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Integrable Systems, Random Matrices and Random Processes

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Introduction

Random matrix theory, began in the 1950's, when E. Wigner [58] proposed that the local statistical behavior of scattering resonance levels for neutrons of heavy nucleii could be modeled by the statistical behavior of eigenvalues of a large random matrix, provided the ensemble had orthogonal, unitary or symplectic invariance. The approach was developed by many others, like Dyson [30, 31], Gaudin [34] and Mehta, as documented in Mehta's [44] famous treatise. The field experienced a revival in the 1980's due to the work of M. Jimbo, T. Miwa, Y. Mori, and M. Sato [36, 37], showing the Fredholm determinant involving the sine kernel, which had appeared in random matrix theory for large matrices, satisfied the fifth Painlevé transcendent; thus linking random matrix theory to integrable mathematics. Tracy and Widom soon applied their ideas, using more efficient function-theoretic methods, to the largest eigenvalues of unitary, orthogonal and symplectic matrices in the limit of large matrices, with suitable rescaling. This lead to the Tracy–Widom distributions for the 3 cases and the modern revival of random matrix theory (RMT) was off and running.

This article will focus on integrable techniques in RMT, applying Virasoro gauge transformations and integrable equation (like the KP) techniques for finding Painlevé – like ODE's or PDE's for probabilities that are expressible as Fredholm determinants coming up in random matrix theory and random processes, both for finite and large n–limit cases. The basic idea is simple – just deform the probability of interest by some time parameters, so that, at least as a function of these new time parameters, it satisfies some integrable equations. Since in RMT you are usually dealing with matrix integrals, roughly speaking, it is usually fairly obvious which parameters to "turn on," although it always requires an argument to show you have produced "τ–functions" of an integrable system. Fortunately, to show a system is integrable, you ultimately only have to check bilinear identities and we shall present very general methods to accomplish this. Indeed, the bilinear identities are the actual source of a sequence of integrable PDE's for the τ-functions.

Secondly, because we are dealing with matrix integrals, we may change coordinates without changing the value of the integral (gauge invariance), leading to the matrix integrals being annihilated by partial differential operators in the artificially introduced time and the basic parameters of the problem – so-called Virasoro identities. Indeed, because the most useful coordinate changes are often "S^1–like" and because the tangent space of $\mathrm{Diff}(S^1)$ at the identity is the Virasoro Lie algebra (see [41]), the family of annihilating operators tends to be a subalgebra of the Virasoro Lie algebra. Integrable systems possess vertex algebras which infinitesimally deform them and the Virasoro algebras, as they explicitly appear, turn out to be generated by these vertex algebras. Thus while other gauge transformation are very useful in RMT, the Virasoro generating ones tend to mesh well with the integrable systems. Finally, the PDE's of integrable systems, upon feeding in the Virasoro relations, lead, upon setting the artificially introduced times to zero, to Painlevé-like ODE or PDE for the matrix integrables and hence for the probabilities, but in the original parameters of the problem! Sometimes we may have to introduce "extra parameters," so that the Virasoro relations close up, which we then have to eliminate by some simple means, like compatibility of mixed partial derivatives.

In RMT, one is particularly interested in large n (scaled) limits, i.e. central limit type results, usually called universality results; moreover, one is interested in getting Painlevé type relations for the probabilities in these limiting relations, which amounts to getting Painlevé type ODE's or PDE's for Fredholm determinants involving limiting kernels, analogous to the sine kernel previously mentioned. These relations are analogous to the heat equation for the Gaussian kernel in the central limit theorem (CLT), certainly a revealing relation. There are two obvious ways to derive such a relation; either, get a relation at each finite step for a particularly "computable or integrable" sequence of distributions (like the binomial) approaching the Gaussian, via the CLT, and then take a limit of the relation, or directly derive the heat equation for the actual limiting distribution. In RMT, the same can be said, and the integrable system step and Virasoro step mentioned previously are thus applied directly to the "integrable" finite approximations of the limiting case, which just involve matrix integrals. After deriving an equation at the finite n–step, we must ensure that estimates justify passage to the limit, which ultimately involves estimates of the convergence of the kernel of a Fredholm determinant. If instead we decide to directly work with the limiting case without passing through a limit, it is more subtle to add time parameters to get integrability and to derive Virasoro relations, as we do not have the crutch of finite matrix integrals. Nonetheless, working with the limiting case is usually quite insightful, and we include one such example in this article to illustrate how the limiting cases in RMT faithfully remember their finite $-\ n$ matrix integral ancestry.

In Section 1 we discuss random matrix ensembles and limiting distributions and how they directly link up with KP theory and its vertex operator, leading to PDE's for the limiting distribution. This is the only case where we work only with the limiting distribution. In Section 2 we derive recursion relations in n for n-unitary integrals which include many combinatoric generating functions. The methods involve bi-orthogonal polynomials on the circle and we need to introduce the integrable "Toeplitz Lattice," an invariant subsystem of the semi-infinite 2–Toda lattice of Uéno–Takasaki [55]. In Section 3 we study the coupled random matrix problem, involving bi-orthogonal polynomials for a nonsymmetric $\mathbb{R}^2$ measure, and this system involves the 2–Toda lattice, which leads to a beautiful PDE for the joint statistics of the ensemble. In Section 4 we study Dyson Brownian motion and 2 associated limiting processes – the Airy and Sine processes gotten by edge and bulk scaling. Using the PDE of Section 3, we derive PDE's for the Dyson process and then the 2 limiting processes, by taking a limit of the Dyson case, and then we derive for the Airy process asymptotic long time results. In Section 5 we study the Pearcey process, a limiting process gotten from the GUE with an external source or alternately described by the large n behavior of $2n$-non-intersecting Brownian motions starting at $x = 0$ at time $t = 0$, with n conditioned to go to $+\sqrt{n}$, and the other n conditioned to go to $-\sqrt{n}$ at $t = 1$, and we observe how the motions diverge at $t = \frac{1}{2}$ at $x = 0$, with a microscope of resolving power $n^{-1/4}$. The integrable system involved in the finite n problem is the 3-KP system and now instead of bi-orthogonal polynomials, multiple orthogonal polynomials (MOPS) are involved. We connect the 3-KP system and the associated Riemann–Hilbert (RH) problem and the MOP's.

The philosophy in writing this article, which is based on five lectures delivered at CRM, is to keep as much as possible the immediacy and flow of the lecture format through minimal editing and so in particular, the five sections can read in any order. It should be mentioned that although the first section introduces the basic structure of RMT and the KP equation, it in fact deals with the most sophisticated example. The next point was to pick five examples which maximized the diversity of techniques, both in applying Virasoro relations and using integrable systems. Indeed, this article provides a fair but sketchy introduction to integrable systems, although in point of fact, the only ones used in this particular article are invariant subsystems (reductions) and lattices generated from the n-KP, for $n = 1, 2, 3$. The point being that a lot of the skill is involved in picking precisely the right integrable system to deploy. Fortunately, if some sort of orthogonal polynomials are involved, this amounts to deforming the measure(s) intelligently. For further reading see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

Acknowledgement

These lectures present joint work, in the fullest sense, with Pierre van Moerbeke, Pol Vanhaecke and Taka Shiota, who I must thank for all the fun we had and also for enduring my teasing. I also wish to thank Alexei Borodin, Persi Diaconis, Alexander Its, Kurt Johansson, Ken McLaughlin, Craig Tracy, Harold Widom, and Nick Witte for many helpful conversations and insightful remarks – the kind of remarks which make more sense as time goes on. Let me also thank Pol Vanhaecke for reading the manuscript and making helpful suggestions, in addition to producing the various figures that appear. Finally, I wish to thank Louise Letendre of CRM for her beautiful typing job of a messy and hard to read manuscript.

The support of a National Science Foundation grant #DMS-04-06287 is gratefully acknowledged.

1 Matrix Integrals and Solitons

1.1 Random matrix ensembles

Consider the probability ensemble over $n \times n$ Hermitian matrices

$$P\big(M \in \mathcal{H}(E)\big) = \frac{\int_{\mathrm{sp}(M)\in E} e^{-\operatorname{tr} V(M)}\, dM}{\int_{\mathrm{sp}(M)\in \mathbb{R}} e^{-\operatorname{tr} V(M)}\, dM} = \frac{\int_{E^n} \Delta_n^2(z) \prod_1^n e^{-V(z_i)}\, dz_i}{\int_{\mathbb{R}^n} \Delta_n^2(z) \prod_1^n e^{-V(z_i)}\, dz_i} \,, \tag{1}$$

with $\Delta(z) = \prod_{\substack{i\leqslant i,\\ j\leqslant n}}(z_i - z_j) = \det(z_j^{i-1})_{\substack{i\leqslant i,\\ j\leqslant n}}$ the Vandermonde and with $V(x)$ a "nice" function so the integral makes sense, $\mathrm{sp}(M)$ means the spectrum of M and $E \subset \mathbb{R}$,

$$\mathcal{H}_n(E) = \{M \text{ an Hermitian } n \times n \text{ matrix} \mid \mathrm{sp}(M) \subset E\} \,,$$
$$dM = \prod_{i=1}^n dM_{ii} \prod_{i<j} d \text{ Real } M_{ij} \prod_{i<j} d \text{ Imag } (M_{ij}) \,,$$

the induced Haar measure on $n \times n$ Hermitian matrices, viewed as

$$T\big(\mathrm{SL}(n,\mathbb{C})/\,\mathrm{SU}(n)\big)\Big|_I \,,$$

and where we have used the Weyl integration formula [35] for

$$M = U \operatorname{diag}(z_1, \dots, z_n)\mathcal{U}^{-1} \,,$$
$$dM = \Delta_n^2(z_1, \dots, z_n) \prod_1^n dz_i\, d\mathcal{U} \,.$$

In order to recast $P(M)$ so that we may take a limit for large n and avoid ∞-fold integrals, we follow the reproducing method of Dyson [30, 31].

Let $p_k(z)$ be the monic orthogonal polynomials:

$$\int_{\mathbb{R}} p_i(z)p_j(z)e^{-V(z)}\,dz = h_i\delta_{ij} \tag{2}$$

and remember

$$(\det A)^2 = \det(AA^T)\ . \tag{3}$$

Compute

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Delta_n^2(z)\prod_1^n e^{-V(z_i)}\,dz_i \\
&\qquad = \int_{\mathbb{R}^n} \det\big(p_{i-1}(z_j)\big)_{\substack{1\leqslant i,\\ j\leqslant n}} \det\big(p_{k-1}(z_\ell)\big)_{\substack{1\leqslant k,\\ \ell\leqslant n}} \prod_1^n e^{-V(z_i)}\,dz_i \\
&\qquad = \sum_{\pi,\pi'\in S_n} (-1)^{\pi+\pi'}\prod_1^n \int_{\mathbb{R}} p_{\pi(k)-1}(z_k)p_{\pi'(k)-1}(z_k)e^{-V(z_k)}\,dz_k \\
&\qquad = n!\prod_0^{n-1}\int_{\mathbb{R}} p_k^2(z)e^{-V(z)}\,dz \quad \text{(orthogonality)} \\
&\qquad = n!\prod_0^{n-1} h_k\ ,
\end{aligned} \tag{4}$$

and so using (3) and (4), conclude

$$\begin{aligned}
&P\big(M\in\mathcal{H}_n(E)\big) \\
&\qquad = \frac{1}{n!\prod_1^n h_{i-1}}\int_{E^n} \det\left(\sum_1^n p_{j-1}(z_k)p_{j-1}(z_\ell)\right)_{\substack{1\leqslant k,\\ \ell\leqslant n}} \prod_1^n e^{-V(z_i)}\,dz_i \\
&\qquad = \frac{1}{n!}\int_{E^n}\det\big(K_n(z_k,z_\ell)\big)_{\substack{1\leqslant k,\\ \ell\leqslant n}}\prod_1^n dz_i\ ,
\end{aligned}$$

with the Christoffel–Darboux kernel

$$K_n(y,z) := \sum_{j=1}^n \varphi_j(y)\varphi_j(z) = \frac{h_n}{h_{n-1}}\frac{(\varphi_n(y)\varphi_{n-1}(z) - \varphi_n(z)\varphi_{n-1}(y))}{y-z}\ , \tag{5}$$

and

$$\varphi_j(x) = e^{-V(x)/2}p_{j-1}(x)/\sqrt{h_{j-1}}\ .$$

Thus by (2)

$$\int_{\mathbb{R}} \varphi_i(x)\varphi_j(x)\,dx = \delta_{ij}\ ,$$

and so we have

$$\begin{aligned}&\int_{\mathbb{R}} K_n(y,z)K_n(z,u)\,dz = K_n(y,u)\ ,\\ &\int_{\mathbb{R}} K_n(z,z)\,dz = n\ ,\\ &\text{– the reproducing property,}\end{aligned} \tag{6}$$

which yields the crucial property:

$$\begin{aligned}\int_{\mathbb{R}^{n-m}} &\det\big(K_n(z_i,z_j)\big)_{\substack{1\leqslant i,\\ j\leqslant n}} dz_{m+1},\dots,dz_n\\ &= (n-m)!\det\big(K_n(z_i,z_j)\big)_{\substack{1\leqslant i,\\ j\leqslant m}}\ .\end{aligned} \tag{7}$$

Replacing $E^n \to \prod_1^k dz_i \mathbb{R}^{n-k}$ in (1) and integrating out $z_{k+1},\dots,z_n$ via the producing property yields:

$$\begin{aligned}&\text{“}P\ (\text{one eigenvalue in each } [z_i, z_i+dz_i],\ i=1,\dots,k)\\ &\qquad = \frac{1}{\binom{n}{k}}\det\big(K_n(z_i,z_j)\big)_{\substack{1\leqslant i,\\ j\leqslant k}}\prod_1^k dz_i\ ,\text{”}\end{aligned} \tag{8}$$

heuristically speaking. Setting

$$E = \bigcup_{dz_i\in E} dz_i = \cup E_i\ ,$$

and using Poincaré's formula

$$P(\cup E_i) = \sum_i P(E_i) - \sum_{i<j} P(E_i\cap E_j) + \sum_{i<j<k} P(E_i\cap E_j\cap E_k) + \cdots\ ,$$

yields the Fredholm determinant[1]

$$\begin{aligned}P\big(M\in\mathcal{H}_n(E^c)\big) &= 1+\sum_{k=1}^{\infty}(-\lambda)^k\int_{z_1\leqslant\cdots\leqslant z_k}\det\big(K_n^E(z_i,z_j)\big)_{\substack{1\leqslant i,\\ j\leqslant k}}\prod_1^k dz_i\Big|_{\lambda=1}\\ &= \det(I-\lambda K_n^E)\big|_{\lambda=1}\end{aligned} \tag{9}$$

with kernel

$$K_n^E(y,z) = K_n(y,z)I_E(z)$$

and with $I_E(z)$ the indicator function of the set E, and more generally see [48],

$$P\ (\text{exactly } k\text{-eigenvalues} \in E) = \frac{(-1)^k}{k!}\left(\frac{\partial}{\partial\lambda}\right)^k \det(I-\lambda K_n^E)\big|_{\lambda=1}\ . \tag{10}$$

[1] E^c is the complement of E in $\mathbb{R}$.

1.2 Large n–limits

The Fredholm determinant formulas (9) and (10), enable one to take large n–limits of the probabilities by taking large n–limits of the kernels $K_n(y,z)$. The first and most famous such law is for the Gaussian case $V(x)=x^2$, although it has been extended far beyond the Gaussian case [38, 29].

Wigner's semi-circle law:

The density of eigenvalues converges (see Fig. 1) in the sense of measure:

$$K_n(z,z)\,dz \to \begin{cases} \frac{1}{\pi}\sqrt{2n-z}\,dz, & |z| \leqslant \sqrt{2n}\ , \\ 0, & |z| > \sqrt{2n} \end{cases} \tag{11}$$

and so

$$\mathrm{Exp}(\#\text{ eigenvalues in } E) = \int_E K_n(z,z)\,dz.$$

This is a sort of Law of Large Numbers. Is there more refined universal behavior, a sort of Central Limit Theorem? The answer is as follows:

Bulk scaling limit:

The density of eigenvalues near $z=0$ is $\sqrt{2n}/\pi$ and so $\pi/\sqrt{2n} =$ average distance between eigenvalues. Magnifying at $z=0$ with this rescaling

$$\begin{aligned}\lim_{n\to\infty} K_n\left(\frac{\pi x}{\sqrt{2n}}, \frac{\pi y}{\sqrt{2n}}\right) d\left(\frac{y\pi}{\sqrt{2n}}\right) &= \frac{\sin \pi(x-y)}{\pi(x-y)}\,dy \\ &=: K_{\sin}(x,y)\,dy\ , \end{aligned} \tag{12}$$

with

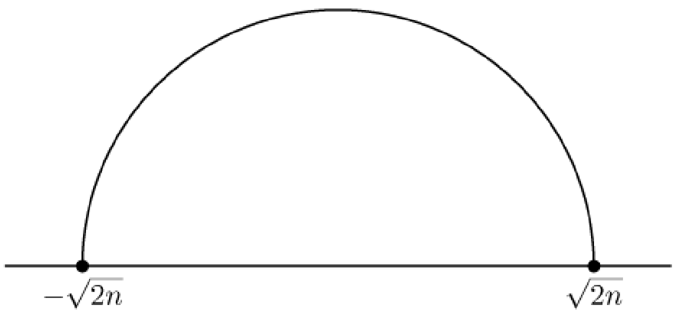

Fig. 1.

$$\lim_{n\to\infty} P\left(\text{exactly } k \text{ eigenvalues} \in \frac{\pi}{\sqrt{2n}}[-2a,2a]\right)$$
$$= \frac{(-1)^k}{k!}\left(\frac{\partial}{\partial\lambda}^k\right)\det(I-\lambda K_{\sin}^{[-2a,2a]})\Big|_{\lambda=1} . \quad (13)$$

Moreover,

$$\det(I-\lambda K_{\sin}^{[-2a,2a]}) = \exp\int_0^{\pi a}\frac{f(x,\lambda)}{x}\,dx$$

with

$$(xf'')^2 + 4(xf'-f)(f'^2+xf'-f) = 0 \qquad \left(' = \frac{d}{dx}\right) , \quad (14)$$

and boundary condition:

$$f(x,\lambda) \cong \frac{-\lambda x}{\pi} - \left(\frac{\lambda}{\pi}\right)^2 x^2 - \cdots , \quad x \sim 0 .$$

The O.D.E. (14) is Painlevé V, and this is the celebrated result of Jimbo, Miwa, Mori, and Sato [36, 37].

Edge scaling limit:

The density of eigenvalues near the edge, $z = \sqrt{2n}$ of the Wigner semi-circle is $\sqrt{2}n^{1/6}$. Magnifying at the edge with the rescaling $1/\sqrt{2}n^{1/6}$:

$$\lim_{n\to\infty} K_n\left(\sqrt{2n}+\frac{u}{\sqrt{2}n^{1/6}}, \sqrt{2n}+\frac{v}{\sqrt{2}n^{1/6}}\right) d\left(\sqrt{2n}+\frac{v}{\sqrt{2}n^{1/6}}\right)$$
$$:= K_{\mathrm{Airy}}(u,v)\,dv , \quad (15)$$

with

$$\begin{aligned} K_{\mathrm{Airy}}(u,v) &= \int_0^\infty A_i(x+u)A_i(x+v)\,dx , \\ A_i(u) &= \frac{1}{\pi}\int_0^\infty \cos\left(\frac{u^3}{3}+xu\right)dv . \end{aligned} \quad (16)$$

Setting $\lambda_{\max} = \sqrt{2n} + u/\sqrt{2}n^{1/6}$

$$\lim_{n\to\infty} P(\sqrt{2}n^{1/6}(\lambda_{\max}-\sqrt{2n}) \leqslant u)$$
$$= \det(I-K_{\mathrm{Airy}}^{[u,\infty]}) = \exp\left(-\int_u^\infty(\alpha-u)g^2(\alpha)\,d\alpha\right)$$
$$\text{– the Tracy–Widom distribution} , \quad (17)$$

with

$$g'' = xg + 2g^3 \quad (18)$$

and boundary condition:

$$g(x) \simeq \frac{\exp(-2/3x^{3/2})}{2\sqrt{\pi}x^{1/4}}, \quad x \to \infty . \tag{19}$$

Equation (18) is Painlevé II and (17) is due to Tracy–Widom [49].

Hard edge scaling limit:

Consider the Laguerre ensemble of $n \times n$ Hermitian matrices:

$$e^{-V(z)}\, dz = z^{\nu/2} e^{-z/2} I_{(0,\infty)}(z)\, dz . \tag{20}$$

Note $z = 0$ is called the hard edge, while $z = \infty$ is called the soft edge. The density of eigenvalues for large n has a limiting shape and the density of eigenvalues near $z = 0$ is $4n$. Rescaling the kernel with this magnification:

$$\begin{aligned}\lim_{n\to\infty} K_n^{(\nu)}\left(\frac{u}{4n}, \frac{v}{4n}\right) d\left(\frac{v}{4n}\right) &=: K_\nu(u,v)\, dv = \frac{1}{2}\int_0^1 sJ_\nu(s\sqrt{u})J_\nu(s\sqrt{v})\, ds\, dv \\ &= \frac{J_\nu(\sqrt{u})\sqrt{v}J_\nu'(\sqrt{v}) - J_\nu(\sqrt{v})\sqrt{u}J_\nu'(\sqrt{u})}{2(u-v)}\, dv \end{aligned} \tag{21}$$

yields the Bessel kernel, while one finds:

$$\begin{aligned}\lim_{n\to\infty} P&\left(\text{no eigenvalue} \in \frac{1}{4n}[0,x]\right) \\ &= \det(I - K_\nu^{[0,x]}) = \exp\left(-\int_0^x \frac{f(v)}{u}\, du\right) , \end{aligned} \tag{22}$$

with

$$(xf'')^2 - 4(xf' - f)f'^2 + \big((x-\nu^2)f' - f\big)f' = 0 , \tag{23}$$

and boundary condition:

$$f(x) = c_\nu x^{1+\nu}\left(1 - \frac{1}{2(2+\nu)}x + \cdots\right), \quad c_\nu = [2^{2\nu+2}\Gamma(1+\nu)\Gamma(2+\nu)]^{-1} .$$

Equation (23) is Painlevé V and is due to Tracy–Widom [50].

1.3 KP hierarchy

We give a quick discussion of the KP hierarchy. A more detailed discussion can be found in [28, 56]. Let $L = L(x,t)$ be a pseudo-differential operator and $\Psi^+(x,t,z)$ its eigenfunction (wave function). The KP hierarchy is an isospectral deformation of L:[2]

[2] Ψ^- is the eigenfunction of L adjoint$:= L^T$, $(A)_+ :=$ differential operator part of A.

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad L = D_x + a_{-1}(x,t)D_x^{-1} + \cdots , \qquad n = 1, 2, \dots , \\ D_x = \frac{\partial}{\partial x}, \quad t = (t_1, t_2, \dots) \tag{24}$$

with Ψ parametrized by Sato's τ-function and satisfying

$$\begin{aligned} \Psi^{\pm}(x,t,z) &= e^{\pm(xz + \sum_1^\infty t_i z^i)} \frac{\tau(t \mp [z^{-1}])}{\tau(t)} \\ &= e^{\pm(xz+\sum_1^\infty t_i z^i)} \big(1 + 0(1/z)\big), \quad z \to \infty \end{aligned} \tag{25}$$

and

$$\begin{aligned} z\Psi^+ &= L\Psi^+, & \frac{\partial \Psi^+}{\partial t_n} &= (L^n)_+ \Psi^+, \\ z\Psi^- &= L^T \Psi^-, & \frac{\partial \Psi^-}{\partial t_n} &= -\big((L^T)^n\big)_+ \Psi^- , \end{aligned} \tag{26}$$

with

$$[x] := \left(x, \frac{x^2}{2}, \frac{x^3}{3}, \dots \right) .$$

Consequently the τ-function satisfies the crucial formal residue identity.

Bilinear identity for τ-function:

$$\oint_\infty e^{\sum_1^\infty (t_i - t_i') z^i} \tau(t - [z^{-1}]) \tau(t' + [z^{-1}])\, dz = 0, \quad \forall t, t' \in \mathbb{C}^\infty , \tag{27}$$

which characterizes the KP τ-function. This is equivalent (see Appendix for proof) to the following generating function of Hirota relations ($a = (a_1, a_2, \dots)$ arbitrary):

$$\sum_{j=0}^\infty s_j(-2a) s_{j+1}(\tilde{\partial}_t) e^{\sum_{\ell=1}^\infty a_\ell \partial/\partial t_\ell} \tau(t) \circ \tau(t) = 0 , \tag{28}$$

where

$$\tilde{\partial}_t = \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \dots \right), \quad \partial_t = \left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \dots \right) , \tag{29}$$

and

$$e^{\sum_1^\infty t_i z_i} := \sum_{j=0}^\infty s_j(t) z^j \quad (s_j(t) \text{ elementary Schur polynomials}) \tag{30}$$

and

$$p(\partial_t) f(t) \circ g(t) := p\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y_2}, \dots \right) f(t+y) g(t-y) \Big|_{y=0} \text{ (Hirota symbol).} \tag{31}$$

This yields the KP hierarchy

$$\left(\frac{\partial^4}{\partial t_1} + 3\left(\frac{\partial}{\partial t_2}\right)^2 - 4\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_3}\right)\tau(t) \circ \tau(t) = 0 \tag{32}$$

$\vdots$

equivalent to

$$\left(\left(\frac{\partial}{\partial t_1}\right)^4 + 3\left(\frac{\partial}{\partial t_2}\right)^2 - 4\frac{\partial^2}{\partial t_1 \partial t_3}\right)\log\tau + 6\left(\frac{\partial^2}{\partial t_1^2}\log\tau\right)^2 = 0$$

$$\text{(KP equation).} \tag{33}$$

$\vdots$

The p-reduced KP corresponds to the reduction:

$$L^p = D_x^p + \cdots = \text{diff. oper.} = (L^p)_+$$

and so

$$\frac{\partial L}{\partial t_{pn}} = [(L^{pn})_+, L] = [L^{pn}, L] = 0\ . \tag{34}$$

$p = 2$: KdV

$$\tau = \tau(t_1, t_3, t_5, \dots) \quad (\text{ignore } t_2,\ t_4, \dots) \tag{35}$$

$$\Psi^{\pm}(x,t,z) = \Psi(x,t,\pm z)\ . \tag{36}$$

1.4 Vertex operators, soliton formulas and Fredholm determinants

Vertex operators in integrable systems generate the tangent space of solutions and Darboux transformations; in other words, they yield the deformation theory. We now present

KP – *vertex operator*:[3]

$$X(t,y,z) = \frac{1}{z-y} e^{\sum_1^\infty (z^i - y^i)t_i} e^{\sum_1^\infty (y^{-i} - z^{-i}) 1/i\ \partial/\partial t_i}\ , \tag{37}$$

and the "τ-space" near τ parametrized: $\tau + \varepsilon X(t,y,z)\tau$. Moreover

$$\tau + \varepsilon X(t,y,z)\tau$$

is a τ-function, *not just infinitesimally*, so it satisfies the bilinear identity. This vertex operator was used in [28] to generate solitons, but it also plays a role in generating Kac–Moody Lie algebras [40]. The identities that follow in Sections 1.4, 1.5, 1.6 were derived by Adler–Shiota–van Moerbeke in [16], carefully and in great detail.

[3] $X(t,y,z)f(t) = \big(1/(z-y)\big)e^{\sum_1^\infty (z^i - y^i)t_i} f(t + [y^{-1}] - [z^{-1}])$, $z \neq y$ and z, y are large complex parameters, and how we expand the operator X shall depend on the situation.

Link with kernels:

We have the differential Fay Identity (Adler–van Moerbeke [1])

$$\frac{1}{\tau(t)}X(t,y,z)\tau(t) = D_x^{-1}\big(\Psi^+(x,t,y)\Psi^-(x,t,z)\big) \ , \tag{38}$$

where since D_x^{-1} is integration, the r.h.s. of (38) has the same structure as the Airy and Bessel kernels of (16) and (21).

If $|y_i|, |z_i| < |y_{i+1}|, |z_{i+1}|, \quad 1 \leqslant i \leqslant n-1$, then we have the Fay Identity:

$$\begin{aligned}\det\Big(D_x^{-1}\big(\Psi^+(x,t,y_i)\Psi^-(x,t,z_j)\big)\Big)_{\substack{1\leqslant i,\\ j\leqslant n}} &= \det\left(\frac{1}{\tau(t)}X(t,y_i,z_j)\tau(t)\right)_{\substack{1\leqslant i,\\ j\leqslant n}}\\ &= \frac{1}{\tau}X(t,y_1,z_1)\cdots X(t,y_n,z_n)\tau \ .\end{aligned} \tag{39}$$

We also have the following

Vertex identities:

$$X(y,z)X(y,z) = 0 \quad \text{and so} \quad e^{aX(y,z)} = 1 + aX(y,z) \ , \tag{40}$$

and

$$[X(\alpha,\beta), X(\lambda,\mu)] = 0 \quad \text{if} \quad \alpha \neq \mu, \ \beta \neq \lambda \ . \tag{41}$$

In addition we have the expansion of the vertex operator along the diagonal:

$$\begin{aligned} &X(t,y,z) = \frac{1}{z-y}\sum_{k=0}^{\infty}\frac{(z-y)^k}{k!}\sum_{\ell=-\infty}^{\infty} y^{-\ell-k}W_\ell^{(k)} \ ,\\ &\text{Heisenberg:}\quad W_n^{(1)} := \frac{\partial}{\partial t_n} + (-n)t_{-n} \quad (W_n^{(0)} = \delta_{n0}) \ ,\\ &\text{Virasoro:}\quad W_n^{(2)} := 2\sum_{i,i+n\geqslant 1} it_i\frac{\partial}{\partial t_{i+n}} + \sum_{i+j=n}\frac{\partial^2}{\partial t_i \partial t_j} + \sum_{i+j=-n}(it_i)(jt_j)\\ &\qquad\qquad - (n+1)\left(\frac{\partial}{\partial t_n} + (-n)t_{-n}\right)\end{aligned} \tag{42}$$

and from the commutation relations:

$$\begin{aligned} [X(\alpha,\beta), X(\lambda,\mu)] &= \big(-n^T(\alpha,\beta,\lambda) + n(\alpha,\beta,\mu)\big)X(\lambda,\mu) \ ,\\ n(\lambda,\mu,z) &:= \delta(\lambda,z)e^{(\mu-\lambda)\ \partial/\partial z} \ ,\\ \delta(\lambda,z) &:= \frac{1}{z}\sum_{-\infty}^{\infty}\left(\frac{z}{\lambda}\right)^{\ell} \ ,\end{aligned} \tag{43}$$

conclude the vertex Lax equation:

$$\frac{\partial}{\partial z}\big(z^{\ell+1}Y(z)\big)=\left[\frac{1}{2}W_\ell^{(2)},Y(z)\right] \tag{44}$$

with

$$Y(z)=X(t,\omega z,\omega' z),\quad \omega,\omega'\in\xi_p\ ,^4 \qquad \ell\geqslant -p \text{ and } p/\ell\ , \tag{45}$$

or a linear combination of such $X(t,\omega z,\omega' z)$.

A Fredholm determinant style soliton formula:

A classical KP soliton formula [28] is as follows:

$$\tau(t)=\frac{1}{\tau_0}\left(\prod_{\substack{1\\ \text{ordered}}}^{n} e^{a_iX(y_i,z_i)}\right)\tau_0\Bigg|_{\tau_0=1}=\det\left(\delta_{ij}+\frac{a_j}{z_j-y_i}e^{\sum_{k=1}^\infty(z_j^k-y_i^k)t_k}\right)_{\substack{1\leqslant i,\\ j\leqslant n}}.$$

We now relax the condition that $\tau_0=1$ and setting $|y_i|,|z_i|<|y_{i+1}|,|z_{i+1}|$, $1\leqslant i\leqslant n-1$, compute using the Fay identity (39) and differential Fay (38):

$$\begin{aligned}
\frac{\tilde\tau(t)}{\tau(t)}&:=\frac{1}{\tau(t)}e^{\sum_1^n a_iX(y_i,z_i)}\tau(t)\\
&=\frac{1}{\tau(t)}\left(\prod_{\substack{1\\ \text{ordered}}}^{n} e^{a_iX(y_i,z_i)}\right)\tau(t)\\
&=\frac{1}{\tau(t)}\left(\prod_{\substack{1\\ \text{ordered}}}^{n}\big(1+a_iX(y_i,z_i)\big)\right)\tau(t)\\
&=\frac{1}{\tau(t)}\Bigg(\tau+\sum_{1\leqslant i\leqslant n}a_iX(y_i,z_i)\tau+\sum_{1\leqslant i<j\leqslant n}a_ia_jX(y_i,z_i)X(y_j,z_j)\tau\\
&\qquad\qquad+\cdots+\prod_1^n a_i\left(\prod_{\substack{1\\ \text{ordered}}}^{n}X(y_i,z_i)\right)\tau\Bigg)\\
&=1+\sum_{1\leqslant i\leqslant n}a_i\frac{X(y_i,z_i)\tau}{\tau}+\sum_{1\leqslant i<j\leqslant n}\det\begin{pmatrix}a_i\frac{X(y_i,z_i)\tau}{\tau}, & a_j\frac{X(y_i,z_j)\tau}{\tau}\\ a_i\frac{X(y_j,z_i)\tau}{\tau}, & a_j\frac{X(y_j,z_j)\tau}{\tau}\end{pmatrix}\\
&\qquad\qquad+\cdots+\det\left(a_j\frac{X(y_i,z_j)\tau}{\tau}\right)_{\substack{1\leqslant i,\\ j\leqslant n}}\\
&=\det\left(\delta_{ij}+a_j\frac{X(y_i,z_j)\tau}{\tau}\right)_{\substack{1\leqslant i,\\ j\leqslant n}}\quad \text{“Fredholm expansion” of determinant}
\end{aligned}$$

[4] ξ_p the pth roots of unity.

$$= \det\Big(\delta_{ij} + a_j D_x^{-1}\big(\Psi^+(x,t,y_i)\Psi^-(x,t,y_j)\big)\Big)_{\substack{1\leqslant i,\\ j\leqslant n}} .$$

Replacing $y_i \to \omega z_i$, $z_i \to \omega' z_i$, $a_i = -\lambda\delta z$

$$z_i = a + (i-1)\delta z, \quad \delta z = \frac{b-a}{n-1}, \quad n \to \infty$$

yields the continuous (via the Riemann Integral).

