.1), (2.2) and (2.3))

$$\gamma(t) \equiv \gamma^-(0) \exp(\Sigma_{j \geqq 1} t_j L^j(z)) = \gamma^-(t) (\text{mod } B^+),\ \gamma^-(t) \in N^-,$$

where $\gamma(t)$ denotes the loop of the flag $\{W_k(t)\}_{k \in \mathbf{Z}}$, becomes by virtue of continuity and (3.6)

$$\text{(3.9)} \qquad \gamma(t^*) \equiv \gamma^-(0) \exp(\Sigma_{j \geqq 1} t_j^* L^j(z)) = \gamma_*^- w (\text{mod } B^+),\ \gamma_*^-(t) \in N^-.$$

We can now state the following complementary result to Theorem 2.1.

Theorem 3.1. *Assume that when $t \to t^*$, the divisor $D(t^*)$ fails to be regular, that is there exists $J \subsetneqq \{0, 1, \ldots, N-1\}$ such that*

$$D(t^*) \in \bigcap_{j \in J} \Theta_j \bigcap_{j \notin J} \Theta_j^c.$$

Then, the loop $\gamma(z, t^*)$ of the flag $\{W_k(t^*)\}_{k \in \mathbf{Z}}$ satisfies

$$\text{(3.10)} \qquad \gamma(z, t^*) \equiv \gamma^-(0) \exp(\Sigma_{j \geqq 1} t_j^* L(z)^j) = \gamma_*^-(z) w_J (\text{mod } B^+),$$

with

$$w_J = \text{the longest affine Weyl group element generated by the } w_j\text{s}, j \in J.$$

Moreover, we have the following formula for the limit loop $\gamma_*^-(z) \in N^-$. For each $0 \leqq k \leqq N-1$, let us write $k = r + l$, with r the biggest integer such that $r \leqq k$ and r mod $N \notin J$, and let

$$g_k(\zeta) \equiv \lambda(\zeta)^l \psi_r^-(t^*, \zeta),$$

then we can pick

$$\text{(3.11)} \qquad \gamma_*^-(z) = [\hat{g}_0, \hat{g}_1, \ldots, \hat{g}_{N-1}].$$

Example. Let $J = \{0, 1, 2, N-1\}$. Then

$$\gamma_*^-(z) = [\lambda^2\psi_{-2}^-(t^*)^\wedge, \lambda^3\psi_{-2}^-(t^*)^\wedge, \lambda^4\psi_{-2}^-(t^*)^\wedge, \psi_3^-(t^*)^\wedge, \ldots, \psi_{N-2}^-(t^*)^\wedge, \lambda\psi_{N-2}^-(t^*)^\wedge] .$$

To prove the theorem, we must first establish three lemmas.

Lemma 3.1. *Let D be a positive divisor of degree g on X such that*

$$\text{(i)} \qquad L(D - kQ + (k-1)P) \neq 0, \quad 1 \leqq k \leqq s, \qquad = 0, \quad k = 0,$$

then

$$D = sQ + \Sigma,$$

with Σ a positive divisor of degree $g - s$.

(ii) *If in addition $L(D - kQ + (k-1)P) = 0$, $k = s+1$, then*

$$\Sigma \not\ni P, Q, x + ix, \quad \text{for any } x \in X; \tag{3.12}$$

also

$$\dim L(\Sigma + (s-k)Q) = 1, \tag{3.13}$$

and

$$\dim L(\Sigma + (s-k)Q + sP) = s - k + 1, \quad 0 \leqq k \leqq s. \tag{3.14}$$

Proof. We first prove part (i). Since $L(D - P) = 0$, D is non special and does not contain P. The proof goes by induction on s. If $s = 1$, since $L(D - Q) \neq 0$, $D \sim Q + \Sigma$, and since D is non special $D = Q + \Sigma$. By induction hypothesis,

$$D = (s-1)Q + \sum_{i=1}^{g-s+1} x_i .$$

Since $L(D - sQ + (s-1)P) \neq 0$,

$$D \sim sQ - (s-1)P + \sum_{i=1}^{g-1} y_i ,$$

which implies that

$$(s-1)P - Q + \sum_{i=1}^{g-s+1} x_i \sim \sum_{i=1}^{g-1} y_i ,$$

and therefore

$$\dim L\left((s-1)P - Q + \sum_{i=1}^{g-s+1} x_i\right) = \dim \Omega\left(-\sum_{i=1}^{g-s+1} x_i - (s-1)P + Q\right)$$
$$= \dim \Omega\left(-\sum_{i=1}^{g-s+1} x_i - (s-1)P\right) \neq 0 .$$

Let $\omega \in \Omega\left(-\sum_{i=1}^{g-s+1} x_i - (s-1)P\right)$, then

$$\omega = \frac{(c\lambda^{g-s} + (\text{lower order terms}))}{\sqrt{P(\lambda)}}\, d\lambda ,$$

with $P(\lambda) = R(\lambda)^2 - 4A$ (see (0.3)). We know that $P \notin \sum_{i=1}^{g-s+1} x_i$. Suppose $Q \notin \sum_{i=1}^{g-s+1} x_i$. Since D is non special, $x + ix \notin \sum_{i=1}^{g-s+1} x_i$, and therefore the numerator of ω must have at least $g - s + 1$ zeroes, which is absurd, so $Q \in \sum_{i=1}^{g-s+1} x_i$ as desired, finishing the proof of part (i).

It remains to establish (3.12), (3.13) and (3.14). We already know that $D = sQ + \Sigma$, and $\Sigma \not\ni P, x + ix$. Now (ii) means that $L(\Sigma + sP - Q) = 0$, and thus $Q \notin \Sigma$. (3.13) and (3.14) are both easy applications of Riemann-Roch. First,

$$\dim L(\Sigma + (s-k)Q) = \dim \Omega(-\Sigma - (s-k)Q) - k + 1 .$$

Let $\omega \in \Omega(-\Sigma - (s-k)Q)$, then

$$\omega = \frac{(c\lambda^{g-(s-k+1)} + (\text{lower order terms}))}{\sqrt{P(\lambda)}} d\lambda .$$

Since the numerator must vanish at $g - s$ points determined by Σ, we have

$$\dim \Omega(-\Sigma - (s-k)Q) = g - (s-k) - (g-s) = k ,$$

whence, $\dim L(\Sigma + (s-k)Q) = 1$. Similarly,

$$\dim L(\Sigma + (s-k)Q + sP) = \dim \Omega(-\Sigma - (s-k)Q - sP) + s - k + 1 .$$

Since $s \geqq s - k$, $\omega \in \Omega(-\Sigma - (s-k)Q - sP)$ can be written as

$$\omega = \frac{(c\lambda^{g-s-1} + (\text{lower order terms}))}{\sqrt{P(\lambda)}} d\lambda .$$

Since the numerator has to vanish at $g - s$ points, $\omega \equiv 0$; this completes the proof of Lemma 3.1.

