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The Intersection of Four Quadrics in $\mathbb{P}^6$, Abelian Surfaces and their Moduli

Mark Adler[1,*] and Pierre van Moerbeke[2,*]

[1] Department of Mathematics, Brandeis University, Waltham, MA 02254, USA
[2] Department of Mathematics, University of Louvain, B-1348 Louvain-la-Neuve, Belgium and Brandeis University, Waltham, MA 02254, USA

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1. Introduction

Since Jacobi and Klein there has been considerable interest in the problem of the intersection of quadrics and the study of their moduli. This problem is particularly interesting when the affine intersection of quadrics completes into an abelian variety. We *conjecture* that the intersection of four quadrics in $\mathbb{P}^6$ completes into an abelian variety if and only if their linear span contains a non-degenerate curve of rank 4 quadrics. In this paper, we show the conjecture for a natural set of quadrics inspired by dynamics; they lead to abelian surfaces with a certain polarization and at the same time we reveal their beautiful underlying geometry. We also show how this idea generalizes to quartics. Other circumstances leading to abelian varieties have arisen in the classical literature. As pointed out by Weil and fully developed by Reid [21], the moduli space of the intersection of two quadrics in $\mathbb{P}^N$ (N odd) coincides with the moduli of hyperelliptic curves of genus $(N-1)/2$; this problem has been related to dynamics, specifically to Jacobi's geodesic motion of ellipsoids, by Moser [18] and Knörrer [13]. For $N=3$, Reid's result leads to the elementary

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fact that an ellipsoid intersects a sphere in $\mathbb{P}^3$ according to an elliptic curve. A dynamical system on this elliptic curve is the famous Euler free rigid body motion, whose invariant surfaces are given by the sphere and the ellipsoid. In a different vein, Klein [12] studied the 3-dimensional variety of spherical triangles on the sphere, which turns out to be the complete intersection of three quadrics in $\mathbb{P}^6$. Tyurin [22] has studied the question of the moduli space of the intersection of three quadrics in $\mathbb{P}^N$ (N even) and Barth [8] has shown that the question is intimately related to classifying stable algebraic rank-2 vector bundles on $\mathbb{P}^2$.

Thus, it seems that Hamiltonian mechanics often provides descriptions of the moduli of abelian varieties. The Arnold-Liouville theorem [6] asserts that a compact (real or complex) connected n-dimensional manifold M having n commuting everywhere independent vector fields is diffeomorphic to a torus. In an integrable Hamiltonian system, the invariants (of sufficient number) define the manifold M and the Hamiltonian vector fields generated by the invariants in involution provide the commuting vector fields on M. In many cases, the problem has much more structure, namely the real invariant surfaces extend to affine complex varieties. Only after adjoining an appropriate divisor can they lead to abelian varieties on which the above flows (run with complex time) are linear motions. These flows are then solvable in terms of abelian integrals. A dynamical system for which this is possible will be called *algebraically completely integrable*, as first defined in Adler and Moerbeke [1, 5].

The Euler rigid body motion mentioned above is nothing else but geodesic motion on $SO(3)$ for a left invariant metric which can be interpreted as geodesic motion on a 2-dimensional ellipsoid. The generalization of this problem to geodesic motion on ellipsoids of arbitrary dimension leads to a description of the moduli space of the intersection of two quadrics in $\mathbb{P}^N$ (N odd) mentioned above. Another way of generalizing the $SO(3)$ problem is to turn to integrable geodesic motion on $SO(4) = SO(3) \otimes SO(3)$, of which special instances have been studied around the turn of the century in the context of rigid body motion in fluids by Clebsch, Steklov, and Lyapounov (see Adler and van Moerbeke [5]), although obstensively in the context of $E_3 = SO(3) \times R^3$. This leads to an example of four quadrics in $\mathbb{P}^6$ whose complete intersection is an abelian surface only after blowing up and down. This has led to a classification of the left-invariant metrics on $SO(4)$ for which geodesic flow is algebraically completely integrable [3], as announced in [2, 16]. This set of metrics consists of three different strata, a first one, for which the invariant surfaces complete to hyperelliptic Jacobians, a second stratum (Manakov metrics) leading to abelian surfaces with a polarization $(1, 2)$ and a third set to abelian surfaces with a polarization $(1, 6)$. The first two cases lead to examples of four quadrics in $\mathbb{P}^6$ whose complete intersection is an abelian surface only after blowing up and down. The third case finally leads to examples of 3 quadrics and 1 quartic in $\mathbb{P}^6$. However a number of systems do not have quadratic invariants, but still they linearize on abelian surfaces with principal and $(1, 2)$-polarizations. For instance, the equations for the Kowalevski top linearize on abelian surfaces of type $(1, 2)$, as does the Hénon-Heiles system and the Manakov geodesic flow. Therefore there must exist a rational map transforming one problem into the other. This relies heavily on finding good normal forms for the quadrics defining abelian surfaces of that polarization. The results contained in this paper have provided the key to realizing both the Kowalevski and the Hénon-Heiles systems as a Manakov geodesic flow on $SO(4)$ and thus providing a Lax pair for these systems. Observe that abelian surfaces are not simply connected, and therefore can never be themselves projective complete intersections. And so, the complete intersection of the four quadrics in $\mathbb{P}^6$, if it is to be related to an abelian surface, must contain a singular locus $\mathscr{E}$. The nature of this singular locus will play an important role in this paper.