Soliton Fredholm determinant:

$$\frac{\tau(t,E)}{\tau(t)} := \frac{1}{\tau(t)} e^{-\lambda\int_E X(t,\omega z,\omega' z)\,dz}\tau(t) = \det(I - \lambda k^E) \tag{46}$$

with kernel

$$k^E(y,z) = D_x^{-1}\big(\Psi^-(t,\omega y)\Psi^+(t,\omega' z)\big)I_E(z), \quad E = [a,b] .$$

More generally, for p-reduced K-P, replace in (46)

$$X(t,y,z) \to Y(t,y,z) := \sum_{\omega,\omega'\in\xi_p} a_\omega b_{\omega'} X(t,\omega y,\omega' z) , \tag{47}$$

$$k^E(y,z) \to D_x^{-1} \sum_{\omega\in\xi_p} a_\omega \Psi^-(x,t,\omega y) \sum_{\omega'\in\xi_p} b_{\omega'}\Psi^+(x,t,\omega' z)I_E(z) \tag{48}$$

with

$$\sum_{\omega\in\xi_p} \frac{a_\omega b_\omega}{\omega} = 0 \quad (\text{so } Y(t,z,z) \text{ is pole free}) \tag{49}$$

and

$$X(t,\omega z,\omega' z) \to Y(t,z,z) := Y(z) , \tag{50}$$

$$E \to \bigcup_{i=1}^{k} [a_{2i-1}, a_{2i}] . \tag{51}$$

1.5 Virasoro relations satisfied by the Fredholm determinant

It is a marvelous fact that the soliton Fredholm determinant satisfies a Virasoro relation as a consequence of the vertex Lax-equation [16]; indeed, compute

$$\begin{aligned}
0 &= \int_a^b \left(\frac{\partial}{\partial z} z^{\ell+1} Y(z) - \left[\frac{1}{2}W_\ell^{(2)}, Y(z)\right]\right) dz \\
&= b^{\ell+1}Y(b) - a^{\ell+1}Y(a) - \left[\frac{1}{2}W_\ell^{(2)}, \int_a^b Y(z)\,dz\right] \\
&= \left(b^{\ell+1}\frac{\partial}{\partial b} + a^{\ell+1}\frac{\partial}{\partial a} - \left[\frac{1}{2}W_\ell^{(2)}, \cdot\right]\right) \int_a^b Y(z)\,dz \\
&:= \delta(U) ,
\end{aligned}$$

with δ a derivation and hence

$$\delta e^{-\lambda U} = 0 \,,$$

or spelled out

$$\left(b^{\ell+1}\frac{\partial}{\partial b} + a^{\ell+1}\frac{\partial}{\partial a} - \left[\frac{1}{2}W_\ell^{(2)}, \cdot\right]\right) e^{-\lambda \int_a^b Y(z)\,dz} = 0 \,. \tag{52}$$

Let τ be a vacuum vector for p-KP:

$$W_\ell^{(2)}\tau = c_\ell \tau \,, \quad \ell = kp \,, \qquad k = -1, 0, 1, \ldots \,. \tag{53}$$

Remembering (46) with $Y(z)$ given by (50), and with $\tau(t)$ taken as a vacuum vector, yields

$$\frac{\tau(t,E)}{\tau(t)} = \frac{1}{\tau} e^{-\lambda \int_E Y(z)\,dz}\tau = \det(I - \lambda k^E) \,, \tag{54}$$

and setting $\ell = kp$, $k = -1, 0, 1, \ldots$, compute using, (52), (53) and (54):

$$\begin{aligned} 0 &= \left(b^{\ell+1}\frac{\partial}{\partial b} + a^{\ell+1}\frac{\partial}{\partial a} - \frac{1}{2}W_\ell^{(2)}\right) e^{-\lambda \int_a^b Y(z)\,dz}\tau + e^{-\lambda \int_a^b Y(z)\,dz}\left(\frac{1}{2}W_\ell^{(2)}\tau\right) \\ &= \left(b^{\ell+1}\frac{\partial}{\partial b} + a^{\ell+1}\frac{\partial}{\partial a} - \frac{1}{2}(W_\ell^{(2)} - c_\ell)\right)\left(e^{-\lambda \int_a^b Y(z)\,dz}\tau\right) \\ &= \left(b^{\ell+1}\frac{\partial}{\partial b} + a^{\ell+1}\frac{\partial}{\partial a} - \frac{1}{2}(W_\ell^{(2)} - c_\ell)\right)\big(\tau(t)\det(I - \lambda k^E)\big) \,. \end{aligned} \tag{55}$$

More generally: setting

$$[a,b] \to E^{1/p} := \bigcup_{i=1}^{k} [a_{2i-1}, a_{2i}] \,,$$

$$b^{\ell+1}\frac{\partial}{\partial b} + a^{\ell+1}\frac{\partial}{\partial a} \to \sum_{j=1}^{2k} a_j^{\ell+1}\frac{\partial}{\partial a_j} := B_\ell(a) \,,$$

deduce

$$\left(B_{kp}(a) - \frac{1}{2}(W_{kp}^{(2)} - c_{kp})\right)\big(\tau(t)\det(I - \lambda k^{E^{1/p}})\big) = 0 \,.$$

with

$$W_{kp}^{(2)}\tau(t) = c_{kp}\tau(t), \quad k \geqslant -1 \,. \tag{56}$$

Since changing integration variables in a Fredholm determinant leaves the determinant invariant, change variables:

$$z \to z^p = \lambda, \quad a_i \to a_i^p = A_i \,, \quad \text{and}$$

$$E^{1/p} \to E = \bigcup_{i=1}^{k} [A_{2i-1}, A_{2i}] \,,$$

and

$$k^{E^{1/p}}(z, z') \to K^E(\lambda, \lambda') := \frac{k^{E^{1/p}}(\lambda^{1/p}, \lambda'^{1/p})}{p\lambda^{(p-1)/2p}\lambda'^{(p-1)/2p}} I_E(\lambda') \,, \tag{57}$$

yielding

$$\begin{aligned} \det(I - \mu K^E) &= \det(I - \mu k^{E^{1/p}}) \\ &= \frac{1}{\tau} e^{-\mu \int_{E^{1/p}} Y(z)\, dz} = \frac{\tau(t, E)}{\tau(t)} \,, \end{aligned} \tag{58}$$

and so (56) yields the p *reduced Virasoro relation*:

$$\big(B_k(A) - \tfrac{1}{2}(W^{(2)}_{kp} - c_{kp})\big)\big(\tau(t) \det(I - \mu K^E)\big) = 0 \,, \tag{59}$$

with

$$W^{(2)}_{kp}\tau(t) = c_{kp}\tau(t), \quad k \geqslant -1 \,.$$

1.6 Differential equations for the probability in scaling limits

Now we shall derive differential equations for the limiting probabilities discussed in Section 1.2 using the integrable tools previously developed.

Airy edge limit:

Remembering the edge limit for Hermitian eigenvalues of (15) and (16):

$$\lim_{n \to \infty} P(\sqrt{2} n^{1/16}(\lambda_{\max} - \sqrt{2n}) \in E^c) = \det(I - K^E_{\text{Airy}}) \tag{60}$$

with

$$\begin{aligned} K_{\text{Airy}}(u, v) &= \int_0^\infty A_i(x + u) A_i(x + v)\, dx \,, \\ A_i(u) &= \frac{1}{\pi} \int_0^\infty \cos\left(\frac{u^3}{3} + xu\right) du \,. \end{aligned} \tag{61}$$

Consider the KdV reduction

$$\left(p = 2, \begin{array}{c} t = (t_1, 0, t_3, 0, t_5, \dots) \\ t_0 = (0, 0, 2/3, 0, 0, \dots) \end{array} \right) \tag{62}$$

with initial conditions:

$$\begin{cases} \Psi(x, t_0, z) = 2\sqrt{\pi z} A(x + (-z)^2) \\ \qquad\qquad\quad = e^{xz + 2z^2/3}\big(1 + 0(1/z)\big) \,, \\ \\ (D_x^2 - x)\Psi(x, t_0, z) = z^2 \Psi(x, t_0, z) \,. \end{cases} \tag{63}$$

Under the KP (KdV) flow:

$$\begin{cases} x = q(x,t_0) \to q(x,t) = \dfrac{2\partial^2}{\partial t_1^2} \ell n\big(\tau(t)\big) , \\ \Psi(x,t_0) \to \Psi(x,t) = e^{xz+\sum_1^\infty t_i z_i} \dfrac{\tau(t-[z^{-1}])}{\tau(t)} , \\ \tau(t_0) \to \tau(t) . \end{cases} \tag{64}$$

where $\tau(t)$ turns out to be the well-known Konsevich integral [16, 18], satisfying a vacuum condition as a consequence of Grassmannian invariance, to wit:

Kontsevich integral:

$$\begin{cases} \tau(t) = \lim_{N\to\infty} \dfrac{\int_{\mathcal{H}_N} dX e^{-\operatorname{Tr}(X^3/3+X^2Z)}}{\int_{\mathcal{H}_N} dX e^{-\operatorname{Tr}(X^2Z)}} , \\ Z \operatorname{diag}\colon t_n = -\dfrac{1}{n} \operatorname{Tr} Z^{-n} + \dfrac{2}{3}\delta_{n,3} . \end{cases} \tag{65}$$

Vacuum condition:

$$W_{2k}^{(2)}\tau = -\tfrac{1}{4}\delta_{k0}\tau, \quad k \geqslant -1 . \tag{66}$$

Grassmannian invariance condition:

$$\begin{cases} W := \operatorname{span}_{\mathbb{C}} \left\{ \left(\dfrac{\partial}{\partial x}\right)^j \Psi(x,t_0,z)\Big|_{x=0} , j = 0,1,2,\dots \right\} , \\ z^2 W \subset W \ (KdV) , \quad AW \subset W , \\ A = \dfrac{1}{2z}\left(\dfrac{\partial}{\partial z} + 2z^2\right) - \dfrac{1}{4z^2} , \quad A^2\Psi(0,t_0,z) = z^2\Psi(0,t_0,z) . \end{cases} \tag{67}$$

We have the initial kernel:

$$\begin{aligned} K_{t_0}^E(\lambda,\lambda') &= \frac{I_E(\lambda')}{2\lambda^{1/4}\lambda'^{1/4}} \int_0^\infty \Psi(x,t_0,-\sqrt{\lambda})\Psi(x,t_0,\sqrt{\lambda'})\, dx \\ &= 2\pi I_E(\lambda') \int_0^\infty A_i(x+\lambda)A_i(x+\lambda')\, dx \\ &\xrightarrow{t_0\to t} K_t^E(\lambda,\lambda') . \end{aligned} \tag{68}$$

Conditions on $\tau(t,E)$:

$$\begin{aligned} \tau(t,E) &= \tau(t)\det\left(I - \frac{1}{2\pi}K_t^E\right) && \text{(by (58))} \\ &= \tau(t_0)\det(I - K_{\text{Airy}}^E) \quad \text{at } t = t_0 && \text{(by (68))} , \end{aligned} \tag{69}$$

which satisfies (33) and (35):

$$\left(\left(\frac{\partial}{\partial t_1}\right)^4 - 4\frac{\partial^2}{\partial t_1 \partial t_3}\right)\log\tau + 6\left(\frac{\partial^2}{\partial t_1^2}\log\tau\right)^2 = 0 \quad \text{(KdV)}\ , \tag{70}$$

and by (59), (42) and (62) we find for $\tau(t,E)$

Virasoro constraints:

$$\begin{aligned} B_{-1}(A)\tau &= \left(\frac{\partial}{\partial t_1} + \frac{1}{2}\sum_{i\geqslant 3} it_i \frac{\partial}{\partial t_{i-2}} + \frac{t_1^2}{4}\right)\tau\ , \\ B_0(A)\tau &= \left(\frac{\partial}{\partial t_3} + \frac{1}{2}\sum_{i\geqslant 1} it_i \frac{\partial}{\partial t_i} + \frac{1}{16}\right)\tau\ . \end{aligned} \tag{71}$$

Replace t-derivatives of $\tau(t,E)$ at t_0 with A derivatives in KdV:

$$B_{-1}\tau = \frac{\partial\tau}{\partial t_1}, \quad B_{-1}^2\tau = \frac{\partial^2}{\partial t_1^2}\tau, \dots, B_{-1}B_0\tau = \left(\frac{\partial^2}{\partial t_1 \partial t_3} + \frac{1}{2}\frac{\partial}{\partial t_1}\right)\tau, \dots\ , \quad \text{at } t = t_0\ , \tag{72}$$

yielding

Theorem 1.1 (Adler–Shiota–van Moerbeke [16]).

$$\begin{aligned} R := B_{-1}\log\lim_{n\to\infty} P(\sqrt{2}n^{1/6}(\lambda_{\max} - \sqrt{2n}) \in E^c) &= B_{-1}\log\det(I - K_{\text{Airy}}^E) \\ &= B_{-1}\log\frac{\tau(t_0,E)}{\tau(t_0)} \\ &= B_{-1}\log\tau(t_0,E) \end{aligned}$$

satisfies

$$\left(B_{-1}^3 - 4(B_0 - \tfrac{1}{2})\right)R + 6(B_{-1}R)^2 = 0\ . \tag{73}$$

Setting $E = (a,\infty)$ yields:

$$R''' - 4aR' + 2R + 6R'^2 = 0\ . \tag{74}$$

Setting

$$R = g'^2 - ag^2 - g^4, \quad R' = g^2$$

yields

$$g'' = 2g^3 + ag \quad \text{(Painlevé II)}\ . \tag{75}$$

Hard edge limit:

Remembering the hard edge limit (21) for the Hermitian Laguerre ensemble (20):

$$\lim_{n\to\infty} P\left(\text{no eigenvalues } \in \frac{1}{4n}E\right) = \det(I - K_\nu^E) \, , \qquad K_\nu(u,v) = \frac{1}{2}\int_0^1 sJ_\nu(s\sqrt{u})J_\nu(s\sqrt{v})\,ds \tag{76}$$

defined in terms of Bessel functions, consider the KdV reduction with initial conditions:

$$\begin{cases} \Psi(x,0,z) = e^{xz}B\big((1-x)z\big) = e^{xz}\big(1+0(1/z)\big) \, , \\ \left(D_x^2 - \dfrac{(\nu^2-\frac{1}{4})}{(x-1)^2}\right)\Psi(x,0,z) = z^2\Psi(x,0,z) \end{cases} \tag{77}$$

with (see [19])

$$B(z) := \varepsilon\sqrt{z}H_\nu(iz) = \frac{e^z 2^{\nu+1/2}}{\Gamma(-\nu+\frac{1}{2})}\int_1^\infty \frac{z^{-\nu+1/2}e^{-uz}}{(u^2-1)^{\nu+1/2}}\,du \, ,$$

where

$$\varepsilon = i\sqrt{\frac{\pi}{2}}e^{i\pi\nu/2}, \quad -\frac{1}{2} < \nu < \frac{1}{2} \, .$$

Under the KdV flow:

$$\left(\frac{\nu^2-\frac{1}{4}}{x^2-1}, \Psi(x,0,z), \tau(0)\right) \xrightarrow{t} \big(q(x,t), \Psi(x,t,z), \tau(t)\big) \, , \tag{78}$$

where $\tau(t)$ is both a Laplace matrix integral and a vacuum vector [16, 18] due to Grassmannian invariance:

Laplace integral:

$$\tau(t) = \lim_{N\to\infty} S_1(t)\frac{\int_{\mathcal{H}_N^+} dX\ \det X^{\nu-1/2}e^{-\operatorname{Tr} Z^2X}\int_{\mathcal{H}_N^+} dY\ S_0(Y)e^{-\operatorname{Tr} XY^2}}{\int_{\mathcal{H}_N^+} dX\, e^{-\operatorname{Tr} X^2 Z}}$$

with Z diag: $t_n = -1/n \operatorname{tr} Z^{-n}$, $\mathcal{H}_n^+ = \mathcal{H}_n \cap$ (matrices with non-negative spectrum) and $S_1(t)$, $S_0(Y)$ are symmetric functions,

Vacuum condition:

$$W_{2k}^{(2)}\tau = \big((2\nu)^2 - 1\big)\tau\delta_{k0}, \quad k \geqslant -1 \, , \tag{79}$$

Grassmannian invariance:

$$\begin{cases} z^2 W \subset W, \quad AW \subset W, \\ A = \frac{1}{2} z\left(\frac{\partial}{\partial z} - 1\right), \quad \left(4A^2 - 2A - \nu^2 + \frac{1}{4}\right)\Psi(0,0,z) = z^2 \Psi(0,0,z)\,. \end{cases}$$

We have the initial kernel:

$$\begin{aligned} &K^E_{(x,t)}(\lambda,\lambda') \\ &\qquad = \frac{I_E(\lambda')}{2\lambda^{1/4}\lambda'^{1/4}} D_x^{-1} Y(x,t,\sqrt{\lambda},\sqrt{\lambda'}) \quad \text{(see (48), (49) and (57))} \\ &\qquad = \frac{I_E(\lambda')}{2\lambda^{1/4}\lambda'^{1/4}} D_x^{-1} \left(\sum_{\pm} a_{\pm} \Psi^{-}(x,t,\pm\sqrt{\lambda}) \sum_{\pm} b_{\pm} \Psi^{+}(x,t,\pm\sqrt{\lambda'}) \right) \\ &\qquad := \frac{I_E(\lambda')}{2\lambda^{1/4}\lambda'^{1/4}} \int_1^x \left(\frac{ie^{i\pi\nu/2}}{\sqrt{2\pi}} \Psi(x,t,-\sqrt{\lambda}) + \frac{e^{-i\pi\nu/2}}{\sqrt{2\pi}} \Psi(x,t,\sqrt{\lambda}) \right) \\ &\qquad\qquad \times \left(-\frac{e^{-i\pi\nu/2}}{\sqrt{2\pi}} \Psi(x,t,\sqrt{\lambda'}) - \frac{ie^{i\pi\nu/2}}{\sqrt{2\pi}} \Psi(x,t,-\sqrt{\lambda'}) \right) dx \\ &\qquad = I_E(\lambda') \frac{1}{2} \int_0^1 s J_\nu(s\sqrt{\lambda}) J_\nu(s\sqrt{\lambda'})\, ds \quad \text{at } (x,t_0) = (1+i,-e_1)\,, \end{aligned} \tag{80}$$

and under the t-flow

$$K^E_{(1+i,-e_1)}(\lambda,\lambda') \xrightarrow{t_0 \to t} K^E_{(x,t)}(\lambda,\lambda')\ .$$

Conditions on $\tau(t,E)$:

$$\tau(t,E) = \tau(t)\det(I - K^E_{(x,t)}) = \tau(t_0)(I - K^E_\nu) \quad \text{at } (x,t_0) = (1+i,e_1) \quad \text{(by (58) and (80))}\,, \tag{81}$$

satisfies (33) and (35) :

$$\left(\left(\frac{\partial}{\partial t_1}\right)^4 - 4\frac{\partial^2}{\partial t_1 \partial t_3} \right) \log\tau + 6\left(\frac{\partial^2}{\partial t_1^2}\log\tau\right)^2 = 0 \quad \text{(KdV)}\,, \tag{82}$$

and by (59), (42), (62) and (81), the

Virasoro constraints:

$$\begin{aligned} B_0(A)\tau &= \frac{1}{2}\left(\sum_{i \geqslant 1} it_i \frac{\partial}{\partial t_i} + \sqrt{-1}\frac{\partial}{\partial t_1} + 2\left(\frac{1}{4} - \nu^2\right) \right)\tau\,, \\ B_1(A)\tau &= \frac{1}{2}\left(\sum_{i \geqslant 1} it_i \frac{\partial}{\partial t_{i+2}} + \frac{1}{2}\frac{\partial^2}{\partial t_1^2} + \sqrt{-1}\frac{\partial}{\partial t_3} \right)\tau\ . \end{aligned}$$

Replace t-derivatives of $\tau(t,E)$ at $(1+i,-e_1)$ with A derivatives in KdV:

$$B_0(A)\tau = \frac{i}{2}\frac{\partial\tau}{\partial t_1} + \left(\frac{1}{4}-\nu^2\right)\tau\ , \quad B_1(A) = \frac{1}{4}\frac{\partial^2\tau}{\partial t_1^2} + \frac{i}{2}\frac{\partial\tau}{\partial t_3}, \dots \quad \text{at } (1+i,-e_1)$$

yielding

Theorem 1.2 (Adler–Shiota–van Moerbeke [16]).

$$\begin{aligned} R := \log \lim_{n\to\infty} P\left(\textit{no eigenvalues} \in \frac{E}{4n}\right) &= \log\det(I - K_\nu^E) \\ &= \log\frac{\tau(i,0,0,\dots;E)}{\tau(i.0,0,\dots;R)} \end{aligned}$$

satisfies

$$\left(B_0^4 - 2B_0^3 + (1-\nu^2)B_0^2 + B_1(B_0 - \tfrac{1}{2})\right)R - 4(B_0R)(B_0^2R) + 6(B_0^2R)^2 = 0\ .$$

Setting:

$$E = (0,x), \quad f = -x\frac{\partial R}{\partial x}$$

yields

$$f''' + \frac{f''}{x} - \frac{6f'^2}{x} + \frac{4ff'}{x^2} + \frac{(x-\nu^2)f'}{x^2} - \frac{f}{2x^2} = 0 \quad \text{(Painlevé V)}\ .$$

2 Recursion Relations for Unitary Integrals

2.1 Results concerning unitary integrals

Many generating functions in a parameter t for combinatorial problems are expressible in the form of unitary integrals $I_n(t)$ over $U(n)$ (see [20, 22, 47, 51]). Our methods can be used to either get a differential equation for $I_n(t)$ in t [2] or a recursion relation in n [3] and in the present case we concentrate on the latter. Borodin first got such results [26] using Riemann–Hilbert techniques. Consider the following basic objects ($\Delta_n(z)$ is the Vandermonde determinant, $i=\sqrt{-1}$):

Unitary integral:

$$\begin{aligned} I_n^{(\varepsilon)} &= \int_{U(n)} \det\left(M^\varepsilon \rho(M)\right) dM \\ &= \frac{1}{n!}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(z_k^\varepsilon \rho(z_k)\frac{dz_k}{2\pi i z_k}\right) \\ &= \det\left(\int_{S^1} z^{\varepsilon+i'-j'}\rho(z)\frac{dz}{2\pi i z}\right)_{\substack{1\leqslant i',\\ j'\leqslant n}}\ , \end{aligned} \tag{83}$$

with weight $\rho(z)$:

$$\rho(z) = e^{P_1(z)+P_2(z^{-1})} z^{\gamma}(1-d_1 z)^{\gamma_1'}(1-d_2 z)^{\gamma_2'}(1-d_1^{-1}z^{-1})^{\gamma_1''} \times (1-d_2^{-1}z^{-1})^{\gamma_2''} , \tag{84}$$

$$P_1(z) := \sum_1^{N_1} \frac{u_i z^i}{i}, \quad P_2(z) := \sum_1^{N_2} \frac{u_{-i} z^i}{i} , \tag{85}$$

and we introduce the

Basic recursion variables:

$$x_n := (-1)^n \frac{I_n^+}{I_n^{(0)}} , \quad y_n := (-1)^n \frac{I_n^-}{I_n^{(0)}} , \tag{86}$$

$$v_n := 1 - x_n y_n = \frac{I_{n-1}^{(0)} I_{n+1}^{(0)}}{I_n^{(0)2}} , \tag{87}$$

and so

$$I_n^{(0)} = (I_1^{(0)})^n \prod_1^{n-1} (1-x_i y_i)^{n-i} , \tag{88}$$

thus

$$(x,y) \text{ recursively yields } \{I_n^{(0)}\} .$$

We also introduce the fundamental semi-infinite matrices:

Toeplitz matrices:

$$L_1(x,y) := \begin{pmatrix} -x_1 y_0 & 1-x_1 y_1 & 0 & 0 & \dots \\ -x_2 y_0 & -x_2 y_1 & 1-x_2 y_2 & 0 & \dots \\ -x_3 y_0 & -x_3 y_1 & -x_3 y_2 & 1-x_3 y_3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \tag{89}$$
$$L_2 := L_1^T(y,x) ,$$

in terms of which we write the following:

Recursion matrices:[5]

$$\begin{aligned} \mathcal{L}_1^{(n)} &:= (aI + bL_1 + cL_1^2)P_1'(L_1) + c(n + \gamma_1' + \gamma_2' + \gamma)L_1 , \\ \mathcal{L}_2^{(n)} &:= (cI + bL_2 + aL_2^2)P_2'(L_2) + a(n + \gamma_1'' + \gamma_2'' - \gamma)L_2 . \end{aligned} \tag{90}$$

There exists the following basic involution $\sim$:

[5] I in (90) is the semi-infinite identity matrix and $P_i'(z) = dP_i(z)/dz$.

Basic involution:

$$
\begin{gathered}
\sim : z \to z^{-1}, \quad \rho(z) \to \rho(z^{-1}) \;, \\
I_n^{(0)} \leftrightarrow I_n^{(0)}, \quad I_n^+ \leftrightarrow I_n^- \;, \\
x_n \leftrightarrow y_n, \quad a \leftrightarrow c, \quad b \leftrightarrow b, \quad \gamma \to -\gamma \\
L_1 \leftrightarrow L_2^T, \quad \mathcal{L}_1^{(n)} \leftrightarrow \mathcal{L}_2^{(n)T} \;.
\end{gathered} \tag{91}
$$

Self-dual case:

$$
\rho(z) = \rho(z^{-1}) \;, \quad x_n = y_n \Longrightarrow L_1 = L_2^T, \quad \mathcal{L}_1^{(n)} = \mathcal{L}_2^{(n)^T} \;. \tag{92}
$$

Let us define the "total discrete derivative":[6]

$$
\partial_n f(n) = f(n+1) - f(n) \;. \tag{93}
$$

We now state the main theorems of Adler–van Moerbeke [3]:

Rational relations for (x, y):

Theorem 2.1. *The* $(x_k, y_k)_{k\geqslant 1}$ *satisfy* 2 *finite step relations*:

Case 1.

In the generic situation, to wit:

$$
d_1, d_2, d_1 - d_2, |\gamma_1'| + |\gamma_1''|, |\gamma_2'| + |\gamma_2''| \neq 0 \quad \textit{in } \rho(z) \;, \tag{94}
$$

we find for $n \geqslant 1$ *that*

$$
(+) \begin{cases} \partial_n(\mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)})_{n,n} + (cL_1 - aL_2)_{n,n} = 0 \\ \partial_n(v_n\mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)})_{n+1,n} + (cL_1^2 + bL_1)_{n+1,n+1} = (same)_{n=0} \;, \end{cases}
$$

with

$$
(a, b, c) = \frac{1}{\sqrt{d_1 d_2}}(1, -d_1, -d_2, d_1 d_2) \;, \tag{95}
$$

and the dual equations $(\tilde{+})$*, also hold.*

Case 2.

Upon rescaling,

$$
\rho(z) = z^{\gamma}(1+z)^{\gamma_1'} e^{P_1(z) + P_2(z^{-1})} \;, \tag{96}
$$

and then $(+)$, $(\tilde{+})$ *are satisfied with* $(a, b, c) = (1, 1, 0)$ *or* $(0, 1, 1)$.

---

[6] The total derivative means you must take account, in writing $f(n)$, of *all* the places n appears either implicitly or explicitly, so $f(n) = g(n, n, n, \dots)$ in reality.

Case 3.

Upon rescaling,

$$\rho(z) = z^{\gamma} e^{P_1(z)+P_2(z^{-1})} , \tag{97}$$

and then (+), $(\tilde{+})$ *are satisfied for* a, b, c *arbitrary and in addition finer relations hold*:

$$\Gamma_n(x,y) = 0 , \quad \widetilde{\Gamma}_n(x,y) = 0 ,$$

$$\Gamma_n(x,y) := \frac{v_n}{y_n} \begin{pmatrix} -\big(L_1 P_1'(L_1)\big)_{n+1,n+1} & -\big(L_2 P_2'(L_2)\big)_{n,n} \\ +\big(P_1'(L_1)\big)_{n+1,n} & +P_2'(L_2)_{n\,,n+1} \end{pmatrix} + n x_n . \tag{98}$$

If $|N_1 - N_2| \leqslant 1$, where N_i = degree P_i, the rational relations of Theorem 2.1 become, upon setting $z_n := (x_n, y_n)$:

Rational recursion relations:

Theorem 2.2. *For* $N_1 = N_2 \pm 1$ *or* $N_1 = N_2$, *the rational relations of Theorem* 2.1 *become recursion relations as follows*:

Case 1.

Yields inductive rational $N_1 + N_2 + 4$ *step relations*:

$$z_n = F_n(z_{n-1}, z_{n-2}, \ldots, z_{n-N_1-N_2-3}) .$$

Case 2.

Yields inductive rational $N_1 + N_2 + 3$ *step relations*:

$$z_n = F_n(z_{n-1}, z_{n-2}, \ldots, z_{n-N_1-N_2-2}) ,$$

with either

$$N_1 = N_2 \quad \textit{or} \quad N_2 + 1 \colon \ (a,b,c) = (1,1,0) ,$$

or

$$N_1 = N_2 \quad \textit{or} \quad N_2 - 1 \colon \ (a,b,c) = (0,1,1) .$$

Case 3.

Yields inductive rational $N_1 + N_2 - 1$ *step relations*:

$$z_n = F_n(z_{n-1}, z_{n-2}, \ldots, z_{n-N_1-N_2}),$$

upon using Γ_n *and* $\widetilde{\Gamma}_n$ *and in the case of the*

Self dual weight:

$$\rho(z) = e^{\sum_1^{N_1} u_i(z^i+z^{-i})/i} .$$

One finds recursion relations:

$$x_n = F_n(x_{n-1}, x_{n-2}, \dots, x_{n-2N}) .$$

2.2 Examples from combinatorics

In this section, we give some well-known examples from combinatorics in the notation of the previous section. In the case of a permutation π_k, of k–numbers, $L(\pi_k)$ shall denote the length of the largest increasing subsequence of π_k. If π_k is only a word of size k from an alphabet of α numbers, $L^{(w)}(\pi_k)$ shall denote the length of the largest weakly increasing subsequence in the word π_k. We will also consider odd permutations π on $(-k, \dots, -1, 1, \dots, k)$ or $(-k, \dots, -1, 0, 1, \dots, k)$, which means $\pi(-j) = -\pi(j)$, for all j.

Example 1. $\rho(z) = e^{t(z+z^{-1})}$ (self-dual case)

$$I_n^{(0)}(t) := \sum_{k=0}^{\infty} \frac{t^{2k}}{k!} P(\pi_k \in S_k) \mid L(\pi_k) \leqslant n) = \int_{U(n)} e^{t\,\mathrm{Tr}(M+M^{-1})}\, dM ,$$

the latter equality due to I. Gessel, with

$$x_n(t) = (-1)^n \frac{\int_{U(n)} \det(M) e^{t\,\mathrm{Tr}(M+M^{-1})}\, dM}{\int_{U(n)} e^{t\,\mathrm{Tr}(M+M^{-1})}\, dM} , \quad \text{(as in (86))}$$

and

$$I_n^{(0)}(t) = \left(I_1^{(0)}(t)\right)^n \prod_{i=1}^{n-1} (1-x_i^2)^{n-i} .$$

One finds a

3-*step recursion relation for* x_i:

$$\begin{aligned} 0 &= n x_n - \frac{(1-x_n^2)}{x_n} (t(L_1)_{n+1,n+1} + t(L_1)_{nn}) \quad \text{(as in (98))} \\ &= n x_n + t(1-x_n^2)(x_{n+1} + x_{n-1}) \quad \text{(Borodin [26])} \end{aligned}$$

which possesses an

$$\begin{gathered} \text{Invariant} : \Phi(x_{n+1}, x_n) = \Phi(x_n, x_{n-1}) , \\ \Phi(y,z) = (1-y^2)(1-z^2) - \frac{n}{t} yz \quad \text{(McMillan [43])} . \end{gathered}$$

The initial conditions in the recursion relation are as follows:

$$x_0 = 1, \quad x_1 = -\frac{1}{2}\frac{d}{dt} \log I_0(2t), \quad I_1^{(0)} = I_0(2t) ,$$

with I_0 the hyperbolic Bessel function of the first kind (see [19]).

Example 2. $\rho(z) = e^{t(z+z^{-1})+s(z^2+z^{-2})}$ (self-dual case)
Set

$$S_{2k}^{\mathrm{odd}} = \{\pi_{2k} \in S_{2k} \text{ acts on } (-k,\dots,-1,1,\dots,k) \text{ oddly}\}\ ,$$
$$S_{2k+1}^{\mathrm{odd}} = \{\pi_{2k+1} \in S_{2k+1} \text{ acts on } (-k,\dots,-1,0,1,\dots,k) \text{ oddly}\}$$

and then one finds:

$$\sum_{k=0}^{\infty} \frac{(\sqrt{2}s)^{2k}}{k!} P(\pi_{2k} \in S_{2k}^{\mathrm{odd}} \mid L(\pi_{2k}) \leqslant n) = \int_{U(n)} e^{s\,\mathrm{Tr}(M^2+M^{-2})}\, dM\ ,$$
$$\sum_{k=0}^{\infty} \frac{(\sqrt{2}s)^{2k}}{k!} P(\pi_{2k+1} \in S_{2k+1}^{\mathrm{odd}} \mid L(\pi_{2k+1}) \leqslant n)$$
$$= \frac{1}{4}\left(\frac{\partial}{\partial t}\right)^2 \left(\int_{U(n)} e^{\mathrm{Tr}(t(M+M^{-1})+s(M^2+M^{-2}))}\, dM + \text{same } (t,-s) \right)\Bigg|_{t=0}\ ,$$

as observed by M. Rains [47] and Tracy–Widom [51]. Moreover,

$$x_n(t,x) = (-1)^n \frac{\int_{U(n)} \det(M) e^{\mathrm{Tr}(t(M+M^{-1})+s(M^2+M^{-2}))}\, dM}{\int_{U(n)} e^{\mathrm{Tr}(t(M+M^{-1})+s(M^2+M^{-2}))}\, dM}\ ,$$

and

$$I_n^{(0)}(t) = \left(I_1^{(0)}(t)\right)^n \prod_{i=1}^{n-1} (1-x_i^2)^{n-i}\ .$$

One finds a

5-*step recursion relation for* x_i:

$$nx_n + tv_n(x_{n-1}+x_{n+1}) + 2sv_n(x_{n+2}v_{n+1} + x_{n-2}v_{n-1} - x_n(x_{n+1}+x_{n-1})^2) = 0$$
$$(v_n = 1 - x_n^2)$$

which possesses the

$$\text{Invariant}\ : \Phi(x_{n-1},x_n,x_{n+1},x_{n+2}) = \text{same}\Big|_{n\to n+1}\ ,$$
$$\Phi(x,y,z,u) := nyz - (1-y^2)(1-z^2)\Big(t + 2s\big(x(u-y) - z(u+y)\big)\Big)\ ,$$

analogous to the McMillan invariant of the previous example.

Example 3. $\rho(z) = (1+z)e^{sz^{-1}}$ (Case 2 of Theorem 2.1)
Set

$$S_{k,\alpha} = \{\text{words } \pi_k \text{ of length } k \text{ from alphabet of size } \alpha\}\ ,$$

with

$$I_n^{(0)}(s) = \sum_{k=0}^{\infty} \frac{(\alpha s)^k}{k!} P(\pi_k \in S_{k,\alpha} \mid L^{(w)}(\pi_k) \leqslant n)$$
$$= \int_{U(n)} \det(I+M)^\alpha e^{s \operatorname{tr} M^{-1}} \, dM \ ,$$

the latter identity observed by Tracy–Widom [52]. Then setting in

Case 2.