Lemma 3.2. *Assume that when $t \to t^*$, $D(t^*) \notin \Theta_r$, then*

$$W_{r+k}^{\text{alg}}(t^*) = \psi_r^-(t^*)\, W^{\text{alg}}([D_r(t^*) - kQ], \varphi_k(t^*)) \tag{3.15}$$
$$= \psi_r^-(t^*) \bigcup_{j\in\mathbb{Z}} L(D_r(t^*) - kQ + jP) ,$$

and

$$W_r(t^*) \in \Sigma_{\{r, r+1, r+2, \dots\}} , \tag{3.16}$$

with $\psi_r^-(t^*)$ having a pole of order r at P.

Proof. Remember from (3.2) that $D(t) \sim D_r(t) - rP + rQ$, and so by (3.3)

$$D(t^*) \notin \Theta_r \Leftrightarrow D_r(t^*) \notin \Theta_0$$
$$\Leftrightarrow D_r(t^*) \text{ is non special and does not contain } P .$$

Then for $t \neq t^*$, we may write (see (1.3) and (1.4))

$$W_{r+k}^{\text{alg}}(t) = \psi_r^-(t) \left\{ \frac{f_{r+k}(t)}{f_r(t)}, \frac{f_{r+k+1}(t)}{f_r(t)}, \dots \right\}$$
$$= \psi_r^-(t)\, W^{\text{alg}}([D_r(t) - kQ], \varphi_k(t)) ,$$

with $\varphi_k(t)$ the trivialization of $[D_r(t) - kQ]$ about P, which is defined by the unique meromorphic section s (up to a constant) of this line bundle such that $(s) + kQ \geqq 0$. Since $D_r(t^*)$ is non special and does not contain P, it follows respectively that both $\psi_r^-(t)$ and $\varphi_k(t)$ make still sense at the limit $t = t^*$, so that (using (1.4)) we have shown (3.15).

To prove (3.16), from (3.15) deduce

$$W_r^{\mathrm{alg}}(t^*) = \psi_r^-(t^*) \bigcup_{j\in\mathbf{Z}} L(D_r(t^*) + jP) . \tag{3.17}$$

Since $\dim L(D_r(t^*) - P) = 0$ and $\dim L(D_r(t^*) + jP) \geqq j + 1$, it follows that

$$\dim L(D_r(t^*) + jP) = j + 1, \quad j = 0, 1, 2, \ldots ; \tag{3.18}$$

so by (3.17), if we set δ equal to the order of the pole of $\psi_r^-(t^*)$ at P, (3.17) yields

$$W_r(t^*) \in \Sigma_{\{\delta, \delta+1, \delta+2, \delta+3, \ldots\}} . \tag{3.19}$$

Since $W_r(t)$ has virtual dimension $-r$ for most t, and since this dimension condition defines a connected set, $W_r(t^*)$ also has virtual dimension $-r$, and so by (3.19), δ must be equal r, proving the lemma.

Lemma 3.3. *Assume that when $t \to t^*$*

$$D(t^*) \in \Theta_r^c \cap \Theta_{r+1} \cap \ldots \cap \Theta_{r+s} \cap \Theta_{r+s+1}^c .$$

Then, for $0 \leqq k \leqq s$, $W_{r+k}(t^*) \in \Sigma_{S_{r+k}}$ with

$$S_{r+k} = \{r, r+1, \ldots, r+s-k, r+s+1, r+s+2, \ldots\} . \tag{3.20}$$

Moreover, for $0 \leqq k \leqq s$,

$$\lambda^{s-k} \psi_r^-(\zeta, t^*) \tag{3.21}$$

spans $W_{r+k}(t^*)/W_{r+k+1}(t^*)$.

Proof. Since from (3.2) we have that $D_r(t) \sim D(t) - r(Q - P)$, it follows from (3.3) that

$$D(t^*) \in \Theta_r^c \cap \Theta_{r+1} \cap \ldots \cap \Theta_{r+s} \cap \Theta_{r+s+1}^c \Leftrightarrow D_r(t^*) \in \Theta_0^c \cap \Theta_1 \cap \ldots \cap \Theta_s \cap \Theta_{s+1}^c .$$

From Lemma 3.1, it follows that

$$D_r(t^*) = sQ + \Sigma, \text{ with } \Sigma \text{ a positive divisor of degree } g - s ,$$

$$\text{such that } P, Q, x + ix \notin \Sigma ,$$

and from (3.15) we get

$$W_{r+k}^{\mathrm{alg}}(t^*) = \psi_r^-(t^*) \bigcup_{j\in\mathbf{Z}} L(\Sigma + (s-k)Q + jP) . \tag{3.22}$$

We now need to establish:

$$\begin{aligned}
&\dim L(\Sigma + (s-k)Q - P) = 0 \quad &&\text{(from (3.12) and (3.13))}\\
&\dim L(\Sigma + (s-k)Q) = 1 &&\text{(from (3.13))}\\
&\dim L(\Sigma + (s-k)Q + P) = 2 &&\\
&\quad\vdots &&\\
&\dim L(\Sigma + (s-k)Q + (s-k-1)P) = s - k &&
\end{aligned}$$

$$\left.\begin{array}{l}\dim L(\Sigma + (s-k)Q + (s-k)P) \\ \vdots \\ \dim L(\Sigma + (s-k)Q + sP)\end{array}\right\rangle = s - k + 1 \quad \text{(from (3.14))}$$

$$\dim L(\Sigma + (s-k)Q + (s+1)P) = s - k + 2$$

$$\vdots$$

$$\dim L(\Sigma + (s-k)Q + (s+j)P) = s - k + j + 1$$

$$\vdots$$

From (3.13), the fact that adding a pole can increase the dimension by at most one and the fact that

(3.23) $$\{1, \lambda, \ldots, \lambda^j\} \in L(\Sigma + (s-k)Q + jP), \quad 1 \leqq j \leqq s - k,$$

we get that $\dim L(\Sigma + (s-k)Q + jP) = j + 1$, $1 \leqq j \leqq s - k$. From (3.14), it then follows that $\dim L(\Sigma + (s-k)Q + jP) = s - k + 1$, $s - k \leqq j \leqq s$. Since, by Riemann-Roch, $\dim L(\Sigma + (s-k)Q + (s+j)P) \geqq s - k + j + 1$, and adding a pole can increase the dimension by at most one, it follows that $\dim L(\Sigma + (s-k)Q + (s+j)P) = s - k + j + 1$, $j \geqq 1$. Thus, we have shown that, for $0 \leqq k \leqq s$, the function $h(j) = \dim L(\Sigma + (s-k)Q + jP)$ experiences jumps at

$$j = 0, 1, 2, \ldots, s - k, s + 1, s + 2, \ldots,$$

while by Lemma 3.2, $\psi_r^-(t^*)$ has a pole of order r at P; so by formula (3.15), $W_{r+k}(t^*) \in \Sigma_{S_{r+k}}$, with S_{r+k} defined by (3.20). In fact, remembering (3.23), we have found an algebraic basis of $W_{r+k}(t^*)$, namely

(3.24) $$W_{r+k}^{\mathrm{alg}}(t^*) = \psi_r^-(t^*)\left\{1, \lambda, \lambda^2, \ldots, \lambda^{s-k}, \frac{\psi_{r+s+1}^-(t^*)}{\psi_r^-(t^*)}, \ldots\right\}, \quad 0 \leqq k \leqq s,$$

which establishes (3.21), ending the proof of Lemma 3.3.