The main purpose of this paper is to describe a natural class of four quadrics $\bar{Q}_1, \ldots, \bar{Q}_4$ in $\mathbb{P}^6$ of the block form

$$\sum_1^3 (\gamma_i^2 x_i^2 + \gamma_{i+3}^2 x_{i+3}^2 + 2\gamma_{i,i+3} x_i x_{i+3}) - c x_0^2$$

leading to abelian surfaces and provide their moduli. This class is specified by requiring that the linear span $\bar{V}$ of the quadrics contains *a non-degenerate curve* $\mathscr{C}$ *of rank* 4 *quadrics* (sum of four squares), rather than a discrete set of points, as would be the case for a generic set of 4 quadrics. This set of quadrics can equally well be described, as is done in Theorem 1 (Sect. 2), by requiring that their intersection in $\mathbb{P}^6$ be *singular along some component* $\mathscr{E}$ at infinity (i.e., at $x_0 = 0$) of genus $\neq 3$; then $\mathscr{E}$ is a natural $4-1$ unramified cover of $\mathscr{C}$. The *abelian surface* is then obtained by blowing up the intersection $\bigcap_1^4 \{Q_i = c_i x_0^2\}$ in $\mathbb{P}^6$ along $\mathscr{E}$ and blowing it down along the complementary locus at infinity.

In Sect. 3, we provide a full description of the family of linear spans $\bar{V}$ described above and we show that it splits into two definite classes, according to whether $\mathscr{C}$ has genus 0 or 1 (Theorem 2). With regard to a fixed set of variables, each of these classes is parametrized by high-dimensional varieties $\mathscr{V}$ spelled out in Theorem 3. However, allowing linear changes of variables, these classes, away from the branch locus of $\mathscr{V}$, can be described by a set of four *canonical quadrics* (normal forms) depending on 3 parameters in the first case and 4 parameters in the second case; this is carried out in Sects. 4–6. Theorem 7 of Sect. 6 is the main result of this paper. In Sect. 7, we assume that $\mathscr{C}$ contains a *degenerate* curve component; this leads naturally to $K3$ surfaces. The parameters mentioned above produce moduli not only for the intersection of quadrics, but also for abelian surfaces of *principal polarization* (hyperelliptic Jacobians) and *polarization* $(1,2)$ (Prym varieties of double covers of elliptic curves). Moreover these normal forms yield the set of invariants for the algebraically *completely integrable geodesic flows* on the group $SO(4)$ for specific families of left-invariant metrics. The latter have been classified by us in [2, 4] and the integration of the associated geodesic flows as linear motions on abelian surfaces has been carried out in [5], using the normal forms exhibited in this paper; the latter yield a handy set of coordinates in which to perform the linearizations. Finally, that discussion has led to a new left-invariant metric on $SO(4)$ whose geodesic flow has three quadratic invariants Q_i, $1 \leqq i \leqq 3$, and one quartic invariant Q_4. Upon blowing up the intersection in $\mathbb{P}^6$ defined by these equations along one component $\mathscr{E}$ of the singular locus at infinity and upon blowing it down along the other components, one obtains an abelian surface of *polarization* $(1,6)$. Then $\mathscr{E}$ is a natural $4-1$ cover of a *curve* of "*rank* 4 *quartics*" in the 6-dimensional projective space of quartic invariants generated by $Q_i Q_j$, $1 \leqq i \leqq j \leqq 3$, and Q_4. A quartic has "rank 4" when it has the form $\sum_0^3 x_i^2$ (quadric)$_i$. Most of the ideas for quadrics can then be extended to quartics. This is the object of Sect. 8.

As has been remarked, the methods employed are strongly guided by dynamics. We very much hope that these results will inspire the algebraic geometer to turn them into a more algebraic theory. Along these lines, there are some natural questions and extensions of these results. Given the intersection of four quadrics Q_i in $\mathbb{P}^6$ of the most general form, it is plausible to conjecture that the affine intersection completes into an abelian surface if and only if the linear span $\bar{V}$ contains a curve of rank 4 quadrics. Moreover what is the proper algebraic-geometrical framework in which to generalize the results about quartics in $\mathbb{P}^6$. Does it lead to descriptions of abelian surfaces with other polarizations?