$P_1(z) = 0$, $P_2(z) = sz$, $N_1 = 0$, $N_2 = 1$, $(a, b, c) = (0, 1, 1)$,

$$\mathcal{L}_1^{(n)} = (n+\alpha) L_1, \quad \mathcal{L}_2^{(n)} = s(I + L_2) \ ,$$

one finds the

Recursion relations:

$$\frac{\partial_n \big((n+\alpha)L_1 - sL_2\big)_{nn} + (L_1)_{nn}}{\partial_n \big((n-1)+\alpha\big) v_{n-1} L_1 - sL_2\big)_{n,n-1} + (L_1^2 + L_1)_{n,n}} = (\text{same})\Big|_{n=1}$$
$$(v_n = 1 - x_n y_n) \ ,$$

with

$$\underset{+}{x_n}, \underset{-}{y_n} = \frac{(-1)^n \int_{U(n)} (\det M)^{\pm} \det(I+M)^\alpha e^{s \operatorname{Tr} M^{-1}} \, dM}{\int_{U(n)} \det(I+M)^\alpha e^{s \operatorname{tr} M^{-1}} \, dM}$$

and

$$I_n^{(0)}(s) = \big(I_1^{(0)}(s)\big)^n \prod_{i=1}^{n-1} (1 - x_i y_i)^{n-i} \ ,$$

leading to the

3 *and* 4 *step relations for* (x_i, y_i):

$$-(n+\alpha+1)x_{n+1}y_n + s x_n y_{n+1} + (n+\alpha-1)x_n y_{n-1} - s x_{n-1} y_n = 0 \ ,$$
$$- v_n\big((n+\alpha+1)x_{n+1}y_{n-1} + s\big) + v_{n-1}\big((n+\alpha-2)x_n y_{n-2} + s\big)$$
$$+ x_n y_{n-1}(x_n y_{n-1} - 1) = -v_1\big((2+\alpha)x_2 + s\big) + x_1(x_1 - 1)$$
$$(x_0 = y_0 = 1) \ .$$

2.3 Bi-orthogonal polynomials on the circle and the Toeplitz lattice

It turns out that the appropriate integrable system for our problem is the Toeplitz lattice, an invariant subsystem of the 2–Toda lattice. Indeed x_n and y_n of Section 2.1 turn out to be dual Hamiltonian variables for the integrable system; moreover x_n and y_n are nothing but the constant term of the nth bi-orthogonal polynomials on the circle, generated by a natural time deformation of our measure $\rho(z)$ of (84). These things are discussed by Adler–van Moerbeke in [2, 3] in detail. Consider the bi-orthogonal polynomials and inner product generated by the following measure on S^1:

$$\rho(z,t,s)\frac{dz}{2\pi iz} = e^{\sum_1^\infty (t_k z^k - s_k z^{-k})}\frac{dz}{2\pi iz}\ , \tag{99}$$

with

Inner product:

$$\langle f(z), g(z)\rangle := \oint_{S^1} \frac{dz}{2\pi iz} f(z)g(z^{-1})e^{\sum_1^\infty (t_i z^i - siz^{-i})}\ . \tag{100}$$

Bi-orthogonal polynomials:

$$\langle p_n^{(1)}, p_m^{(2)}\rangle := \delta_{n,m}h_n, \quad h_n = \frac{\tau_{n+1}(t,s)}{\tau_n(t,s)}, \quad n,m = 0,1,\dots\ . \tag{101}$$

The polynomials are parametrized by

2-*Toda* τ *functions*:

$$\begin{aligned}\tau_n(t,s) &= \det(\langle z^k, z^\ell\rangle)_{\substack{0\leqslant k,\\ \ell\leqslant n-1}} \qquad \text{(Toeplitz determinant)}\\ &= \frac{1}{n!}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n e^{\sum_{j=1}^\infty (t_j z_k^j - s_j z_k^{-j})}\frac{dz_k}{2\pi iz_k}\\ &= \int_{U(n)} e^{\operatorname{Tr}\sum_1^\infty (t_i M^i - s_i M^{-i})}\,dM, \quad n \geqslant 1,\ \tau_0 \equiv 1\ ,\end{aligned} \tag{102}$$

as follows:

$$\begin{aligned}(p_n^{(1)}, p_n^{(2)})(u;t,s) &= \frac{u^n}{\tau_n(t,s)}\big(\tau_n(t-[u^{-1}],s), \tau_n(t,s+[u^{-1}])\big)\\ &= \left(\frac{\int_{U(n)}\det(uI-M)e^{\operatorname{Tr}(\sum_1^\infty t_iM^i - s_iM^{-i})}\,dM}{\int_{U(n)} e^{\operatorname{Tr}(\sum t_iM_i - s_iM^{-i})}\,dM}\ ,\right.\\ &\qquad \left.\frac{\int_{U(n)}\det(uI-M^{-1})e^{\operatorname{Tr}(\sum_1^\infty t_iM^i - s_iM^{-i})}\,dM}{\int_{U(n)} e^{\operatorname{Tr}(\sum t_iM_i - s_iM^{-i})}\,dM}\right),\\ &\qquad [x] = (x, x^2/2, x^3/3, \dots)\ .\end{aligned} \tag{103}$$

The constant term of our polynomials yield the

Dynamical variables:

$$x_n(t,x) = p_n^{(1)}(0;t,s), \quad y_n(t,s) = p_n^{(2)}(0;t,s)\ , \tag{104}$$

$$v_n = 1 - x_n y_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}\ ; \tag{105}$$

giving rise to an invariant subsystem of the

Toda equations:

$$L_1(x,y) := \begin{pmatrix} -x_1y_0 & 1-x_1y_1 & 0 & 0 & \dots \\ -x_2y_0 & -x_2y_1 & 1-x_2y_2 & 0 & \dots \\ -x_3y_0 & -x_3y_1 & -x_3y_2 & 1-x_3y_3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{106}$$

$$L_2(x,y) := L_1^T(y,x)\ , \quad h_n = \tau_{n+1}/\tau_n\ , \quad h = \mathrm{diag}(h_0, h_1\ , \dots)\,, \tag{107}$$

$$\tilde{L}_1 = hL_1h^{-1}\ , \quad \tilde{L}_2 = L_2\ , \tag{108}$$

$$(T)\begin{cases} \dfrac{\partial \tilde{L}_i}{\partial t_n} & = [(\tilde{L}_1^n)_+, \tilde{L}_i]^7 \\ \dfrac{\partial \tilde{L}_i}{\partial s_n} & = [(\tilde{L}_2^n)_-, \tilde{L}_i] \end{cases}, \quad i = 1,2,\ n = 1,2,\dots \tag{109}$$

with

$$\begin{cases} (A)_+ & = \text{upper tri}(A) + \text{diag}(A)\ , \\ (A)_- & = \text{lower tri}(A)\ . \end{cases}$$

We find, $(T) \Leftrightarrow$ *Toeplitz lattice*:[7]

$$\begin{cases} \dfrac{\partial x_n}{\partial t_i} = v_n \dfrac{\partial H_i^{(1)}}{\partial y_n}, & \dfrac{\partial y_n}{\partial t_i} = -v_n \dfrac{\partial H_i^{(1)}}{\partial x_n} \\ \dfrac{\partial x_n}{\partial s_i} = v_n \dfrac{\partial H_i^{(2)}}{\partial y_n}, & \dfrac{\partial y_n}{\partial s_i} = -v_n \dfrac{\partial H_i^{(2)}}{\partial x_n} \end{cases}, \quad i = 1,2,\dots\ , \tag{110}$$

$$H_i^{(j)} := -\frac{\mathrm{tr}\, L_j^i}{i}, \quad \omega = \sum_{k=1}^{\infty} \frac{dx_k \wedge dy_k}{v_k}, \quad v_n = 1 - x_n y_n\ , \tag{111}$$

with

$$\begin{cases} x_n(0,0) = y_n(0,0) = 0, & n \geqslant 1\ , \\ x_0(t,s) = y_0(t,s) = 1, & \forall t,s\ , \end{cases} \quad \text{(see [2, 3]).} \tag{112}$$

---

[7] Equations (T) are the 2–Toda equations of Ueno–Takasaki [55], but equations (T) with precisely the initial conditions (106) and (107) are an invariant subsystem of the 2-Toda equations which are equivalent to the Toeplitz lattice, (110), (111). The latter equations are Hamiltonian, with ω the symplectic form.

Example

$$\begin{aligned} \frac{\partial x_n}{\partial t_1} &= (1 - x_n y_n) x_{n+1} \ , \quad \frac{\partial y_n}{\partial t_1} = -(1 - x_n y_n) y_{n-1} \ , \\ \frac{\partial x_n}{\partial s_1} &= (1 - x_n y_n) x_{n-1} \ , \quad \frac{\partial y_n}{\partial s_1} = -(1 - x_n y_n) y_{n+1} \ , \end{aligned} \tag{113}$$

yielding the

Ladik–Ablowitz lattice:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_1} - \frac{\partial}{\partial s_1} \ . \tag{114}$$

2.4 Virasoro constraints and difference relations

In this section, using a 2–Toda vertex operator, which generates a subspace of the tangent space of the sequence of 2–Toda τ-functions $\{\tau_n(t,s)\}$, we derive Virasoro relations in our special case. Indeed, deriving a Lax-equation for the vertex operator leads to a fixed point theorem for our particular sequence of matrix integral τ-functions, fixed under an integrated version of the vertex operator, integrated over the unit circle. The Virasoro relations coupled with the Toeplitz lattice identities then lead to difference relations after some manipulation.

We present the following operators:

Toda vertex operator:

$$\begin{aligned} &X(u,v)\big(f_n(t,s)\big)_{n\geqslant 0} \\ &\qquad := (e^{\sum_1^\infty (t_i u^i - s_i v^{-i})} (uv)^{n-1} f_{n-1}(t - [u^{-1}], s + [v^{-1}])_{n\geqslant 0} \ . \end{aligned} \tag{115}$$

Integrated version:

$$Y^\gamma := \int_{S^1} \frac{du}{2\pi i u} u^\gamma X(u, u^{-1}) \ . \tag{116}$$

Virasoro operator:

$$\begin{aligned} \mathbb{V}_k^\gamma = (\mathbb{V}_{k,n}^\gamma)_{n\geqslant 0} &:= \mathbb{J}_k^{(2)}(t) - \mathbb{J}_{-k}^{(2)}(-s) - (k-\gamma)\big(\theta \mathbb{J}_k^{(1)}(t) + (1-\theta)\mathbb{J}_{-k}^{(1)}(-s)\big) \\ &\qquad \text{(vector differential operator in } t,\ s\text{, acting diagonally} \\ &\qquad\qquad \text{as defined explicitly in (123), (124), and (125))} \ . \end{aligned} \tag{117}$$

Main facts:

$$\text{Lax-equation:} \quad u^{-\gamma} u \frac{d}{du} u^{k+\gamma} X(u, u^{-1})) = [\mathbb{V}_k^\gamma, X(u, u^{-1})] \,, \tag{118}$$

$$\text{Commutativity:} \quad [\mathbb{V}_k^\gamma, Y^\gamma] = 0 \,, \tag{119}$$

$$\text{Fixed Point:} \quad Y^\gamma I = I, \quad I := \left(n! \tau_n^{(\gamma)}(t,s)\right)_{n \geqslant 0} \,, \tag{120}$$

where

$$\tau_n^{(\gamma)}(t,s) = \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(z_k^\gamma e^{\sum_{j=1}^\infty (t_j z_k^j - s_j z_k^{-j})} \frac{dz_k}{2\pi i z_k} \right) \,, \quad (\tau_0 = 1) \tag{121}$$

and $\{\tau_n^{(\gamma)}(t,s)\}$ *satisfy an* $s\ell(2,\mathbb{Z})$ *algebra of Virasoro constraints*:

$$\mathbb{V}_{k,n}^\gamma \tau_n^{(\gamma)}(t,s) = 0 \quad \text{for} \quad \begin{cases} k = -1, & \theta = 0 \,, \\ k = \ \ 0, & \theta \text{ arbitrary} \,, \\ k = \ \ 1, & \theta = 1, \end{cases} \quad (n \geqslant 0) \,. \tag{122}$$

Proof of main facts.

The proof of the Lax-equation is a lengthy calculation (see [3]). Integrating the Lax-equation with regard to

$$\int_{S^1} \frac{du\, u^\gamma}{2\pi i u}$$

immediately leads to the commutativity statement. To see the fixed point statement, compute (setting $u = z_n$)

$$\begin{aligned} I_n(t,s) &= n! \tau_n^{(\gamma)}(t,s) \\ &= \int_{S^1} \frac{u^\gamma du}{2\pi i u} e^{\sum_1^\infty (t_j u^j - s_j u^{-j})} u^{n-1} u^{-n+1} \\ &\quad \left(\int_{(S^1)^{n-1}} \Delta_{n-1}(z) \overline{\Delta_{n-1}(z)} \prod_{k=1}^{n-1} \left(1 - \frac{z_k}{u}\right)\left(1 - \frac{u}{z_k}\right) e^{\sum_1^\infty (t_j z_k^j - s_j z_k^{-j})} \frac{z_k^\gamma dz_k}{2\pi i z_k} \right) \\ &= \int_{S^1} \frac{u^\gamma du}{2\pi i u} e^{\sum_1^\infty (t_j u^j - s_j u^{-j})} (u u^{-1})^{n-1} e^{-\sum_1^\infty (u^{-j}/j\, \partial/\partial t_j - u^j/j\, \partial/\partial s_j)} \\ &\quad \left(\int_{(S^1)^{n-1}} \Delta_{n-1}(z) \overline{\Delta_{n-1}(z)} \prod_{k=1}^{n-1} e^{\sum_1^\infty (t_j z_k^j - s_j z_k^{-j})} \frac{z_k^\gamma dz_k}{2\pi i z_k} \right) \\ &= (Y^\gamma I)_n \,, \end{aligned}$$

yielding the fixed point statement. Finally, to see the $s\ell(2,Z)$ Virasoro statement, observe that from the other main facts, we have

$$
\begin{aligned}
0 &= ([\mathbb{V}_k^\gamma, (Y^\gamma)^n] I)_n \\
&= (\mathbb{V}_k^\gamma (Y^\gamma)^n I - (Y^\gamma)^n \mathbb{V}_k^\gamma I)_n \\
&= (\mathbb{V}_k^\gamma I - (Y^\gamma)^n \mathbb{V}_k^\gamma I)_n \\
&= \mathbb{V}_{k,n}^\gamma I_n - \int_{S^1} u^\gamma \frac{du}{2\pi i u} e^{\sum_1^\infty (t_j u^j - s_j u^{-j})} e^{-\sum_1^\infty (u^{-j}/j\,\partial/\partial t_j - u^j/j\,\partial/\partial s_j)} \\
&\qquad \cdots \int_{S^1} u^\gamma \frac{du}{2\pi i u} e^{\sum_1^\infty (t_j u^j - s_j u^{-j})} e^{-\sum_1^\infty (u^{-j}/j\,\partial/\partial t_j - u^j/j\,\partial/\partial s_j)} V_k^\gamma I_0 \ ,
\end{aligned}
$$

upon using the backward shift (see (115)) present in Y^γ. Note $I_0 = 1$, and so it follows from the explicit Virasoro formulas given below, that $V_{k,0}^\gamma 1 = 0$ precisely if $k = -1,\, 0,\, 1$ and θ is as specified in (122), yielding the Virasoro constraints (122). We now make explicit the Virasoro relations.

Explicit Virasoro formulas:

$$
\begin{cases}
\mathbb{J}_k^{(1)} = (J_k^{(1)} + n\delta_{0k})_{n\geqslant 0} \ , \\
\mathbb{J}_k^{(2)} = \frac{1}{2}(J_k^{(2)} + (2n+k+1)J_k^{(1)} + n(n+1)\delta_{0k})_{n\geqslant 0} \ .
\end{cases}
\tag{123}
$$

$$
\left.\begin{cases}
\mathbb{J}_k^{(1)} = \dfrac{\partial}{\partial t_k} + (-k)t_{-k} \\
\mathbb{J}_k^{(2)} = 2\sum i t_i \dfrac{\partial}{\partial t_{i+k}}
\end{cases}\right. , \quad \text{if} \quad k = 0, \pm 1 \ .
\tag{124}
$$

Virasoro relations:

$$
\begin{cases}
\mathbb{V}_{-1,n}^\gamma \tau_n^{(\gamma)} = \left(\frac{1}{2}J_{-1}^{(2)}(t) - \frac{1}{2}J_1^{(2)}(s) + n\left(t_1 + \frac{\partial}{\partial s_1}\right) - \gamma\frac{\partial}{\partial s_1}\right)\tau_n^{(\gamma)} = 0 \ , \\
\mathbb{V}_{0,n}^\gamma \tau_n^{(\gamma)} = \left(\frac{1}{2}J_0^{(2)}(t) - \frac{1}{2}J_0^{(2)}(s) + n\gamma\right)\tau_n^{(\gamma)} = 0 \ , \\
\mathbb{V}_{1,n}^\gamma \tau_n^{(\gamma)} = \left(-\frac{1}{2}J_{-1}^{(2)}(s) + \frac{1}{2}J_1^{(2)}(t) + n\left(s_1 + \frac{\partial}{\partial t_1}\right) + \gamma\frac{\partial}{\partial t_1}\right)\tau_n^{(\gamma)} = 0 \ .
\end{cases}
\tag{125}
$$

We now derive the first difference relation (the second is similar) in Case 2 of Theorem 2.1. Setting for arbitrary a, b, c, t, and s,

$$
\begin{aligned}
\alpha_i(t) &:= a(i+1)t_{i+1} + bit_i + c(i-1)t_{i-1} + c(n+\gamma)\delta_{i1} \ , \\
\beta_i(s) &:= a(i-1)s_{i-1} + bis_i + c(i+1)s_{i+1} - a(n-\gamma)\delta_{i1} \ ,
\end{aligned}
\tag{126}
$$

$$
\mathcal{L}_1^{(n)} := \sum_{i\geqslant 1} \alpha_i(t) L_1^i \quad \text{and} \quad \mathcal{L}_2^{(n)} := -\sum_{i\geqslant 1} \beta_i(t) L_2^i \ ,
\tag{127}
$$

and remembering

$$
(x_n, y_n) = (-1)^n \left(\frac{\tau_n^+}{\tau_n^0}, \frac{\tau_n^-}{\tau_n^0}\right) , \quad v_n = 1 - x_n y_n \ ,
$$

compute, using the Virasoro relations and the Toeplitz flow:

$$
\begin{aligned}
0 &= \frac{x_n y_n}{v_n}\Bigg\{\frac{1}{\tau_n^+}(a\mathbb{V}^+_{-1,n}+b\mathbb{V}^+_{0,n}+c\mathbb{V}^+_{1,n})\tau_n^+ + \frac{1}{\tau_n^-}(a\mathbb{V}^-_{-1,n}+b\mathbb{V}^-_{0,n}+c\mathbb{V}^-_{1,n})\tau_n^- \\
&\qquad\qquad - \frac{2}{\tau_n}(a\mathbb{V}_{-1,n}+b\mathbb{V}_{0,n}+c\mathbb{V}_{1,n})\tau_n\Bigg\} \\
&= \frac{x_n y_n}{v_n}\left(\sum_{i\geqslant 1}\left(a_i(t)\frac{\partial}{\partial t_i}-\beta_i(s)\frac{\partial}{\partial s_i}\right)\log x_n y_n \right. \\
&\qquad\qquad \left. +\left(c\frac{\partial}{\partial t_1}-a\frac{\partial}{\partial s_1}\right)\log\frac{x_n}{y_n}\right) \\
&= \frac{x_n y_n}{v_n}\left(\sum_{i\geqslant 1}\left(a_i\frac{\partial}{\partial t_i}-\beta_i\frac{\partial}{\partial s_i}\right)(\log x_n+\log y_n)\right. \\
&\qquad\qquad \left. +\left(c\frac{\partial}{\partial t_1}-a\frac{\partial}{\partial s_1}\right)(\log x_n-\log y_n)\right) \\
&= (\mathcal{L}_1^{(n)}-\mathcal{L}_2^{(n)}+aL_2-cL_1)_{nn} \\
&\qquad\qquad -(\mathcal{L}_1^{(n)}-\mathcal{L}_2^{(n)}-aL_2+cL_1)_{n+1,n+1}\ , \\
&= -\partial_n(\mathcal{L}_1^{(n)}-\mathcal{L}_2^{(n)})_{n,n}+(aL_2-cL_1)_{nn}\ .
\end{aligned}
\tag{128}
$$

We then find our result by specializing the above identity to the precise locus $\mathcal{L}$ in t, s space corresponding to the measure $\rho(z)$ of Case 2 of Theorem 2.1:

$$
\mathcal{L} := \left\{
\begin{array}{l}
it_i = it_i^{(0)} := \begin{cases} u_i-(\gamma_1' d_1^i+\gamma_2' d_2^i), & \text{for } 1\leqslant i\leqslant N_1 \\ -(\gamma_1' d_1^i+\gamma_2' d_2^i), & \text{for } N_1+1\leqslant i<\infty \end{cases} \\[2ex]
is_i = is_i^{(0)} := \begin{cases} -u_{-i}+(\gamma_1'' d_1^{-i}+\gamma_2'' d_2^{-i}), & \text{for } 1\leqslant i\leqslant N_2 \\ (\gamma_1'' d_1^{-i}+\gamma_2'' d_2^{-i}), & \text{for } N_2+1\leqslant i<\infty \end{cases}
\end{array}
\right\}
\tag{129}
$$

with

$$
(a,b,c) = \frac{1}{\sqrt{d_1 d_2}}(1,-d_1-d_2,d_1d_2)\ ,
\tag{130}
$$

and so in (129) all $\alpha_i(t)=0$ for all $i\geqslant N_1+2$ and all $\beta_i(t)=0$ for $i\geqslant N_2+2$.

2.5 Singularity confinement of recursion relations

Since for the combinatorial examples the unitary integral $I_n^{(0)}(t)$ satisfies Painlevé differential equations in t, it is natural to expect they satisfy a discrete version of the Painlevé property regarding the development of poles. For instance, algebraically integrable systems (a.c.i.) $\dot z = f(z)$, $z\in\mathbb{C}^n$, admit Laurent solutions depending on the maximal number, $n-1$, of free parameters [see [14]). An analogous property for rational recursion relations

$$
z_n = F(z_{n-1},\dots,z_{n-\delta}),\quad z_n\in\mathbb{C}^k\ ,
$$

would be that there exists solutions of the recursion relation $\{z_i(\lambda)\}$ which are "formal Laurent" solutions in λ developing a pole which disappears after a finite number of steps, and such that these "formal Laurent" solutions depend on the maximal number of free parameters $\delta \times k$ (counting λ) and moreover the coefficients of the expansions depend rationally on these free parameters. We shall give results for Case 2 of Theorem 2.1 and 2.2, where $N_1 = N_2 = N$. The results in this section, Theorem 2.3–2.6, are due to Adler–van Moerbeke–Vanhaecke and can be found with proofs in [13].

Self-dual case:

$$\rho(z) = e^{\sum_1^N u_i(z^i + z^{-i})/i} . \tag{131}$$

Theorem 2.3 (Singularity confinement: self-dual case). *For any $n \in \mathbb{Z}$,*[8] *the difference equations $\Gamma_k(x) = 0$, ($k \in \mathbb{Z}$) admit two formal Laurent solution $x = \big(x_k(\lambda)\big)_{k\in\mathbb{Z}}$ in a parameter λ, having a (simple) pole at $k = n$ only and $\lambda = 0$. These solutions depend on $2N$ non-zero free parameters*

$$\alpha = (a_{n-2N}, \dots, a_{n-2}) \quad \textit{and} \quad \lambda .$$

Explicitly, these series with coefficients rational in α are given by ($\varepsilon = \pm 1$):

$$\begin{aligned} x_k(\lambda) &= \sum_{i=0}^{\infty} x_k^{(i)}(\alpha)\lambda^i , \quad k < n - 2N , \\ x_k(\lambda) &= \alpha_k , \quad n - 2N \leqslant k \leqslant n - 2 , \\ x_{n-1}(\lambda) &= \varepsilon + \lambda , \\ x_n(\lambda) &= \frac{1}{\lambda} \sum_{i=0}^{\infty} x_n^{(i)}(\alpha)\lambda^i , \\ x_{n+1}(\lambda) &= -\varepsilon + \sum_{i=1}^{\infty} x_{n+1}^{(i)}(\alpha)\lambda^i , \\ x_k(\lambda) &= \sum_{i=0}^{\infty} x_k^{(i)}(\alpha)\lambda^i , \quad n + 1 < k . \end{aligned}$$

General case:

$$\rho(z) = e^{\sum_1^N (u_i z^i + u_{-i} z^{-i})/i} .$$

Theorem 2.4 (Singularity confinement: general case). *For any $n \in \mathbb{Z}$, the difference equations $\Gamma_k(x,y) = \tilde{\Gamma}_k(x,y) = 0$, ($k \in \mathbb{Z}$) admit a formal Laurent solution $x = \big(x_k(\lambda)\big)_{k\in\mathbb{Z}}$ and $y = \big(y_k(\lambda)\big)_{k\in\mathbb{Z}}$ in a parameter λ, having*

[8] We consider the obvious *bi*-infinite extension of $L_i(x,y)$ (89) which through (98) defines a *bi*-infinite extension of $\Gamma_k(x,y)$, $\widetilde{\Gamma}_k(x,y)$.

a (simple) pole at $k = n$ and $\lambda = 0$, and no other singularities. These solutions depend on $4N$ non-zero free parameters

$$\alpha_{n-2N}, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-2N}, \ldots, \beta_{n-2} \quad \textit{and} \quad \lambda \ .$$

Setting $z_n := (x_n, y_n)$ and $\gamma_i := (\alpha_i, \beta_i)$, and $\gamma := (\gamma_{n-2N}, \ldots, \gamma_{n-2}, \alpha_{n-1})$, the explicit series with coefficients rational in γ read as follows:

$$\begin{aligned} z_k(\lambda) &= \sum_{i=0}^{\infty} z_k^{(i)}(\gamma)\lambda^i \ , \quad k < n - 2N \ , \\ z_k(\lambda) &= \gamma_k \ , \quad n - 2N \leqslant k \leqslant n - 2 \ , \\ x_{n-1}(\lambda) &= \alpha_{n-1} \ , \\ y_{n-1}(\lambda) &= \frac{1}{\alpha_{n-1}} + \lambda \ , \\ z_n(\lambda) &= \frac{1}{\lambda} \sum_{i=0}^{\infty} z_n^{(i)}(\gamma)\lambda^i \ , \\ z_k(\lambda, \gamma) &= \sum_{i=0}^{\infty} z_k^{(i)}(\gamma)\lambda^i \ , \quad n < k \ . \end{aligned}$$

Singularity confinement is consequence of:

(1) Painlevé property of a.c.i. Toeplitz lattice.
(2) Rational difference relations as a whole define an invariant manifold of the Toeplitz lattice.
(3) Formal Laurent solutions of Toeplitz lattice with maximal parameters restrict to the above invariant manifold, restricting the parameters.
(4) Reparametrizing the "restricted" Laurent solutions by $t \to \lambda$ and "restricted parameters" $\to \gamma$ yields the confinement result.

We discuss (1) and (2). Indeed, consider the Toeplitz lattice with the Hamiltonian $H = H_1^{(1)} - H_1^{(2)}$, yielding the flow of (114):

General case:

$$\left.\begin{aligned} \frac{\partial x_k}{\partial t} &= (1 - x_k y_k)(x_{k+1} - x_{k-1}) \ , \\ \frac{\partial y_k}{\partial t} &= (1 - x_k y_k)(y_{k+1} - x_{k-1}) \ , \end{aligned}\right. \quad k \in \mathbb{Z} \ .$$

Self-dual case:

$$\frac{\partial x_k}{\partial t} = (1 - x_k^2)(x_{k+1} - x_{k-1}) \ , \quad k \in \mathbb{Z} \ .$$

Then we have

Maximal formal Laurent solutions:

Theorem 2.5. *For arbitrary but fixed n, the first Toeplitz lattice vector field* (114) *admits the following formal Laurent solutions,*

$$x_n(t) = \frac{1}{(a_{n-1}-a_{n+1})t}\big(a_{n-1}a_{n+1}(1+at)+O(t^2)\big) ,$$
$$y_n(t) = \frac{1}{(a_{n-1}-a_{n+1})t}\left(-1+\left(a+\frac{a_{n+1}a_+ - a_{n-1}a_-}{a_{n+1}-a_{n-1}}\right)t+O(t^2)\right) ,$$
$$x_{n\pm1}(t) = a_{n\pm1}+a_{n\pm1}a_\pm t+O(t^2) ,$$
$$y_{n\pm1}(t) = \frac{1}{a_{n\pm1}}-\frac{a_\pm t}{a_{n\mp1}}+O(t^2)$$

whereas for all remaining k such that $|k-n| \geqslant 2$,

$$x_k(t) = a_k+(1-a_kb_k)(a_{k+1}-a_{k-1})t+O(t^2) ,$$
$$y_k(t) = b_k+(1-a_kb_k)(b_{k+1}-b_{k-1})t+O(t^2) ,$$

where a, $a_\pm$, $a_{n\pm1}$ and all a_i, b_i, with $i \in \mathbb{Z}\backslash\{n-1,n,n+1\}$ and with $b_{n\pm1} = 1/a_{n\pm1}$, are arbitrary free parameters, and with $(a_{n-1}-a_{n+1})a_{n-1}a_{n+1} \neq 0$. In the self-dual case it admits the following two formal Laurent solutions, parametrized by $\varepsilon = \pm 1$,

$$x_n(t) = -\frac{\varepsilon}{2t}\big(1+(a_+-a_-)t+O(t^2)\big) ,$$
$$x_{n\pm1}(t) = \varepsilon\big(\mp 1+4a_\pm t+O(t^2)\big) ,$$
$$x_k(t) = \varepsilon\big(a_k+(1-a_k^2)(a_{k+1}-a_{k-1})t+O(t^2)\big) , \quad |k-n| \geqslant 2 ,$$

where a_+, a_- and all a_i, with $i \in \mathbb{Z}\backslash\{n-1,n,n+1\}$ are arbitrary free parameter and $a_{n-1} = -a_{n+1} = 1$.

Together with time t these parameters are in bijection with the phase space variables; we can put for the general Toeplitz lattice for example $z_k \leftrightarrow (a_k,b_k)$ for $|k-n| \geqslant 1$ and $x_{n\pm1} \leftrightarrow a_{n\pm1}$ and $y_{n\pm1}$, x_n, $y_n \leftrightarrow a_\pm$, a, t. Consider the locus $\hat{\mathcal{L}}$ defined by the difference relations (98) of Case 3 of Theorem 2.1, namely:

General case:

$$\hat{\mathcal{L}} = \{(x,y) \mid \Gamma_n(x,y,u) = 0 , \widetilde{\Gamma}_n(x,y,u) = 0 , \forall n\} .$$

Self-dual case:

$$\hat{\mathcal{L}} = \{x \mid \Gamma_n(x,y,u) = 0, \ \forall n\} ,$$

where we now explicitly exhibit the dependence of Γ_n, $\widetilde{\Gamma}_n$ on the coefficients of the measure, namely u. The point of the following theorem is that $\hat{\mathcal{L}}$ is an invariant manifold for the flow generated by $H = H_1^{(1)} - H_1^{(2)}$, upon our imposing the following u dependence on t ($v_n = 1 - x_ny_n$).

Theorem 2.6. *Upon setting $\partial u_{\pm i}/\partial t = \delta_{1i}$, the recursion relations satisfy the following differential equations*

$$\dot{\Gamma}_k = v_k(\Gamma_{k+1} - \Gamma_{k-1}) + (x_{k+1} - x_{k-1})(x_k \tilde{\Gamma}_k - y_k \Gamma_k) \ ,$$
$$\dot{\tilde{\Gamma}}_k = v_k(\tilde{\Gamma}_{k+1} - \tilde{\Gamma}_{k-1}) - (y_{k+1} - y_{k-1})(x_k \tilde{\Gamma}_k - y_k \Gamma_k) \ ,$$

which specialize in the self-dual case to

$$\dot{\Gamma}_k = v_k(\Gamma_{k+1} - \Gamma_{k-1}) \ .$$

Sketch of proof:

The proof is based on the crucial identities:

$$\Gamma_n = \mathcal{V}_0 x_n + n x_n \quad \text{and} \quad \tilde{\Gamma}_n = -\mathcal{V}_0 y_n + n y_n \ ,$$

where

$$\mathcal{V}_0 := \sum_{i \geqslant 1} \left(u_i \frac{\partial}{\partial t_i} + u_{-i} \frac{\partial}{\partial s_i} \right) ,$$

and

$$\frac{\partial x_n}{\partial \begin{Bmatrix} t_1 \\ s_1 \end{Bmatrix}} = v_n x_{n \pm 1} \ , \quad \frac{\partial y_n}{\partial \begin{Bmatrix} t_1 \\ s_1 \end{Bmatrix}} = -v_n y_{\mp 1} \ , \quad \frac{\partial u_k}{\partial \begin{Bmatrix} t_1 \\ s_1 \end{Bmatrix}} = \pm \delta_{k, \pm 1} \ ,$$

which implies

$$\frac{\partial \Gamma_n}{\partial \begin{Bmatrix} t_1 \\ s_1 \end{Bmatrix}} = v_n \Gamma_{n \pm 1} + x_{n \pm 1}(x_n \tilde{\Gamma}_n - y_n \Gamma_n) \ ,$$
$$\frac{\partial \tilde{\Gamma}_n}{\partial \begin{Bmatrix} t_1 \\ s_1 \end{Bmatrix}} = -v_n \tilde{\Gamma}_{n \mp 1} + y_{n \mp 1}(x_n \tilde{\Gamma}_n - y_n \Gamma_n) \ ,$$

which, upon using $\partial/\partial t = \partial/\partial t_1 - \partial/\partial s_1$, yields the theorem.

3 Coupled Random Matrices and the 2–Toda Lattice

3.1 Main results for coupled random matrices

The study of coupled random matrices will lead us to the 2–Toda lattice and bi-orthogonal polynomials, which are essentially 2 of the 4 wave functions for the 2–Toda lattice. This problem will lead to many techniques which will come up again, as well as a PDE for the basic probability in coupled random matrices.

Let $M_1, M_2 \in \mathcal{H}_n$, Hermitian $n \times n$ matrices and consider the probability ensemble of

Coupled random matrices:

$$P\big((M_1, M_2) \subset S\big) = \frac{\int_S dM_1\, dM_2 e^{-1/2\,\mathrm{Tr}(M_1^2+M_2^2-2cM_1M_2)}}{\int_{\mathcal{H}_n \times \mathcal{H}_n} dM_1\, dM_2 e^{-1/2\,\mathrm{Tr}(M_1^2+M_2^2-2cM_1M_2)}} \tag{132}$$

with

$$dM_1 = \Delta_n^2(x) \prod_1^n dx_i\, dU_1, \quad dM_2 = \Delta_n^2(y) \prod_1^n dy_i\, dU_2 \,.$$

Given $E = E_1 \times E_2 = \bigcup_1^r [a_{2i-1}, a_{2i}] \times \bigcup_1^s [b_{2i-1}, b_{2i}]$, define the boundary operators:

$$\begin{gathered}\mathcal{A}_1 = \frac{1}{1-c^2}\left(\sum_1^{2r} \frac{\partial}{\partial a_j} + c \sum_1^{2s} \frac{\partial}{\partial b_j}\right), \quad \mathcal{A}_2 = \sum_1^{2r} a_j \frac{\partial}{\partial a_j} - c\frac{\partial}{\partial c}\,,\\ \mathcal{B}_1 = \mathcal{A}_1\Big|_{a \leftrightarrow b}, \quad \mathcal{B}_2 = \mathcal{A}_2\Big|_{a \leftrightarrow b},\end{gathered} \tag{133}$$

which form a Lie algebra:

$$\begin{aligned}&[\mathcal{A}_1, \mathcal{B}_1] = 0\,, \quad [\mathcal{A}_1, \mathcal{A}_2] = \frac{1+c^2}{1-c^2}\mathcal{A}_1\,, \quad [\mathcal{A}_2, \mathcal{B}_1] = -\frac{2c}{1-c^2}\mathcal{A}_1\,,\\ &[\mathcal{A}_2, \mathcal{B}_2] = 0\,, \quad [\mathcal{B}_1, \mathcal{B}_2] = \frac{1+c^2}{1-c^2}\mathcal{B}_1\,, \quad [\mathcal{B}_2, \mathcal{A}_1] = -\frac{2c}{1-c^2}\mathcal{B}_1\,,\end{aligned} \tag{134}$$

We can now state the main theorem of Section 3:

Theorem 3.1 (Adler–van Moerbeke [4]). *The statistics*

$$\begin{aligned}F_n := \frac{1}{n}\log P_n(E) := \frac{1}{n}\log P(&\textit{all } (M_1\textit{–eigenvalues}) \in E_1,\\ &\textit{all } (M_2\textit{–eigenvalues}) \in E_2)\end{aligned} \tag{135}$$

satisfies the third order nonlinear PDE:

$$\mathcal{A}_1\left(\frac{\mathcal{B}_2\mathcal{A}_1 F_n}{\mathcal{B}_1\mathcal{A}_1 F_n + c/(1-c^2)}\right) = \mathcal{B}_1\left(\frac{\mathcal{A}_2\mathcal{B}_1 F_n}{\mathcal{A}_1\mathcal{B}_1 F_n + c/(1-c^2)}\right) = 0\,. \tag{136}$$

In particular for $E = (-\infty, a] \times (-\infty, b]$, setting: $x := -a + cb$, $y := -ac + b$, $\mathcal{A}_1 \to -\partial/\partial x$, $\mathcal{B}_1 \to \partial/\partial y$, (136) *becomes*

$$\begin{aligned}&\frac{\partial}{\partial x}\left(\frac{(c^2-1)^2\partial^2 F_n/\partial x \partial c + 2cx - (1+c^2)y}{(c^2-1)\partial^2 F_n/\partial x\partial y + c}\right)\\ &\qquad = \frac{\partial}{\partial y}\left(\frac{(c^2-1)^2\partial^2 F_n/\partial y \partial c + 2cy - (1+c^2)x}{(c^2-1)\partial^2 F_n/\partial y\partial x + c}\right)\,.\end{aligned} \tag{137}$$

3.2 Link with the 2–Toda hierarchy

In this section we deform the coupled random matrix problem in a natural way to introduce the 2–Toda hierarchy into the problem. We first need the celebrated Harish–Chandra–Itzkson–Zuber formula [23]:

HCIZ: $x = \operatorname{diag}(x_1, \ldots, x_n)$, $y = \operatorname{diag}(y_1, \ldots, y_n)$

$$\int_{U(n)} dU\, e^{c \operatorname{Tr} xUyU^{-1}} = \left(\frac{2\pi}{c}\right)^{n(n-1)/2} \frac{\det(e^{cx_i y_j})_{1\leqslant i,j\leqslant n}}{n!\Delta_n(x)\Delta_n(y)} \,. \tag{138}$$

Compute, using HCIZ:

$$\iint_{\mathcal{H}_n(E_1)\times\mathcal{H}_n(E_2)} dM_1\, dM_2 e^{c \operatorname{Tr} M_1 M_2} e^{\operatorname{Tr}(\sum_1^\infty (t_i M_1^i - s_i M_2^i))} \quad (t,\, s \text{ deformation})$$

$$\begin{aligned}
&= \int_{E_1^n E_2^n} \left\{ \Delta_n^2(x)\Delta_n^2(y) \prod_{k=1}^n e^{\sum_1^\infty (t_i x_k^i - s_i y_k^i)} dx_k\, dy_k \right\} \\
&\qquad \times \iint_{U(n)\times U(n)} e^{c \operatorname{Tr} U_1 \times U_1^{-1} U_2 y U_2^{-1}}\, dU_1\, dU_2 \\
&= \int_{E_1^n E_2^n} \left\{ \Delta_n^2(x)\Delta_n^2(y) \prod_{k=1}^n e^{\sum_1^\infty (t_i x_k^i - s_i y_k^i)} dx_k\, dy_k \right\} \\
&\qquad \times \int_{U(n)} dU_1 \int_{U(n)} e^{c \operatorname{Tr} \times U_1^{-1} U_2 y U_2^{-1} U_1} dU_2 \\
&= \int_{E_1^n} \int_{E_2^n} \left\{ \Delta_n^2(x)\Delta_n^2(y) \prod_{k=1}^n e^{\sum_1^\infty (t_i x_k^i - s_i y_k^i)} dx_k\, dy_k \right\} \\
&\qquad \times \int_{U(n)} dU_1 \int_{U(n)} e^{c \operatorname{Tr} \times U y U^{-1}} dU \quad \begin{pmatrix} U = U_1^{-1} U_2 \\ dU = dU_2 \end{pmatrix} \\
&= c_n \int_{E_1^n} \int_{E_2^n} \Delta_n(x)\Delta_n(y) (\det e^{cx_i y_j})_{\substack{1\leqslant i,\\ j\leqslant n}} \prod_1^n e^{\sum_1^\infty (t_i x_k^i - s_i y_k^i)} dx_k\, dy_k \\
&:= c_n \int_{E_1^n} \int_{E_2^n} \Delta_n(x)\Delta_n(y) \left(\sum_{\sigma\in S_n} (-1)^\sigma \prod_1^n e^{cx_i y_{\sigma(i)}} \right) \times \prod_1^n d\mu(x_k)\, d\psi(y_k) \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{setting: } y_{\sigma(i)} \to y_i,\ E_2^n \to E_2^n) \\
&= n! c_n \int_{E_1^n} \int_{E_2^n} \Delta_n(x)\Delta_n(y) \prod_1^n e^{cx_i y_i} \prod_1^n d\mu(x_k) d\psi(y_k) \\
&= n! c_n \int_{E^n} \Delta_n(x)\Delta_n(y) \prod_{k=1}^n e^{\sum_1^\infty (t_i x_k^i - s_i y_k^i) + c x_k y_k} dx_k\, dy_k,\ \ E = E_1 \times E_2 \,,
\end{aligned}$$

where we have used in the above that Haar measure is translation invariant.