Proof of Theorem 3.1. We first deal with the case of an intersection of consecutive translates of the theta divisor, i.e.

$$\begin{aligned} L(D(t^*) - kQ + (k-1)P) &\neq 0 \text{ for } k = r + 1, \ldots, r + s \bmod N \\ &= 0 \text{ otherwise}. \end{aligned}$$

By Lemma 3.2, when $k \neq r + 1, \ldots, r + s \bmod N$, $W_k(t^*) \in \Sigma_{S_k}$ with

(3.25) $$S_k = \{k, k + 1, k + 2, \ldots\}.$$

From Lemma 3.3, we know that for $1 \leqq k \leqq s$,

(3.26) $$S_{r+k} = \{r, r + 1, \ldots, r + s - k, r + s + 1, r + s + 2, \ldots\},$$

and thus, from the periodicity condition $S_{k+N} = N + S_k$, we have determined all the sequences S_k. Let $\pi = \{\pi_k\}_{k \in \mathbf{Z}}$ $(\pi_{k+N} = \pi_k + N)$ be the permutation obtained by picking the unique element in $S_k \backslash S_{k+1}$. From (3.25) and (3.26), one easily checks that π is given by

$$\pi \quad \begin{array}{cc|ccccc|cc} \cdots & r-1 & r & r+1 & \cdots & r+(s-1) & r+s & r+s+1 & \cdots \\ & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \\ \cdots & r-1 & r+s & r+(s-1) & \cdots & r+1 & r & r+s+1 & \cdots \end{array}$$

Thus the permutation π amounts to reversing all the sequences $\{r, r+1, \ldots, r+s\} \bmod N$, and to leave the rest of the sequence unchanged, i.e. it is the longest permutation generated by the elementary transpositions $\{(r, r+1), (r+1, r+2), \ldots, (r+s-1, r+s)\}$. This precisely means that the corresponding affine Weyl group element via the procedure explained in (3.5) and (3.7) is the longest Weyl group element generated by the elementary reflections w_{j_k}, with $0 \leqq j_k \leqq N-1$ such that $j_k = k \mod N$, $r+1 \leqq k \leqq r+s$. For instance if $0 \leqq r < r+s \leqq N-1$, from (3.7) we compute

$$w_J = \begin{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} & & \\ & \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix} & \\ & & \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \end{pmatrix} \begin{array}{l} 0^{\text{th}} \text{ row} \\ \\ r^{\text{th}} \text{ row} \\ \\ (r+s)^{\text{th}} \text{ row} \\ \\ (N-1)^{\text{th}} \text{ row} \end{array}$$

and if for instance $r = -1$, we find

$$\pi_0 = s-2, \ldots, \pi_{s-2} = 0, \qquad \pi_{s-1} = -1 = (-1)N + (N-1)$$

$$\pi_s = s, \ldots, \pi_{N-2} = N-2, \quad \pi_{N-1} = s-1+N = 1 \cdot N + (s-1)$$

and thus w_J is given by

$$w_J = \begin{pmatrix} \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix} & & \\ & 0 & & z \\ & & \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} & \\ & z^{-1} & & 0 \end{pmatrix} \begin{array}{l} 0^{\text{th}} \text{ row} \\ \\ (s-1)^{\text{th}} \text{ row} \\ \\ \\ (N-1)^{\text{th}} \text{ row} \end{array}$$

Also it follows from Lemma 3.3, formula (3.21) and $z^j W_k = W_{k+jN}$, that the 1-dimensional space $W_k(t^*)/W_{k+1}(t^*)$ is spanned by

$$\text{(3.27)} \quad \begin{cases} v_k = \lambda^{r+s-l} z^j \psi_r^-(\zeta, t^*), \text{ if } k = l + jN, \ l \in \{r, r+1, \ldots, r+s\} \\ v_k = \psi_k^-(\zeta, t^*), \text{ if } k \bmod N \notin \{r, r+1, \ldots, r+s\} , \end{cases}$$

with v_k an element of order π_k. Check that (3.27) amounts to

$$\gamma(z) \equiv [\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_{N-1}] = \gamma_*^-(z) w_J ,$$

with $\gamma_*^-(z)$ defined as in (3.11) and w_J the Weyl group element associated with $\{\pi_k\}$, that we just have computed, and this with (3.9) yields (3.10).

It remains to deal by concatenation with the general case

$$L(D(t^*) - kQ + (k-1)P)$$

$\neq 0$, for $k \bmod N \in \{r_1 + 1, \ldots, r_1 + s_1\} \cup \{r_2 + 1, \ldots, r_2 + s_2\} \cup \ldots$

with $r_2 > r_1 + s_1, \ldots$

$= 0$, otherwise .

It follows from (3.16) and (3.20) that the permutation π is given by

$\ldots$	r_1	$\ldots$	$r_1 + s_1$	$\ldots$	j	$\ldots$	r_2	$\ldots$	$r_2 + s_2$	$\ldots$
π	$\downarrow$		$\downarrow$		$\downarrow$		$\downarrow$		$\downarrow$	
$\ldots$	$r_1 + s_1$	$\ldots$	r_1	$\ldots$	j	$\ldots$	$r_2 + s_2$	$\ldots$	r_2	$\ldots$

which corresponds again to the longest affine Weyl group element generated by the reflections w_j such that $D(t^*) \in \Theta_j, j \in J$. Again one finds (3.11) (and hence (3.10) via (3.9)), since by (3.21) again one proves a concatenated version of (3.27) which amounts to (3.11), thus concluding the proof.

4 Limit matrices

An important ingredient in this chapter is the existence of a pair of algebraic bases*

$$W_0^{\text{alg}}(t) = \{\psi_0^-(t), \psi_1^-(t), \psi_2^-(t), \ldots\}$$

$$= \{\psi_0^-(t), \nabla\psi_0^-(t), \nabla^2\psi_0^-(t), \ldots\}, \nabla = \frac{\partial}{\partial t_1} - \lambda ,$$

related to one another by means of a change of coordinates of the type

$$\text{(4.1)} \qquad \psi_k^-(t) = \sum_{l=0}^{k} \gamma_l^{(k)}(t)\, \nabla^{k-l}\psi_0^-(t) .$$

Indeed, (4.1) defines a change of basis $B_s(t)$ given by polynomials in (a, b) as follows:

$$B_s(t): \{\varphi_j\}_{j\in\mathbf{Z}} \curvearrowright \{\psi_k^-\}_{k\in\mathbf{Z}} ,$$