Notations. Consider the linear span

$$V(Q_1,\dots,Q_N)=\left\{\sum_1^N \lambda_i Q_i,(\lambda_1,\dots,\lambda_N)\in\mathbb{P}^{N-1}\right\}\simeq\mathbb{P}^{N-1}$$

of N quadrics $Q_1,\dots,Q_N$ and its discriminant variety

$$\Delta(Q_1,\dots,Q_N)=\left\{\sum_1^N \lambda_i Q_i \quad \text{such that determinant} \quad \left(\sum\lambda_i Q_i\right)=0\right\}$$
$$\subseteq V(Q_1,\dots,Q_N)\simeq\mathbb{P}^{N-1}.$$

This paper deals with the situation of 4 quadrics $Q_1,\dots,Q_4$ in x of the block form [1]

$$\sum_1^3(\gamma_i^2x_i^2+\gamma_{i+3}^2x_{i+3}^2+2\gamma_{i,i+3}x_ix_{i+3});$$

define $\bar{Q}_i\equiv Q_i-c_ix_0^2$. Let a_i, a_{i+3} and $a_{i,i+3}$ $(1\leqq i\leqq 3)$ be linear functions of X, Y, Z, U defined by

$$XQ_1+YQ_2+ZQ_3+UQ_4\equiv\sum_1^3(a_ix_i^2+a_{i+3}x_{i+3}^2+2a_{i,i+3}x_ix_{i+3}). \tag{1}$$

Then, one checks that

$$\Delta(\bar{Q}_1,\dots,\bar{Q}_4)=\Delta(Q_1,\dots,Q_4)\cup H,$$

where H is the hyperplane

$$H=\{p=(X,Y,Z,U)\in\mathbb{P}^3 \quad \text{such that} \quad c(p)\equiv Xc_1+Yc_2+Zc_3+Uc_4=0\}.$$

It is easily seen that

$$\Delta(Q_1,\dots,Q_4)=K_1\cup K_2\cup K_3\subseteq\mathbb{P}^3,$$

where the K_i are three quadratic cones in $\mathbb{P}^3$

$$K_i=\{p\in\mathbb{P}^3,\ H_i(p)\equiv a_ia_{i+3}-a_{i,i+3}^2=0\}; \tag{2}$$

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[1] Throughout this paper, the vectors x, y or $z\in\mathbb{R}^6$ have the form $x=(x_1,x_4,x_2,x_5,x_3,x_6)=(x',x'')$ with $x'=(x_1,x_2,x_3)$ and $x''=(x_4,x_5,x_6)$

they describe the locus of $Q \in V(Q_1, \ldots, Q_4)$ such that the i^{th} term of (1) is a perfect square.

Henceforth, let

$$V = V(Q_1, \ldots, Q_4) \simeq \mathbb{P}^3, \qquad \bar{V} = V(\bar{Q}_1, \ldots, \bar{Q}_4) \simeq \mathbb{P}^3$$
$$\Delta = \Delta(Q_1, \ldots, Q_4), \qquad \bar{\Delta} = \Delta(\bar{Q}_1, \ldots, Q_4).$$

The space V will be required to satisfy the following three non-degeneracy conditions[2]:

Condition C0. After a linear change of variables not mixing up blocks there are two quadrics Q_1 and Q_2 in V having the form $Q_1 = x_1^2 + x_2^2 + x_3^2$ and $Q_2 = x_4^2 + x_5^2 + x_6^2$.

Condition C1. The three cones K_i are irreducible (i.e., for every $i = 1, 2, 3$, there is some quadric in V, containing the term $x_i x_{i+3}$).

Condition C2. The three cones K_i have distinct vertices (i.e., no quadric in V has the form $\gamma_i^2 x_i^2 + \gamma_{i+3}^2 x_{i+3}^2 + 2\gamma_{i,i+3} x_i x_{i+3}$).

As a consequence of condition C0, one can pick a basis of V having the following form:

$$\begin{aligned}
Q_1 &= x_1^2 + x_2^2 + x_3^2 \\
Q_2 &= x_4^2 + x_5^2 + x_6^2 \\
Q_3 &= \sum_1^3 (\alpha_i^2 x_i^2 + \alpha_{i+3}^2 x_{i+3}^2 + 2\alpha_{i,i+3} x_i x_{i+3}) \\
Q_4 &= \sum_1^3 (\beta_i^2 x_i^2 + \beta_{i+3}^2 x_{i+3}^2 + 2\beta_{i,i+3} x_i x_{i+3})
\end{aligned} \tag{3}$$

with fixed α_i and $\beta_i \in \mathbb{C}$. With regard to this basis

$$a_i = X + \alpha_i^2 Z + \beta_i^2 U, \quad a_{i+3} = Y + \alpha_{i+3}^2 Z + \beta_{i+3}^2 U, \quad a_{i,i+3} = \alpha_{i,i+3} Z + \beta_{i,i+3} U$$

and therefore the quadratic equations $H_i = 0$ defining the cones K_i are linear in the variables X and Y. Define (see Fig. 1)

$$\mathscr{C} = K_1 \cap K_2 \cap K_3$$

$$\cong \left\{ \begin{array}{l} \text{the set of quadrics in } \bar{V} \\ \text{of rank 4 having the form} \\ \sum_1^3 (\gamma_i x_i + \gamma_{i+3} x_{i+3})^2 - c_i x_0^2 \end{array} \right\} = \bigcap_1^3 \{p \in \mathbb{P}^3 \quad \text{such that} \quad H_i(p) = 0\}$$