We now make a further c-deformation of this matrix integral.

Define τ-function:

$$\tau_n(t,s,C,E)$$
$$:= \frac{1}{n!}\int_{E^n} \Delta_n(x)\Delta_n(y)\prod_{k=1}^n e^{\sum_1^\infty (t_i x_k^i - s_i y_k^i)+\sum_{i,j\geqslant 1} c_{ij} x_k^i y_k^j} dx_k\, dy_k\ ,$$
$$(\tau_0 = 1) \text{ which is } \textit{not} \text{ a matrix integral!} \quad (139)$$

Thus
$$\frac{\tau_n(t,s,C,E)}{\tau_n(t,s,C,\mathbb{R})} \text{ is } t,s,C \text{ deformation of } P_n(E)\ .$$

It is quite crucial to recast the τ-function using the de Bruijn trick:

Moment matrix form of τ-function:

$$\begin{aligned}
&\tau_n(t,s,c,E)\\
&= \frac{1}{n!}\int_{E^n} \Delta_n(x)\Delta_n(y)\prod_{k=1}^n e^{\sum_1^\infty (t_i x_k^i - s_i y_k^i)+\sum_{i,j\geqslant 1} c_{ij} x_k^i y_k^j} dx_k\, dy_k\\
&= \frac{1}{n!}\int_{E^n} \det\big(f_i(x_j)\big)_{\substack{1\leqslant i,\\ j\leqslant n}} \det\big(g_i(y_j)\big)_{\substack{1\leqslant i,\\ j\leqslant n}} \prod_1^n d\psi(x_k,y_k)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad (f_i(x) = g_i(x) = x^{i-1})\\
&= \frac{1}{n!}\sum_{\sigma,\mu\in S_n}(-1)^{\sigma+\mu}\int_{E^n}\prod_1^n f_{\sigma(i)}(x_i)g_i(y_{\mu(i)})d\psi(x_{\mu(i)},y_{\mu(i)})\\
&= \frac{1}{n!}\sum_{\sigma,\mu\in S_n}(-1)^{\sigma+\mu}\int_{E^n}\prod_1^n f_{\sigma\circ\mu(i)}(x_{\mu(i)})g_i(y_{\mu(i)})d\psi(x_{\mu(i)},y_{\mu(i)})\\
&= \frac{1}{n!}\sum_{\mu\in S_n}\sum_{\sigma'\in S_n}(-1)^{\sigma'}\prod_1^n\int_E f_{\sigma'(i)}(x)g_i(y)d\psi(x,y)\\
&= \frac{1}{n!}\sum_{\mu\in S_n}\det\left(\int_E f_i(x)g_j(y)d\psi(x,y)\right)_{\substack{1\leqslant i,\\ j\leqslant n}}\\
&= \det(\mu_{ij})_{\substack{0\leqslant i,\\ j\leqslant n-1}}\ ,
\end{aligned} \tag{140}$$

with

$$\mu_{ij}(t,s,C,E) = \int_E x^i y^j e^{\sum_{k=1}^\infty (t_k x^k - s_k y_k^k)+\sum_{\alpha,\beta\geqslant 1} c_{\alpha\beta}x^\alpha y^\beta} dx\, dy := \langle x^i, y^j\rangle\ . \tag{141}$$

Thus we have shown:

$$\tau_n(t,s,C,E) = \det m_n\ , \quad m_n = (\mu_{ij})_{\substack{0\leqslant i,\\ j\leqslant n-1}}\ . \tag{142}$$

This immediately leads to the

2–*Toda differential equations – Moment form*:

$$\frac{\partial \mu_{ij}}{\partial t_k} = \mu_{i+k,j} \ , \quad \frac{\partial \mu_{ij}}{\partial s_k} = -\mu_{i,j+k} \ , \tag{143}$$

which we reformulate in terms of the moment matrix m_∞

$$\frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty \ , \quad \frac{\partial m_\infty}{\partial s_k} = -m_\infty (\Lambda^T)^k \ , \tag{144}$$

or equivalently

$$m_\infty(t,s) = e^{\sum_1^\infty t_k \Lambda^k} m_\infty(0,0) e^{-\sum_1^\infty s_k (\Lambda^T)^k} \ , \tag{145}$$

with the semi-infinite shift matrix

$$\Lambda := \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} . \tag{146}$$

Thus

$$\tau_n(t,s,C,E) = \det m_n(t,s) = \det \big((E_n(t) m_\infty(0,0) E_n^T(-s) \big) \ , \tag{147}$$

with

$$E_n(t) = (I + s_1(t)\Lambda + s_2(t)\Lambda^2 + \cdots)_{n \times \infty}$$

and

$$e^{\sum_1^\infty t_i z^i} := \sum_0^\infty s_i(t) z^i \ , \tag{148}$$

with $s_i(t)$ the elementary Schur polynomials.

3.3 $L - U$ decomposition of the moment matrix, bi-orthogonal polynomials and 2–Toda wave operators

The $L-U$ decomposition of m_∞ is equivalent to bi-orthogonal polynomials; indeed, consider the $L-U$ decomposition of m_∞, as follows (see [9]):

$$m_\infty = LhU := S^{-1}(m_\infty) \big(h(m_\infty) \big(S^{-1}(m_\infty^T) \big)^T \big) := S_1^{-1} S_2 \tag{149}$$

where we define the string orthogonal polynomials

$$\left(p_n^{(1)}(y)\right)_{n\geqslant 0} := S(m_\infty)\begin{pmatrix}1\\ y\\ y^2\\ \vdots\end{pmatrix} := \left(\frac{\det\left(\begin{array}{ccc|c} & m_n & & \begin{matrix}1\\ y\\ \vdots\end{matrix}\\ \hline \mu_{n0} & \dots & \mu_{n,n-1} & y^n\end{array}\right)}{\det m_n}\right)_{n\geqslant 0} \tag{150}$$

and

$$\left(p_n^{(2)}(y)\right)_{n\geqslant 0} := S(m_\infty^T)\begin{pmatrix}1\\ y\\ y^2\\ \vdots\end{pmatrix} . \tag{151}$$

Setting

$$\langle\ ,\ \rangle\colon \langle x^i, y^j\rangle := \mu_{ij}(t,s,C) , \tag{152}$$

conclude (as a tautology) the defining relations of the monic bi-orthogonal polynomials, namely

$$\langle p_i^{(1)}, p_j^{(2)}\rangle = h_i\delta_{ij} \Longleftrightarrow S(m_\infty)m_\infty\left(S(m_\infty^T)\right)^T = h(m_\infty) , \tag{153}$$

with the first identity a consequence of (150) and (151), which also implies

$$h(m_\infty) := \operatorname{diag}\left(h_0, h_1, \dots, h_i = \frac{\det m_{i+1}}{\det m_i}, \dots\right) , \tag{154}$$

and by (149)

$$S_1 = S(m_\infty) , \quad S_2 = h(m_\infty)\left(S^{-1}(m_\infty^T)\right)^T . \tag{155}$$

We now define the 2–Toda operators:

$$L_1 = S_1 \Lambda S_1^{-1} , \quad L_2 = S_2\Lambda^T S_2^{-1} . \tag{156}$$

Since $m_\infty = S_1^{-1}S_2$, with

$$S_1 \in I + g_- , \quad S_2 \in g_+ ,$$

then

$$\dot{S}_1 S_1^{-1} \in g_- , \quad \dot{S}_2 S_2^{-1} \in g_+ ,$$

with g_- strictly lower triangular matrices and g_+ upper triangular matrices, including the diagonal, and $g_- + g_+ = g :=$ all semi-infinite matrices. Compute

$$S_1\dot{m}_\infty S_2^{-1} = S_1(S_1^{-1}S_2)^{\cdot} S_2^{-1} = -\dot{S}_1S_1^{-1} + \dot{S}_2 S_2^{-1} \in g_- + g_+ , \tag{157}$$

where we have used $(S_1^{-1})^{\dot{}} = -S_1^{-1}\dot{S}_1 S_1^{-1}$. On the other hand, one computes, using (144), for $\partial/\partial t_n$ or $\partial/\partial s_n$ separately, that:

$$
\begin{aligned}
S_1 \frac{\partial m_\infty}{\partial t_n} S_2^{-1} &= S_1 \Lambda^n m_\infty S_2^{-1} = S_1 \Lambda^n S_1^{-1} S_2 S_2^{-1} = S_1 \Lambda^n S_1^{-1} \\
&= L_1^n := (L_1^n)_- + (L_1^n)_+ \in g_- + g_+ \ ,
\end{aligned}
\tag{158}
$$

and

$$
\begin{aligned}
S_1 \frac{\partial m_\infty}{\partial s_n} S_2^{-1} &= -S_1 m_\infty (\Lambda^T)^n S_2^{-1} = -S_1 S_1^{-1} S_2 (\Lambda^T)^n S_2^{-1} \\
&= -S_2 (\Lambda^T)^n S_2^{-1} \\
&= -L_2^n := -(L_2^n)_- - (L_2^n)_+ \in g_- + g_+ \ ,
\end{aligned}
\tag{159}
$$

and so (157), (158), and (159) yield the differential equations

$$
\begin{aligned}
\frac{\partial S_1}{\partial t_n} S_1^{-1} &= -(L_1^n)_- \ , \quad & \frac{\partial S_2}{\partial t_n} S_2^{-1} &= \ \ (L_1^n)_+ \ , \\
\frac{\partial S_1}{\partial s_n} S_1^{-1} &= \ \ (L_2^n)_- \ , \quad & \frac{\partial S_2}{\partial s_n} S_2^{-1} &= -(L_2^n)_- \ .
\end{aligned}
\tag{160}
$$

Setting $\chi(x) = (1, z, z^2, \dots)^T$ we now connect the bi-orthogonal polynomials with the 2–Toda wave functions:

2–*Toda wave functions*:

$$
\begin{cases}
\Psi_1(z) & := e^{\sum_1^\infty t_k z^k} p^{(1)}(z) = e^{\sum_1^\infty t_k z^k} S_1 \chi(z) \ , \\
\Psi_2^*(z) & := e^{-\sum_1^\infty s_k z^{-k}} h^{-1} p^{(2)}(z^{-1}) = e^{-\sum_1^\infty s_k z^{-k}} (S_2^{-1})^T \chi(z^{-1}) \ .
\end{cases}
$$

Eigenfunction identities:

$$
L_1 \Psi_1(z) = z \Psi_1(z), \quad L_2^T \Psi_2^*(z) = z^{-1} \Psi_2^*(z) \ .
$$

Formulas (160) and (156) yield,

$t - s$ *flows for* L_i *and* Ψ:

$$
\begin{cases}
\dfrac{\partial L_i}{\partial t_n} = [(L_1^n)_+, L_i] \ , & \dfrac{\partial L_i}{\partial s_n} = [(L_2^n)_-, L_i] \ , \qquad i = 1, 2, \ n = 1, 2, \dots \\
\dfrac{\partial \Psi_1}{\partial t_n} = (L_1^n)_+ \Psi_1 \ , & \dfrac{\partial \Psi_1}{\partial s_n} = (L_2^n)_- \Psi_1 \ , \\
\dfrac{\partial \Psi_2^*}{\partial t_n} = -(L_1^n)_+^T \Psi_1^* \ , & \dfrac{\partial \Psi_2^*}{\partial s_n} = -(L_2^n)_-^T \Psi_2^* \ .
\end{cases}
$$

Wave operators:

$$W_1 := S_1 e^{\sum_1^\infty t_k \Lambda^k} , \quad W_2 := S_2 e^{\sum_1^\infty s_k (\Lambda^T)^k} , \tag{161}$$

satisfy

$$W_1(t,s)W_1(t',s')^{-1} = W_2(t,s)W_2(t',s')^{-1} , \quad \forall t,s,t',s' . \tag{162}$$

All the data in 2–Toda is parametrized by τ-functions, to wit:

L_1, L_2, Ψ_1, Ψ_2^* parametrized by τ-functions:

$$\left\{\begin{aligned} &\Psi_1(z,t,s) \\ &\qquad = \left(\frac{\tau_n(t-[z^{-1}],s)}{\tau_n(t,s)} e^{\sum_1^\infty t_i z^i} z^n\right)_{n\geqslant 0} \\ &\Psi_2^*(z,t,s) \\ &\qquad = \left(\frac{\tau_n(t,s+[z])}{\tau_{n+1}(t,s)} e^{-\sum_1^\infty s_i z^{-i}} z^{-n}\right)_{n\geqslant 0} \end{aligned}\right. , \quad [x] = (x, x^2/2, \dots) , \tag{163}$$

$$\left\{\begin{aligned} &L_1^k = \sum_{\ell=0}^\infty \operatorname{diag}\left(\frac{s_\ell(\tilde{\partial}_t)\tau_{n+k-\ell+1} \circ \tau_n}{\tau_{n+k-\ell+1}\tau_n}\right)_{n\geqslant 0} \Lambda^{k-\ell} , \\ &(h(L_2^T)^k h^{-1}) = \textstyle\sum_{\ell=0}^\infty \operatorname{diag}\left(\dfrac{s_\ell(\tilde{\partial}_t)\tau_{n+k-\ell+1} \circ \tau_n}{\tau_{n+k-\ell+1}\tau_n}\right)\Bigg|_{\tilde{\partial}_t \to -\tilde{\partial}_s} , \end{aligned}\right. \tag{164}$$

with

$$s_\ell(t) \text{ the elementary Schur polynomials, } \tilde{\partial}_t = \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \dots\right) ,$$

and the

Hirota symbol:

$$p\left(\frac{\partial}{\partial t}\right) f \circ g := p\left(\frac{\partial}{\partial y}\right) f(t+y)g(t-y)\Big|_{y=0} . \tag{165}$$

3.4 Bilinear identities and τ-function PDE's

Just like in KP theory (see Section 1.3), where the bilinear identity generates the KP hierarchy of PDE's for the τ-function, the same situation holds for the 2–Toda Lattice. In general 2–Toda theory (see [17]) the bilinear identity is a consequence of (163) and (162), but in the special case of 2–Toda being generated from bi-orthogonal polynomials, we can and will, at the end of this section, sketch a quick direct alternate proof of Adler–van Moerbek–Vanhaecke [15] based on the bi-orthogonal polynomials, which has been vastly generalized. Since all we ever need of integrability in any problem is the PDE hierarchy, it is clearly of great practical use to have in general a quick proof of just the bilinear identities, but without all the usual integrable baggage. We now give the

2–*Toda bilinear identities*:

$$\oint_{z=\infty} \tau_n(t-[z^{-1}],s)\tau_{m+1}(t'+[z^{-1}],s')e^{\sum_1^\infty (t_i-t_i')z^i} z^{n-m-1}\,dz$$
$$= \oint_{z=0} \tau_{n+1}(t,s-[z^{-1}])\tau_m(t',s'+[z^{-1}])e^{\sum_1^\infty (s_i-s_i')z^i} z^{m-n-1}\,dz\ ,$$
$$\forall s,t,s',t',m,n\ . \quad (166)$$

The identities are a consequence of (162) and (163) and they yield, as in Section 1.3 (see Appendix) a generating function involving elementary Schur polynomials $s_j(\cdot)$ and arbitrary parameters a, b, in the following[9]

Hirota form:

$$0 = -\sum_{j=0}^{\infty} s_{m-n+j}(-2a)s_j(\tilde\partial_t)e^{\sum_1^\infty (a_k\partial/\partial t_k+b_k\partial/\partial s_k)}\tau_{m+1}\circ\tau_n$$
$$+\sum_{j=0}^{\infty} s_{-m+n+j}(-2b)s_j(\tilde\partial_s)e^{\sum_1^\infty (a_k\partial/\partial t_k+b_k\partial/\partial s_k)}\tau_m\circ\tau_{n+1} \quad (167)$$
$$= a_{j+1}\left(2s_j(\tilde\partial_t)\tau_{n+2}\circ\tau_n + \frac{\partial^2}{\partial s_1\partial t_{j+1}}\tau_{n+1}\circ\tau_{n+1}\right) + 0(a_{j+1}^2)\ , \quad (168)$$

upon setting $m=n+1$, and all b_k, $a_k=0$, except a_{j+1}. Note:

$$s_0(t)=1\ ,\quad s_1(t)=t_1\ ,\quad s_1(\tilde\partial_t)f\circ g = g\frac{\partial f}{\partial t_1} - f\frac{\partial g}{\partial t_1}\ ,$$

and $s_k(t) = t_k +$ poly. $(t_1,\dots,t_{k-1})$. This immediately yields the

2–*Toda τ-function identities*:

$$-\frac{\partial^2}{\partial s_1\partial t_k}\log\tau_{n+1} = \frac{s_{k-1}(\tilde\partial_t)\tau_{n+2}\circ\tau_n}{\tau_{n+1}^2} \quad (169)$$

$$= \begin{cases} \dfrac{\tau_{n+2}\tau_n}{\tau_{n+1}^2}\ , & k=1\ ,\\[2ex] \dfrac{\tau_{n+2}\tau_n}{\tau_{n+1}^2}\dfrac{\partial}{\partial t_1}\log\dfrac{\tau_{n+2}}{\tau_n}\ , & k=2\ ,\end{cases} \quad (170)$$

from which we deduce, by forming the ratio of the $k=1,2$ identities, and using (164):

[9] Hopefully there will be no confusion in this section between the elementary Schur polynomials, $s_j(\cdot)$, which are *functions*, and the time variables s_j, which are parameters, but the situation is not ideal.

Fundamental identities:

$$(L_1^2)_{n-1,n} = \frac{\partial}{\partial t_1} \log \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{\partial^2/\partial s_1 \partial t_2 \log \tau_n}{\partial^2/\partial s_1 \partial t_1 \log \tau_n} \;, \tag{171}$$

(and by duality $t \leftrightarrow -s$, $L_1 \leftrightarrow hL_2^T h^{-1}$)

$$(hL_2^T h^{-1})^2_{n-1,n} = -\frac{\partial}{\partial s_1} \log \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{\partial^2/\partial t_1 \partial s_2 \log \tau_n}{\partial^2/\partial t_1 \partial s_1 \log \tau_n} \;. \tag{172}$$

As promised we now give, following Adler–van Moerbeke–Vanhaecke [15]:

Sketch of alternate proof of bilinear identities:

The proof is based on the following identities concerning the bi-orthogonal polynomials and their Cauchy transforms with regard to the measure $d\rho$ defining the moments of (141) and (152) μ_{ij}:

$$d\rho(x, y, t, s, c) = e^{\sum_1^\infty (t_i x^i - s_i y^i) + \sum_{\alpha,\beta \geqslant 1} c_{\alpha\beta} x^\alpha y^\beta} \, dx \, dy \;.$$

Namely, the bi-orthogonal polynomials of (150) and (151) and their formal Cauchy transforms with regard to $d\rho$ can be expressed in terms of τ-functions as follows:[10]

$$\begin{cases} p_n^{(1)}(z, t, s) = z^n \dfrac{\tau_n(t - [z^{-1}], s)}{\tau_n(t, s)} \;, \\ \qquad\qquad\qquad\qquad\qquad \text{(suppressing } c \text{ and } E \text{ in } p_n^{(i)} \text{ and } \tau_n) \;, \\ p_m^{(2)}(z, t, s) = z^m \dfrac{\tau_m(t, s + [z^{-1}])}{\tau_m(t, s)} \;, \\ \left\langle p_n^{(1)}(x), \dfrac{1}{z - y} \right\rangle = z^{-n-1} \dfrac{\tau_{n+1}(t, s - [z^{-1}])}{\tau_n(t, s)} \;, \\ \left\langle \dfrac{1}{z - x}, p_m^{(2)}(y) \right\rangle = z^{-m-1} \dfrac{\tau_{m+1}(t + [z^{-1}], s)}{\tau_m(t, s)} \;. \end{cases} \tag{173}$$

These formulas are not hard to prove, they depend on substituting

$$e^{\pm \sum_{i \geqslant 1} (x/z)^i / i} = \left(1 - \frac{x}{z}\right)^{\mp 1} = \sum_{i \geqslant 0} \left(\frac{x}{z}\right)^i \quad \text{or} \quad 1 - \frac{x}{z} \tag{174}$$

into formula (142), which express $\tau_n(t, s, C, E)$ in terms of the moments $\mu_{ij}(t, s, c, E)$ of (141). Then one must expand the rows or columns of the ensuing determinants using (174) and make the identification of (173), using (150) and (151), an amusing exercise. If one then substitutes (173) into the bilinear identity (166) divided by $\tau_n(t, s)\tau_m(t, s)$, thus eliminating τ-functions, and if we make use of the following self-evident *formal* residue identities:

[10] $1/(z - x) := 1/z \sum_{i=0}^\infty (x/z)^i$, etc., $1/(z - y)$, and thus z is viewed as large.

$$\oint_{z=\infty} f(z)\left\langle \frac{h(x)}{z-x}, g(y)\right\rangle \frac{dz}{2\pi i} = \langle f(x)h(x), g(y)\rangle ,$$
$$\oint_{z=\infty} f(z)\left\langle g(x), \frac{h(y)}{z-y}\right\rangle \frac{dz}{2\pi i} = \langle g(x), f(y)h(y)\rangle , \tag{175}$$

with

$$f(z) = \sum_{i=0}^{\infty} a_i z^i , \tag{176}$$

the bilinear identity becomes a tautology.

3.5 Virasoro constraints for the τ-functions

In this section we derive the Virasoro constraints for our τ-functions $\tau(t, s, C, E)$ using their integral form:

$$V_k^{(1)}\tau(t, s, C, E) = 0 , \quad V_k^{(2)}\tau(t, s, C, E) = 0 , \qquad k \geqslant -1 . \tag{177}$$

Explicitly:

$$\begin{cases} \displaystyle\sum_1^r a_i^{k+1}\frac{\partial}{\partial a_i}\tau_n^E = \left(\mathbb{J}_{k,n}^{(2)}(t) + \sum_{i,j\geqslant 1} ic_{ij}\frac{\partial}{\partial c_{i+k,j}}\right)\tau_n^E := \mathbb{V}_k\tau_n^E \\ \displaystyle\sum_1^s b_i^{k+1}\frac{\partial}{\partial b_i}\tau_n^E = \left(\mathbb{J}_{k,n}^{(2)}(-s) + \sum_{i,j\geqslant 1} jc_{ij}\frac{\partial}{\partial c_{i,j+k}}\right)\tau_n^E := \widetilde{\mathbb{V}}_k\tau_n^E \end{cases} ,$$
$$k \geqslant -1 , \ n \geqslant 0 , \tag{178}$$

with

$$E = \bigcup_1^r [a_{2i-1}, a_{2i}] \times \bigcup_1^s [b_{2i-1}, b_{2i}] , \tag{179}$$

$$\begin{aligned} \tau_n^E &= \tau_n(t, s, C, E) \\ &= \frac{1}{n!}\int_{E^n}\left(\Delta_n(x)\prod_{k=1}^n e^{\sum_1^\infty t_i x_k^i} dx_k\right) \\ &\qquad \times \left(\Delta_n(y)\prod_{k=1}^n e^{-\sum_1^\infty s_i y_k^i}\, dy_k\right)\prod_{k=1}^n e^{\sum_{i,j\geqslant 1} c_{ij}x_k^i y_k^j} , \end{aligned} \tag{180}$$

and

$$
\begin{aligned}
\mathbb{J}_k^{(1)}(t) &:= \left(\mathbb{J}_{k,n}^{(1)}(t)\right)_{n\geqslant 0} := (J_k^{(1)}(t) + nJ_k^{(0)})_{n\geqslant 0}\ ,\\
\mathbb{J}_k^{(2)}(t) &:= \left(\mathbb{J}_{k,n}^{(2)}(t)\right)_{n\geqslant 0}\\
&:= \frac{1}{2}(J_k^{(2)}(t) + (2n+k+1)J_k^{(1)}(t) + n(n+1)J_k^{(0)})_{n\geqslant 0}\ ,\\
J_k^{(1)}(t) &:= \frac{\partial}{\partial t_k} + (-k)t_{-k}, \quad J_k^{(0)} = \delta_{0k}\ ,\\
J_k^{(2)}(t) &:= \sum_{i+j=k} \frac{\partial^2}{\partial t_i \partial t_j} + 2\sum_{i\geqslant 1} it_i \frac{\partial}{\partial t_{i+k}} + \sum_{i+j=-k} (it_i)(jt_j)\ .
\end{aligned}
\tag{181}
$$

Main fact:

$\mathbb{J}_k^{(2)}$ forms a Virasoro algebra of charge $c=-2$,

$$
[\mathbb{J}_k^{(2)}, \mathbb{J}_\ell^{(2)}] = (k-\ell)\mathbb{J}_{k+\ell}^{(2)} + (-2)\frac{(k^3-k)}{12}\delta_{k,-\ell}\ . \tag{182}
$$

To prove (178) we need the following lemma:

Lemma 3.1 (Adler–van Moerbeke [5]). *Given*

$$
\rho = e^{-V} \quad \textit{with} \quad -\frac{\rho'}{\rho} = V' = \frac{g}{f} = \frac{\sum_0^\infty \beta_i z^i}{\sum_0^\infty \alpha_i z^i}\ ,
$$

the integrand

$$
dI_n(x) := \Delta_n(x) \prod_{k=1}^n (e^{\sum_1^\infty t_i x^i} \rho(x_k) dx_k)\ , \tag{183}
$$

satisfies the following variational formula:

$$
\frac{d}{d\varepsilon}\, dI_n(x_i \mapsto x_i + \varepsilon f(x_i) x_i^{m+1})\Big|_{\varepsilon=0} = \sum_{\ell=0}^\infty (\alpha_\ell \mathbb{J}_{m+\ell,n}^{(2)} - \beta_\ell \mathbb{J}_{m+\ell+1,n}^{(1)})\, dI_n\ . \tag{184}
$$

The contribution coming from $\prod_1^n dx_j$ *is given by*

$$
\sum_{\ell=0}^\infty a_\ell (\ell+m+1)\mathbb{J}_{m+\ell,n}^{(1)}\, dI_n\ . \tag{185}
$$

Proof of (178).

First make the change of coordinates $x_i \to x_i + \varepsilon x_i^{k+1}$, $1 \leqslant i \leqslant n$, in the integral (180), which remains unchanged, and then differentiate the results by ε, at $\varepsilon = 0$, which of course yields 0, i.e.,

$$
\frac{d}{d\varepsilon}\tau_n^E\big|_{x_i \to x_i + \varepsilon x_i^{k+1}}\Big|_{\varepsilon=0} = 0\ . \tag{186}
$$

Now, by (180), there are precisely 3 contributions to the ℓ.h.s. of (186), namely one of the form (184), with $\rho(x)=1$, yielding $\mathbb{J}^{(2)}_{k,n}(t)\tau_n^E$, one coming from

$$\frac{d}{d\varepsilon}\left(\prod_{\ell=1}^{n} e^{\sum_{i,j\geqslant 1} c_{ij}x_\ell^i y_\ell^j}\right)\Bigg|_{x_s\to x_s+\varepsilon x_s^{k+1}}\Bigg|_{\varepsilon=0}$$
$$= \sum_{i,j\geqslant 1} ic_{ij}\sum_{\ell=1}^{n} x_\ell^{i+k}y_\ell^j \prod_{\ell=1}^{n} e^{\sum_{i,j\geqslant 1} c_{ij}x_\ell^i y_\ell^j}$$
$$= \sum_{i,j\geqslant 1} ic_{ij}\frac{\partial}{\partial c_{i+k,j}}\prod_{\ell=1}^{n} e^{\sum_{i,j\geqslant 1} c_{ij}x_\ell^i y_\ell^j}\ ,$$

yielding $\sum_{i,j\geqslant 1} ic_{ij}\partial/\partial c_{i+k,j}\tau_n^E$.[11] Finally, we have a third contribution, since the limits of integration the integral (180) must change:

$$a+i\to a_i-\varepsilon a_i^{k+1}+0(\varepsilon^2)\ ,\quad 1\leqslant i\leqslant 2r\ .$$

Upon differentiating τ_n^E with respect to the ε in these *new* limits of integration, we have by the chain rule, the contribution $-\sum_1^{2r} a_i^{k+1}\partial/\partial a_i\tau_n^E$. Thus altogether we have:

$$0=\frac{d}{d\varepsilon}\tau_n^E\Big|_{x_i\to x_i+\varepsilon x_i^{k+1}}\Big|_{\varepsilon=0}=-\sum_1^{2r} a_i^{k+1}\frac{\partial}{\partial a_i}\tau_n^E+\mathbb{J}^{(2)}_{k,n}(t)\tau_n^E+\sum_{i,j\geqslant 1} ic_{ij}\frac{\partial}{\partial c_{i+k,n}}\tau_n^E\ ,$$

yielding the first expression (178). The second expression follows from the first by duality, $t\leftrightarrow -s$, $a\leftrightarrow b$, $c_{ij}\leftrightarrow c_{ji}$.