* In this section t denotes a vector, $t = (t_1, \ldots, t_g)$

where

$$\varphi_j = (-\nabla)^{s-j}\psi_0^-, \text{ for } 0 \leqq j \leqq s\,,$$
$$= \psi_j^-, \text{ for } s+1 \leqq j \leqq N-1\,,$$
$$\varphi_{j+N} = z\varphi_j\,,$$

such that if

$$D(t^*) \in \Theta_0^c \cap \Theta_1 \cap \ldots \cap \Theta_s \cap \Theta_{s+1}^c \cap \ldots \cap \Theta_{N-1}^c\,, \tag{4.2}$$

the operator $L(t)$, expressed in this new basis, has a finite limit $L(t^*)$, when $t \to t^*$, where the upper $s+1$ by $s+1$ square is a *companion matrix* conjugated to the Toda matrix L_s

$$L_s \equiv \begin{pmatrix} b_1 & a_1 & & \bigcirc \\ 1 & b_2 & a_2 & \\ & \ddots & \ddots & a_s \\ \bigcirc & & 1 & b_{s+1} \end{pmatrix}; \tag{4.3}$$

the rest of the matrix is *almost* unchanged from the usual N-periodic Toda matrix (see Theorem 4.1, for precision). In general, the situation will be a concatenation of the above state of affairs, and we will just consider the case (4.2) for simplicity of notation. To prove Theorem 4.1, which is the main result, we embark upon a set of lemmas.

Lemma 4.1. $W_0^{\text{alg}}(t) = \{\psi_0^-(t), \nabla\psi_0^-(t), \nabla^2\psi_0^-(t), \ldots\}$, $\nabla = \dfrac{\partial}{\partial t_1} - \lambda$.

Proof. This fact is due to Sato [S] (see also [S-W]); by the definition of the Baker function of $W_0(0)$

$$\psi_0^+(t) = \exp\left(-\sum_{j\geqq 1} t_j\lambda^j\right)\psi_0^-(t) \in W_0(0), \quad \text{for all } t\,,$$

and thus

$$W_0(0) \ni \frac{\partial^l}{\partial t_1^l}\psi_0^+(t) = \exp\left(-\sum_{j\geqq 1} t_j\lambda^j\right)\nabla^l\psi_0^-(t)\,. \tag{4.4}$$

Therefore

$$\nabla^l\psi_0^-(t) \in \exp\left(\sum_{j\geqq 1} t_j\lambda^j\right)W_0(0) = W_0(t)\,,$$

and since $\nabla^l\psi_0^-(t) = (-\lambda)^l(1+O(\lambda^{-1}))$, these functions form an algebraic basis of $W_0(t)$.

Lemma 4.2. *If* $D(t^*) \in \Theta_0^c \cap \Theta_1 \cap \Theta_2 \cap \ldots \cap \Theta_s \cap \Theta_{s+1}^c \cap \ldots \cap \Theta_{N-1}^c$

$$\nabla^l\psi_0^-(t^*) = (-\lambda)^l\psi_0^-(t^*), \quad 0 \leqq l \leqq s\,. \tag{4.5}$$

Proof. From (2.16) and (A.11) compute

$$\frac{\partial \psi_k^+}{\partial t_1} = -b_{k+1}\psi_k^+ - \psi_{k+1}^+ \tag{4.6}$$
$$= -\lambda\psi_k^+ + a_k\psi_{k-1}^+ ,$$

and setting $k = 0$, operating on both sides with $\exp\left(\sum_{j\geqq 1} t_j\lambda^j\right)\left(\frac{\partial}{\partial t_1}\right)^l$ and using (4.4), conclude

$$\nabla^{l+1}\psi_0^- = -\lambda\nabla^l\psi_0^- + \sum_{j=0}^{l}\binom{l}{j}\left(\frac{\partial}{\partial t_1}\right)^{l-j} a_0\,\nabla^j\psi_{-1}^- , \tag{4.7}$$

and since (see Lemma 4.5) $a_0(t) = c(t_1 - t_1^*)^s + \cdots$, $c \neq 0$, when $t \to t^*$, we get

$$\nabla^{l+1}\psi_0^-(t^*) = -\lambda\nabla^l\psi_0^-(t^*), \quad 0 \leqq l \leqq s-1 ,$$

proving the lemma.

Lemma 4.3. *If*

$$\det(\lambda - L_k) \equiv \lambda^{k+1} - I_1^{(k)}\lambda^k + I_2^{(k)}\lambda^{k-1} + \cdots + (-1)^{k+1} I_{k+1}^{(k)} , \tag{4.8}$$

and $I_0^{(k)} = 1$, $I_l^{(k)} = 0$, $l < 0$ *or* $l > k+1$, *then*

$$I_l^{(k)} = I_l^{(k-1)} + b_{k+1} I_{l-1}^{(k-1)} - a_k I_{l-2}^{(k-2)}, \quad 1 \leqq l \leqq k+1 , \tag{4.9}$$

and

$$\frac{\partial I_l^{(k)}}{\partial t_1} = -a_{k+1} I_{l-1}^{(k-1)} + a_0 J_{l-1}^{(k-1)} \tag{4.10}$$

where $J_{l-1}^{(k-1)}$ *is* $I_{l-1}^{(k-1)}$, *but with all the indices shifted up one and* $I_l^{(k)}$ *has weight* l, *if all* a_i *have weight* 2 *and all* b_i *have weight* 1.

Proof. Equation (4.9) is obtained by expanding $\det(\lambda - L_k)$ along the last column and using (4.3). To see (4.10) observe

$$I_l^{(k)} = \sum b_{i_1} b_{i_2} \ldots b_{i_r}(-a_{j_1})(-a_{j_2}) \ldots (-a_{j_s})$$

where

$$1 \leqq i_1, \ldots, i_r \leqq k+1; \quad 1 \leqq j_1, \ldots, j_s \leqq k, \quad r + 2s = l ,$$

and

$$i_1, \ldots, i_r, j_1, \ldots, j_s, j_1 + 1, \ldots, j_s + 1 \text{ are all distinct}$$

and so $I_l^{(k)}$ has weight l and

$$I_l^{(k)} = b_{k+1} I_{l-1}^{(k-1)} + b_1 J_{l-1}^{(k-1)} + (\text{terms which do not involve } b_1 \text{ and } b_{k+1}) .$$

Since the $I_l^{(k)}$ are first integrals of the finite Toda lattice going with L_k, the only terms which will survive in $\frac{\partial I_l^{(k)}}{\partial t_1}$ are those which come from the periodic $SL(N)$

Toda lattice which differ from the equations of the finite $SL(k+1)$ Toda, i.e.

$$\dot{b}_1 = a_0 - a_1 \text{ and } \dot{b}_{k+1} = a_k - a_{k+1}, \left(\dot{} = \frac{\partial}{\partial t_1}\right).$$

Therefore we get that

$$\dot{I}_l^{(k)} = -a_{k+1} I_{l-1}^{(k-1)} + a_0 J_{l-1}^{(k-1)},$$

as claimed.