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[2] The three conditions C0, C1, C2 are generically satisfied

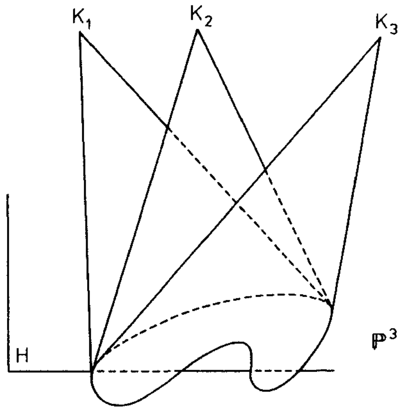


Fig. 1

Observe that the generic intersection $\mathscr{C} = K_1 \cap K_2 \cap K_3$ is finite and non-empty. Whenever $\mathscr{C}$ is a curve, it will be called *non-degenerate*, if $\mathscr{C} \nsubseteq$ hyperplane in $V \cong \mathbb{P}^3$. Also define

$$I \equiv \bigcap_{1}^{4} \{Q_i - c_i = 0,\ x \in \mathbb{C}^6\},$$

$$\bar{I} \equiv \bigcap_{1}^{4} \{Q_i - c_i x_0^2 = 0,\ x \in \mathbb{P}^6\},$$

$$\mathscr{E} \equiv \{p \in \bar{I} \cap \{x_0 = 0\} \text{ where } \bar{I} \text{ is singular}\} \subseteq \bar{I} \cap \{x_0 = 0\}.$$

Whenever $\mathscr{E}$ is a curve, it will be called non-degenerate, if $\mathscr{E} \nsubseteq$ a 2-dimensional plane of the form

$$\bigcap_{i=1}^{3} \{A_i x_i + A_{i+3} x_{i+3} = 0\} \cap \{x_0 = 0\} \subset \mathbb{P}^6.$$

Let $\bar{\mathscr{V}}$ be the space of linear spans $\bar{V}$ with regard to a fixed set of variables x. It is naturally acted upon by the subgroup

$$g = S_3 \otimes GL(2) \otimes GL(2) \otimes GL(2) \otimes GL(1) \subset GL(7),$$

which induces linear maps on x_0 and each block (x_i, x_{i+3}) for $1 \leqq i \leqq 3$ and which permutes the order of the blocks. Thus it is natural to define $\bar{\mathscr{M}} \equiv \bar{\mathscr{V}}/g$.

2. Curves of Rank 4 Quadrics and Singular Curves at Infinity

In this section we exhibit a natural 1–4 map from the set of rank 4 quadrics $\mathscr{C}$ to the singular locus $\mathscr{E}$ of $\bar{I}$ at ∞. Then under some mild conditions, the set of rank 4 quadrics and the singular locus will be curves at the same time; therefore if $\mathscr{C}$ is a curve, $\mathscr{E}$ is a four-fold cover of $\mathscr{C}$, which is sometimes unramified and sometimes ramified. As discussed later, the unramified case appears whenever the affine intersection of the quadrics completes into an abelian surface, while the ramified situation occurs whenever the affine intersection completes into a $K3$ surface; the former case is discussed in Sects. 4–6 and the latter in Sect. 7.

Throughout this section, the space V of quadrics satisfies conditions C0, C1, C2 and we pick a basis $Q_1, \dots, Q_4$ of the block form; see the notations in Sect. 1. Define

$$\mathscr{C}' \equiv \mathscr{C} \setminus \{\text{line and point components}\} \subset V$$
$$\mathscr{E}' \equiv \mathscr{E} \setminus \{\text{its degenerate and point components}\} \subseteq \bar{I} \cap \{x_0 = 0\}.$$

Then $\mathscr{C}'$ is parametrized as mentioned in Sect. 1 by a family of rank 4 quadrics having the form

$$\bar{Q}(p) = Q(p) - c(p)x_0^2 = \sum (\gamma_i(p)x_i + \gamma_{i+3}(p)x_{i+3})^2 - c(p)x_0^2, \quad p \in \mathscr{C}'.$$

Theorem 1. *There is a 1–4 map from $\mathscr{C}'$ to $\mathscr{E}'$ given by*

$$p \curvearrowright \Gamma_p \cap \bar{I} = 4 \text{ points},$$

where

$$\Gamma_p = 2\text{-dimensional plane}: \bigcap_{i=1}^{3} \left\{ \begin{matrix} \gamma_i(p)x_i + \gamma_{i+3}(p)x_{i+3} = 0 \\ x_0 = 0, \quad 1 \leqq i \leqq 3 \end{matrix} \right\},$$

while the map

$$\mathscr{E}' \curvearrowright \mathscr{C}'$$
$$q \curvearrowright \sum_1^4 n_i(q)Q_i, \quad \text{where} \quad (n_1(q), \dots, n_4(q))^T \in \mathit{kernel} \left[\frac{\partial Q_i}{\partial x_j}(q) \right]^T_{\substack{1 \leqq i \leqq 4 \\ 0 \leqq j \leqq 6}}$$

provides the inverse. Hence $\mathscr{C}'$ and $\mathscr{E}'$ are curves simultaneously.