3.6 Consequences of the Virasoro relations

Observe from (132), (139) that $\big(e_2=(0,1,0,0,\dots)\big)$

$$P_n(E)=\frac{\tau_n^E(t-e_2/2,s+e_2/2,C)}{\tau_n^{\mathbb{R}}(t-e_2/2,s+e_2/2,C)}\Bigg|_{\mathcal{L}}\ , \tag{187}$$
$$\mathcal{L}:=\{t=s=0,\ \text{all } c_{ij}=0,\ \text{but } c_{11}=c\}\ , \tag{188}$$

and so computing (178) for $\tau_n^E(t-\frac12 e_2,s+\frac12 e_2,C)$ requires us to shift the t, s in $\mathbb{J}^{(2)}_{k,n}(t)$, $\mathbb{J}^{(2)}_{k,n}(-s)$ accordingly, and we find from (181) shifted, the following:

$$\mathcal{A}_k\tau_n=\mathcal{V}_k\tau_n,\quad \mathcal{B}_k\tau_n=\mathcal{W}_k\tau_n,\quad k=1,2 \tag{189}$$

with

$$\tau_n=\tau_n^E\left(t-\tfrac12 e_2,s+\tfrac12 e_2,C\right)\ ,$$

and

[11] It must be noted that: $\partial/(\partial c_{0,n})=-\partial/(\partial s_n)$, $\partial/(\partial c_{n,0})=\partial/(\partial t_n)$.

$$\mathcal{A}_1 = \frac{1}{1-c^2}\left(\sum_1^{2r}\frac{\partial}{\partial a_j} + c\sum_1^{2s}\frac{\partial}{\partial b_j}\right), \ \mathcal{B}_1 = \frac{1}{1-c^2}\left(\sum_1^{2s}\frac{\partial}{\partial b_j} + c\sum_1^{2r}\frac{\partial}{\partial a_j}\right),$$
$$\mathcal{A}_2 = \sum_1^{2r} a_j\frac{\partial}{\partial a_j}, \qquad \mathcal{B}_2 = \sum_1^{2s} b_j\frac{\partial}{\partial b_j}, \tag{190}$$

with

$$\begin{aligned}
\mathcal{V}_1 &:= \frac{1}{1-c^2}(\mathbb{V}_{-1} + c\tilde{\mathbb{V}}_{-1}) = \hat{\mathcal{V}}_1 + v_1 \\
&:= -\frac{\partial}{\partial t_1} - \frac{n(t_1 - cs_1)}{c^2-1} \\
&\quad - \frac{1}{c^2-1}\left(\sum_{i\geqslant 2} i\left(t_i\frac{\partial}{\partial t_{i-1}} + cs_i\frac{\partial}{\partial s_{i-1}}\right) + \sum_{\substack{i,j\geqslant 1,\\ i,j\neq(1,1)}} c_{ij}\left(i\frac{\partial}{\partial c_{i-1,j}} + jc\frac{\partial}{\partial c_{i,j-1}}\right)\right),\\
\mathcal{W}_1 &:= \frac{1}{1-c^2}(c\mathbb{V}_{-1} + \tilde{\mathbb{V}}_{-1}) = \widehat{\mathcal{W}}_1 + w_1 \\
&:= \frac{\partial}{\partial s_1} - \frac{n(ct_1 - s_1)}{c^2-1} \\
&\quad - \frac{1}{c^2-1}\left(\sum_{i\geqslant 2} i\left(ct_i\frac{\partial}{\partial t_{i-1}} + s_i\frac{\partial}{\partial s_{i-1}}\right) + \sum_{\substack{i,j\geqslant 1,\\ i,j\neq(1,1)}} c_{ij}\left(ci\frac{\partial}{\partial c_{i-1,j}} + j\frac{\partial}{\partial c_{i,j-1}}\right)\right),\\
\mathcal{V}_2 &:= \mathbb{V}_0 - c\frac{\partial}{\partial c} := \hat{\mathcal{V}}_2 + v_2 \\
&:= -\frac{\partial}{\partial t_2} + \sum_{i\geqslant 1} it_i\frac{\partial}{\partial t_i} + \frac{n(n+1)}{2} + \sum_{\substack{i,j\geqslant 1,\\ (i,j)\neq(1,1)}} ic_{ij}\frac{\partial}{\partial c_{ij}},\\
\mathcal{W}_2 &:= \tilde{\mathbb{V}}_0 - c\frac{\partial}{\partial c} := \widehat{\mathcal{W}}_2 + w_2 \\
&:= \frac{\partial}{\partial s_2} + \sum_{i\geqslant 1} is_i\frac{\partial}{\partial s_i} + \frac{n(n+1)}{2} + \sum_{\substack{i,j\geqslant 1,\\ (i,j)\neq(1,1)}} jc_{ij}\frac{\partial}{\partial c_{ij}}.
\end{aligned}$$

Note $\hat{\mathcal{V}}_1$, $\widehat{\mathcal{W}}_1$, $\hat{\mathcal{V}}_2$, $\widehat{\mathcal{W}}_2$ are first order operators such that (and this is the point):

$$\hat{\mathcal{V}}_1\Big|_{\mathcal{L}} = -\frac{\partial}{\partial t_1}, \quad \widehat{\mathcal{W}}_1\Big|_{\mathcal{L}} = \frac{\partial}{\partial s_1}, \quad \hat{\mathcal{V}}_2\Big|_{\mathcal{L}} = -\frac{\partial}{\partial t_2}, \quad \widehat{\mathcal{W}}_2\Big|_{\mathcal{L}} = \frac{\partial}{\partial s_2}, \tag{191}$$

and

$$v_1 = \frac{n(t_1 - cs_1)}{1-c^2}, \quad w_1 = \frac{n(ct_1 - s_1)}{1-c^2}, \quad v_2 = w_2 = \frac{n(n+1)}{2}. \tag{192}$$

Because of (191) we call this a principal symbol analysis. Hence

$$\mathcal{A}_k \log \tau_n = \hat{\mathcal{V}}_k \log \tau_n + v_k \ , \quad \mathcal{B}_k \log \tau_n = \widehat{\mathcal{W}}_k \log \tau_n + w_k \ , \qquad k = 1, 2 \ , \tag{193}$$

and so on the locus $\mathcal{L}$ using (191) – (193) we find:

$$\begin{aligned} \frac{\partial}{\partial t_1} \log \tau_n \Big|_{\mathcal{L}} &= -\mathcal{A}_1 \log \tau_n \Big|_{\mathcal{L}} \ , & \frac{\partial}{\partial s_1} \log \tau_n \Big|_{\mathcal{L}} &= \mathcal{B}_1 \log \tau_n \Big|_{\mathcal{L}} \ , \\ \frac{\partial}{\partial t_2} \log \tau_n \Big|_{\mathcal{L}} &= -\mathcal{A}_2 \log \tau_n \Big|_{\mathcal{L}} \ , & \frac{\partial}{\partial s_2} \log \tau_n \Big|_{\mathcal{L}} &= \mathcal{B}_2 \log \tau_n \Big|_{\mathcal{L}} \\ &\quad + \frac{n(n+1)}{2} & &\quad - \frac{n(n+1)}{2} \ . \end{aligned} \tag{194}$$

Extending this analysis to second derivatives, compute:

$$\begin{aligned} \mathcal{B}_1 \mathcal{A}_1 \log \tau_n \Big|_{\mathcal{L}} &= \mathcal{B}_1 (\hat{\mathcal{V}}_1 \log \tau_n + v_1) \Big|_{\mathcal{L}} = \mathcal{B}_1 \hat{\mathcal{V}}_1 \log \tau_n \Big|_{\mathcal{L}} + \mathcal{B}_1(v_1) \Big|_{\mathcal{L}} \\ &\overset{(x)}{=} \hat{\mathcal{V}}_1 \mathcal{B}_1 \log \tau_n \Big|_{\mathcal{L}} + \mathcal{B}_1(v_1) \Big|_{\mathcal{L}} \\ &\overset{(\frac{x}{x})}{=} -\frac{\partial}{\partial t_1} (\widehat{\mathcal{W}}_1 \log \tau_n + w_1) \Big|_{\mathcal{L}} + \mathcal{B}_1(v_1) \Big|_{\mathcal{L}} \\ &= -\frac{\partial}{\partial t_1} \left(\frac{\partial}{\partial s_1} + \cdots \right) \log \tau_n \Big|_{\mathcal{L}} - \frac{\partial}{\partial t_1} w_1 \Big|_{\mathcal{L}} + \mathcal{B}_1(v_1) \Big|_{\mathcal{L}} \end{aligned} \tag{195}$$

where we have used in (x) that $[\mathcal{B}_1, \hat{\mathcal{V}}_1]\Big|_{\mathcal{L}} = 0$ and in $(\frac{x}{x})$ that $\hat{\mathcal{V}}_1\Big|_{\mathcal{L}} = -\partial/\partial t_1$. So we must compute

$$-\frac{\partial}{\partial t_1} \widehat{\mathcal{W}}_1 \Big|_{\mathcal{L}} = -\frac{\partial^2}{\partial t_1 \partial s_1} \ , \quad -\frac{\partial}{\partial t_1} w_1 \Big|_{\mathcal{L}} = \frac{nc}{c^2 - 1} \ , \quad \mathcal{B}_1(v_1) \Big|_{\mathcal{L}} = 0 \ , \tag{196}$$

and so conclude that $\left(\tau_n = \tau_n^E(t - \frac{1}{2} e_2, s + \frac{1}{2} e_2, C)\right)$

$$\frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_n \Big|_{\mathcal{L}} = -\mathcal{B}_1 \mathcal{A}_1 \log \tau_n + \frac{nc}{c^2 - 1} \ . \tag{197}$$

The crucial points in this calculation were:

$$[\mathcal{B}_1, \hat{\mathcal{V}}_1] \Big|_{\mathcal{L}} = 0 \ , \quad \hat{\mathcal{V}}_1 \Big|_{\mathcal{L}} = -\frac{\partial}{\partial t_1}, \quad \widehat{\mathcal{W}}_1 = \frac{\partial}{\partial s_1} + \cdots \ , \tag{198}$$

and indeed this is a model calculation, which shall be repeated over and over again. And so in the same fashion, conclude:

$$\frac{\partial^2}{\partial t_1 \partial s_2} \log \tau_n \Big|_{\mathcal{L}} = -\mathcal{B}_2 \mathcal{A}_1 \log \tau_n \ , \quad \frac{\partial^2}{\partial s_1 \partial t_2} \log \tau_n \Big|_{\mathcal{L}} = -\mathcal{A}_2 \mathcal{B}_1 \log \tau_n \ , \tag{199}$$

where we have used $\partial/\partial t_1 (n(n+1)/2) = \partial/\partial s_1 \ (n(n+1)/2) = 0$.

3.7 Final equations

We have derived in Section 3.6, the following

Relations on Locus $\mathcal{L}$:

$$\begin{aligned}
&\frac{\partial}{\partial t_1}\log\tau_n^E=-\mathcal{A}_1\log\tau_n^E, &&\frac{\partial}{\partial t_2}\log\tau_n^E=-\mathcal{A}_2\log\tau_n^E+\frac{n(n+1)}{2}\ ,\\
&\frac{\partial}{\partial s_1}\log\tau_n^E=\mathcal{B}_1\log\tau_n^E, &&\frac{\partial}{\partial s_2}\log\tau_n^E=\mathcal{B}_2\log\tau_n^E-\frac{n(n+1)}{2}\ ,\\
&\frac{\partial^2}{\partial t_1\partial s_2}\log\tau_n^E=-\mathcal{B}_2\mathcal{A}_1\log\tau_n^E\ , &&\frac{\partial^2}{\partial s_1\partial t_2}\log\tau_n^E=-\mathcal{A}_2\mathcal{B}_1\log\tau_n^E\ ,
\end{aligned} \tag{200}$$

$$\frac{\partial^2}{\partial t_1\partial s_1}\log\tau_n^E = -\mathcal{B}_1\mathcal{A}_1\log\tau_n^E+\frac{nc}{c^2-1}\ .$$

Remember from Section 3.4:

2–*Toda relations*

$$\begin{aligned}
\frac{\partial}{\partial t_1}\log\frac{\tau_{n+1}^E}{\tau_{n-1}^E} &= \frac{\partial^2/\partial s_1\partial t_2\log\tau_n^E}{\partial^2/\partial t_1\partial s_1\log\tau_n^E}\ ,\\
-\frac{\partial}{\partial s_1}\log\frac{\tau_{n+1}^E}{\tau_{n-1}^E} &= \frac{\partial^2/\partial t_1\partial s_2\log\tau_n^E}{\partial^2/\partial t_1\partial s_1\log\tau_n^E}\ ,
\end{aligned} \tag{201}$$

Substitute the relations on $\mathcal{L}$ into the 2–Toda relations, which yields:

Pure boundary relations on the locus $\mathcal{L}$

$$\begin{aligned}
-\mathcal{A}_1\log\frac{\tau_{n+1}^E}{\tau_{n-1}^E} &= \frac{\mathcal{A}_2\mathcal{B}_1\log\tau_n^E}{\mathcal{A}_1\mathcal{B}_1\log\tau_n^E+nc/(1-c^2)}\ ,\\
-\mathcal{B}_1\log\frac{\tau_{n+1}^E}{\tau_{n-1}^E} &= \frac{\mathcal{B}_2\mathcal{A}_1\log\tau_n^E}{\mathcal{B}_1\mathcal{A}_1\log\tau_n^E+nc/(1-c^2)}\ .
\end{aligned} \tag{202}$$

Since $\mathcal{A}_1\mathcal{B}_1=\mathcal{B}_1\mathcal{A}$, conclude that

$$\mathcal{A}_1\left(\frac{\mathcal{B}_2\mathcal{A}_1\log\tau_n^E}{\mathcal{B}_1\mathcal{A}_1\log\tau_n^E+nc/(1-c^2)}\right)=\mathcal{B}_1\left(\frac{\mathcal{A}_2\mathcal{B}_1\log\tau_n^E}{\mathcal{A}_1\mathcal{B}_1\log\tau_n^E+nc/(1-c^2)}\right)\ . \tag{203}$$

Notice that since $\tau_n^{\mathbb{R}}$ is independent of a_i and b_i, we find that

$$\begin{aligned}
\mathcal{A}_1\log\tau_n^E &= \mathcal{A}_1\log\tau_n^E-\mathcal{A}_1\log\tau_n^{\mathbb{R}}\\
&= \mathcal{A}_1\log\frac{\tau_n^E}{\tau_n^{\mathbb{R}}}=\mathcal{A}_1\log P_n(E)\ ,
\end{aligned} \tag{204}$$

and so (203) is true with $\log\tau_n^E\to\log P_n(E)$, yielding the final equation for $F_n(E)=1/n\log P_n(E)$, and proving Theorem 3.1.

4 Dyson Brownian Motion and the Airy Process

4.1 Processes

The joint distribution for the Dyson process at 2–times deforms naturally to the 2–Toda integrable system, as it is described by a coupled Hermitian matrix integral, analyzed in the previous section. Taking limits of the Dyson process leads to the Airy and Sine processes. We describe the processes in this section in an elementary and intuitive fashion. A good reference for this discussion would be [33] and Dyson's celebrated papers [30, 31] on Dyson diffusion.

A random walk corresponds to a particle moving either left or right at time n with probability p. If the particle is totally drunk, one may take $p = \frac{1}{2}$. In that case, if X_n is its location after n steps,

$$E(X_n) = 0, \quad E(X_n^2) = n \ , \tag{205}$$

and in any case, this discrete process has no memory:

$$P\big(X_{n+1} = j \mid (X_n = i) \cap (\text{arbitrary past event})\big) = P(X_1 = j \mid X_0 = i) \ ,$$

i.e. it is Markovian. In the continuous version of this process (say with $p = \frac{1}{2}$), $[t/\delta]$ steps are taken in time t and each step is of magnitude $\sqrt{\delta}$, consistent with the scaling of (205). By the central limit theorem (CLT) for the binomial distribution, or in other words by Stirlings formula, it follows immediately that

$$\lim_{\delta \to 0} P(X_t \in (X, X+dX) \mid X_0 = \overline{X}) = \frac{e^{-(X-\overline{X})^2/2t}}{\sqrt{2\pi t}} dX =: P(t, \overline{X}, X) dX \ . \tag{206}$$

Note that $P(t, \overline{X}, X)$ satisfies the (heat) diffusion equation

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2 P}{\partial X^2} \ . \tag{207}$$

The limiting motion where the particle moves $\pm\sqrt{\delta}$ with equal probability $\frac{1}{2}$ in time δ, is in the limit, as $\delta \to 0$, Brownian motion. The process is scale invariant and so infinitesimally its fluctuations in t are no larger than $\sqrt{\Delta(t)}$ and hence while the paths are continuous, they are nowhere differentiable (for almost all initial conditions.) We may also consider Brownian motion in n directions, all independent, and indeed, it was first observed in $n = 2$ directions, under the microscope by Robert Brown, an English botanist, in 1828. In general, by independence,

$$P(t, \overline{X}, X) = \prod_1^n P(t, \overline{X}_i, X_i) = \frac{1}{((2\pi t)/\beta)^{n/2}} e^{-\sum_1^n (X_i - \overline{X}_i)^2/2t/\beta} \ , \tag{208}$$

hence

$$\frac{\partial P}{\partial t} = \frac{1}{2\beta}\sum_1^n \frac{\partial^2}{\partial X_i^2}P \ , \tag{209}$$

where we have changed the variance and hence the diffusion constant from $1 \to \beta$.

In addition (going back to $n = 1$), besides changing the rate of diffusion, we may also subject the diffusing particle located at X, to a harmonic force $-\rho X$, pointing toward the origin. Thus you have a drunken particle executing Brownian motion under the influence of a steady wind pushing him towards the origin — the Ornstein–Uhlenbeck (see [33]) process — where now the probability density $P(t, \overline{X}, X)$ is given by the diffusion equation:

$$\frac{\partial P}{\partial t} = \left(\frac{1}{2\beta}\frac{\partial^2}{\partial X^2} - \frac{\partial}{\partial X}(-\rho X)\right)P. \tag{210}$$

This can immediately be transformed to the case $\rho = 0$, yielding ($c = e^{-\rho t}$)

$$P(t, \overline{X}, X) = \frac{1}{\sqrt{2\pi}((1-c^2)/2\rho\beta)^{1/2}} e^{-(X-c\overline{X})^2/((1-c^2)/\rho\beta)} \ , \tag{211}$$

which as $\rho \to 0$, $(1-c^2)/2\rho \to t$, transforms to the old case. This process becomes stationary, i.e. the probabilities at a fixed time do not change in t, if and only if the initial distribution of $\overline{X}$ is given by the limiting $t \to \infty$ distribution of (211):

$$\frac{e^{-\rho\beta\overline{X}^2}}{\sqrt{\pi/\rho\beta}} d\overline{X} \ , \tag{212}$$

and this is the only "normal Markovian process" with this property.

Finally, consider a Hermitian matrix $B = (B_{ij})$ with n^2 real quantities B_{ij} undergoing n^2 *independent* Ornstein–Uhlenbeck processes with $\rho = 1$, but

$$\begin{cases} \beta = 1 & \text{for} \quad B_{ii} \qquad \text{(on diagonal)} \ , \\ \beta = 2 & \text{for} \quad B_{ij} \qquad \text{(off diagonal)} \ , \end{cases}$$

and so the respective probability distribution P_{ii}, P_{ij} satisfy by (210):

$$\begin{cases} \dfrac{\partial P_{ii}}{\partial t} = \left(\dfrac{1}{2}\dfrac{\partial^2}{\partial B_{ii}^2} - \dfrac{\partial}{\partial B_{ii}}(-B_{ii})\right)P_{ii} \ , \\ \dfrac{\partial P_{ij}}{\partial t} = \left(\dfrac{1}{2\times 2}\dfrac{\partial^2}{\partial B_{ij}^2} - \dfrac{\partial}{\partial B_{ij}}(-B_{ij})\right)P_{ij} \ , \end{cases} \tag{213}$$

with solution, by (211) ($c = e^{-t}$), given by:

$$\begin{cases} P_{ii}(t, \overline{B}_{ii}, B_{ii}) = \dfrac{1}{\sqrt{2\pi}\sqrt{(1-c^2)/2}} e^{-(B_{ii}-c\overline{B}_{ii})^2/(1-c^2)} \ , \\ P_{ij}(t, \overline{B}_{ij}, B_{ij}) = \dfrac{1}{\sqrt{2\pi}\sqrt{(1-c^2)/4}} e^{-(B_{ij}-c\overline{B}_{ij})^2/((1-c^2)/2)} \ . \end{cases} \tag{214}$$

By the independence of the processes the joint probability distribution is given by:

$$P(t,\overline{B},B)=\prod_{i=1}^{n}P_{ii}\prod_{\substack{1\leqslant i,\\ j\leqslant n,\\ i\neq j}}P_{ij}=\frac{Z^{-1}}{(1-c^2)^{n^2/2}}e^{-\operatorname{Tr}(B-c\overline{B})^2/(1-c^2)}\ , \tag{215}$$

with $Z=(2\pi)^{n^2/2}2^{(-n^2+n/2)}$, which by (213) and (215) evolves by the Ornstein–Uhlenbeck process:

$$\begin{aligned}\frac{\partial P}{\partial t}&=\sum_{i,j=1}^{n}\left(\frac{1}{4}(1+\delta_{ij})\frac{\partial^2}{\partial B_{ij}^2}+\frac{\partial}{\partial B_{ij}}B_{ij}\right)P\\ &=\sum_{i,j=1}^{n}\left(\frac{1}{4}(1+\delta_{ij})\frac{\partial}{\partial B_{ij}}\Phi(\mathbb{B})\frac{\partial}{\partial B_{ij}}\frac{1}{\Phi(B)}\right)P(B)\ ,\end{aligned} \tag{216}$$

with $\Phi(B)=\exp(-\operatorname{tr}B^2)$. Note that the *most general solution* (215) to (216) is invariant under the unitary transformation

$$(\overline{B},B)\to(U\overline{B}U^{-1},UBU^{-1})\ , \tag{217}$$

which forces the actual process (216) to possess this unitary invariance and in fact (216) induces a random motion purely on the spectrum of B. This motion, discovered by Dyson in [30, 31], is called Dyson diffusion, and indeed the Ornstein–Uhlenbeck process

$$B(t)=\big(B_{ij}(t)\big)$$

given by (216) with solution (215), induces Dyson Brownian motion: $\big(\lambda_1(t),\dots,\lambda_n(t)\big)\in\mathbb{R}^n$ on the eigenvalues of $B(t)$.

The transition probability $P(t,\bar{\lambda},\lambda)$ satisfies the following diffusion equation:

$$\begin{aligned}\frac{\partial P}{\partial t}&=\sum_{1}^{n}\left(\frac{1}{2}\frac{\partial^2}{\partial\lambda_i^2}-\frac{\partial}{\partial\lambda_i}\frac{\partial\log\sqrt{\Phi(\lambda)}}{\partial\lambda_i}\right)P\\ &=\frac{1}{2}\sum_{i=1}^{n}\frac{\partial}{\partial\lambda_i}\Phi(\lambda)\frac{\partial}{\partial\lambda_i}\frac{1}{\Phi(\lambda)}P\end{aligned} \tag{218}$$

with

$$\Phi(\lambda)=\Delta_n^2(\lambda)\prod_{1}^{n}e^{-\lambda_i^2}\ ,$$

which is a Brownian motion, where instead of the particle at λ_i feeling only the harmonic restoring force $-\lambda_i$, as in the Ornstein–Uhlenbeck process, it feels the full force

$$F_i(\lambda):=\frac{\partial\log\sqrt{\Phi(\lambda)}}{\partial\lambda_i}=\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}-\lambda_i\ , \tag{219}$$

which acts to keep the particles apart. In short, the Vandermonde in $\Phi(\lambda)$ creates n–repelling Brownian motions, while the exponential term keeps them from flying out to infinity. The equation (218) was shown by Dyson in [30, 31] by observing that Brownian motion with a force term $F = (F_i)$ is, in general, completely characterized infinitesimally by the dynamics:

$$E(\delta\lambda_i) = F_i(\lambda)\delta t, \quad E\big((\delta\lambda_i)^2\big) = \delta t \ , \tag{220}$$

and so in particular (216) yields

$$E(\delta B_{ij}) = -B_{ij}\delta t, \quad E\big((\delta B_{ij})^2\big) = \tfrac{1}{2}(1+\delta_{ij})\delta t \ . \tag{221}$$

Then by the unitary invariance (217) of the process (216), one may set at time t: $B(t) = \operatorname{diag}\big(\lambda_1(t), \dots, \lambda_n(t)\big)$, and then using the perturbation formula:

$$\delta\lambda_i\Big|_t = \delta B_{ii} + \sum_{j\neq i} \frac{(\delta B_{ji})^2 + (\delta B_{ji})^2}{(\lambda_i - \lambda_j)} + \cdots \ ,$$

compute $E(\delta\lambda_i)$ and $E\big((\delta\lambda_i)^2\big)$ by employing (221), immediately yielding (220) with $F_i(\lambda)$ given by (219). Thus by the characterization of (218) by (220), we have verified Dysons result (218).

Remember an Ornstein–Uhlenbeck process has a stationary measure precisely if we take for the initial measure the equilibrium measure at $t \to \infty$. So consider our Ornstein–Uhlenbeck transition density (215) with $t \to \infty$ stationary distribution:

$$Z^{-1} e^{-\operatorname{tr} B^2}\, dB \ ,$$

and with this invariant measure as initial condition, one finds for the joint distribution ($c = e^{-(t_2 - t_1)}$)

$$\begin{aligned} &P(B(t_1) \in dB_1, B(t_2) \in dB_2) \\ &\qquad = Z^{-1} \frac{dB_1 dB_2}{(1-c^2)^{n^2/2}} e^{-1/(1-c^2)\operatorname{Tr}(B_1^2 - 2cB_1B_2 + B_2^2)} \ , \end{aligned} \tag{222}$$

and similarly ($c_i = e^{-(t_{i+1} - t_i)}$) compute

$$\begin{aligned} &P(B(t_1) \in dB_1, \dots, B(t_k) \in dB_k) \\ &\quad = Z_k^{-1} e^{-\operatorname{tr} B_1^2} \prod_{i=2}^{k} e^{-1/(1-c_{i-1}^2)\operatorname{Tr}(B_i - c_{i-1}B_{i-1})^2} dB_1, \dots, dB_k \\ &\quad = Z_k^{-1} \prod_{i=1}^{k} e^{-(1/(1-c_{i-1}^2) + c_i^2/(1-c_i^2)) \sum_{j=1}^{n} \lambda_{j,i}^2} \prod_{j=1}^{n} d\lambda_{j,i} \\ &\quad \prod_{i=1}^{k-1} \det(e^{(2c_i/(1-c_i^2))\, \lambda_{\ell,i+1}\lambda_{m,i}})_{\substack{1\leqslant \ell, \\ m\leqslant n}} \Delta_n(\lambda_1)\Delta_n(\lambda_k), \\ &\quad \big(\lambda_i = (\lambda_{1,i}, \lambda_{2,i}, \dots, \lambda_{n,i})\big) \ , \end{aligned} \tag{223}$$

using the HCIZ formula (138).

The distribution of the eigenvalues for GUE is expressible as a Fredholm determinant (9) involving the famous *Hermite kernel* (5) and Eynard and Mehta [32] showed that you have for the Dyson process an analogous extended Hermite kernel, specifically the matrix kernel [39]:

$$K^{H,n}_{t_i t_j} := \begin{cases} \displaystyle\sum_{k=1}^{\infty} e^{-k(t_i-t_j)} \varphi_{n-k}(x)\varphi_{n-k}(y), & t_i \geqslant t_j\ , \\ \displaystyle -\sum_{k=-\infty}^{0} e^{k(t_j-t_i)} \varphi_{n-k}(x)\varphi_{n-k}(y), & t_i < t_j\ , \end{cases} \tag{224}$$

with

$$\int_{\mathbb{R}} \varphi_i(x)\varphi_j(x)\,dx = \delta_{ij}, \quad \varphi_i(x) = p_i(x)e^{-x^2/2}\ ,$$

where

$$\varphi_k(x) = \begin{cases} e^{-x^2/2} p_k(x), & \text{for } k \geqslant 0, \qquad \text{with} \quad p_k(x) = \dfrac{H_k(x)}{2^{k/2}\sqrt{k!}\pi^{1/4}}\ , \\ 0, & \text{for } k < 0\,; \end{cases}$$

so $p_k(x)$ are the normalized Hermite polynomials. Then we have

$$\text{Prob (all } B(t_i) \text{ eigenvalues} \notin E_i,\ 1 \leqslant i \leqslant m) = \det(I - K^{H,E}): \\ K^{H,E}_{ij}(x,y) = I_{E_i}(x) K^{H,n}_{t_i t_j}(x,y) I_{E_j}(y)\ , \tag{225}$$

the above being a Fredholm determinant with a matrix kernel.

Remark. In general such a Fredholm determinant is given by:

$$\begin{aligned} &\det(I - z(K_{t_i t_j})_{1\leqslant i,j\leqslant m})\Big|_{z=1} \\ &\quad = 1 + \sum_{N=1}^{\infty} (-z)^N \sum_{\substack{0\leqslant r_i \leqslant N, \\ \sum_1^m r_i = N}} \int_{\mathbb{R}} \prod_1^{r_1} d\alpha_i^{(1)} \cdots \prod_1^{r_m} d\alpha_i^{(m)} \\ &\qquad \times \det\left(\left(K_{t_k t_\ell}(\alpha_i^{(k)}, \alpha_j^{(\ell)}) \right)_{\substack{1\leqslant i \leqslant r_k, \\ 1 \leqslant j \leqslant r_\ell}} \right)_{1\leqslant k\ ,\ell\leqslant m}\Bigg|_{z=1}\ , \end{aligned}$$

where the N–fold integral above is taken over the range

$$\mathbb{R} = \left\{ \begin{matrix} -\infty < \alpha_1^{(1)} \leqslant \cdots \leqslant \alpha_{r_1}^{(1)} < \infty \\ \vdots \\ -\infty < \alpha_1^{(m)} \leqslant \cdots \leqslant \alpha_{r_m}^{(m)} < \infty \end{matrix} \right\}\ ,$$

with integrand equal to the determinant of an $N\times N$ matrix, with blocks given by the $r_k\times r_\ell$ matrices $\big(K_{t_kt_\ell}(\alpha_i^{(k)},\alpha_j^{(\ell)})\big)_{\substack{1\leqslant i\leqslant r_k,\\ 1\leqslant j\leqslant r_\ell}}$. In particular, for $m=2$, we have

$$1+\sum_{N=1}^{\infty}(-z)^N\sum_{\substack{0\leqslant r,s\leqslant N,\\ r+s=N}}\int_{\left\{\substack{-\infty<\alpha_1\leqslant\cdots\leqslant\alpha_r<\infty\\ -\infty<\beta_1\leqslant\cdots\leqslant\beta_s<\infty}\right\}}\prod_1^r d\alpha_i\prod_1^s d\beta_i$$

$$\times\det\left.\begin{pmatrix}\big(\widehat{K}_{t_1t_1}(\alpha_i,\alpha_j)\big)_{\substack{1\leqslant i,\\ j\leqslant r}} & \big(\widehat{K}_{t_1t_2}(\alpha_i,\beta_j)\big)_{\substack{1\leqslant i\leqslant r,\\ 1\leqslant j\leqslant s}}\\ \big(\widehat{K}_{t_2t_1}(\beta_i,\alpha_j)\big)_{\substack{1\leqslant i\leqslant s,\\ 1\leqslant j\leqslant r}} & \big(\widehat{K}_{t_2t_2}(\beta_i,\beta_j)\big)_{\substack{1\leqslant i,\\ j\leqslant s}}\end{pmatrix}\right|_{z=1}. \tag{226}$$

These processes have scaling limits corresponding to the bulk and edge scaling limits in the GUE.

The *Airy process* is defined by rescaling in the extended Hermite kernel:

$$x=\sqrt{2n}+\frac{u}{\sqrt{2}n^{1/6}}\,,\quad y=\sqrt{2n}+\frac{v}{\sqrt{2}n^{1/6}}\,,\quad t=\frac{\tau}{n^{1/3}}\,, \tag{227}$$

and the *Sine process* by rescaling in the extended Hermite kernel:

$$x=\frac{u\pi}{\sqrt{2n}}\,,\quad y=\frac{v\pi}{\sqrt{2n}}\,,\quad t=\pi^2\frac{\tau}{2n}\,. \tag{228}$$

This amounts to following, in slow time, the eigenvalues at the edge and in the bulk, but with a microscope specified by the above rescalings. Then the extended kernels have well-defined limits as $n\to\infty$:

$$K^A_{t_it_j}(x,y)=\begin{cases}\displaystyle\int_0^\infty e^{-z(t_i-t_j)}A_i(x+z)A_i(y+z)\,dz, & t_i\geqslant t_j\,,\\ \displaystyle-\int_{-\infty}^0 e^{z(t_j-t_i)}A_i(x+z)A_i(y+z)\,dz, & t_i<t_j\,,\end{cases} \tag{229}$$

$$K^S_{t_it_j}=\begin{cases}\displaystyle\frac{1}{\pi}\int_0^\pi e^{(z^2/2)(t_i-t_j)}\cos z(x-y)\,dz, & t_i\geqslant t_j\,,\\ \displaystyle-\frac{1}{\pi}\int_\pi^\infty e^{-(z^2/2)(t_j-t_i)}\cos z(x-y)\,dz, & t_i<t_j\,,\end{cases} \tag{230}$$

with A_i the Airy function. Letting $A(t)$ and $S(t)$ denote the Airy and Sine processes, we define them below by

$$\begin{aligned}\operatorname{Prob}(A(t_i)\notin E_i,1\leqslant i\leqslant k)&=\det(I-K^{A,E})\,,\\ \operatorname{Prob}(S(t_i)\notin E_i,1\leqslant i\leqslant k)&=\det(I-K^{S,E})\,,\end{aligned} \tag{231}$$

where the determinants are matrix Fredholm determinants defined by the matrix kernels (229) and (230) in the same fashion as (225). The Airy process was first defined by Prähofer and Spohn in [46] and the Sine process was first defined by Tracy and Widom in [53].

4.2 PDE's and asymptotics for the processes

It turns out that the 2–time joint probabilities for all three processes, Dyson, Airy and Sine satisfy PDE's, which moreover lead to long time $t = t_2 - t_1$ asymptotics in, for example, the Airy case. In this section we state the results of Adler and van Moerbeke [6], sketching the proofs in the next section. The first result concerns the Dyson process:

Theorem 4.1 (Dyson process). *Given $t_1 < t_2$ and $t = t_2 - t_1$, the logarithm of the joint distribution for the Dyson Brownian motion $\big(\lambda_1(t), \dots, \lambda_n(t)\big)$,*

$$G_n(t; a_1, \dots a_{2r}; b_1, \dots, b_{2s}) := \log P(\textit{all } \lambda_i(t_1) \in E_1,\ \textit{all } \lambda_i(t_2) \in E_2) \ ,$$

satisfies a third-order nonlinear PDE *in the boundary points of E_1 and E_2 and t, which takes on the simple form, setting $c = e^{-t}$,*

$$\mathcal{A}_1 \frac{\mathcal{B}_2 \mathcal{A}_1 G_n}{\mathcal{B}_1 \mathcal{A}_1 G_n + 2nc} = \mathcal{B}_1 \frac{\mathcal{A}_2 \mathcal{B}_1 G_n}{\mathcal{A}_1 \mathcal{B}_1 G_n + 2nc} \ . \tag{232}$$

The sets E_1 and E_2 are the disjoint union of intervals

$$E_1 := \bigcup_{i=1}^{r} [a_{2i-1}, a_{2i}] \quad \textit{and} \quad E_2 := \bigcup_{i=1}^{s} [b_{2i-1}, b_{2i}] \subseteq \mathbb{R} \ ,$$

which specify the linear operators

$$\begin{aligned}
\mathcal{A}_1 &= \sum_1^{2r} \frac{\partial}{\partial a_j} + c \sum_1^{2s} \frac{\partial}{\partial b_j} \ , \\
\mathcal{B}_1 &= c \sum_1^{2r} \frac{\partial}{\partial a_j} + \sum_1^{2s} \frac{\partial}{\partial b_j} \ , \\
\mathcal{A}_2 &= \sum_1^{2r} a_j \frac{\partial}{\partial a_j} + c^2 \sum_1^{2s} b_j \frac{\partial}{\partial b_j} + (1 - c^2) \frac{\partial}{\partial t} - c^2 \ , \\
\mathcal{B}_2 &= c^2 \sum_1^{2r} a_j \frac{\partial}{\partial a_j} + \sum_1^{2s} b_j \frac{\partial}{\partial b_j} + (1 - c^2) \frac{\partial}{\partial t} - c^2 \ .
\end{aligned}$$

The duality $a_i \leftrightarrow b_j$ reflects itself in the duality $\mathcal{A}_i \leftrightarrow \mathcal{B}_i$.

The next result concerns the Airy process:

Theorem 4.2 (Airy process). *Given $t_1 < t_2$ and $t = t_2 - t_1$, the joint distribution for the Airy process $A(t)$,*

$$G(t; u_1, \dots u_{2r}; v_1, \dots, v_{2s}) := \log P(A(t_1) \in E_1, A(t_2) \in E_2) \ ,$$

satisfies a third-order nonlinear PDE *in the u_i, v_i and t, in terms of the Wronskian $\{f(y), g(y)\}_y := f'(y) g(y) - f(y) g'(y)$,*

$$\big((L_u+L_v)(L_uE_v-L_vE_u)+t^2(L_u-L_v)L_uL_v\big)G \\ =\tfrac{1}{2}\{(L_u^2-L_v^2)G,(L_u+L_v)^2G\}_{L_u+L_v}\ . \tag{233}$$

The sets E_1 and E_2 are the disjoint union of intervals

$$E_1:=\bigcup_{i=1}^{r}[u_{2i-1},u_{2i}] \quad \textit{and} \quad E_2:=\bigcup_{i=1}^{s}[v_{2i-1},v_{2i}]\subseteq\mathbb{R}\ ,$$

which specify the set of linear operators

$$L_u:=\sum_1^{2r}\frac{\partial}{\partial u_i}\ , \qquad L_v:=\sum_1^{2s}\frac{\partial}{\partial v_i}\ ,$$
$$E_u:=\sum_1^{2r}u_i\frac{\partial}{\partial u_i}+t\frac{\partial}{\partial t}\ , \quad E_v:=\sum_1^{2s}v_i\frac{\partial}{\partial v_i}+t\frac{\partial}{\partial t}\ .$$

The duality $v_i\leftrightarrow v_j$ reflects itself in the duality $L_u\leftrightarrow L_v$, $E_u\leftrightarrow E_v$.