The next lemma gives a precise description of the $\gamma_l^{(k)}(t)$ in (4.1):

Lemma 4.4. *For $k \geqq 0$, we have*

$$\psi_k^-(t) = (-1)^k \sum_{l=0}^{k} [I_l^{(k-1)} + a_0 F_{l-2}] \nabla^{k-l} \psi_0^-(t), \tag{4.11}$$

where F_{l-2} denotes some weight homogeneous polynomial in the a_i's and b_i's of degree $l-2$, $F_{-2} = 0$, $F_{-1} = 0$, and $F_0 = 1$.

Proof. By multiplying (4.11) by $\exp(-\sum_{j \geqq 1} t_j \lambda^j)$, it is equivalent (using 4.4) to establish the same identity with $-$ replaced by $+$, and ∇ replaced by $\dot{} = \dfrac{\partial}{\partial t_1}$. The proof is by induction on k. From (4.6) we have

$$\psi_1^+ = -\left(b_1 \psi_0^+ + \frac{\partial \psi_0^+}{\partial t_1}\right),$$

which proves the case $k = 1$. Assuming we have shown (4.11) up to k, substitute it into (4.6):

$$\psi_{k+1}^+ = -b_{k+1} \psi_k^+ - \frac{\partial \psi_k^+}{\partial t_1},$$

and then use (4.10)

$$\dot{I}_l^{(k-1)} = -a_k I_{l-1}^{(k-2)} + a_0 J_{l-1}^{(k-2)},$$

to yield

$$\psi_{k+1}^+ = (-1)^{k+1} \sum_{l=0}^{k+1}$$

$$\left\{ \begin{array}{l} (I_l^{(k-1)} + b_{k+1} I_{l-1}^{(k-1)} - a_k I_{l-2}^{(k-2)}) \\ + a_0((b_{k+1} + b_0 - b_1) F_{l-3} + F_{l-2} + \dot{F}_{l-3} + J_{l-2}^{(k-2)}) \end{array} \right\} \left(\frac{\partial}{\partial t_1}\right)^{k+1-l} \psi_0^+$$

which, using (4.9), establishes the lemma.

Lemma 4.5. *If $D(t^*) \in \Theta_0^c \cap \Theta_1 \cap \Theta_2 \cap \ldots \cap \Theta_s \cap \Theta_{s+1}^c \cap \ldots \cap \Theta_{N-1}^c$, then as $t \to t^*$ in the t_1-direction*

$$\begin{cases} a_0(t) = c(t_1 - t_1^*)^s + \cdots, \; a_{s+1} = c(t_1 - t_1^*)^s + \cdots, \\ a_i(t) = \dfrac{c'}{(t_1 - t_1^*)^2} + \cdots, \; 1 \leqq i \leqq s, \quad b_i(t) = \dfrac{c''}{t_1 - t_1^*} + \cdots, \; 1 \leqq i \leqq s+1, \\ F_l(a, b) = \dfrac{d}{(t_1 - t_1^*)^l} + \cdots, \end{cases}$$

F_l a weight homogeneous polynomial of degree l, with $c, c', c'' \neq 0$ and d possibly zero.

Proof. See [A-vM1, Theorem 3.I].

Theorem 4.1. *The change of basis $\psi^- \curvearrowright \varphi = B(a, b)^{-1} \psi^-$*

(4.12)
$$\psi^- = (\psi_0^-, \psi_1^-, \ldots, \psi_{N-1}^-) \curvearrowright \varphi = (\varphi_0 = (-\nabla)^s \psi_0^-, \varphi_1 = (-\nabla)^{s-1} \psi_0^-, \ldots, \varphi_s = \psi_0^-, \varphi_{s+1} = \psi_{s+1}^-, \varphi_{s+2} = \psi_{s+2}^-, \ldots \varphi_{N-1} = \psi_{N-1}^-),$$

given by (see Lemma 4.4, (4.11))

(4.13)
$$B(a, b): \begin{cases} \psi_k^- = (-1)^k \sum_{l=0}^k (-1)^l [I_{k-l}^{(k-1)} + a_0 F_{k-l-2}] \varphi_{s-l}, & 0 \leqq k \leqq s, \\ \psi_k^- = \varphi_k, & s+1 \leqq k \leqq N-1, \end{cases}$$

is a polynomial map of the block form

$$B(a, b) = \left(\begin{array}{ccc|c} \bigcirc & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \bigcirc \\ 1 & & * & \\ \hline & \bigcirc & & I \end{array} \right)$$

with polynomial inverse. Then expressing L in the new basis

(4.14)
$$L^T \curvearrowright L_J^T = B^{-1} L^T B$$

yields in the limit, when $t \to t^*$, the finite basis

(4.15)
$$\varphi(t^*) = (\lambda^s \psi_0^-(t^*), \lambda^{s-1} \psi_0^-(t^*), \ldots, \psi_0^-(t^*), \psi_{s+1}^-(t^*), \psi_{s+2}^-(t^*), \ldots, \psi_{N-1}^-(t^*))$$

and for $L_J^T(t^*)$ the finite limit matrix:

For $1 \leqq s \leqq N-3$.

(4.16) $L_J^T(t^*) =$

$$\left(\begin{array}{cccc|cccccc} I_1^{(s)} & -I_2^{(s)} & \ldots & (-1)^s I_{s+1}^{(s)} & 1 & 0 & \ldots & 0 & (-1)^{s+1} \sum_{l=1}^{s} I_l^{(s-1)} a_0^{(s-l)} z^{-1} & \\ 1 & 0 & & \bigcirc & & & & & & \\ & & & & & & \bigcirc & & & \\ \bigcirc & & 1 & 0 & & & & & & \\ \hline & & & (-1)^s a_{s+1} I_s^{(s-1)} & b_{s+2} & 1 & & & & \\ & \bigcirc & & 0 & a_{s+2} & b_{s+3} & 1 & \bigcirc & & \\ & & & \vdots & & & & & & \\ & & & 0 & \bigcirc & & a_{N-2} & b_{N-1} & 1 & \\ & & & z & & & & a_{N-1} & b_N & \end{array} \right).$$

For $s = N - 2$,

$L_J^T(t^*) =$

$$\left(\begin{array}{ccccc|c} I_1^{(N-2)} & -I_2^{(N-2)} & \cdots & (-1)^{N-2} I_{N-1}^{(N-2)} & & 1 + (-1)^{N-1} \sum_{l=1}^{N-2} I_l^{(N-3)} a_0^{(N-2-l)} z^{-1} \\ 1 & 0 & & & & 0 \\ & \ddots & \ddots & \bigcirc & & \vdots \\ \bigcirc & & 1 & 0 & & 0 \\ \hline 0 & \cdots & 0 & [(-1)^{N-2} a_{N-1} I_{N-2}^{(N-3)} + z] & & b_N \end{array}\right)$$

and for $s = N - 1$,

$L_J^T(t^*) =$

$$\begin{pmatrix} I_1^{(N-1)} & -I_2^{(N-1)} & \cdots & (-1)^{N-2} I_{N-1}^{(N-1)} & Az^{-1} + z + (-1)^{N-1} I_N^{(N-1)} \\ 1 & 0 & & & \\ & 1 & & & \bigcirc \\ & & \ddots & \ddots & \\ & \bigcirc & & 1 & 0 \end{pmatrix}$$

with $A = \prod_{i=0}^{N-1} a_i,\ a_0^{(k)} = \left(\frac{\partial}{\partial t_1}\right)^k a_0,\ I_l^{(s-1)} a_0^{(s-l)} = \lim_{t \to t^*} I_l^{(s-1)} a_0^{(s-l)}$, etc.