The surface $\bar{I}$ experiences a two-fold normal crossing along $\mathscr{E}'$, with pinch points occuring at $c(p) = 0$; unless[3] *$u_1^2 = u_2^2 = u_3^2$ on $\mathscr{C}$, $\mathscr{E}'$ can be given by*

$$\mathscr{E}' = \left\{ \begin{matrix} \mathscr{C}' \\ \dfrac{x_1^2}{u_3^2 - u_2^2} = \dfrac{x_2^2}{u_1^2 - u_3^2} = \dfrac{x_3^2}{u_2^2 - u_1^2} \end{matrix} \right\}, \tag{1}$$

where $u_i \equiv -a_{i,i+3}/a_{i+3} = -\gamma_i(p)/\gamma_{i+3}(p)$ are rational functions on $\mathscr{C}'$. Upon normalizing $\bar{I}$ along $\mathscr{E}'$, the curve $\mathscr{E}'$ turns into the curve

$$\tilde{D} = \left\{ \begin{matrix} \mathscr{E}' \\ U^2 = c(p) \end{matrix} \right\}.$$

Proof. Let $\mathscr{C}''$ be a component of $\mathscr{C}'$. Then referring to the notations in Sect. 1, the linear functions of X, Y, Z, U

$$\gamma_i^2(p) = a_i, \gamma_{i+3}^2(p) = a_{i+3}, \gamma_i\gamma_{i+3}(p) = a_{i,i+3} \quad \text{and} \quad c(p)$$

are meromorphic on $\mathscr{C}''$; for each $1 \leqq i \leqq 3$, γ_i^2 or $\gamma_{i+3}^2 \not\equiv 0$ on $\mathscr{C}''$, or else $\mathscr{C}'' \subseteq (\text{vertex } K_i) = \text{point}$; this shows that Γ_p is indeed a two-dimensional plane. Next we show that

$$\Gamma_p \cap \bar{I}_c = \Gamma_p \cap \{Q' = 0\} \cap \{Q'' = 0\} \tag{2}$$

[3] This actually happens when $\mathscr{C}$ is degenerate; this case is discussed in Sect. 7

for appropriately chosen quadrics Q' and Q'' in V. Indeed, for most points $p \in \mathscr{C}''$, $\bar{V}$ can be spanned by the four quadrics

$$Q(p) - c(p)x_0^2, \ \frac{\partial Q}{\partial p}(p) - \frac{\partial c}{\partial p}(p)x_0^2, \ Q' - c'x_0^2 \quad \text{and} \quad Q'' - c''x_0^2;$$

their intersection is $\bar{I}$ as well. Since, evidently

$$\Gamma_p \subseteq \{\bar{Q}(p) = 0\} \cap \left\{\frac{\partial \bar{Q}}{\partial p}(p) = 0\right\},$$

we have that (2) holds.

To show that the image of the map above is in $\mathscr{E}$, notice that the gradient $\partial(Q(p) - c(p)x_0^2)/\partial p$ of one of the defining relations $Q(p) - c(p)x_0^2 = 0$ clearly vanishes along the plane Γ_p. More is true: the image is in $\mathscr{E}'$, as x_{i+3}/x_i (constant on Γ_p) is a non-constant function of $p \in \mathscr{C}''$ for some $1 \leqq i \leqq 3$. To see this, we must distinguish between two cases:

(i) When $\mathscr{C}''$ is nondegenerate. Then

$$\left(\frac{x_{i+3}}{x_i}\right)^2\Bigg|_{\Gamma_p \cap I_c} = \left(\frac{\gamma_i(p)}{\gamma_{i+3}(p)}\right)^2 = \frac{a_i(X,Y,Z,U)}{a_{i+3}(X,Y,Z,U)} \neq \text{const (independent of } p \in \mathscr{C}''),$$

since otherwise $\mathscr{C}'' \subset$ hyperplane $\subset \mathbb{P}^3$, violating the nondegeneracy of $\mathscr{C}''$.