Corollary 4.1. *In the case of semi-infinite intervals E_1 and E_2, the* PDE *for the Airy joint probability*

$$H(t;x,y):=\log P\left(A(t_1)\leqslant\frac{y+x}{2},A(t_2)\leqslant\frac{y-x}{2}\right)\ ,$$

takes on the following simple form in x, y and t^2, with $t=t_2-t_1$, also in terms of the Wronskian,

$$2t\frac{\partial^3 H}{\partial t\partial x\partial y}=\left(t^2\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 H}{\partial x^2}-\frac{\partial^2 H}{\partial y^2}\right)+8\left\{\frac{\partial^2 H}{\partial x\partial y},\frac{\partial^2 H}{\partial y^2}\right\}_y\ . \tag{234}$$

Remark. Note for the solution $H(t;x,y)$,

$$\lim_{t\searrow 0}H(t;x,y)=\log F_2\left(\min\left(\frac{y+x}{2},\frac{y-x}{2}\right)\right)\ .$$

The following theorem concerns the Sine process and uses the same sets and operators as Theorem 4.2:

Theorem 4.3 (Sine process). *For $t_1<t_2$, and compact E_1 and $E_2\subset\mathbb{R}$, the* log *of the joint probability for the Sine processes $S_i(t)$,*

$$G(t;u_1,\dots u_{2r};v_1,\dots,v_{2s}:=\log P(\textit{all } S_i(t_1)\in E_1^c,\ \textit{all } S_i(t_2)\in E_2^c)\ ,$$

satisfies

$$L_u\frac{(2E_vL_u+(E_v-E_u-1)L_v)G}{(L_u+L_v)^2G+\pi^2} \\ =L_v\frac{(2E_uL_v+(E_u-E_v-1)L_u)G}{(L_u+L_v)^2G+\pi^2}\ . \tag{235}$$

Corollary 4.2. *In the case of a single interval, the logarithm of the joint probability for the Sine process,*

$$H(t;x,y) = \log P(S(t_1) \notin [x_1+x_2, x_1-x_2], S(t_2) \notin [y_1+y_2, y_1-y_2])$$

satisfies

$$\frac{\partial}{\partial x_1} \frac{(2E_y \partial/\partial x_1 + (E_y - E_x - 1)\partial/\partial y_1)H}{(\partial/\partial x_1 + \partial/\partial y_1)^2 H + \pi^2} = \frac{\partial}{\partial y_1} \frac{(2E_x \partial/\partial y_1 + (E_x - E_y - 1)\partial/\partial x_1)H}{(\partial/\partial x_1 + \partial/\partial y_1)^2 H + \pi^2} \,. \tag{236}$$

Asymptotic consequences:

Prähofer and Spohn showed that the Airy process is a stationary process with continuous sample paths; thus the probability $P(A(t) \leqslant u)$ is independent of t, and is given by the Tracy–Widom distribution

$$P(A(t) \leqslant u) = F_2(u) := \exp\left(- \int_u^\infty (\alpha - u) q^2(\alpha)\, d\alpha \right) , \tag{237}$$

with $q(\alpha)$ the solution of the *Painlevé* II equation,

$$q'' = \alpha q + 2q^3 \quad \text{with} \quad q(\alpha) \simeq \begin{cases} -\dfrac{e^{-(2/3)\alpha^{3/2}}}{2\sqrt{\pi}\alpha^{1/4}} \,, & \text{for } \alpha \nearrow \infty \,, \\ \sqrt{-\alpha/2} \,, & \text{for } a \searrow -\infty \,. \end{cases} \tag{238}$$

The PDE's obtained above provide a very handy tool to compute large time asymptotics for these different processes, with the disadvantage that one usually needs, for justification, a nontrivial assumption concerning the interchange of sums and limits, which can be avoided upon directly using the Fredholm determinant formula for the joint probabilities (see Widom [57]) the latter method, however, tends to be quite tedious and quickly gets out of hand. We now state the following asymptotic result:

Theorem 4.4 (Large time asymptotics for the Airy process). *For large $t = t_2 - t_1$, the joint probability admits the asymptotic series*

$$P(A(t_1) \leqslant u, A(t_2) \leqslant v) = F_2(u)F_2(v) + \frac{F_2'(u)F_2'(v)}{t^2} + \frac{\Phi(u,v) + \Phi(v,u)}{t^4} + O\left(\frac{1}{t^6}\right) , \tag{239}$$

with the function $q = q(\alpha)$ given by (238) *and*

$$\Phi(u,v) := F_2(u)F_2(v)\left(\frac{1}{4}\left(\int_u^\infty q^2 d\alpha\right)^2 \left(\int_v^\infty q^2 d\alpha\right)^2 + q^2(u)\left(\frac{1}{4}q^2(v) - \frac{1}{2}\left(\int_v^\infty q^2 d\alpha\right)^2\right)\right)$$

$$+\int_v^\infty d\alpha(2(v-\alpha)q^2+q'^2-q^4)\int_u^\infty q^2 d\alpha\Bigg) . \quad (240)$$

Moreover, the covariance for large $t = t_2 - t_1$ behaves as

$$E\big(A(t_2)A(t_1)\big) - E\big(A(t_2)\big)E\big(A(t_1)\big) = \frac{1}{t^2} + \frac{c}{t^4} + \cdots , \quad (241)$$

where

$$c := 2\iint_{\mathbb{R}^2} \Phi(u,v)\,du\,dv .$$

Conjecture. The Airy process satisfies the nonexplosion condition for fixed x:

$$\lim_{z\to\infty} P(A(t) \geqslant x+z \mid A(0) \leqslant -z) = 0 . \quad (242)$$

4.3 Proof of the results

In this section we sketch the proofs of the results of the prior section, but all of these proofs are ultimately based on a fundamental theorem that we have proven in Section 3, which we now restate.

Let $M_1, M_2 \in \mathcal{H}_n$, Hermitian $n \times n$ matrices and consider the ensemble:

$$P\big((M_1,M_2)\subset S\big) = \frac{\int_S dM_1 dM_2 e^{-1/2\,\mathrm{Tr}(M_1^2+M_2^2-2cM_1M_2)}}{\int_{\mathcal{H}_n\times\mathcal{H}_n} dM_1 dM_2 e^{-1/2\,\mathrm{Tr}(M_1^2+M_2^2-2cM_1M_2)}} , \quad (243)$$

with

$$dM_1 = \Delta_n^2(x)\prod_1^n dx_i\,dU_1 , \quad dM_2 = \Delta_n^2(y)\prod_1^n dy_i\,dU_2 .$$

Given

$$E = E_1 \times E_2 = \bigcup_1^r [a_{2i-1}, a_{2i}] \times \bigcup_1^s [b_{2i-1}, b_{2i}] ,$$

define the boundary operators:

$$\tilde{\mathcal{A}}_1 = -\frac{1}{c^2-1}\left(\sum_1^r \frac{\partial}{\partial a_j} + c\sum_1^s \frac{\partial}{\partial b_j}\right) , \quad \tilde{\mathcal{A}}_2 = \sum_1^r a_j\frac{\partial}{\partial a_j} - \frac{\partial}{\partial c} ,$$

$$\tilde{\mathcal{B}}_1 = \tilde{\mathcal{A}}_1\Big|_{a\leftrightarrow b} , \qquad \tilde{\mathcal{B}}_2 = \tilde{\mathcal{A}}_2\Big|_{a\leftrightarrow b} .$$

Note $\tilde{\mathcal{A}}_1\tilde{\mathcal{B}}_1 = \tilde{\mathcal{B}}_1\tilde{\mathcal{A}}_1$.

The following theorem was proven in Section 3:

Theorem 4.5. *The statistics*

$$F_n(c; a_1,\ldots,a_{2r}; b_1,\ldots,b_{2s}) := \log P_n(E)$$

$$:= \log P(\mathit{all}\ (M_1\text{–}\mathit{eigenvalues}) \in E_1,\ \ \mathit{all}\ (M_2\text{–}\mathit{eigenvalues}) \in E_2)\ ,$$

satisfies the third order nonlinear PDE:

$$\tilde{\mathcal{A}}_1\left(\frac{\tilde{\mathcal{B}}_2\tilde{\mathcal{A}}_1 F_n}{\tilde{\mathcal{B}}_1\tilde{\mathcal{A}}_1 F_n + nc/(1-c^2)}\right) = \tilde{\mathcal{B}}_1\left(\frac{\tilde{\mathcal{A}}_2\tilde{\mathcal{B}}_1 F_n}{\tilde{\mathcal{A}}_1\tilde{\mathcal{B}}_1 F_n + nc/(1-c^2)}\right) . \tag{244}$$

Proof of Theorem 4.1.

Changing limits of integration in the integral F_n defined by the measure (243) to agree with the integral G_n defined by the measure (222), we find the function G_n of Theorem 4.1 is related to the function F_n of Theorem 4.5 by a trivial rescaling:

$$\begin{aligned} G_n(t; a_1, \ldots, a_{2r}; b_1, \ldots, b_{2s}) = F_n\Bigg(c; \frac{a_1}{\sqrt{(1-c^2)/2}}, \ldots, \frac{a_{2r}}{\sqrt{(1-c^2)/2}} ;\\ \frac{b_1}{\sqrt{(1-c^2)/2}}, \ldots, \frac{b_{2s}}{\sqrt{(1-c^2)/2}}\Bigg) \end{aligned} \tag{245}$$

and applying the chain rule to (244) using (245) leads to Theorem 4.1 immediately, upon clearing denominators.

In order to prove the theorems concerning the Airy and Sine processes, we need a rigorous statement concerning the asymptotics of our Dyson, Airy and Sine kernels. To that end, letting

$$\begin{aligned} \mathcal{S}_1 &:= \left\{ t \mapsto \frac{t}{n^{1/3}}\ , s \mapsto \frac{s}{n^{1/3}}\ , \begin{array}{l} x \mapsto \sqrt{2n+1} + \dfrac{u}{\sqrt{2}n^{1/6}} \\ y \mapsto \sqrt{2n+1} + \dfrac{v}{\sqrt{2}n^{1/6}} \end{array} \right\} \\ \mathcal{S}_2 &:= \left\{ t \mapsto \frac{\pi^2 t}{2n}\ , s \mapsto \frac{\pi^2 s}{2n}\ , x \mapsto \frac{\pi u}{\sqrt{2n}}\ , y \mapsto \frac{\pi v}{\sqrt{2n}} \right\}\ , \end{aligned} \tag{246}$$

we have:

Proposition 4.1. *Under the substitutions $\mathcal{S}_1$ and $\mathcal{S}_2$, the extended Hermite kernel tends with derivative,respectively, to the extended Airy and Sine kernel, when $n \to \infty$, uniformly for $u, v \in$ compact subsets $\subset \mathbb{R}$:*

$$\begin{aligned} \lim_{n\to\infty} K^{H,n}_{t,s}(x,y)\,dy\Big|_{\mathcal{S}_1} &= K^{A}_{t,s}(u,v)\,dv\ , \\ \lim_{n\to\infty} K^{H,n}_{t,s}(x,y)\,dy\Big|_{\mathcal{S}_2} &= e^{-(\pi^2/2)(t-s)} K^{S}_{t,s}(u,v)\,dv\ . \end{aligned} \tag{247}$$

Remark. The proof involves careful estimating and Riemann–Hilbert techniques and is found in [6].

Proof of Theorem 4.2.

Rescale in Theorem 4.1

$$a_i = \sqrt{2n} + \frac{u_i}{\sqrt{2}n^{1/6}}\ , \quad b_i = \sqrt{2n} + \frac{v_i}{\sqrt{2}n^{1/6}}\ , \quad t = \frac{\tau}{n^{1/3}} \tag{248}$$

and then from Proposition 4.1 it follows that, with derivatives that,

$$G_n\left(\frac{\tau}{n^{1/3}}, a; b\right) = G(\tau, u, v) + O\left(\frac{1}{k}\right)\ , \quad k = n^{1/6}\ . \tag{249}$$

We now do large n asymptotics on the operators $\mathcal{A}_i$, $\mathcal{B}_i$, setting $L = L_u + L_v$, $E = E_u + E_v$, with L_u, L_v, E_u, E_v defined in Theorem 4.2; we find:

$$\begin{aligned}
\mathcal{A}_1 &= \sqrt{2}k\left(L - \left(\frac{\tau}{k^2} - \frac{\tau^2}{2k^4} + \frac{\tau^3}{6k^6}\right)L_v + O\left(\frac{1}{k^8}\right)\right)\ ,\\
\mathcal{B}_1 &= \sqrt{2}k\left(L - \left(\frac{\tau}{k^2} - \frac{\tau^2}{2k^4} + \frac{\tau^3}{6k^6}\right)L_u + O\left(\frac{1}{k^8}\right)\right)\ ,\\
\mathcal{A}_2 &= 2k^4\left(L - \frac{2\tau}{k^2}L_v + \frac{1}{2k^4}(E - 1 + 4\tau^2 L_v)\right.\\
&\qquad\qquad\left. - \frac{\tau}{k^6}\left(E_v - 1 + \frac{4}{3}\tau^2 L_v\right) + O\left(\frac{1}{k^8}\right)\right)\ ,\\
\mathcal{B}_2 &= 2k^4\left(L - \frac{2\tau}{k^2}L_u + \frac{1}{2k^4}(E - 1 + 4\tau^2 L_u)\right.\\
&\qquad\qquad\left. - \frac{\tau}{k^6}\left(E_u - 1 + \frac{4}{3}\tau^2 L_u\right) + O\left(\frac{1}{k^8}\right)\right)\ ,
\end{aligned} \tag{250}$$

and consequently

$$\begin{aligned}
&\frac{1}{2\sqrt{2}k^5}\mathcal{B}_2\mathcal{A}_1\\
&= L^2 - \frac{\tau}{k^2}(L + L_u)L + \frac{1}{2k^4}\big(L(E-2) + \tau^2\big(4L_u(L + L_v) + LL_v\big)\big)\\
&\quad - \frac{\tau}{k^6}\left(L(E_u - 2) + \frac{1}{2}L_v(E+2) + \frac{\tau^2}{6}(8LL_u + 18L_uL_v + LL_v)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad + O\left(\frac{1}{k^8}\right)
\end{aligned} \tag{251}$$

$$\frac{1}{2k^2}\mathcal{B}_1\mathcal{A}_1 = L^2 - \frac{\tau}{k^2}L^2 + \frac{\tau^2}{k^4}\left(\frac{1}{2}L^2 + L_uL_v\right) - \frac{\tau^3}{k^6}\left(\frac{1}{6}L^2 + L_uL_v\right) + O\left(\frac{1}{k^8}\right)\ .$$

Feeding these estimates and (249) into the relation (232) of Theorem 4.1, multiplied by $(\mathcal{B}_1\mathcal{A}_1G_n + 2nc)^2$, which by the quotient rule becomes an identity involving Wronskians, we then find ($\{f, g\}_X = gXf - fXg$)

$$
\begin{aligned}
0 &= \left\{ \frac{1}{2\sqrt{2}k^5} \mathcal{B}_2 \mathcal{A}_1 G_n, \frac{1}{2k^2} (\mathcal{B}_1 \mathcal{A}_1 G_n + 2k^6 e^{-\tau/k^2}) \right\}_{\mathcal{A}_1/(\sqrt{2}k)} \\
&\qquad - \left\{ \frac{1}{2\sqrt{2}k^5} \mathcal{A}_2 \mathcal{B}_1 G_n, \frac{1}{2k^2} (\mathcal{A}_1 \mathcal{B}_1 G_n + 2k^6 e^{-\tau/k^2}) \right\}_{\mathcal{B}_1/(\sqrt{2}k)} \\
&= \frac{2\tau}{k^2} \Big[\big((L_u + L_v)(L_u E_v - L_v E_u) + \tau^2 (L_u - L_v) L_u L_v \big) G_n \\
&\qquad - \frac{1}{2} \{ (L_u^2 - L_v^2) H_n, (L_u + L_v)^2 G_n \}_{L_u + L_v} \Big] + O\left(\frac{1}{k^3} \right) \\
&= \frac{2\tau}{k^2} \Big[\big((L_u + L_v)(L_u E_v - L_v E_u) + \tau^2 (L_u - L_v) L_u L_v \big) G \\
&\qquad - \frac{1}{2} \{ (L_u^2 - L_v^2) G, (L_u + L_v)^2 G \}_{L_u + L_v} \Big] + O\left(\frac{1}{k^3} \right) .
\end{aligned}
$$

In this calculation, we used the linearity of the Wronskian $\{X, Y\}_Z$ in its three arguments and the following commutation relations:

$$
[L_u, E_u] = L_u, \quad [L_u, E_v] = [L_u, L_v] = [L_u, \tau] = 0, \quad [E_u, \tau] = \tau ,
$$

including their dual relations by $u \leftrightarrow v$; also we have $\{L^2 G, 1\}_{L_u - L_v} = \{L(L_u - L_v)G, 1\}_L$. It is also useful to note that the two Wronskians in the first expression are dual to each other by $u \leftrightarrow v$. The point of the computation is to preserve the Wronskian structure up to the end. This proves Theorem 4.2, upon replacing $\tau \to t$.

Proof of Corollary 4.1.

Equation (233) for the probability

$$
G(\tau; u, v) := \log P(A(\tau_1) \leqslant u, A(\tau_2) \leqslant v), \quad \tau = \tau_2 - \tau_1 ,
$$

takes on the explicit form

$$
\begin{aligned}
\tau \frac{\partial}{\partial \tau} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) G &= \frac{\partial^3 G}{\partial u^2 \partial v} \left(2 \frac{\partial^2 G}{\partial v^2} + \frac{\partial^2 G}{\partial u \partial v} - \frac{\partial^2 G}{\partial u^2} + u - v - \tau^2 \right) \\
&\quad - \frac{\partial^3 G}{\partial v^2 \partial u} \left(2 \frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial u \partial v} - \frac{\partial^2 G}{\partial v^2} - u + v - \tau^2 \right) \\
&\quad + \left(\frac{\partial^3 G}{\partial u^3} \frac{\partial}{\partial v} - \frac{\partial^3 G}{\partial v^3} \frac{\partial}{\partial u} \right) \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) G . \qquad (252)
\end{aligned}
$$

This equation enjoys an obvious $u \leftrightarrow v$ duality. Finally the change of variables in the statement of Corollary 4.1 leads to (234).

The proof of Theorem 4.3 is done in the same spirit as that of Theorem 4.2 and Corollary 4.2 follows immediately by substitution in Theorem 4.3. Next, we need some preliminaries to prove Theorem 4.4. The first being:

Proposition 4.2. *The following ratio of probabilities admits the asymptotic expansion for large $t > 0$ in terms of functions $f_i(u,v)$, symmetric in u and v*

$$\frac{P(A(0) \leqslant u, A(t) \leqslant v)}{P(A(0) \leqslant u)P(A(t) \leqslant v)} = 1 + \sum_{i \geqslant 1} \frac{f_i(u,v)}{t^i} , \tag{253}$$

from which it follows that

$$\lim_{t \to \infty} P(A(0) \leqslant u, A(t) \leqslant v) = P(A(0) \leqslant u)P(A(t) \leqslant v) = F_2(u)F_2(v) ,$$

this means that the Airy process decouples at ∞.

The proof necessitates using the extended Airy kernel. Note, since the probabilities in (253) are symmetric in u and v, the coefficients f_i are symmetric as well. The last equality in the formula above follows from stationarity and (237).

Conjecture. The coefficients $f_i(u,v)$ have the property

$$\lim_{u \to \infty} f_i(u,v) = 0 \qquad \textit{for fixed } v \in \mathbb{R} , \tag{254}$$

and

$$\lim_{z \to \infty} f_i(-z, z+x) = 0 \quad \textit{for fixed } x \in \mathbb{R} . \tag{255}$$

The justification for this plausible conjecture will now follow. First, considering the following conditional probability:

$$\begin{aligned} P(A(t) \leqslant v \mid A(0) \leqslant u) &= \frac{P(A(0) \leqslant u, A(t) \leqslant v)}{P(A(0) \leqslant u)} \\ &= F_2(v) \left(1 + \sum_{i \geqslant 1} \frac{f_i(u,v)}{t^i} \right) , \end{aligned}$$

and letting $v \to \infty$, we have automatically

$$\begin{aligned} 1 = \lim_{v \to \infty} P(A(t) \leqslant v \mid A(0) \leqslant u) &= \lim_{v \to \infty} \left[F_2(v) \left(1 + \sum_{i \geqslant 1} \frac{f_i(u,v)}{t^i} \right) \right] \\ &= 1 + \lim_{v \to \infty} \sum_{i \geqslant 1} \frac{f_i(u,v)}{t^i} , \end{aligned}$$

which would imply, assuming the interchange of the limit and the summation is valid,

$$\lim_{v \to \infty} f_i(u,v) = 0 , \tag{256}$$

and, by symmetry

$$\lim_{u\to\infty} f_i(u,v) = 0 \ .$$

To deal with (255) we assume the following *nonexplosion* condition for any fixed $t > 0$, $x \in \mathbb{R}$, namely, that the conditional probability satisfies

$$\lim_{z\to\infty} P(A(t) \geqslant x + z \mid A(0) \leqslant -z) = 0 \ .$$

Hence, the conditional probability satisfies, upon setting

$$v = z + x \ , \quad u = -z \ ,$$

and using $\lim_{z\to\infty} F_2(z+x) = 1$, the following:

$$1 = \lim_{z\to\infty} P(A(t) \leqslant z + x \mid A(0) \leqslant -z) = 1 + \lim_{z\to\infty} \sum_{i\geqslant 1} \frac{f_i(-z, z+x)}{t^i} \ ,$$

which, assuming the validity of the same interchange, implies that

$$\lim_{z\to\infty} f_i(-z, z+x) = 0 \quad \text{for all } i \geqslant 1 \ .$$

Proof of Theorem 4.4.

Putting the log of the expansion (253)

$$\begin{aligned} G(t;u,v) &= \log P(A(0) \leqslant u, A(t) < v) \\ &= \log F_2(u) + \log F_2(v) + \sum_{i\geqslant 1} \frac{h_i(u,v)}{t^i} \qquad (257) \\ &= \log F_2(u) + \log F_2(v) + \frac{f_1(u,v)}{t} + \frac{f_2(u,v) - f_1^2(u,v)/2}{t^2} + \cdots \ , \end{aligned}$$

into (252) leads to:

(i) a leading term of order t, given by

$$\mathcal{L} h_1 = 0 \ , \qquad (258)$$

where

$$\mathcal{L} := \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) \frac{\partial^2}{\partial u \partial v} \ . \qquad (259)$$

The most general solution to (258) is given by

$$h_1(u,v) = r_1(u) + r_3(v) + r_2(u+v) \ ,$$

with arbitrary functions r_1, r_2, r_3. Hence,

$$P(A(0) \leqslant u, A(t) \leqslant v) = F_2(u)F_2(v)\left(1 + \frac{h_1(u,v)}{t} + \cdots\right)$$

with $h_1(u,v) = f_1(u,v)$ as in (253). Applying (254)

$$r_1(u) + r_3(\infty) + r_2(\infty) = 0 \quad \text{for all } u \in \mathbb{R} ,$$

implying

$$r_1(u) = \text{ constant } = r_1(\infty) ,$$

and similarly

$$r_3(u) = \text{ constant } = r_3(\infty) .$$

Therefore, without loss of generality, we may absorb the constants $r_1(\infty)$ and $r_3(\infty)$ in the definition of $r_2(u+v)$. Hence, from (257),

$$f_1(u,v) = h_1(u,v) = r_2(u+v)$$

and using (255),

$$0 = \lim_{z\to\infty} f_1(-z, z+x) = r_2(x) ,$$

implying that the $h_1(u,v)$–term in the series (257) vanishes.

(ii) One computes that the term $h_2(u,v)$ in the expansion (257) of $G(t;u,v)$ sastisfies

$$\mathcal{L}h_2 = \frac{\partial^3 g}{\partial u^3}\frac{\partial^2 g}{\partial v^2} - \frac{\partial^3 g}{\partial v^3}\frac{\partial^2 g}{\partial u^2} \quad \text{with } g(u) := \log F_2(u) . \tag{260}$$

This is the term of order t^0, by putting the series (257) in (252). The most general solution to (260) is

$$h_2(u,v) = g'(u)g'(v) + r_1(u) + r_3(v) + r_2(u+v) .$$

Then

$$\begin{aligned} P(A(0) \leqslant u, A(t) \leqslant v) &= e^{G(t;u,v)} \\ &= F_2(u)F_2(v) \exp \sum_{i\geqslant 2} \frac{h_i(u,v)}{t^i} \\ &= F_2(u)F_2(v)\left(1 + \frac{h_2(u,v)}{t^2} + \cdots\right) . \end{aligned} \tag{261}$$

In view of the explicit formula for the distribution F_2 (237) and the behavior of $q(\alpha)$ for $\alpha \nearrow \infty$, we have that

$$\begin{aligned} \lim_{u\to\infty} g'(u) &= \lim_{u\to\infty} \left(\log F_2(u)\right)' \\ &= \lim_{u\to\infty} \int_u^\infty q^2(\alpha)\, d\alpha = 0 . \end{aligned}$$

Hence

$$0 = \lim_{u\to\infty} f_2(u,v) = \lim_{u\to\infty} h_2(u,v) = r_1(\infty) + r_3(v) + r_2(\infty) \ ,$$

showing r_3 and similarly r_1 are constants. Therefore, by absorbing $r_1(\infty)$ and $r_3(\infty)$ into $r_2(u+v)$, we have

$$f_2(u,v) = h_2(u,v) = g'(u)g'(v) + r_2(u+v) \ .$$

Again, by the behavior of $q(x)$ at $+\infty$ and $-\infty$, we have for large $z > 0$,

$$g'(-z)g'(z+x) = \int_{-z}^{\infty} q^2(\alpha)\,d\alpha \int_{z+x}^{\infty} q^2(\alpha)\,d\alpha \leqslant cz^{3/2}e^{-2z/3} \ .$$

Hence

$$0 = \lim_{z\to\infty} f_2(-z, z+x) = r_2(x)$$

and so

$$f_2(u,v) = h_2(u,v) = g'(u)g'(v) \ ,$$

yielding the $1/t^2$ term in the series (257), and so it goes.

Finally, to prove (241), we compute from (239), after integration by parts and taking into account the boundary terms using (238):

$$\begin{aligned} E\big(A(0)A(t)\big) &= \iint\limits_{\mathbb{R}^2} uv \frac{\partial^2}{\partial u \partial v} P(A(0) \leqslant u, A(t) \leqslant v)\,du\,dv \\ &= \int_{-\infty}^{\infty} uF_2'(u)\,du \int_{-\infty}^{\infty} vF_2'(v)\,dv \\ &\qquad + \frac{1}{t^2}\int_{-\infty}^{\infty} F_2'(u)\,du \int_{-\infty}^{\infty} F_2'(v)\,dv \\ &\qquad + \frac{1}{t^4}\iint\limits_{\mathbb{R}^2} \big(\Phi(u,v) + \Phi(v,u)\big)\,du\,dv \\ &\qquad + O\left(\frac{1}{t^6}\right) \\ &= \Big(E\big(A(0)\big)\Big)^2 + \frac{1}{t^2} + \frac{c}{t^4} + O\left(\frac{1}{t^6}\right) \ , \end{aligned}$$

where

$$c := \iint\limits_{\mathbb{R}^2} \big(\Phi(u,v) + \Phi(v,u)\big)\,du\,dv = 2\iint\limits_{\mathbb{R}^2} \Phi(u,v)\,du\,dv \ ,$$

thus ending the proof of Theorem 4.4.

5 The Pearcey Distribution

5.1 GUE with an external source and Brownian motion

In this section we discuss the equivalence of GUE with an external source, introduced by Brézin–Hikami [27] and a conditional Brownian motion, following Aptkarev, Bleher and Kuijlaars [21].

Non-intersecting Brownian paths:

Consider n–non-intersecting Brownian paths with predetermined endpoints at $t = 0, 1$, as specified in Figure 2.

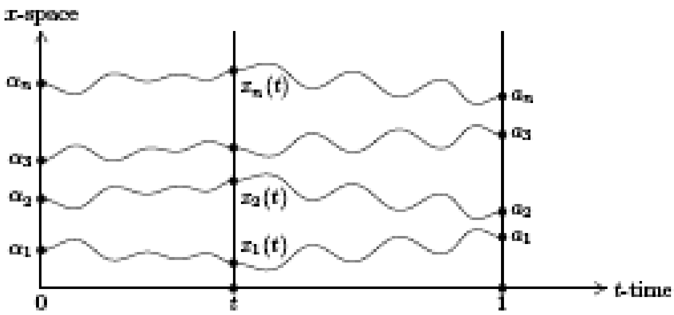

Fig. 2.

By the Karlin–McGregor formula [42], the above situation has probability density

$$\begin{aligned} p_n(t, x_1, \ldots, x_n) &= \frac{1}{Z_n} \det \big(p(\alpha_i, x_j, t)\big)_{\substack{1 \leqslant i, \\ j \leqslant n}} \det \big(p(x_i, a_j, 1-t)\big)_{\substack{1 \leqslant i, \\ j \leqslant n}} \\ &= \frac{1}{Z'_n} \prod_1^n e^{-x_i^2/t(1-t)} \det(e^{2a_i x_j/t})_{\substack{1 \leqslant i, \\ j \leqslant n}} \det(e^{2a_i x_j/1-t})_{\substack{1 \leqslant i, \\ j \leqslant n}} , \end{aligned} \tag{262}$$

with[12]

$$p(x, y, t) = \frac{e^{-(x-y)^2/t}}{\sqrt{\pi t}} . \tag{263}$$

For example,[13] let all the particles start out at $x = 0$, at $t = 0$, with n_1 particles ending up at a, n_2 ending up at $-a$ at $t = 1$, with $n = n_1 + n_2$.

[12] Here $Z_n = Z_n(a, \alpha)$.

[13] Obviously, implicit in this example is a well-defined limit as the endpoints come together.

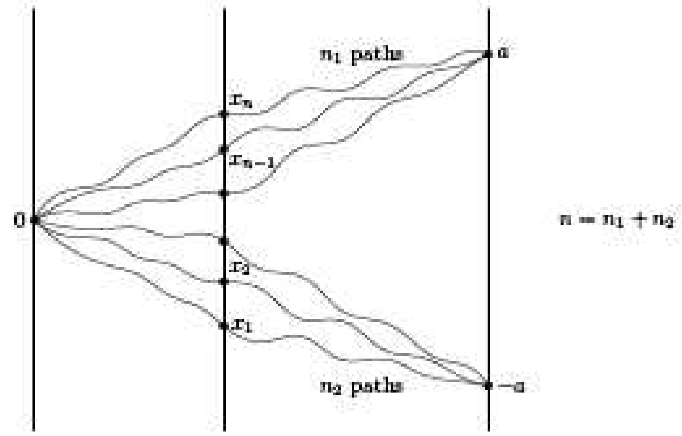

Fig. 3.

Here

$$p_{n_1,n_2}(t, x_1, \ldots, x_n) = \frac{1}{Z_{n_1,n_2,a}} \Delta_{n_1+n_2}(x) \det \begin{pmatrix} \left(\psi^+_{i_+}(x_j)\right)_{\substack{1\leqslant i_+\leqslant n_1,\\ 1\leqslant j\leqslant n_1+n_2}} \\ \left(\psi^-_{i_-}(x_j)\right)_{\substack{1\leqslant i_-\leqslant n_2,\\ 1\leqslant j\leqslant n_1+n_2}} \end{pmatrix} ,$$

with

$$\psi^{\pm}_i(x) = x^{i-1} e^{-x^2/t(1-t) \pm 2ax/(1-t)} .$$

So setting $E = \bigcup_{1\leqslant i\leqslant r} [b_{2i-1}, b_{2i}]$, we find

$$\begin{aligned} P^a_{n_1,n_2}(t, b) &:= \operatorname{Prob}^a_{n_1,n_2}(\text{all } x_i(t) \subset E) \\ &:= \operatorname{Prob}\left(\text{all } x_i(t) \subset E \;\middle|\; \begin{array}{l} n_1 \text{ left-most paths end up at } a \\ n_2 \text{ right-most paths end up at } -a \\ \text{and all start at 0, with all paths} \\ \text{non-intersecting.} \end{array} \right) \\ &= \frac{\int_{E^n} \prod_1^n dx_i \Delta_n(x) \det \begin{pmatrix} \left(\psi^+_{i_+}(x_j)\right)_{\substack{1\leqslant i_+\leqslant n_1,\\ 1\leqslant j\leqslant n_1+n_2}} \\ \left(\psi^-_{i_-}(x_j)\right)_{\substack{1\leqslant i_-\leqslant n_2,\\ 1\leqslant j\leqslant n_1+n_2}} \end{pmatrix}}{\int_{\mathbb{R}^n} \prod_1^n dx_i \Delta_n(x) \det \begin{pmatrix} \left(\psi^+_{i_+}(x_j)\right)_{\substack{1\leqslant i_+\leqslant n_1,\\ 1\leqslant j\leqslant n_1+n_2}} \\ \left(\psi^-_{i_-}(x_j)\right)_{\substack{1\leqslant i_-\leqslant n_2,\\ 1\leqslant j\leqslant n_1+n_2}} \end{pmatrix}} . \end{aligned} \tag{264}$$

Random matrix with external source:

Consider the ensemble, introduced by Brézin–Hikami [27], on $n \times n$ Hermitian matrices $\mathcal{H}_n$

$$P\big(M \in (M, M+dM)\big) = \frac{1}{Z_n} e^{\operatorname{tr}(-V(M)+AM)}\, dM \,, \tag{265}$$

with

$$A = \operatorname{diag}(a_1, \ldots, a_n) \,.$$

By HCIZ (see (138)), we find

$$\begin{aligned} P\big(\text{spec } (M) \subset E\big) &= \frac{1}{Z_n} \int_{E^n} \Delta_n^2(z) \prod_1^n e^{-V(z_i)}\, dz_i \int_{U(n)} e^{\operatorname{tr} AUAU^{-1}}\, dU \\ &= \frac{1}{Z_n'} \int_{E^n} \Delta_n^2(z) \prod_1^n e^{-V(z_i)}\, dz_i \frac{\det[e^{a_i z_j}]_{\substack{1\leqslant i,\\ j\leqslant n}}}{\Delta_n(z)\Delta_n(a)} \\ &= \frac{1}{Z_n''} \int_{E^n} \Delta_n(z) \prod_1^n e^{-V(z_i)}\, dz_i \det[e^{a_i z_j}]_{\substack{1\leqslant i,\\ j\leqslant n}} \,. \end{aligned} \tag{266}$$

For example: consider the limiting case

$$A = \operatorname{diag}\big(\underbrace{-a, -a, \ldots, -a}_{n_2}, \underbrace{a, a, \ldots, a}_{n_1}\big) \,, \quad E = \bigcup_{i=1}^r [b_{2i-1}, b_{2i}] \,, \tag{267}$$

$$\begin{aligned} &P(\operatorname{spec}(M) \subset E) := P_{n_1,n_2}(a,b) \\ &= \frac{\int_{E^n} \prod_1^n dx_i \Delta_n(z) \det \begin{pmatrix} \big(\rho^+_{i_+}(x_j)\big)_{\substack{1\leqslant i_+\leqslant n_1,\\ 1\leqslant j\leqslant n_1+n_2}} \\ \big(\rho^-_{i_-}(x_j)\big)_{\substack{1\leqslant i_-\leqslant n_2,\\ 1\leqslant j\leqslant n_1+n_2}} \end{pmatrix}}{\int_{\mathbb{R}^n} \prod_1^n dx_i \Delta_n(z) \det \begin{pmatrix} \big(\rho^+_{i_+}(x_j)\big)_{\substack{1\leqslant i_+\leqslant n_1,\\ 1\leqslant j\leqslant n_1+n_2}} \\ \big(\rho^-_{i_-}(x_j)\big)_{\substack{1\leqslant i_-\leqslant n_2,\\ 1\leqslant j\leqslant n_1+n_2}} \end{pmatrix}} \,. \end{aligned} \tag{268}$$

where

$$\rho_i^{\pm}(z) = z^{i-1} e^{-V(z) \pm az}, \quad n = n_1 + n_2 \,. \tag{269}$$

Then we have by Aptkarev, Bleher, Kuijlaars [21]:

Non-intersecting Brownian motion $\Leftrightarrow$ GUE *with external source*

$$P^a_{n_1,n_2}(t,b) = P_{n_1,n_2}\left(\sqrt{\frac{2t}{1-t}}a, \sqrt{\frac{2}{t(1-t)}}b\right)\Big|_{V(z)=z^2/2} \,, \tag{270}$$

so the two problems: non-intersecting Brownian motion and GUE with an external source, are equivalent!

5.2 MOPS and a Riemann–Hilbert problem

In this section, we introduce multiple orthogonal polynomials (MOPS), following Bleher and Kuijlaars [24]. This lead them to a determinal m–point correlation function for the GUE with external source, in terms of a "Christoffel–Darboux" kernel for the MOPS, as in the pure GUE case. In addition, they formulated a Riemann–Hilbert (RH) problem for the MOPS, analogous to that for classical orthogonal polynomials, thus enabling them to understand universal behavior for the MOPS and hence universal behavior for the GUE with external source (see [21, 25]).