Proof of Theorem 4.1. First observe that formula (4.11) (Lemma 4.4) leads at once to the change of basis (4.13), while formula (4.15) is just a consequence of Lemma 4.2. Since $\psi^- L = \lambda \psi^-$, $\varphi L_J = \lambda \varphi$, we have

$$\Phi L_J = \Lambda \Phi$$

with (ω is a N^{th} root of unity) $\Phi(t)_{ij} = [\varphi_j(t, \omega^i \zeta)]_{0 \leq i, j \leq N-1}$ and $\Lambda = \text{diag}\,(\lambda(\zeta), \ldots, \lambda(\omega^i \zeta), \ldots)$; so in particular $L_J(t^*) = \Phi^{-1}(t^*) \Lambda \Phi(t^*)$ is finite, since $\Phi(t^*)$ is finite.

It follows from Lemma 4.4 applied to $k = s$ and $s - 1$ that

$$\psi_s^-(t) = \alpha_0 \nabla^s \psi_0^- + \alpha_1 \nabla^{s-1} \psi_0^- + \cdots + \alpha_s \psi_0^- , \tag{4.17}$$

and

$$\psi_{s-1}^-(t) = \beta_0 \nabla^{s-1} \psi_0^- + \beta_1 \nabla^{s-2} \psi_0^- + \cdots + \beta_{s-1} \psi_0^- . \tag{4.18}$$

Rewriting (4.17) as

$$\alpha_0 \nabla^s \psi_0^- = \psi_s^- - \sum_{l=1}^{s} \alpha_l \nabla^{s-l} \psi_0^- ,$$

and using

$$\lambda\psi_s^- = a_s\psi_{s-1}^- + b_{s+1}\psi_s^- + \psi_{s+1}^- ,$$

we obtain

$$\lambda(\alpha_0 \nabla^s \psi_0^-) = a_s\psi_{s-1}^- + b_{s+1}\psi_s^- + \psi_{s+1}^- - \sum_{l=1}^{s} \alpha_l \lambda \nabla^{s-l}\psi_0^- .$$

By substituting (4.17), (4.18) and (4.7) into this we find

$$\text{(4.19)}\quad \lambda(\alpha_0 \nabla^s \psi_0^-) = a_s \sum_{l=0}^{s-1} \beta_l \nabla^{s-1-l}\psi_0^- + b_{s+1}\sum_{l=0}^{s} \alpha_l \nabla^{s-l}\psi_0^- + \psi_{s+1}^- + \sum_{l=1}^{s} \alpha_l \left\{ \nabla^{s-l+1}\psi_0^- - \sum_{j=0}^{s-l} \binom{s-l}{j} \left(\frac{\partial}{\partial t_1}\right)^{s-l-j} a_0 \nabla^j \psi_{-1}^- \right\}.$$

We now compute the limit of (4.19) when $t \to t^*$. Since the first Toda flow starting at a point of the stratum $\sum_{w_j}$ enters the main stratum immediately (see Lemma 4.5), it makes good sense to approach $\sum_{w_j}$ along the t_1-direction. So all the limits below are taken in the t_1-direction. Since from (4.11) one sees α_l is weight homogeneous of degree l, it follows from Lemma 4.5 that

$$\lim_{t\to t^*} \alpha_l \left(\frac{\partial}{\partial t_1}\right)^{s-l-j} a_0 = 0, \text{ for } j \geqq 1 .$$

Therefore, taking the limit for $t \to t^*$ in (4.19) yields

$$\text{(4.20)}\quad \lim_{t\to t^*} \lambda(\alpha_0 \nabla^s \psi_0^-) = \psi_{s+1}^-(t^*) + \lim_{t\to t^*} \left\{ - \left[\sum_{l=1}^{s} \alpha_l \left(\frac{\partial}{\partial t_1}\right)^{s-l} a_0 \right] \psi_{-1}^- + \sum_{l=0}^{s} [b_{s+1}\alpha_l + a_s\beta_{l-1} + \alpha_{l+1}] \nabla^{s-l}\psi_0^- \right\},$$

with the convention that $\beta_{-1} = 0$ and $\alpha_{s+1} = 0$. From the explicit formula for the α's and β's (4.11) and Lemma 4.5, we have the estimates

$$\begin{cases} \alpha_l \left(\dfrac{\partial}{\partial t_1}\right)^{s-l} a_0 = (-1)^s I_l^{(s-1)} \left(\dfrac{\partial}{\partial t_1}\right)^{s-l} a_0 + O((t-t^*)^{s+2}) , \\ b_{s+1}\alpha_l = (-1)^s b_{s+1} I_l^{(s-1)} + O((t-t^*)^{s-l+1}), \\ a_s\beta_{l-1} = (-1)^{s-1} a_s I_{l-1}^{(s-2)} + O((t-t^*)^{s-l+1}), \\ \alpha_{l+1} = (-1)^s I_{l+1}^{(s-1)} + O((t-t^*)^{s-l+1}) . \end{cases}$$

Therefore, using the definition of φ (4.12), $\alpha_0 = (-1)^s$, $z\psi_{-1}^- = \psi_{N-1}^-$, and the above estimates, (4.20) becomes

$$\text{(4.21)}\quad \lambda\varphi_0(t^*) = \sum_{l=0}^{s} (-1)^l [b_{s+1} I_l^{(s-1)} - a_s I_{l-1}^{(s-2)} + I_{l+1}^{(s-1)}] \varphi_l(t^*) + \psi_{s+1}^-(t^*) + (-1)^{s+1}\left[\sum_{l=1}^{s} I_l^{(s-1)} \left(\frac{\partial}{\partial t_1}\right)^{s-l} a_0 \right] z^{-1} \psi_{N-1}^-(t^*) .$$

Since $\varphi_l(t^*) = \lambda^{s-l}\psi_0^-(t^*)$ for $0 \leqq l \leqq s$ (4.5), and $\varphi_l(t^*) = \psi_l^-(t^*)$ for $s+1 \leqq l \leqq N-1$, by substituting (4.9) into (4.21) and $\psi^- L = \lambda\psi^-$ and $\psi_N^-(t^*) = z\psi_0^-(t^*) = z\varphi_s(t^*)$, we find

$$(4.22)\quad \begin{cases} \lambda\varphi_0(t^*) = \displaystyle\sum_{l=0}^{s}(-1)^l I_{l+1}^{(s)}\varphi_l(t^*) + \varphi_{s+1}(t^*) \\ \qquad + (-1)^{s+1}\left[\displaystyle\sum_{l=1}^{s} I_l^{(s-1)}\left(\frac{\partial}{\partial t_1}\right)^{s-l} a_0\right] z^{-1}\varphi_{N-1}(t^*)\,, \\ \lambda\varphi_l(t^*) = \varphi_{l-1}(t^*),\ 1 \leqq l \leqq s\,, \\ \lambda\varphi_{s+1}(t^*) = (-1)^s a_{s+1} I_s^{(s-1)}\varphi_s(t^*) + b_{s+2}\varphi_{s+1}(t^*) + \varphi_{s+2}(t^*)\,, \\ \lambda\varphi_l(t^*) = a_l\varphi_{l-1}(t^*) + b_{l+1}\varphi_l(t^*) + \varphi_{l+1}(t^*),\ s+2 \leqq l \leqq N-2\,, \\ \lambda\varphi_{N-1}(t^*) = a_{N-1}\varphi_{N-2}(t^*) + b_N\varphi_{N-1}(t^*) + z\varphi_s(t^*)\,. \end{cases}$$