(ii) When $\mathscr{C}''$ is degenerate. This case is completely spelled out in Sect. 7 and Appendix 1; from that analysis it turns out that for some i

$$\left(\frac{x_{i+3}}{x_i}\right)^2\Bigg|_{\Gamma_p \cap I_c} = \left(\frac{\gamma_i(p)}{\gamma_{i+3}(p)}\right)^2 \neq \text{const.}$$

Having defined a map from $\mathscr{C}'$ to $\mathscr{E}'$, we now show that it is 1–4; by a degree count, it suffices to show that the set (2) is finite. In (2) we may pick $Q' = Q_1$ and Q'' a generic element in V of the form

$$Q'' = \sum_1^3 (\alpha_i^2 x_i^2 + \alpha_{i+3}^2 x_{i+3}^2 + 2\alpha_{i,i+3} x_i x_{i+3}). \tag{3}$$

Then observing that $u_i(p) \neq \text{const}$ (for some i) on $\mathscr{C}''$ [see (i) and (ii) above], we get

$$\begin{aligned}\Gamma_p \cap \bar{I} &= \Gamma_p \cap \{Q_1 = 0\} \cap \{Q'' = 0\} \\ &= \left\{\begin{matrix} x_1^2 + x_2^2 + x_3^2 = 0 \\ x_1^2 P_1(u_1) + x_2^2 P_2(u_2) + x_3^2 P_3(u_3) = 0 \\ x_{i+3} = u_i(p) x_i, \quad i = 1,2,3 \end{matrix}\right\},\end{aligned} \tag{4}$$

where

$$P_i(u_i) = \alpha_i^2 + \alpha_{i+3}^2 u_i^2 + 2\alpha_{i,i+3} u_i.$$

The right hand side of (4) consists of four points for most $p \in \mathscr{C}''$ unless the two first equations in the brackets are proportional, i.e.,

$$P_1(u_1(p)) \equiv P_2(u_2(p)) \equiv P_3(u_3(p)), \quad \text{for most } p \in \mathscr{C}''. \tag{5}$$

In particular picking $Q''=Q_2$ forces $u_1(p)\equiv u_2(p)\equiv u_3(p)\equiv u(p)$ for all $p\in\mathscr{C}''$, by possibly flipping the signs of some x_i. Since $u(p)$ is nonconstant along $\mathscr{C}''$, the polynomials P_1, P_2 and P_3 agree for a continuum of values by (5) and hence $P_1(u)\equiv P_2(u)\equiv P_3(u)$. This implies that the generic element Q'' in (2) would have the form

$$Q''=\alpha_1^2 Q_1+\alpha_4^2 Q_2+2\alpha_{14}(x_1x_4+x_2x_5+x_3x_6),$$

showing that V is three-dimensional, which is absurd. Thus $\Gamma_p\cap\bar{I}$ is a finite set and hence it consists of four points. This ends the proof that the map $\mathscr{C}'\to\mathscr{E}'$ is a 1–4 map, as it holds on each component $\mathscr{C}''$ of $\mathscr{C}'$.

Conversely, we now proceed to construct the inverse map $\mathscr{E}'\curvearrowright\mathscr{C}'$ in the following fashion. For $q\in\mathscr{E}''$, a component of $\mathscr{E}'$, define

$$(\lambda_1(q),\dots,\lambda_4(q))\in\operatorname{kernel}\left(\frac{\partial(\bar{Q}_1,\dots,\bar{Q}_4)}{\partial(x_0,\dots,x_6)}\right)^T, \tag{6}$$

the rank of that Jacobian matrix being <4 (and $=3$ as will be shown in Remark 1 at the end of the proof) along $\mathscr{E}''$ and define accordingly the unique quadric

$$Q(q)\equiv\sum_1^4\lambda_i(q)Q_i(x)\equiv\sum_1^3(\gamma_i^2(q)x_i^2+\gamma_{i+3}^2(q)x_{i+3}^2+2\gamma_{i,i+3}(q)x_ix_{i+3})\in V. \tag{7}$$

Statement (6) expresses that

$$\left.\frac{\partial Q(q)}{\partial x_i}\right|_{x=q}=0,\qquad i=0,1,\dots,6,$$

amounting to the linear system

$$\left.\begin{cases}\gamma_i^2(q)x_i+\gamma_{i,i+3}(q)x_{i+3}=0\\ \gamma_{i,i+3}(q)x_i+\gamma_{i+3}^2(q)x_{i+3}=0\end{cases}\right\}\qquad i=1,2,3.$$