Let us first order the spectrum of A of (267) in some definite fashion, for example

$$\text{Ordered spectrum of } (A) = (-a, a, a, -a, \ldots, -a) := (\alpha_1, \alpha_2, \ldots, \alpha_n)\ .$$

For each $k = 0, 1, \ldots, n$, let $k = k_1 + k_2$, k_1, k_2 defined as follows:

$$\begin{aligned} k_1 &:= \#\text{ times } \quad a \text{ appears in } \alpha_1, \ldots, a_k\ , \\ k_2 &:= \#\text{ times } -a \text{ appears in } \alpha_1, \ldots, a_k\ . \end{aligned} \tag{271}$$

We now define the 2 kinds of MOPS.

MOP II: Define a unique monic kth degree polynomial $p_k = p_{k_1,k_2}$:

$$\begin{aligned} p_k(x) = p_{k_1,k_2}(x)\colon \int_{\mathbb{R}} p_{k_1,k_2}(x)\rho^{\pm}_{i_\pm}(x)\,dx = 0 \\ \rho_i^{\pm}(x) = x^{i-1}e^{-V(x)\pm ax}\ , \quad 1 \leqslant i_+ \leqslant k_1\ ,\ 1 \leqslant i_- \leqslant k_2\ , \end{aligned} \tag{272}$$

MOP I: Define unique polynomials $q^+_{k_1-1,k_2}(x)$, $q^-_{k_1,k_2-1}(x)$ of respective degrees $k_1 - 1$, $k_2 - 1$:

$$\begin{aligned} q_{k-1}(x) := q_{k_1,k_2}(x) = q^+_{k_1-1,k_2}(x)\rho_1^+(x) + q^-_{k_1,k_2-1}(x)\rho_1^-(x) \\ : \int_{\mathbb{R}} x^j q_{k-1}(x)\,dx = \delta_{j,k-1}\ , \quad 0 \leqslant j \leqslant k-1\ , \end{aligned} \tag{273}$$

which immediately yields:

Bi-orthonal polynomials:

$$\int_{\mathbb{R}} p_j(x)q_k(x)\,dx = \delta_{j,k} \quad j, k = 0, 1, \ldots, n-1\ . \tag{274}$$

This leads to a Christoffel–Darboux kernel, as in (5):

$$K^{(a)}_{n_1,n_2}(x,y) := K_n(x,y) := e^{-1/2V(x)+1/2V(y)} \sum_0^{n-1} p_k(x)q_k(y)\ , \tag{275}$$

which is independent of the ad hoc ordering of the spectrum of A and which, due to bi-orthonality, has the usual reproducing properties:

$$\int_{-\infty}^{\infty} K_n(x,x)\,dx = n \ , \quad \int_{-\infty}^{\infty} K_n(x,y)K_n(y,z)\,dy = K_n(x,z) \ . \tag{276}$$

The joint probability density can be written in terms of K_n,

$$\frac{1}{Z_n} e^{\mathrm{Tr}(-V(\Lambda)+A\Lambda)} \Delta_n(\lambda) = \frac{1}{n!} \det[K_n(\lambda_i,\lambda_j)]_{\substack{1\leqslant i,\\ j\leqslant n}} \ , \tag{277}$$

with $\Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$, yielding the m–point correlation function:

$$R_m(\lambda_1,\dots,\lambda_m) = \det[K_n(\lambda_i,\lambda_j)]_{\substack{1\leqslant i,\\ j\leqslant m}} \ , \tag{278}$$

and we find the usual Fredholm determinant formula:

$$P(\mathrm{spec}(M) \subset E^c) = \det\big(I - K_n(x,y)I_E(y)\big) \ . \tag{279}$$

Finally, we have a Riemann–Hilbert (RH) problem for the MOPS.

Riemann–Hilbert problem for MOPS:

MOP II:

$$Y(z) := \begin{bmatrix} p_{n_1,n_2}(z) & C_+p_{n_1,n_2} & C_-p_{n_1,n_2} \\ c_1p_{n_1-1,n_2}(z) & c_1C_+p_{n_1-1,n_2} & c_1C_-p_{n_1-1,n_2} \\ c_2p_{n_1,n_2-1}(z) & c_2C_+p_{n_1,n_2-1} & c_2C_-p_{n_1,n_2-1} \end{bmatrix} \tag{280}$$

with $C_\pm$ Cauchy transforms:

$$C_\pm f(z) := \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(s)\rho_1^\pm(s)\,ds}{s-z} \ , \quad \rho_1^\pm(z) = e^{-V(z)\pm az} \ . \tag{281}$$

Then $Y(z)$ satisfies the RH problem:

1. $Y(z)$ analytic on $\mathbb{C}\backslash\mathbb{R}$.
2. Jump condition for $x \in \mathbb{R}$:

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & \rho_1^+(x) & \rho_1^-(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

3. Behavior as $z \to \infty$

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^{n_1+n_2} & & \\ & z^{-n_1} & \\ & & z^{-n_2} \end{pmatrix} . \tag{282}$$

MOP I: A dual RH problem for $q^\pm_{k_1,k_2}$ and $(Y^{-1})^T$.

Finally we have a Christoffel–Darboux type formula (see (5)) for the kernel $K^{(a)}_{n,n}(x,y)$ of (275) expressed in terms of the RH matrix (280):

$$K_{n,n}^{(a)}(x,y) = \frac{e^{-1/4(x^2+y^2)}}{2\pi i(x-y)}(0, e^{ay}, e^{-ay})Y^{-1}(y)Y(x)\begin{pmatrix}1\\0\\0\end{pmatrix}. \tag{283}$$

Thus to understand the large n asymptotics of the GUE with external source, from (277), (278), and (279), it suffices to understand the asymptotics of $K_{n,n}^{(a)}(x,y)$ given by (275). Thus by (283) it suffices to understand the asymptotics of the solution $Y(z)$ to the RH problem of (282), which is the subject of [21] and [25].

5.3 Results concerning universal behavior

In this section we will first discuss universal behavior for the equivalent random matrix and Brownian motion models, leading to the Pearcey process. We will then give a PDE of Adler–van Moerbeke [7] governing this behavior, and finally a PDE for the n-time correlation function of this process, deriving the first PDE in the following sections. The following pictures illustrate the situation.

Universal behavior:

I. *Brownian motion*: $2n$ paths, $a = \sqrt{n}$

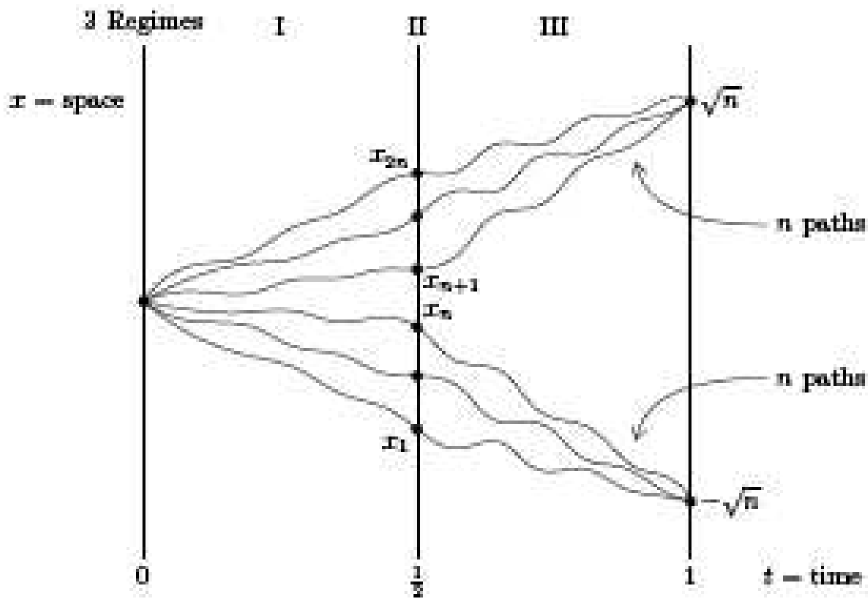

Fig. 4.

At $t = \frac{1}{2}$ the Brownian paths start to separate into 2 distinct groups.

II. *Random matrices*: $n_1 = n_2 = n$, $V(z) = z^2/2$, $a := \hat{a}\sqrt{2n}$.

Density of eigenvalues: $\rho(x) := \lim_{n\to\infty}(K_{n,n}^{\hat{a}\sqrt{2n}}(\sqrt{2n}x, \sqrt{2n}x))/2n$

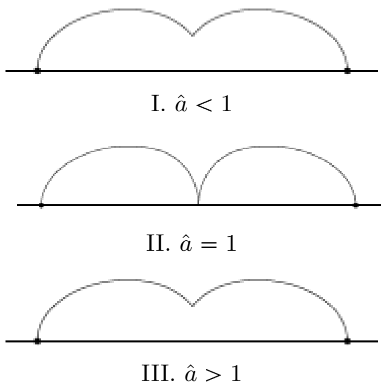

Fig. 5.

The 3 corresponding regimes, I, II and III, for the random matrix density of states $\rho(x)$ and thus the corresponding situation for the Brownian motion, are explained by the following theorem:

Theorem 5.1 (Aptkarev, Bleher, Kuijlaars [21]). *For the* GUE *with external source, the limiting mean density of states for* $\hat{a} > 0$ *is:*

$$\rho(x) := \lim_{n\to\infty} \frac{K_{n,n}^{\hat{a}\sqrt{2n}}(\sqrt{2n}x, \sqrt{2n}x)}{2n} = \frac{1}{\pi}|\operatorname{Im}\xi(x)| \, , \tag{284}$$

with

$$\xi(x)\colon \xi^3 - x\xi^2 - (\hat{a}^2 - 1)\xi + x\hat{a}^2 = 0 \quad (\textit{Pastur's equation } [45]) \, ,$$

yielding the density of eigenvalues pictures.

It is natural to look for universal behavior near $\hat{a} = 1$ by looking with a microscope about $x = 0$. Equivalently, thinking in terms of the $2n$–Brownian motions, one sets $a = \sqrt{n}$ and about $t = \frac{1}{2}$ one looks with a microscope near $x = 0$ to see the $2n$–Brownian motions slowly separating into two distinct groups. Rescale as follows:

$$t = \frac{1}{2} + \frac{\tau}{\sqrt{n}}, \quad x = \frac{u}{n^{1/4}}, \quad a = \sqrt{n} \, . \tag{285}$$

Remembering the equivalence between Brownian motion and the GUE with external source, namely (270), the Fredholm determinant formula (279), for the GUE with external source, yields:

$$\mathrm{Prob}^a_{n_1,n_2}(\text{all } x_i(t) \subset E^c) = \det(I - \widetilde{K}^E_n) \; ,$$
$$\widetilde{K}^E_n(x,y) = \sqrt{\frac{2}{t(1-t)}} K^{\sqrt{2ta/(1-t)}}_{n_1,n_2}\left(\sqrt{\frac{2}{t(1-t)}}\, x, \sqrt{\frac{2}{t(1-t)}}\, y\right) I_E(y) \; . \tag{286}$$

Universal behavior for $n \to \infty$ amounts to understanding $\widetilde{K}^E_n(x,y)$ for $n \to \infty$, under the rescaling (285) and indeed we have the results:

Universal Pearcey limiting behavior:

Theorem 5.2 (Tracy–Widom [54]). *Upon rescaling the Brownian kernel, we find the following limiting behavior, with derivates*:

$$\lim_{n\to\infty} n^{-1/4} \sqrt{\frac{2}{t(1-t)}} K^{\sqrt{2ta/(1-t)}}_{n,n}\left(\sqrt{\frac{2}{t(1-t)}}\, x, \sqrt{\frac{2}{t(1-t)}}\, y\right)\Bigg|_{\substack{a=\sqrt{n},\\ t=\frac{1}{2}+\tau/\sqrt{n},\\ (x,y)=(u,v)/n^{1/4}}} = K^P_\tau(u,v) \; ,$$

with the Pearcy kernel $K_\tau(u,v)$ of Brézin–Hikami [27] defined as follows:

$$\begin{aligned} K_\tau(x,y) &:= \frac{p(x)q''(y) - p'(x)q'(y) + p''(x)q(y) - \tau p(x)q(y)}{x-y} \\ &= \int_0^\infty p(x+z)q(y+z)\,dz \; , \end{aligned} \tag{287}$$

where (*note* $\omega = e^{i\pi/4}$)

$$\begin{aligned} p(x) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-u^4/4-\tau u^2/2-iux}\,du \; , \\ q(y) &:= \frac{1}{2\pi}\int_X e^{u^4/4-\tau u^2/2+uy}\,du \\ &\qquad = \mathrm{Im}\left[\frac{\omega}{\pi}\int_0^\infty du e^{-u^4/4-i\tau u^2/2}(e^{\omega uy} - e^{-\omega uy})\right] \end{aligned} \tag{288}$$

satisfy the differential equations (*adjoint to each other*)

$$p''' - \tau p' - xp = 0 \quad \text{and} \quad q''' - \tau q' + yq = 0 \; .$$

The contour X is given by the ingoing rays from $\pm\infty e^{i\pi/4}$ to 0 and the outgoing rays from 0 to $\pm\infty e^{-i\pi/4}$.

Theorem 5.2 allows us to define the Pearcey process $\mathcal{P}(\tau)$ as the motion of an infinite number of non-intersecting Brownian paths, near $t = \frac{1}{2}$, upon taking a limit in (286), using the precise scaling of (285), to wit:

$$\lim_{n\to\infty} \mathrm{Prob}_{n,n}^{\sqrt{n}}\left(\text{all } n^{1/4}x_i\left(\frac{1}{2}+\frac{\tau}{\sqrt{n}}\right)\notin E\right) = \det(I-K_\tau^P I_E)$$
$$=: \mathrm{Prob}(\mathcal{P}(\tau)\notin E)\ , \quad (289)$$

which defines for us the Pearcey process. Note the pathwise interpretation of $\mathcal{P}(\tau)$ certainly needs to be resolved. The Pearcey distribution with the parameter τ can also be interpreted as the transitional probability for the Pearcey process. We now give a PDE for the distribution, which shall be derived in the following section:

Theorem 5.3 (Adler–van Moerbeke [7]).
For compact $E=\bigcup_{i=1}^r[u_{2i-1},u_{2i}]$,

$$F(\tau;u_1,\dots,u_{2r}) := \log \mathrm{Prob}(\mathcal{P}(\tau)\notin E) \quad (290)$$

satisfies the following 4*th order,* 3*rd degree* PDE *in* τ *and the* u_i:

$$B_{-1}\left(\frac{\frac{1}{2}(\partial^3F)/(\partial\tau^3)+(B_0-2)B_{-1}^2F+\frac{1}{16}\{B_{-1}\partial F/\partial\tau, B_{-1}^2F\}_{B_{-1}}}{B_{-1}^2\partial F/\partial\tau}\right) = 0\ , \quad (291)$$

where

$$B_{-1}=\sum_1^{2r}=\frac{\partial}{\partial u_i}\ , \quad B_0=\sum_1^{2r}=u_i\frac{\partial}{\partial u_i}\ . \quad (292)$$

It is natural to ask about the joint Pearcey distribution involving k–times, namely:

$$\lim_{n\to\infty} \mathrm{Prob}_{n,n}^{\sqrt{n}}\left(\text{all } n^{1/4}x_i\left(\frac{1}{2}+\frac{\tau_j}{\sqrt{n}}\right)\notin E_j, 1\leqslant j\leqslant k\right)$$
$$= \mathrm{Prob}(\mathcal{P}(\tau_j)\notin E_j, 1\leqslant i\leqslant k)$$
$$= \det\left(I-(I_{E_i}K^P_{\tau_i\tau_j}I_{E_j})_{\substack{1\leqslant i,\\ j\leqslant k}}\right)\ , \quad (293)$$

where the above is a Fredholm determinant involving a matrix kernel, and the extended Pearcey kernel of Tracy–Widom [54] $K^P_{\tau_i\tau_j}$, is given by

$$K^P_{\tau_i\tau_j}(x,y) = -\frac{1}{4\pi^2}\int_X\int_{-i\infty}^{i\infty} e^{-s^4/4+\tau_j s^2/2-ys+t^4/4-\tau_i t^2/2+xt}\frac{ds\,dt}{s-t}\ , \quad (294)$$

with X the same contour of (288). We note $K^P_{\tau\tau}(x,y)=K^P_\tau(x,y)$ of (287), the Brézin–Hikami Pearcey kernel. We then have the analogous theorem to Theorem 5.3 (which we will not prove here) namely:

Theorem 5.4 (Adler–van Moerbeke [12]).
For compact $E_j=\bigcup_{i=1}^{r_j}[u^{(j)}_{2i-1}u^{(j)}_{2i}]$, $1\leqslant j\leqslant k$,

$$F(\tau_1,\tau_2,\dots,\tau_k;u^{(1)},u^{(2)},\dots,u^{(k)}) := \log \operatorname{Prob}(P(\tau_j)\notin E_j, 1\leqslant j\leqslant k) \tag{295}$$

satisfies the following 4*th order,* 3*rd degree* PDE *in* τ_i *and* $u^{(j)}$: $\mathcal{D}_{-1}X=0$, *with*

$$X := \left(\frac{4(E_{-1}^2-\tilde{\mathcal{D}}_{-1}\mathcal{D}_{-1})E_{-1}F+(2E_0+\mathcal{D}_0-2)\mathcal{D}_{-1}^2F+\frac{1}{8}\{\mathcal{D}_{-1}E_{-1}F,\mathcal{D}_{-1}^2F\}_{\mathcal{D}_{-1}}}{\mathcal{D}_{-1}^2E_{-1}F}\right) \tag{296}$$

where

$$\mathcal{D}_j := \sum_{i=1}^k B_j(u^{(i)}) , \qquad \tilde{\mathcal{D}}_{-1} := \sum_{i=1}^k \tau_i B_{-1}(u^{(i)}) ,$$
$$B_j(u^{(i)}) := \sum_{\ell=1}^{2r_i}(u_\ell^{(i)})^{j+1}\frac{\partial}{\partial u_\ell^{(i)}} , \quad E_j := \sum_{i=1}^k \tau_i^{j+1}\frac{\partial}{\partial\tau_i} .$$

5.4 3-KP deformation of the random matrix problem

In this section we shall deform the measures (269) in the random matrix problem, for $V(z)=z^2/2$, so as to introduce 3-KP τ–functions into the picture, and using the bilinear identities, we will derive some useful 3-KP PDE for these τ–functions. The probability distribution for the GUE with external source was given by (268), to wit:

$$P(\operatorname{spec}(M)\subset E) := P_{n_1,n_2}(E) = \frac{1}{Z_n}\int_{E^n}\prod_1^n dz_i \varDelta_n(z)\det\begin{pmatrix}\left(\rho^+_{i_+}(z_j)\right)_{\substack{1\leqslant i_+\leqslant n_1,\\ 1\leqslant j\leqslant n_1+n_2}}\\ \left(\rho^-_{i_-}(z_j)\right)_{\substack{1\leqslant i_-\leqslant n_2,\\ 1\leqslant j\leqslant n_1+n_2}}\end{pmatrix} \text{ where}$$
$$\rho_i^{\pm}(z) = z^{i-1}e^{-z^2/2\pm az} . \tag{297}$$

Let us deform $\rho_i^{\pm}(z)$ as follows:

$$\rho_i^{\pm}(z)\to\hat{\rho}_i^{\pm}(z) := z^{i-1}e^{-z^2/2\pm az\pm\beta z^2}e^{\sum_{k=1}^\infty(t_k-s_k^{\pm})z^k} , \tag{298}$$

yielding a deformation of the probability:

$$P_{n_1,n_2}(E)\to\frac{\tau_{n_1,n_2}(t,s^+,s^-,E)}{\tau_{n_1,n_2}(t,s^+,s^-,\mathbb{R})} . \tag{299}$$

Where, by the same argument used to derive (140),

$$\tau_{n_1,n_2}(t,s^+,s^-,E) := \det m_{n_1,n_2}(t,s^+,s^-,E) , \tag{300}$$

with

$$m_{n_1,n_2}(t,s^+,s^-,E) := \left\{ \begin{matrix} [\mu^+_{ij}]_{1\leqslant i_+\leqslant n_1, \atop 0\leqslant j\leqslant n_1+n_2-1} \\ [\mu^-_{ij}]_{1\leqslant i_-\leqslant n_2, \atop 0\leqslant j\leqslant n_1+n_2-1} \end{matrix} \right\} , \tag{301}$$

and

$$\mu^{\pm}_{ij}(t,s^+,s^-) := \int_E \hat{\rho}^{\pm}_{i+j}(z)\, dz \ .$$

We also need the identity ($n = n_1 + n_2$)

$$\begin{aligned} \tau_{n_1,n_2}(t,s^+,s^-,E) &:= \det m_{n_1,n_2}(t,s^+,s^-,E) \\ &= \frac{1}{n_1!n_2!} \int_{E^n} \Delta_n(x,y) \prod_{j=1}^{n_1} e^{\sum_1^\infty t_i x_j^i} \prod_{j=1}^{n_2} e^{\sum_1^\infty t_i y_j^i} \\ &\quad \times \left(\Delta_{n_1}(x) \prod_{j=1}^{n_1} e^{-x_j^2/2+ax_j+\beta x_j^2} e^{-\sum_1^\infty s_i^+ x_j^i}\, dx_j \right) \\ &\quad \times \left(\Delta_{n_2}(y) \prod_{j=1}^{n_2} e^{-y_j^2/2-ay_j-\beta y_j^2} e^{-\sum_1^\infty s_i^- y_j^i}\, dy_j \right) . \end{aligned} \tag{302}$$

That the above is a 3-KP deformation is the content of the following theorem.

Theorem 5.5 (Adler–van Moerbeke–Vanhaecke [15]). *Given the functions τ_{n_1,n_2} as in* (300)*, the wave matrices*

$$\begin{aligned} &\Psi^+_{n_1,n_2}(\lambda;t,s^+,s^-) \\ &:= \frac{1}{\tau_{n_1,n_2}(t,s^+,s^-)} \begin{pmatrix} \psi^{(1)+}_{n_1,n_2} & (-1)^{n_2}\psi^{(2)+}_{n_1+1,n_2} & \psi^{(3)+}_{n_1,n_2+1} \\ (-1)^{n_2}\psi^{(1)+}_{n_1-1,n_2} & \psi^{(2)+}_{n_1,n_2} & (-1)^{n_2}\psi^{(3)+}_{n_1-1,n_2+1} \\ \psi^{(1)}_{n_1,n_2-1} & (-1)^{n_2+1}\psi^{(2)+}_{n_1+1,n_2-1} & \psi^{(3)+}_{n_1,n_2} \end{pmatrix} , \\ &\Psi^-_{n_1,n_2}(\lambda;t,s^+,s^-) \\ &:= \frac{1}{\tau_{n_1,n_2}(t,s^+,s^-)} \begin{pmatrix} \psi^{(1)-}_{n_1,n_2} & (-1)^{n_2+1}\psi^{(2)-}_{n_1-1,n_2} & -\psi^{(3)-}_{n_1,n_2-1} \\ (-1)^{n_2+1}\psi^{(1)-}_{n_1+1,n_2} & \psi^{(2)-}_{n_1,n_2} & (-1)^{n_2}\psi^{(3)-}_{n_1+1,n_2-1} \\ -\psi^{(1)-}_{n_1,n_2+1} & (-1)^{n_2+1}\psi^{(2)-}_{n_1-1,n_2+1} & \psi^{(3)-}_{n_1,n_2} \end{pmatrix} , \end{aligned} \tag{303}$$

with wave functions

$$\begin{aligned} \psi^{(1)\pm}_{n_1,n_2}(\lambda;t,s^+,s^-) &:= \lambda^{\pm(n_1+n_2)} e^{\pm\sum_1^\infty t_i\lambda^i} \tau_{n_1,n_2}(t\mp[\lambda^{-1}],s^+,s^-) , \\ \psi^{(2)\pm}_{n_1,n_2}(\lambda;t,s^+,s^-) &:= \lambda^{\mp n_1} e^{\pm\sum_1^\infty s_i^+\lambda^i} \tau_{n_1,n_2}(t,s^+\mp[\lambda^{-1}],s^-) , \\ \psi^{(3)}_{n_1,n_2}(\lambda;t,s^+,s^-) &:= \lambda^{\mp n_2} e^{\pm\sum_1^\infty s_i^-\lambda^i} \tau_{n_1,n_2}(t,s^+,s^-\mp[\lambda^{-1}]) , \end{aligned} \tag{304}$$

satisfy the bilinear identity,

$$\oint_{\infty} \Psi^{+}_{k_1,k_2}\Psi^{-T}_{\ell_1,\ell_2}\,d\lambda = 0\ , \quad \forall k_1 k_2, \ell_1, \ell_2,\ \forall t, s^{\pm}, \underline{t}, \underline{s}^{\pm}\ , \tag{305}$$

of which the $(1,1)$ *component spelled out is:*

$$\begin{aligned}
&\oint_{\infty} \tau_{k_1,k_2}(t-[\lambda^{-1}],s^+,s^-)\tau_{\ell_1,\ell_2}(\underline{t}+[\lambda^{-1}],\underline{s}^+,\underline{s}^-)\lambda^{k_1+k_2-\ell_1-\ell_2}e^{\sum_1^\infty (t_i-\underline{t}_i)\lambda^i}\,d\lambda \\
&-(-1)^{k_2+\ell_2}\oint_{\infty} \tau_{k_1+1,k_2}(t,s^+-[\lambda^{-1}],s^-)\tau_{\ell_1-1,\ell_2}(\underline{t},\underline{s}^+ +[\lambda^{-1}],\underline{s}^-) \\
&\qquad\qquad\qquad\qquad \times\lambda^{\ell_1-k_1-2}e^{\sum_1^\infty (s_i^+-\underline{s}_i^+)\lambda^i}\,d\lambda \\
&-\oint_{\infty} \tau_{k_1,k_2+1}(t,s^+,s^- -[\lambda^{-1}])\tau_{\ell_1,\ell_2-1}(\underline{t},\underline{s}^+,\underline{s}^- +[\lambda^{-1}]) \\
&\qquad\qquad\qquad\qquad \times\lambda^{\ell_2-k_2-2}e^{\sum_1^\infty (s_i^- -\underline{s}_i^-)\lambda^i}\,d\lambda = 0\ .
\end{aligned} \tag{306}$$

Sketch of Proof:

The proof is via the MOPS of Section 5.2. We use the formal Cauchy transform, thinking of z as large:

$$C_{\pm}f(z) := \int_{\mathbb{R}} \frac{f(s)\hat{\rho}_1^{\pm}(s)}{z-s}\,ds := \sum_{i\geqslant 1}\frac{1}{z^i}\int \hat{\rho}_1^{\pm}(s)s^{i-1}\,ds\ , \tag{307}$$

which should be compared with the Cauchy transform of (281), which we used in the Riemann Hilbert problem involving MOPS, and let C_0 denote the Cauchy transform with 1 instead of $\hat{\rho}_1^{\pm}$. We now make the following identification between the MOPS of (272), (273) defined with $\rho_i^{\pm} \to \hat{\rho}_i^{\pm}$ (and so dependent on t, s^+, s^-) and their Cauchy transforms and shifted τ–functions, namely:

$$\begin{aligned}
p_{k_1,k_2}(\lambda) &= \lambda^{k_1+k_2}\frac{\tau_{k_1,k_2}(t-[\lambda^{-1}],s^+,s^-)}{\tau_{k_1,k_2}(t,s^+,s^-)}\ , \\
C_+p_{k_1,k_2}(\lambda) &= \lambda^{-k_1-1}\frac{\tau_{k_1+1,k_2}(t,s^+-[\lambda^{-1}],s^-)(-1)^{k_2}}{\tau_{k_1,k_2}(t,s^+,s^-)}\ , \\
C_-p_{k_1,k_2}(\lambda) &= \lambda^{-k_2-1}\frac{\tau_{k_1,k_2+1}(t,s^+,s^- -[\lambda^{-1}])}{\tau_{k_1,k_2}(t,s^+,s^-)}\ , \\
C_0q_{k_1,k_2}(\lambda) &= \lambda^{-k_1-k_2}\frac{\tau_{k_1,k_2}(t+[\lambda^{-1}],s^+,s^-)}{\tau_{k_1,k_2}(t,s^+,s^-)}\ , \\
q^+_{k_1-1,k_2}(\lambda) &= \lambda^{k_1-1}\frac{\tau_{k_1-1,k_2}(t,s^+ +[\lambda^{-1}],s^-)(-1)^{k_2+1}}{\tau_{k_1,k_2}(t,s^+,s^-)}\ , \\
q^-_{k_1,k_2-1}(\lambda) &= -\lambda^{k_2-1}\frac{\tau_{k_1,k_2-1}(t,s^+,s^- +[\lambda^{-1}])}{\tau_{k_1,k_2}(t,s^+,s^-)}\ ,
\end{aligned} \tag{308}$$

where $p_{k_1,k_2}(\lambda)$ was the MOP of the second kind and $q_{k_1,k_2}(\lambda) = q^+_{k_1-1,k_2}(\lambda)\hat\rho_1^+ + q^-_{k_1,k_2-1}(\lambda)\hat\rho_1^-$ was the MOP of the first kind. This in effect identifies all the elements in the RH matrix $Y(\lambda)$ given in (280) and $(Y^{-1})^T$, the latter which also satisfies a dual RH problem in terms of ratio's of τ–functions; indeed, $\Psi^+_{k_1,k_2}(\lambda)$ without the exponenials is precisely $Y(\lambda)$, etc. for $\Psi^-_{k_1,k_2}$. Then using a self-evident formal residue identity, to wit:

$$\frac{1}{2\pi i}\oint_\infty \left(f(z) \times \int_{\mathbb{R}} \frac{g(s)}{s-z}\, d\mu(s)\right) dz = \int_{\mathbb{R}} f(s)g(s)\, d\mu(s)\ , \tag{309}$$

with $f(z) = \sum_0^\infty a_i z^i$, and designating $f(t,s^+,s^-)' := f(\underline{t},\underline{s}^+,\underline{s}^-)$, we immediately conclude that

$$\begin{aligned}
&\oint_\infty e^{\sum_1^\infty (t_i-\underline{t}_i)\lambda^i} p_{k_1,k_2}\big(C_0 q_{\ell_1,\ell_2}(\lambda)\big)'\, d\lambda \\
&\qquad = \int_{\mathbb{R}} e^{\sum_1^\infty (t_i-\underline{t}_i)\lambda^i} p_{k_1,k_2}(\lambda,t,s^+,s^-) q_{\ell_1,\ell_2}(\lambda,\underline{t},\underline{s}^+,\underline{s}^-)\, d\lambda \\
&\qquad = \int_{\mathbb{R}} e^{\sum_1^\infty (t_i-\underline{t}_i)\lambda^i} p_{k_1,k_2}(\lambda,t,s^+,s^-)(q^+_{\ell_1-1,\ell_2}(\lambda,\underline{t},\underline{s}^+,\underline{s}^-)\hat\rho_1^+ \\
&\qquad\qquad + q^-_{\ell_1,\ell_2-1}(\lambda,\underline{t},\underline{s}^+,\underline{s}^-)\hat\rho_1^-)\, d\lambda \\
&\qquad = \oint_\infty \big(C_+ p_{k_1,k_2}(\lambda)\big)\big(q^+_{\ell_1-1,\ell_2}(\lambda)\big)' e^{\sum (s_i^+-\underline{s}_i^+)\lambda^i}\, d\lambda \\
&\qquad\qquad + \oint_\infty \big(C_- p_{k_1,k_2}(\lambda)\big)\big(q^-_{\ell_1,\ell_2-1}(\lambda)\big)' e^{\sum (s_i^--\underline{s}_i^-)\lambda^i}\, d\lambda\ .
\end{aligned} \tag{310}$$

By (308), this is nothing but the bilinear identity (306). In fact, all the other entries of (305) are just (306) with its subscripts shifted. To say a quick word about (308), all that really goes into it is solving explicitly the linear systems defining p_{k_1,k_2}, $q^+_{k_1-1,k_2}$ and $q^-_{k_1,k_2-1}$, namely (272) and (273), and making use of the identity $\exp(\pm\sum_1^\infty x^i/i) = (1-x)^{\mp 1}$ for x small in the formula (300) for $\tau_{k_1,k_2}(t,s^+,s^-)$.

An immediate consequence of Theorem 5.5 is the following:

Corollary 5.1. *Given the above τ-functions $\tau_{k_1,k_2}(t,s^+,s^-)$, they satisfy the following bilinear identities*[14]

[14] With $e^{\sum_1^\infty t_i z^i} =: \sum_0^\infty \mathbf{s}_i(t) z^i$ defining the elementary Schur polynomials.

$$
\begin{aligned}
&\sum_{j=0}^{\infty} \mathbf{s}_{\ell_1+\ell_2-k_1-k_2+j-1}(-2a)\mathbf{s}_j(\tilde{\partial}_t)e^{\sum_1^\infty (a_k\,\partial/\partial t_k+b_k\,\partial/\partial s_k^+ +c_k\,\partial/\partial s_k^-)} \\
&\qquad\qquad\qquad \tau_{\ell_1,\ell_2}\circ\tau_{k_1,k_2} \\
&-\sum_{j=0}^{\infty} \mathbf{s}_{k_1-\ell_1+1+j}(-2b)\mathbf{s}_j(\tilde{\partial}_{s^+})e^{\sum_1^\infty (a_k\,\partial/\partial t_k+b_k\,\partial/\partial s_k^+ +c_k\,\partial/\partial s_k^-)} \\
&\qquad\qquad\qquad \tau_{\ell_1-1,\ell_2}\circ\tau_{k_1+1,k_2}(-1)^{k_2+\ell_2} \\
&-\sum_{j=0}^{\infty} \mathbf{s}_{k_2-\ell_2+1+j}(-2c)\mathbf{s}_j(\tilde{\partial}_{s^-})e^{\sum_1^\infty (a_k\,\partial/\partial t_k+b_k\,\partial/\partial s_k^+ +c_k\,\partial/\partial s_k^-)} \\
&\qquad\qquad\qquad \tau_{\ell_1,\ell_2-1}\circ\tau_{k_1,k_2+1} = 0\ ,
\end{aligned}
\tag{311}
$$

with $a, b, c \in \mathbb{C}^\infty$ *arbitrary.*

Upon specializing, these identities imply PDE*'s expressed in terms of Hirota's symbol for* $j = 1, 2\ldots$:

$$
\mathbf{s}_j(\tilde{\partial}_t)\tau_{k_1+1,k_2}\circ\tau_{k_1-1,k_2} = -\tau_{k_1,k_2}^2\frac{\partial^2}{\partial s_1^+\partial t_{j+1}}\log\tau_{k_1,k_2}\ , \tag{312}
$$

$$
\mathbf{s}_j(\tilde{\partial}s^+)\tau_{k_1-1,k_2}\circ\tau_{k_1+1,k_2} = -\tau_{k_1,k_2}^2\frac{\partial^2}{\partial t_1\partial s_{j+1}^+}\log\tau_{k_1,k_2}\ , \tag{313}
$$

yielding

$$
\frac{\partial^2\log\tau_{k_1,k_2}}{\partial t_1\partial s_1^+} = -\frac{\tau_{k_1+1,k_2}\tau_{k_1-1,k_2}}{\tau_{k_1,k_2}^2}\ , \tag{314}
$$

$$
\frac{\partial}{\partial t_1}\log\frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} = \frac{\partial^2/\partial t_2\partial s_1^+\log\tau_{k_1,k_2}}{\partial^2/\partial t_1\partial s_1^+\log\tau_{k_1,k_2}}\ , \tag{315}
$$

$$
-\frac{\partial}{\partial s_1^+}\log\frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} = \frac{\partial^2/\partial t_1\partial s_2^+\log\tau_{k_1,k_2}}{\partial^2/\partial t_1\partial s_1^+\log\tau_{k_1,k_2}}\ , \tag{316}
$$

Proof. Applying Lemma A.1 to the bilinear identity (306) immediately yields (311). Then Taylor expanding in a, b, c and setting in equation (311) all $a_i, b_i, c_i = 0$, except a_{j+1}, and also setting $\ell_1 = k_1+2$, $\ell_2 = k_2$, equation (311) becomes

$$
a_{j+1}\left(-2\mathbf{s}_j(\tilde{\partial}_t)\tau_{k_1+2,k_2}\circ\tau_{k_1,k_2} - \frac{\partial^2}{\partial s_1^+\partial t_{j+1}}\tau_{k_1+1,k_2}\circ\tau_{k_1+1,k_2}\right) + \mathbf{O}(a_{j+1}^2) = 0,
$$

and the coefficient of a_{j+1} must vanish identically, yielding equation (312) upon setting $k_1 \to k_1-1$. Setting in equation (311) all $a_i, b_i, c_i = 0$, except b_{j+1}, and $\ell_1 = k_1$, $\ell_2 = k_2$, the vanishing of the coefficient of b_{j+1} in equation (311) yields equation (313). Specializing equation (312) to $j = 0$ and 1 respectively yields (since $\mathbf{s}_1(t) = t_1$ implies $\mathbf{s}_1(\tilde{\partial}_t) = \partial/\partial t_1$; also $\mathbf{s}_0 = 1$):

$$
\frac{\partial^2\log\tau_{k_1,k_2}}{\partial t_1\partial s_1^+} = -\frac{\tau_{k_1+1,k_2}\tau_{k_1-1,k_2}}{\tau_{k_1,k_2}^2}
$$

and

$$\frac{\partial^2}{\partial s_1^+ \partial t_2} \log \tau_{k_1,k_2} = -\frac{1}{\tau_{k_1,k_2}^2}\left[\left(\frac{\partial}{\partial t_1}\tau_{k_{1+1},k_2}\right)\tau_{k_1-1,k_2} - \tau_{k_1+1,k_2}\left(\frac{\partial}{\partial t_1}\tau_{k_1-1,k_2}\right)\right] .$$

Upon dividing the second equation by the first, we find equation (315) and similarly equation (316) follows from equation (313).