Only the third equation needs explanation, and since $\psi^- L = \lambda\psi^-$, it amounts to the assertion that

$$\lim_{t\to t^*} a_{s+1}(t)\psi_s^-(t) = (-1)^s\left(\lim_{t\to t^*} a_{s+1}(t) I_s^{(s-1)}(t)\right)\psi_0^-(t^*)\,,$$

which follows from multiplying (4.11) for $k = s$ by $a_{s+1}(t)$ and then using the estimates of Lemma 4.5 to compute the limit. Clearly (4.22) yields (4.16) and the other two cases follow in the same fashion, concluding the proof of Theorem 4.1.

Appendix

In this appendix we construct (Proposition A) the Baker functions $\psi_k^{\pm}(x, t)$ using tools of algebraic geometry, thus globalizing these constructions and interpreting *both* $b^{\mp}(z, t)$ of Theorem 2.1. First we need a technical lemma. From (0.7) deduce $(\zeta^{-1} = z^{-1/N})$

$$(A.1)\qquad \lambda^j = \zeta^j + c_{j-2,j}\zeta^{j-2} + \cdots + c_{1,j}\zeta + c_{0,j} + O(\zeta^{-1}),\quad \text{near } P\,.$$

Let (A_i, B_i) be a canonical homology basis on X, and let us denote by $v_j (j \geqq 1)$ the differentials of the second kind with unique pole of order $j+1$ at P, normalized as

$$v_j = d(\zeta^j) + (\text{holomorphic})d\zeta^{-1},$$

with zero A-periods. Similarly, we denote by $\eta_j (j \geqq 1)$ the differentials of the second kind with unique pole of order $j+1$ at P, normalized as

$$\eta_j = d(\lambda^j) + (\text{holomorphic})d\lambda^{-1}\,,$$

with zero A-periods. Clearly,

$$(A.2)\qquad \eta_j = v_j + c_{j-2,j}v_{j-2} + \cdots + c_{1,j}v_1\,.$$

Lemma A. *Near* P,

$$(A.3)\qquad \int_Q^{x(\zeta)} v_j = \zeta^j + O(\zeta^{-1}),\ \textit{modulo periods}\,,$$

and

(A.4) $$\int_Q^{x(\lambda)} \eta_j = \lambda^j - c_{0j} + O(\lambda^{-1}), \text{ modulo periods}.$$

Proof. Let us denote by Δ, the interior of the canonical polygon obtained by dissecting our Riemann surface X along the canonical cycles (A_i, B_i). On Δ, the function $\varphi(x) = \int_Q^x \nu_j$ is single valued and near P,

$$\varphi(x) = \frac{1}{s^j} + d_j + O(s), \quad s = \zeta^{-1}.$$

We want to show that $d_j = 0$. Let ω_{QP} denote the normalized differential of the third kind with a simple pole at Q with residue 1, a simple pole at P with residue -1, and zero A-periods. One knows that

$$\omega_{QP} = d \log \frac{E(x, Q)}{E(x, P)},$$

where $E(x, y)$ denotes the prime form of X (see [Mu, p. 3.212]). Since $(z) = -NP + NQ$,

(A.5) $$z = \text{constant} \times \left[\frac{E(x, Q)}{E(x, P)}\right]^N,$$

and therefore

$$\omega_{QP} = \frac{1}{N} d \log z = -\frac{ds}{s}, \quad \text{near } P.$$

By integrating $\varphi\omega_{QP}$ over the boundary of Δ, we obtain

$$\int_{\partial\Delta} \varphi\omega_{QP} = \sum_i \int_{A_i} \nu_j \int_{B_i} \omega_{QP} - \sum_i \int_{B_i} \nu_j \int_{A_i} \omega_{QP} = 0.$$

On the other hand, evaluating this integral by residues yields

$$\int_{\partial\Delta} \varphi\omega_{QP} = 2\pi\sqrt{-1}\, \mathrm{Res}_P(\varphi\omega_{QP}), \text{ since } \mathrm{Res}_Q(\varphi\omega_{QP}) = 0$$

$$= -2\pi\sqrt{-1}\, d_j,$$

which establishes (A.3). From this formula and (A.2), it follows now that

$$\int_Q^{x(\lambda)} \eta_j = \zeta^j + c_{j-2,j}\zeta^{j-2} + \cdots + c_{1,j}\zeta + O(\zeta^{-1})$$

$$= \lambda^j - c_{0j} + O(\lambda^{-1}),$$

which establishes (A.4).

Let $t = (t_1, t_2, \ldots)^T$, with almost all t_i's equal to zero.

Proposition A. *Let $D = \sum_{i=1}^g x_i$ be a regular divisor on X. There exist uniquely determined functions $\psi_k^+(x, t)$, $k \in \mathbb{Z}$, such that*

(i) *As a function of x, for each t small enough, $\psi_k^+(x,t)$ is a meromorphic function on $X\backslash\{P\}$ satisfying*

$$(\psi_k^+(x,t))|_{X\backslash\{P\}} + D - kQ \geqq 0\,,$$

(ii) *Around P, $\psi_k^+(x,t)$ admits the following expansion*

$$\psi_k^+(x,t) = \exp\left(-\sum_j t_j\lambda^j\right)\lambda^k(1+O(\lambda^{-1}))\,. \tag{A.6}$$

Sketch of the proof. We first prove the existence of $\psi_k^+(x,t)$. Let $\omega = (\omega_1,\ldots,\omega_g)^{\top}$ be a normalized basis of holomorphic differentials on X, such that $\int_{A_i}\omega_j = \delta_{ij}$, and let $A = (a_{ij})$ be a $g\times\infty$ matrix with entries $a_{ij}\in\mathbb{C}$ defined by

$$\omega_i = (a_{i1} + a_{i2}\lambda^{-1} + a_{i3}\lambda^{-2} + \cdots)d\lambda^{-1}, \quad \text{near } P\,.$$