Hence the determinants $(\gamma_i^2\gamma_{i+3}^2-\gamma_{i,i+3}^2)(q)$, $i=1,2,3$ all must vanish, unless $x_i=x_{i+3}=x_0=0$ for some i along $\mathscr{E}''$, which is ruled out by condition C2. Hence

$$Q(q)=\sum_1^3(\gamma_i(q)x_i+\gamma_{i+3}(q)x_{i+3})^2$$

is a rank 3 quadric for every $q\in\mathscr{E}''$. By the non-degeneracy of $\mathscr{E}'$, for some i and along each component of $\mathscr{E}'$, the ratio of $Q(q)$ coefficients satisfies

$$\frac{\gamma_{i,i+3}(q)}{\gamma_i^2(q)}=\pm\frac{\gamma_{i+3}(q)}{\gamma_i(q)}=-\left.\left(\frac{x_i}{x_{i+3}}\right)\right|_q=\text{non-constant function of } q\in\mathscr{E}',$$

and so $Q(q)$ defines a curve of rank 3 quadrics in $V\simeq\mathbb{P}^3$ as q varies along $\mathscr{E}''$. This curve is not a union of lines, for if it were, by Appendix 1, each line would be of the form $x_1^2+(1+t)x_2^2+tx_3^2$, and so $\mathscr{E}'$ would be a union of points which is contradictory. By following through the maps $\mathscr{C}'\curvearrowright\mathscr{E}'\curvearrowright\mathscr{C}'$, it is easily seen that we have constructed the inverse of the map from $\mathscr{C}'\curvearrowright\mathscr{E}'$, thus proving the first part of Theorem 1.

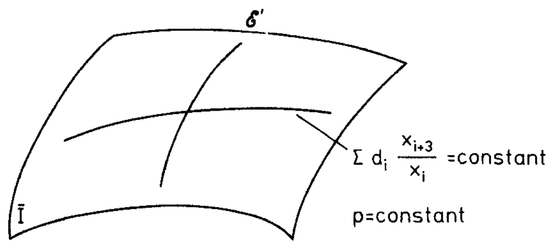


Fig. 2

The curve $\mathcal{E}'$ is given by (1), namely by adjoining the functions x_1/x_3 and x_2/x_3 to $\mathscr{C}'$, as a consequence of putting $x_{i+3}=u_i(p)x_i$ into Q_1 and Q_2:

$$x_1^2+x_2^2+x_3^2=0, \qquad u_1^2x_1^2+u_2^2x_2^2+u_3^2x_3^2=0\,.$$

Then the remaining functions x_4, x_5 and x_6 viewed projectively are rational on $\mathcal{E}'$.

Next we show that $\bar{I}$ experiences a two-fold normal crossing along $\mathcal{E}'$. The formula (1) provides a parametrization of $\mathcal{E}'\subseteq\bar{I}$ in terms of $p\in\mathscr{C}'$. We seek to parametrize the surface $\bar{I}$, in the neighborhood of $\mathcal{E}'$, by p and x_0; to do this, we extend the function p to a neighborhood of $\mathcal{E}'$ in $\bar{I}$, by solving the following equation for p:

$$\sum d_iu_i(p)=\sum d_i\frac{x_{i+3}}{x_i}, \qquad d_i\in\mathbb{C}, \text{ fixed,} \tag{8}$$

the right hand side being a function defined on $\bar{I}$. This can be done, since the $u_i(p)$ are non-constant meromorphic functions on $\mathcal{E}'$. Consider the local change of variables in $\bar{I}\setminus\mathcal{E}'$ near $\mathcal{E}'$

$$(x_0,\ldots,x_6) \ \text{ with } \ x_3=1\to(x_0,p,y_1,y_2,\ldots,y_5), \ \text{ with } \sum_1^3 d_iy_i=0$$

defined by

$$\frac{x_{i+3}}{x_i}(p,x_0)=u_i(p)+x_0y_i \qquad i=1,2,3,$$

$$x_i(p,x_0)=\frac{x_i}{x_3}(p,0)+x_0y_{i+3} \qquad i=1,2,$$

with $\sum d_iy_i=0$ imposed to make it compatible with (8). Using this change of variables in the quadrics $\bar{Q}(p)$ and $\partial\bar{Q}/\partial p$ yields

$$\sum_{i=1}^3 x_i^2(p,0)\gamma_{i+3}^2(p)y_i^2-c(p)=O(x_0)$$

and

$$\sum_{i=1}^3 x_i^2(p,0)\{\gamma_i,\gamma_{i+3}\}(p)y_i=O(x_0)\,,$$

in addition to the customary relation defining the coordinates,

$$\sum_1^3 d_i y_i = 0 .$$

Here, the Wronskian $\{\gamma_i, \gamma_{i+3}\} \equiv \gamma_3^2 \partial(\gamma_i/\gamma_{i+3})/\partial p$ is a meromorphic function on $\mathscr{C}$ and $\mathscr{E}$. By picking, for instance, $d_1 = d_2 = 0$ and $d_3 = 1$, the three relations above have the solution

$$\begin{aligned} &\frac{f_1^2 + f_2^2}{f_2^2} (\gamma_4 x_1)^2 y_1^2 = c(p) + O(x_0) \\ &y_2 = -\left(\frac{x_1}{x_2}\right)^2 \frac{\{\gamma_1, \gamma_4\}}{\{\gamma_2, \gamma_5\}} y_1 + O(x_0) \\ &y_3 = 0, \end{aligned} \tag{9}$$

where the ratios

$$f_i^2 \equiv \frac{x_i^2 \{\gamma_i, \gamma_{i+3}\}^2}{\gamma_{i+3}^2}$$

are meromorphic functions on $\mathscr{C}$. Using the same change of variables in the equations Q_1 and Q_2 yields a linear system in y_4 and y_5

$$\begin{pmatrix} 1 & 1 \\ u_1^2 & u_2^2 \end{pmatrix} \begin{bmatrix} y_4 \dfrac{x_1}{x_3} \\ y_5 \dfrac{x_2}{x_3} \end{bmatrix} = \text{linear function in } (y_1, y_2) + O(x_0) .$$

The upshot is that the y_i are all rational functions of y_1, p and x_0.