5.5 Virasoro constraints for the integrable deformations

Given the Heisenberg and Virasoro operators, for $m \geqslant -1$, $k \geqslant 0$:

$$\begin{aligned}
\mathbb{J}_{m,k}^{(1)} &= \frac{\partial}{\partial t_m} + (-m)t_{-m} + k\delta_{0,m} \ , \\
\mathbb{J}_{m,k}^{(2)}(t) &= \frac{1}{2}\left(\sum_{i+j=m} \frac{\partial^2}{\partial t_i \partial t_j} + 2\sum_{i \geqslant 1} it_i \frac{\partial}{\partial t_{i+m}} + \sum_{i+j=-m} it_i jt_j\right) \\
&\quad + \left(k + \frac{m+1}{2}\right)\left(\frac{\partial}{\partial t_m} + (-m)t_{-m}\right) + \frac{k(k+1)}{2}\delta_{m,0} \ ,
\end{aligned} \tag{317}$$

we now state (explicitly exhibiting the dependence of τ_{k_1,k_2} on β):

Theorem 5.6. *The integral $\tau_{k_1,k_2}(t, s^+, s^-; \beta; E)$, given by* (302) *satisfies*

$$\mathcal{B}_m \tau_{k_1,k_2} = \mathbb{V}_m^{k_1,k_2} \tau_{k_1,k_2} \quad \textit{for } m \geqslant -1 \ , \tag{318}$$

where $\mathcal{B}_m$ and $\mathbb{V}_m$ are differential operators:

$$\mathcal{B}_m = \sum_1^{2r} b_i^{m+1} \frac{\partial}{\partial b_i}, \quad \textit{for } E = \bigcup_1^{2r} [b_{2i-1}, b_{2i}] \subset \mathbb{R} \tag{319}$$

and

$$\begin{aligned}
\mathbb{V}_m^{k_1,k_2} := \{&\mathbb{J}_{m,k_1+k_2}^{(2)}(t) - (m+1)\mathbb{J}_{m,k_1+k_2}^{(1)}(t) \\
&+ \mathbb{J}_{m,k_1}^{(2)}(-s^+) + a\mathbb{J}_{m+1,k_1}^{(1)}(-s^+) - (1-2\beta)\mathbb{J}_{m+2,k_1}^{(1)}(-s^+) \\
&+ \mathbb{J}_{m,k_2}^{(2)}(-s^-) - a\mathbb{J}_{m+1,k_2}^{(1)}(-s) - (1+2\beta)\mathbb{J}_{m+2,k_2}^{(1)}(-s^-)\} \ .
\end{aligned} \tag{320}$$

Lemma 5.1. *Setting*

$$\begin{aligned}
dI_n = \Delta_n(x,y) &\prod_{j=1}^{k_1} e^{\sum_1^\infty t_i x_j^i} \prod_{j=1}^{k_2} e^{\sum_1^\infty t_i y_j^i} \\
&\times \left(\Delta_{k_1}(x) \prod_{j=1}^{k_1} e^{-x_j^2/2 + ax_j + \beta x_j^2} e^{-\sum_1^\infty s_i^+ x_j^i} \, dx_j\right)
\end{aligned}$$

$$\times \left(\Delta_{k_2}(y) \prod_{j=1}^{k_2} e^{-y_j^2/2 - a y_j - \beta y_j^2} e^{-\sum_1^\infty s_i^- y_j^i} \, dy_j \right) ,$$

the following variational formula holds for $m \geqslant -1$:

$$\frac{d}{d\varepsilon} dI_n \begin{pmatrix} x_i \mapsto x_i + \varepsilon x_i^{m+1} \\ y_i \mapsto y_i + \varepsilon y_i^{m+1} \end{pmatrix} \Bigg|_{\varepsilon=0} = \mathbb{V}_m^{k_1,k_2}(dI_n) \, . \tag{321}$$

Proof. The variational formula (321) is an immediate consequence of applying the variational formula (184) separately to the three factors of dI_n, and in addition applying formula (185) to the first factor, to account for the fact that $\prod_{j=1}^{k_1} dx_j \prod_{j=1}^{k_2} dy_j$ is missing from the first factor.

Proof of Theorem 5.6:

Formula (318) follows immediately from formula (321) by taking into account the variation of ∂E under the change of coordinates.

From (317) and (320) and from the identity when acting on on τ_{k_1,k_2},

$$\frac{\partial}{\partial t_n} = -\frac{\partial}{\partial s_n^+} - \frac{\partial}{\partial s_n^-} \, , \tag{322}$$

compute that

$$\begin{aligned}
\mathbb{V}_m^{k_1,k_2} &= \frac{1}{2} \sum_{i+j=m} \left(\frac{\partial^2}{\partial t_i \partial t_j} + \frac{\partial^2}{\partial s_i^+ \partial s_j^+} + \frac{\partial^2}{\partial s_i^- \partial s_j^-} \right) \\
&+ \sum_{i \geqslant 1} \left(i t_i \frac{\partial}{\partial t_{i+m}} + i s_i^+ \frac{\partial}{\partial s_{i+m}^+} + i s_i^- \frac{\partial}{\partial s_{i+m}^-} \right) \\
&+ (k_1 + k_2) \left(\frac{\partial}{\partial t_m} + (-m) t_{-m} \right) - k_1 \left(\frac{\partial}{\partial s_m^+} + (-m) s_{-m}^+ \right) \\
&- k_2 \left(\frac{\partial}{\partial s_m^-} + (-m) s_{-m}^- \right) + (k_1^2 + k_1 k_2 + k_2^2) \delta_{m,0} + a(k_1 - k_2) \delta_{m+1,0} \\
&+ \frac{m(m+1)}{2} (-t_{-m} + s_{-m}^+ + s_{-m}^-) - \frac{\partial}{\partial t_{m+2}} \\
&+ a \left(-\frac{\partial}{\partial s_{m+1}^+} + \frac{\partial}{\partial s_{m+1}^-} + (m+1)(s_{-m-1}^+ - s_{-m-1}^-) \right) \\
&+ 2\beta \left(\frac{\partial}{\partial s_{m+2}^-} - \frac{\partial}{\partial s_{m+2}^+} \right) , \quad m \geqslant -1 \, .
\end{aligned} \tag{323}$$

The following identities, valid when acting on $\tau_{k_1,k_2}(t, s^+, s^-; \beta, E)$, will also be used:

$$\begin{aligned}
\frac{\partial}{\partial s_1^+} &= -\frac{1}{2} \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial a} \right) , \quad \frac{\partial}{\partial s_2^+} = -\frac{1}{2} \left(\frac{\partial}{\partial t_2} + \frac{\partial}{\partial \beta} \right) , \\
\frac{\partial}{\partial s_1^-} &= -\frac{1}{2} \left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial a} \right) , \quad \frac{\partial}{\partial s_2^-} = -\frac{1}{2} \left(\frac{\partial}{\partial t_2} - \frac{\partial}{\partial \beta} \right) .
\end{aligned} \tag{324}$$

Corollary 5.2. *The τ-function $\tau = \tau_{k_1,k_2}(t, s^+, s^-; \beta, E)$ satisfies the following differential identities, with $\mathcal{B}_m = \sum_1^{2r} b_i^{m+1}\partial/\partial b_i$.*

$$
\begin{aligned}
-B_{-1}\tau = \mathcal{V}_1\tau + v_1\tau =:& \left(\frac{\partial}{\partial t_1} - 2\beta\frac{\partial}{\partial a}\right)\tau \\
&- \sum_{i\geqslant 2}\left(it_i\frac{\partial}{\partial t_{i-1}} + is_i^+\frac{\partial}{\partial s_{i-1}^+} + is_i^-\frac{\partial}{\partial s_{i-1}^-}\right)\tau \\
&+ (a(k_2-k_1) + k_1 s_1^+ + k_2 s_1^- - (k_1+k_2)t_1)\tau\ , \\
\frac{1}{2}\left(B_{-1} - \frac{\partial}{\partial a}\right)\tau = \mathcal{W}_1\tau + w_1\tau =:& \left(\frac{\partial}{\partial s_1^+} + \beta\frac{\partial}{\partial a}\right)\tau \\
&+ \frac{1}{2}\sum_{i\geqslant 2}\left(it_i\frac{\partial}{\partial t_{i-1}} + is_i^+\frac{\partial}{\partial s_{i-1}^+} + is_i^-\frac{\partial}{\partial s_{i-1}^-}\right)\tau \\
&+ \left(\frac{a}{2}(k_1-k_2) + \frac{1}{2}(k_1+k_2)t_1 - k_1 s_1^+ - k_2 s_1^-\right)\tau \\
-\left(B_0 - a\frac{\partial}{\partial a}\right)\tau = \mathcal{V}_2\tau + v_2\tau =:& \frac{\partial\tau}{\partial t_2} - 2\beta\frac{\partial\tau}{\partial\beta} \\
&- \sum_{i\geqslant 1}\left(it_i\frac{\partial}{\partial t_i} + is_i^+\frac{\partial}{\partial s_i^+} + is_i^-\frac{\partial}{\partial s_i^-}\right)\tau \\
&- (k_1^2 + k_2^2 + k_1k_2)\tau \\
\frac{1}{2}\left(B_0 - a\frac{\partial}{\partial a} - \frac{\partial}{\partial\beta}\right)\tau = \mathcal{W}_2\tau + w_2\tau =:& \frac{\partial\tau}{\partial s_2^+} + \beta\frac{\partial\tau}{\partial\beta} \\
&+ \frac{1}{2}\sum_{i\geqslant 1}\left(it_i\frac{\partial}{\partial t_i} + is_i^+\frac{\partial}{\partial s_i^+} + is_i^-\frac{\partial}{\partial s_i^-}\right)\tau \\
&+ \frac{1}{2}(k_1^2 + k_2^2 + k_1k_2)\tau
\end{aligned}
\tag{325}
$$

where $\mathcal{V}_1$, $\mathcal{W}_1$, $\mathcal{V}_2$, $\mathcal{W}_2$ are first order operators and v_1, w_1, v_2, w_2 are functions, acting as multiplicative operators.

Corollary 5.3. *On the locus $\mathcal{L} := \{t = s^+ = s^- = 0, \beta = 0\}$, the function $f = \log\tau_{k_1,k_2}(t, s^+, s^-; \beta, E)$ satisfies the following differential identities*:

$$
\begin{aligned}
\frac{\partial f}{\partial t_1} &= -\mathcal{B}_{-1}f + a(k_1 - k_2)\ , \\
\frac{\partial f}{\partial s_1^+} &= \frac{1}{2}\left(\mathcal{B}_{-1} - \frac{\partial}{\partial a}\right)f + \frac{a}{2}(k_2 - k_1)\ , \\
\frac{\partial f}{\partial t_2} &= \left(-\mathcal{B}_0 + a\frac{\partial}{\partial a}\right)f + k_1^2 + k_1k_2 + k_2^2\ , \\
\frac{\partial f}{\partial s_2^+} &= \frac{1}{2}\left(\mathcal{B}_0 - a\frac{\partial}{\partial a} - \frac{\partial}{\partial\beta}\right)f - \frac{1}{2}(k_1^2 + k_2^2 + k_1k_2)\ ,
\end{aligned}
\tag{326}
$$

$$
\begin{aligned}
2\frac{\partial^2 f}{\partial t_1 \partial s_1^+} &= \mathcal{B}_{-1}\left(\frac{\partial}{\partial a} - \mathcal{B}_{-1}\right) f - 2k_1 \ , \\
2\frac{\partial^2 f}{\partial t_1 \partial s_2^+} &= \left(a\frac{\partial}{\partial a} + \frac{\partial}{\partial \beta} - \mathcal{B}_0 + 1\right)\mathcal{B}_{-1} f - 2\frac{\partial f}{\partial a} - 2a(k_1 - k_2) \ , \\
2\frac{\partial^2 f}{\partial t_2 \partial s_1^+} &= \frac{\partial}{\partial a}\left(\mathcal{B}_0 - a\frac{\partial}{\partial a} + a\mathcal{B}_{-1}\right) f - \mathcal{B}_{-1}(\mathcal{B}_0 - 1) f - 2a(k_1 - k_2) \ .
\end{aligned} \tag{327}
$$

Proof. Upon dividing equations (325) by τ and restricting to the locus $\mathcal{L}$, equations (326) follow immediately.

Remembering $f = \log \tau$ and setting

$$
\begin{aligned}
\mathcal{A}_1 &:= -\mathcal{B}_{-1} \ , & \mathcal{B}_1 &:= \frac{1}{2}\left(\mathcal{B}_{-1} - \frac{\partial}{\partial a}\right) \ , \\
\mathcal{A}_2 &:= -\left(\mathcal{B}_0 - a\frac{\partial}{\partial a}\right), & \mathcal{B}_2 &:= \frac{1}{2}\left(\mathcal{B}_0 - a\frac{\partial}{\partial a} - \frac{\partial}{\partial \beta}\right) \ ,
\end{aligned} \tag{328}
$$

we may recast (325) as (compare with (193))

$$
\mathcal{A}_k f = \mathcal{V}_k f + v_k, \quad \mathcal{B}_k f = \mathcal{W}_k f + w_k, \quad k = 1, 2 \ , \tag{329}
$$

where (compare with (191)) we note that

$$
\mathcal{V}_k\big|_{\mathcal{L}} = \frac{\partial}{\partial t_k} \ , \quad \mathcal{W}_k\big|_{\mathcal{L}} = \frac{\partial}{\partial s_k^+} \ , \qquad k = 1, 2 \ . \tag{330}
$$

To prove (327) we will copy the argument of Section 3.6 (see (195)). Indeed, compute

$$
\begin{aligned}
\mathcal{B}_1 \mathcal{A}_1 f\big|_{\mathcal{L}} &= \mathcal{B}_1(\mathcal{V}_1 f + v_1) = \mathcal{B}_1 \mathcal{V}_1 f\big|_{\mathcal{L}} + \mathcal{B}_1(v_1)\big|_{\mathcal{L}} \\
&\overset{(*)}{=} \mathcal{V}_1 \mathcal{B}_1 f\big|_{\mathcal{L}} + \mathcal{B}_1(v_1)\big|_{\mathcal{L}} \\
&\overset{(\substack{*\\*})}{=} \frac{\partial}{\partial t_1}(\mathcal{W}_1 f + w_1)\big|_{\mathcal{L}} + \mathcal{B}_1(v_1)\big|_{\mathcal{L}} \\
&= \frac{\partial}{\partial t_1}\left(\frac{\partial}{\partial s_1^+} + \cdots\right) f\big|_{\mathcal{L}} + \frac{\partial w_1}{\partial t_1}\Big|_{\mathcal{L}} + \mathcal{B}_1(v_1)\big|_{\mathcal{L}} \ ,
\end{aligned} \tag{331}
$$

where we used in $(*)$ that $[\mathcal{B}_1, \mathcal{V}_1]\big|_{\mathcal{L}} = 0$ and in $(\substack{*\\*})$ that $\mathcal{V}_1\big|_{\mathcal{L}} = \partial/\partial t_1$, and so from (331) we must compute

$$
\frac{\partial}{\partial t_1}\mathcal{W}_1\big|_{\mathcal{L}} = \frac{\partial^2}{\partial t_1 \partial s_1^+} \ , \quad \frac{\partial w_1}{\partial t_1}\Big|_{\mathcal{L}} = \frac{1}{2}(k_1 + k_2) \ , \quad \mathcal{B}_1(v_1)\big|_{\mathcal{L}} = \frac{1}{2}(k_1 - k_2) \ . \tag{332}
$$

Now from (331) and (332), we find

$$\mathcal{B}_1\mathcal{A}_1 f\Big|_{\mathcal{L}} = \frac{\partial^2 f}{\partial t_1 \partial s_1^+}\Big|_{\mathcal{L}} + \frac{1}{2}(k_1+k_2) + \frac{1}{2}(k_1-k_2) = \frac{\partial^2 f}{\partial t_2 \partial s_1^+} + k_1 ,$$

and so

$$\frac{\partial^2 f}{\partial t_1 \partial s_1^+}\Big|_{\mathcal{L}} = \mathcal{B}_1\mathcal{A}_1 f\Big|_{\mathcal{L}} - k_1 = -\mathcal{B}_{-1}\frac{1}{2}\left(\mathcal{B}_1 - \frac{\partial}{\partial a}\right) f\Big|_{\mathcal{L}} - k_1 ,$$

which is just the first equation in (327). The crucial point in the calculation being (330) and $[\mathcal{B}_1, \mathcal{V}_1]\Big|_{\mathcal{L}} = 0$. The other two formulas in (327) are done in precisely the same fashion, using the crucial facts (330) and $[\mathcal{B}_2, \mathcal{V}_1]\Big|_{\mathcal{L}} = [\mathcal{A}_2, \mathcal{W}_1]\Big|_{\mathcal{L}} = 0$ and the analogs of (332).

5.6 A PDE for the Gaussian ensemble with external source and the Pearcey PDE

From now on set: $k_1 = k_2 := k$ and restrict to the locus $\mathcal{L}$. From (315) and (316) we have the

3-KP *relations*:

$$\frac{\partial}{\partial t_1} g = \frac{\partial^2 f}{\partial t_2 \partial s_1^+} \Big/ \frac{\partial^2 f}{\partial t_1 \partial s_1^+} , \quad -\frac{\partial}{\partial s_1^+} g = \frac{\partial^2 f}{\partial t_1 \partial s_2^+} \Big/ \frac{\partial^2 f}{\partial t_1 \partial s_1^+} , \tag{333}$$

with

$$f := \log \tau_{k,k} , \quad g := \log(\tau_{k+1,k}/\tau_{k-1,k}) ,$$

while from (326), we find

Virasoro relations on $\mathcal{L}$:

$$\frac{\partial g}{\partial t_1} = -\mathcal{B}_{-1} g + 2a , \quad \frac{\partial g}{\partial s^+} = \frac{1}{2}\left(\mathcal{B}_{-1} - \frac{\partial}{\partial a}\right) g - a . \tag{334}$$

Eliminating $\partial g/\partial t_1$, $\partial g/\partial s_1^+$ from (333) using (334) and then further eliminating $\mathcal{A}_1 g := -\mathcal{B}_- g$ and $\mathcal{B}_1 g := \frac{1}{2}(\mathcal{B}_{-1} - \partial/\partial a) g$ using $\mathcal{A}_1\mathcal{B}_1 g = \mathcal{B}_1\mathcal{A}_1 g$ yields

$$\mathcal{B}_{-1}\left(\frac{\partial^2/\partial t_2 \partial s_1^+ - 2\,\partial^2/\partial t_1 \partial s_2^+) f}{\partial^2 f/\partial t_1 \partial s_1^+}\right) = \frac{\partial}{\partial a}\left(\frac{(\partial^2/\partial t_2 \partial s_1^+ - 2a\,\partial^2/\partial t_1 \partial s_1^+) f}{\partial^2 f/\partial t_1 \partial s_1^+}\right) , \tag{335}$$

while from (327) we find the

Virasoro relations on $\mathcal{L}$:

$$
\begin{aligned}
2\frac{\partial^2}{\partial t_1 \partial s_1^+} f &= \mathcal{B}_{-1}\left(\frac{\partial}{\partial a} - \mathcal{B}_{-1}\right) f - 2k =: F^+ \ , \\
2\left(\frac{\partial^2}{\partial t_2 \partial s_1^+} - 2\frac{\partial^2}{\partial t_1 \partial s_2^+}\right) f &= H_1^+ - 2\mathcal{B}_{-1}\frac{\partial f}{\partial \mathcal{B}} \ , \\
2\left(\frac{\partial^2}{\partial t_2 \partial s_1^+} - 2a\frac{\partial^2}{\partial t_1 \partial s_2^+}\right) f &= H_2^+ \ ,
\end{aligned}
\tag{336}
$$

where the precise formulas for H_i^+ will be given later. Substituting (336) into (335) and clearing the denominator yields[15]

$$
\left\{\mathcal{B}_{-1}\frac{\partial f}{\partial \beta}, F^+\right\}_{\mathcal{B}_{-1}} = \left\{H_1^+, \frac{1}{2}F^+\right\}_{\mathcal{B}_{-1}} - \left\{H_2^+, \frac{1}{2}F^+\right\}_{\partial/\partial a} \ , \tag{337}
$$

and by the involution: $a \to -a$, $\beta \to -\beta$, which by (302), clearly fixes $f = \log \tau_{k,k}$ on $t = s^+ = s^- = 0$, we find ($H_i^- = H^+_{i|a \to -a}$)

$$
-\left\{\mathcal{B}_{-1}\frac{\partial f}{\partial \beta}, F^-\right\}_{\mathcal{B}_{-1}} = \left\{H_1^-, \frac{1}{2}F^-\right\}_{\mathcal{B}_{-1}} - \left\{H_2^-, \frac{1}{2}F^-\right\}_{-\partial/\partial a} \ . \tag{338}
$$

These 2 relations (337) and (338) yield a linear system for:

$$
\mathcal{B}_{-1}\frac{\partial f}{\partial \beta} \ , \quad \mathcal{B}_{-1}^2\frac{\partial f}{\partial \beta} \ .
$$

Solving the system yields:

$$
\mathcal{B}_{-1}\frac{\partial f}{\partial \beta} = R_1 \ , \quad \mathcal{B}_{-1}^2\frac{\partial f}{\partial \beta} = R_2 \ ,
$$

and so

$$
\mathcal{B}_{-1}R_1(f) = R_2(f) \ , \quad f = \log \tau_{k,k}(0,0,0,E)\Big|_{\beta=0} \ .
$$

Since

$$
P(\mathrm{spec}(M) \subset E) = \frac{\tau_{k,k}(0,0,0,E)}{\tau_{k,k}(0,0,0,\mathbb{R}}\Bigg|_{\beta=0} \ ,
$$

with

$$
\tau_{k,k}(0,0,0,\mathbb{R})\Big|_{\beta=0} = \left(\prod_0^{k-1} j!\right)^2 2^{k^2}(-2\pi)^k e^{ka^2} a^{k^2} \ ,
$$

we find the following theorem:

[15] $\{f, g\}_X := gXf - fXg$.

Theorem 5.7 (Adler–van Moerbeke [7]).

For $E = \bigcup_1^r [b_{2i-1}, b_{2i}]$, $A = \mathrm{diag}(\overbrace{-a,\dots,-a}^{k}, \overbrace{a,\dots,a}^{k})$,

$$P(a; b_1, \dots, b_{2r}) = \frac{\int_{\mathcal{H}_{2k(E)}} e^{\mathrm{Tr}(-\frac{1}{2}M^2+AM)}\, dM}{\int_{\mathcal{H}_{2k(\mathbb{R})}} e^{\mathrm{Tr}(-\frac{1}{2}M^2+AM)}\, dM} \tag{339}$$

satisfies a nonlinear $4th$ *order* PDE *in* a, $b_1, \dots b_r$:

$$\begin{aligned}(F^+\mathcal{B}_{-1}G^- + F^-\mathcal{B}_{-1}G^+)(F^+\mathcal{B}_{-1}F^- - F^-\mathcal{B}_{-1}F^+)\\ - (F^+G^- + F^-G^+)(F^+\mathcal{B}_{-1}^2F^- - F^-\mathcal{B}_{-1}^2F^+) = 0\ ,\end{aligned} \tag{340}$$

where

$$\mathcal{B}_{-1} = \sum_1^{2r} \frac{\partial}{\partial b_i}\ , \quad \mathcal{B}_0 = \sum_1^{2r} b_i \frac{\partial}{\partial b_i}\ ,$$

$$\begin{aligned}
F^+ &:= -2k + \mathcal{B}_{-1}\left(\frac{\partial}{\partial a} - \mathcal{B}_{-1}\right)\log P\ ,\\
G^+ &:= \{H_1^+, F^+\}_{\mathcal{B}_{-1}} - \{H_2^+, F^+\}_{\partial/\partial a}\ ,\\
H_1^+ &:= \frac{\partial}{\partial a}\left(\mathcal{B}_0 - a\frac{\partial}{\partial a} - a\mathcal{B}_{-1} + 4\frac{\partial}{\partial a}\right)\log P + \mathcal{B}_0\mathcal{B}_{-1}\log P + 4ak + 4\frac{k^2}{a}\ ,\\
H_2^+ &:= \frac{\partial}{\partial a}\left(\mathcal{B}_0 - a\frac{\partial}{\partial a} - a\mathcal{B}_{-1}\right)\log P + (2a\mathcal{B}_{-1}^2 - \mathcal{B}_0\mathcal{B}_{-1} + 2\mathcal{B}_{-1})\log P\ ,\\
F^- &= F^+\Big|_{a\to -a} \quad \textit{and} \quad G^- = G^+\Big|_{a\to -a}\ .^{16}
\end{aligned} \tag{341}$$

We now show how Theorem 5.7 implies Theorem 5.3. Indeed, remember our picture of $2k$ Brownian paths diverging at $t = \frac{1}{2}$.

Also, remembering the equivalence (270) between GUE with external source and the above Brownian motion, and (286), we find

$$\begin{aligned}P_{k,k}^a(t; b_1, \dots, b_{2r}) &:= \mathrm{Prob}_{k,k}^a(\text{all } x_i(t) \subset E)\\ &= P\left(\sqrt{\frac{2t}{1-t}}a; \sqrt{\frac{2}{t(1-t)}}(b_1, \dots, b_r)\right)\\ &= \det(I - \widetilde{K}_{2k}^{E^c})\ ,\end{aligned} \tag{342}$$

where the *function* $P(*;*)$ is that of (339).

Letting the number of particles $2k \to \infty$ and looking about the location $x = 0$ at time $t = \frac{1}{2}$ with a microscope, and slowing down time as follows:

[16] Note $P(a; b_1, \dots b_{2r}) = P(-a; b_1, \dots, b_r)$.

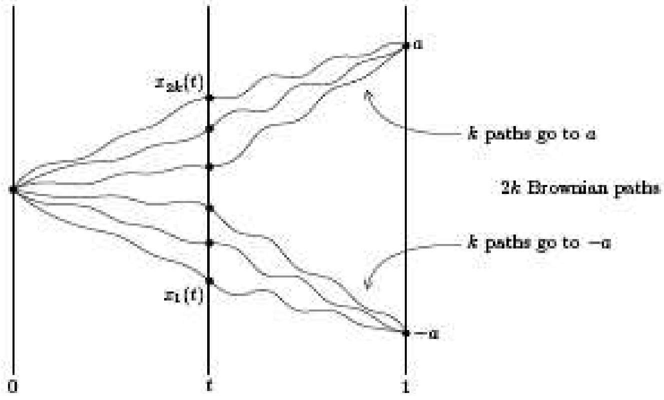

Fig. 6.

$$k = \frac{1}{z^4}\ , \quad \pm a = \pm\frac{1}{z^2}\ , \quad b_i = u_i z\ , \quad t = \tfrac{1}{2} + \tau z^2\ , \qquad z \to 0\ , \tag{343}$$

which is just the Pearcey scaling (285), we find by Theorem 5.2 and (342), that

$$\begin{aligned} &\mathrm{Prob}^{1/z^2}_{(1/z^4,1/z^4)}\left(\text{all } x_i\left(\frac{1}{2}+\tau z^2\right) \in \bigcup_1^r [zu_{2i-1}, zu_{2i}]\right) \\ &\qquad = P\left(\sqrt{\frac{2}{1-t}}a; \sqrt{\frac{2}{t(1-t)}}(b_1,\dots,b_{2r})\right)\Bigg|_{a=1/z^2, b_i=u_iz, t=\frac{1}{2}+\tau z^2} \\ &\qquad = \det(I - K^P_\tau I_{\tilde{E}^c}) + O(z) \\ &\qquad =: Q(\tau; u_1,\dots,u_{2r}) + O(z)\ , \end{aligned} \tag{344}$$

where K^P_τ is the Pearcey kernel (287) and $\tilde{E} = \bigcup_1^r [u_{2i-1}, u_{2i}]$. Taking account of

$$\begin{aligned} P^a_{k,k}(t; b_1,\dots,b_{2r}) &= P\left(\sqrt{\frac{2t}{1-t}}a; \sqrt{\frac{2}{t(1-t)}}(b_1,\dots,b_r\right) \\ &=: P(A, B_1,\dots,B_{2r})\ , \end{aligned} \tag{345}$$

where the function $P(*,*)$ is that of (339), and the scaling (343) of the Pearcey process, we subject the equation (340) to both the change of coordinates involved in the equation (345) and the Pearcey scaling (343) simultaneously:

$$\begin{aligned} 0 = \{&(F^+\mathcal{B}_{-1}G^- + F^-\mathcal{B}_{-1}G^+)(F^+\mathcal{B}_{-1}F^- - F^-\mathcal{B}_{-1}F^+) \\ &-(F^+G^- + F^-G^+)(F^+\mathcal{B}^2_{-1}F^- - F^-\mathcal{B}^2_{-1}F^+)\}\Bigg|_{\substack{A=\frac{\sqrt{2}}{z^2}\sqrt{\frac{\frac{1}{2}+\tau z^2}{\frac{1}{2}-\tau z^2}},\\ B_i=\frac{u_iz\sqrt{2}}{\sqrt{\frac{1}{4}-\tau^2z^4}}}} \end{aligned}$$

$$= \frac{1}{z^{17}}\left(\text{PDE in } \tau \text{ and } u \text{ for } \log P^a_{k,k}(t; b_1, \dots, b_{2r})\Big|_{\text{scaling}}\right) + O\left(\frac{1}{z^{15}}\right)$$
$$= \frac{1}{z^{17}}\big(\text{same PDE for } \log Q(\tau, u_1, \dots, u_{2r})\big) + O\left(\frac{1}{z^{16}}\right) ;$$

the first step is accomplished by the chain rule and the latter step by (344), yielding Theorem 5.3 for $F = \log Q$.

A Hirota Symbol Residue Identity

Lemma A.1. *We have the following formal residue identity*

$$\frac{1}{2\pi i}\oint_\infty f(t' - [z^{-1}], s', u')g(t'' + [z^{-1}], s'', u'')e^{\sum_1^\infty (t'_i - t''_i)z^i} z^r \, dz$$
$$= \sum_{j \geqslant 0} s_{j-1-r}(-2a)s_j(\tilde{\partial}_t)e^{\sum_1^\infty (a_\ell\, \partial/\partial t_\ell + b_\ell\, \partial/\partial s_\ell + c_\ell\, \partial/\partial u_\ell)} g \circ f \ , \quad \text{(A.346)}$$

where

$$\begin{aligned} t' &= t - a \ , \quad s' = s - b \ , \quad u' = u - c \ , \\ t'' &= t + a \ , \quad s'' = s + b \ , \quad u'' = u + c \ , \end{aligned} \quad \text{(A.347)}$$

$$\tilde{\partial}_t = \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \dots\right) , \quad e^{\sum_1^\infty t_i z^i} = \sum_0^\infty s_i(t) z^i \ , \quad \text{(A.348)}$$

and the Hirota symbol

$$p(\partial_t, \partial_s, \partial_u) g \circ f$$
$$:= p(\partial_{t'}, \partial_{s'}, \partial_{u'}) g(t + t', s + s', u + u') f(t - t', s - s', u - u')\Big|_{\substack{t'=0,\\ s'=0,\\ u'=0}} . \quad \text{(A.349)}$$

Proof. By definition,

$$\frac{1}{2\pi i}\oint_\infty \sum_{i=-\infty}^{i=\infty} a_i z^i \, dz = a_{-1} \ , \quad \text{(A.350)}$$

and so by Tayler's Theorem, following [28] compute:

$$\oint_\infty f(t' - [z^{-1}], s', u')g(t'' + [z^{-1}], s'', u'')e^{\sum_1^\infty (t'_i - t''_i)z^i} z^r \frac{dz}{2\pi i}$$
$$= \oint_\infty f(t - a - [z^{-1}], s - b, u - c)g(t + a + [z^{-1}], s + b, u + c)e^{-2\sum_1^\infty a_i z^i} z^r \frac{dz}{2\pi i}$$

$$= \oint_{\infty} e^{\sum_1^\infty (1/iz^i)\partial/\partial a_i} f(t-a, s-b, u-c) g(t+a, s+b, u+c) e^{-2\sum_1^\infty a_i z^i} z^r \frac{dz}{2\pi i}$$

$$= \oint_{\infty} \sum_1^\infty z^{-j} s_j(\tilde{\partial}_a) f(t-a, s-b, u-c) g(t+a, s+b, u+c) \sum_{\ell=0}^\infty z^{\ell+r} s_\ell(-2a) \frac{dz}{2\pi i}$$

(picking out the residue term)

$$= \sum_{j=0}^\infty s_{j-1-r}(-2a) s_j(\tilde{\partial}_a) f(t-a, s-b, u-c) g(t+a, s+b, u+c)$$

$$= \sum_{j=0}^\infty s_{j-1-r}(-2a) s_j(\tilde{\partial}_a) e^{\sum_1^\infty (a_\ell\, \partial/\partial t'_\ell + b_\ell\, \partial/\partial s'_\ell + c_\ell\, \partial/\partial u'_\ell)} f(t-t', s-s', u-u')$$
$$\times\, g(t+t', s+s', u+u') \quad \text{at } t' = s' = u' = 0$$

$$= \sum_{j=0}^\infty s_{j-1-r}(-2a) s_j(\tilde{\partial}_{t'}) e^{\sum_1^\infty (a_\ell\, \partial/\partial t'_\ell + b_\ell\, \partial/\partial s'_\ell + c_\ell\, \partial/\partial u'_\ell)}$$
$$\times\, g(t+t', s+s', u+u') f(t-t', s-s', u-u') \quad \text{at } t' = s' = u' = 0$$

$$= \sum_{j=0}^\infty s_{j-1-r}(-2a) s_j(\tilde{\partial}_t) e^{\sum_1^\infty (a_\ell\, \partial/\partial t_\ell + b_\ell\, \partial/\partial s_\ell + c_\ell\, \partial/\partial u_\ell)} g(t) \circ f(t)\ ,$$

completing the proof.

Proof of (28):

To deduce (28) from (27), observe that since t, t' are arbitrary in (27), when we make the change of coordinates (A.347), a becomes arbitrary and we then apply Lemma A.1, with s and u absent, $r = 0$ and $f = g = \tau$, to deduce (28).

Proof of (166):

To deduce (167) from (166), apply Lemma A.1 to the l.h.s. of (166), setting $f = \tau_n$, $g = \tau_{m+1}$, $r = n - m - 1$, where no u, u' is present, and when we make the change of coordinates (A.347), since t, t', s, s' are arbitrary, so is a and b, while in the r.h.s. of (166), we first need to make the change of coordinates $z \to z^{-1}$, so $z^{n-m-1}\,dz \to z^{m-n-1}\,dz$ (taking account of the switch in orientation) and then we apply Lemma A.1, with $f = \tau_{n+1}$, $g = \tau_m$, $r = m - n - 1$, and so deduce (167) from (166).

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