Let $\theta(z) = \theta(z,\Omega)$, with $\Omega_{ij} = \int_{B_j}\omega_i$, be the Riemann theta function for X, and define the function

$$\psi_k^+(x,t) = \exp\left\{-\sum_j t_j\int_Q^x\eta_j\right\}\exp\left\{-\sum_j t_jc_{0j}\right\}\left[\frac{E(Q,x)}{E(P,x)}\right]^k$$

$$\frac{\theta(At+\xi+k\int_P^Q\omega+\int_P^x\omega)\,\theta(\xi)}{\theta(At+\xi+k\int_P^Q\omega)\,\theta(\xi+\int_P^x\omega)} \tag{A.6'}$$

where $\xi = -\sum_{i=1}^g\int_P^{x_i}\omega - K$, with K the vector of Riemann constants relative to the Abel map based at P, and the path of integration in the integral $\int_P^x\omega$ is defined to be $\int_P^Q\omega + \int_Q^x\omega$, with $\int_Q^x\omega$ taken along the same path as $\int_Q^x\eta_j$. Using the quasi-periodicity properties of the prime form and the theta function, and the fact that $\int_{B_i}\eta_j = -2\pi\sqrt{-1}a_{ij}$, one easily checks that $\psi_k^+(x,t)$ is a single valued function on X. From Riemann's theorem and the definition of the prime form, it satisfies requirement (i) of the proposition. By Lemma A and formula (A.4), it also satisfies condition (ii), up to a (time independent) non-zero constant. The uniqueness of $\psi_k^+(x,t)$ follows easily from the regularity of D. This concludes the proof of the proposition.

Let

$$\begin{cases}\psi_k^-(x,t) = \exp\left(\sum_{j\geqq 1} t_j\lambda^j\right)\psi_k^+(x,t)\\ \psi_k^\pm(x,t) = \psi_k^\pm(x,t,0,0,\ldots),\ t \text{ small enough}\,.\end{cases} \tag{A.7}$$

By definition, $\psi_k^+(x,t)$ is meromorphic on $U^+ = X\backslash\{P\}$, with an essential singularity at P (where $z=\infty$) and a zero of order k at Q. From (A.7), (i) and (ii), it follows that $\psi_k^-(x,t)$ is meromorphic on $U^- = X\backslash\{Q\}$, with an essential singularity at Q (where $z^{-1}=\infty$) and a pole of order k at P. Since ζ is a local coordinate about Q, and ζ^{-1} is a local coordinate about P, we can therefore write

$$\begin{cases}\psi_k^+(x,t) = \zeta^k(*+O(\zeta)), & \text{near } Q\,,\\ \psi_k^-(x,t) = \zeta^k(1+O(\zeta^{-1})), & \text{near } P\,.\end{cases} \tag{A.8}$$

We now introduce the pair of loops

$$\gamma^{\pm}(z,t)=[\hat{\psi}_0^{\pm}(z,t),\hat{\psi}_1^{\pm}(z,t),\ldots,\hat{\psi}_{N-1}^{\pm}(z,t)]$$

with $\gamma^{\pm}$ based on the expansions (A.8) of $\psi^{\pm}$ at Q, P respectively

$$\psi_k^{\pm}(x,t)=\zeta^0\gamma_{0k}^{\pm}(z,t)+\zeta\gamma_{1k}^{\pm}(z,t)+\cdots+\zeta^{N-1}\gamma_{N-1,k}^{\pm}(z,t)\ ;$$

clearly γ^- was introduced in Sect. 2, but not γ^+

$$\gamma^-(z,t)=\begin{pmatrix}1 & & * \\ & \ddots & \\ O & & 1\end{pmatrix}+O(z^{-1})$$

and

$$\gamma^+(z,t)=\begin{pmatrix}* & & O \\ & \ddots & \\ * & & *\end{pmatrix}+O(z)\ .$$

Proposition B. *For $t\in\mathbb{C}$ small enough, the solution of the factorization problem* (2.1) *of Theorem* 2.1 *is given by*

(A.9) $$b^{\pm}(z,t)=\gamma^{\pm}(z,0)^{-1}\gamma^{\pm}(z,t)\ .$$

Sketch of the proof. It suffices to show that if we define $b^{\pm}(z,t)$ by formula (A.9), then $b^-(z,t)b^+(z,t)^{-1}$ satisfies the differential equation of the exponential:

(A.10) $$\frac{d}{dt}b^-(z,t)b^+(z,t)^{-1}=L(z,0)b^-(z,t)b^+(z,t)^{-1}=b^-(z,t)b^+(z,t)^{-1}L(z,0)\ .$$

Let $\psi^{\pm}(x,t)=(\psi_0^{\pm}(x,t),\ldots,\psi_{N-1}^{\pm}(x,t))^{\top}$. One shows (see [vM-Mu]) that there exists a uniquely determined Jacobi matrix $M(z,t)=L(z,t)^{\top}$ such that

(A.11) $$\begin{cases}M(z(x),t)\psi^{\pm}(x,t)=\lambda(x)\psi^{\pm}(x,t)\\ \dot{\psi}^{\pm}(x,t)=M^{\pm}(z(x))\psi^{\pm}(x,t)\ .\end{cases}$$

Again one defines the Baker matrix $\Psi^{\pm}(t)$ as in (2.5) and recalls (2.6) $\Psi^{\pm}(t)=\Delta\gamma^{\pm}(z,t)$ from which it follows that

$$b^{\pm}(z,t)\equiv\gamma^{\pm}(z,0)^{-1}\gamma^{\pm}(z,t)=\Psi^{\pm}(0)^{-1}\Psi^{\pm}(t)\ ,$$

and then the pair (A.11) may be respectively recast as:

$$\begin{aligned}L(z,t)&=\Psi^{\pm}(t)^{-1}\Lambda\Psi^{\pm}(t)\\&=\Psi^{\pm}(t)^{-1}\Psi^{\pm}(0)\,L(z,0)\,\Psi^{\pm}(0)^{-1}\,\Psi^{\pm}(t)\\&=b^{\pm}(z,t)^{-1}\,L(z,0)\,b^{\pm}(z,t)\end{aligned}$$

and

$$\frac{d}{dt}b^{\pm}(z,t)=b^{\pm}(z,t)L^{\pm}(z,t)\ .$$

From these two last equations, it follows immediately that (A.10) is satisfied, which establishes the proposition.

Corollary. *The solution of the Toda lattice equations* (0.2) ($j = 1$) *amounts to the solution of the factorization problem* (2.1) *and is provided by*

(A.12) $$L(z, t) = b^{\pm}(z, t)^{-1} L(z, 0) b^{\pm}(z, t) .$$

Remark. The solution of the higher Toda flows (0.2) $L(z, t)^{\cdot} = [L(z, t), L^j(z, t)^{\pm}]$ is given by the same formula as (A.12), where now $b^{\pm}(z, t)$ solve the factorization problem

$$\exp(tL^j(z)) = b^-(z, t) b^+(z, t)^{-1} .$$

The solution of this factorization problem is again provided by formula (A.9), where now $\psi_k^+(x, t) = \psi_k^+(x, 0, \ldots, 0, t_j = t, 0, \ldots)$ and $\psi_k^-(x, t) = \exp(t\lambda^j(x))\psi_k^+(x, t)$.

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