By picking other $d_i = 0$, one would find two cycled versions of these equations with the same right hand side $c(p)$ whereas the coefficient of y_1^2 in (9) cycles. This implies that $f_1^2 + f_2^2$ is a perfect square on $\mathscr{E}'$, because if not branching would occur at some simple zero of $f_1^2 + f_2^2$, which is not a zero of $c(p)$; hence it would occur in all the cycled equations and thus at such a point $f_1^2 + f_2^2$, $f_2^2 + f_3^2$ and $f_3^2 + f_1^2$ would simultaneously vanish, which is checked to be impossible. Thus the only branching for y_1 occurs at the zeroes of the meromorphic function $c(p)$ on $\mathscr{E}$, which has simple zeroes for generic c_i.

Remark 1. With regard to the Jacobian matrix (6) in the proof above, note that, if its rank <3 for a generic element $q \in \mathscr{E}''$, it would lead to a surface of rank 3 quadrics, which can be eliminated using condition C1.

Remark 2. As pointed out, Theorem 1 also holds for a degenerate component $\mathscr{C}''$; then the map from $\mathscr{C}''$ to $\mathscr{E}$ is worked out explicitly in Sect. 7 and Appendix 1, where it is also shown that $\bar{I}$ is a $K3$-surface after blowing up the surface along $\mathscr{E}$. In this case, the functions u_i satisfy $u_1^2 = u_2^2 = u_3^2$ and thus the expression (1) for the curve $\mathscr{E}'$ must be slightly modified.

3. When Does $\bar{V}$ Contain a Curve of Rank 4 Quadrics?

As before, the space V satisfies conditions C0, C1, and C2 and thus contains a basis of the form (1.3); note that x_0 does not play any role in this section. This section addresses the question how to describe the set of spaces V of quadrics depending on a given set of variables $x_1, \ldots, x_6$ such that V contains a non-degenerate curve of rank 3 quadrics (rank 3 and not 4, since x_0 does not play any role here). It is shown that this set splits up into two very distinct strata, described in detail in Theorems 2 and 3. Each of these strata leads, as will appear in later sections, to different types of Abelian surfaces. Before being able to do so, one must provide a parametrization of the linear spaces V (with regard to the fixed set of variables $x_1, \ldots, x_6$) in terms of some canonical basis in V; this parametrization is only valid in a certain affine patch, while the different patches can be distinguished, one from another, by the configuration of the three vertices of the cones K_i. This indispensable but tedious classification is stated in Lemma 1 and proven in Appendix 3. Finally, for future use, we define the 3×6 matrix of coefficients of the basis (1.3)

$$A = (1 \;\; \alpha_i^2 \;\; \beta_i^2 \;\; (\alpha_{i,i+3} + \beta_{i,i+3})^2 \;\; \alpha_i^2 \alpha_{i+3}^2 - \alpha_{i,i+3}^2 \;\; \beta_i^2 \beta_{i+3}^2 - \beta_{i,i+3}^2)_{i=1,2,3},$$

the square matrix

$$B = (1 \;\; \alpha_{i+3}^2 \;\; \beta_{i+3}^2)_{i=1,2,3}$$

and the 3×9 matrix (A, B).

Lemma 1. *The linear span of quadrics V subject to the customary non-degeneracy conditions* C0, C1, *and* C2, *admits, in appropriate coordinates, a basis either of the normal form*

$$\begin{aligned} NF1: \quad & Q_1 = x_1^2 + x_2^2 + x_3^2 \\ & Q_2 = x_4^2 + x_5^2 + x_6^2 \\ & Q_3 = \alpha_4^2 x_4^2 + (x_2 + \alpha_5 x_5)^2 + (\alpha_3^2 x_3^2 + \alpha_6^2 x_6^2 + 2\alpha_{36} x_3 x_6) \\ & Q_4 = (x_1 + \beta_4 x_4)^2 + \beta_5^2 x_5^2 + (\beta_3^2 x_3^2 + \beta_6^2 x_6^2 + 2\beta_{36} x_3 x_6) \end{aligned}$$

or of the form given by NF2, …, NF9 *as listed in Appendix 2, up to permutations of the Q_i and up to the following permutation of the variables x_i:*

(i) *interchanging the* 3 *blocks (x_i, x_{i+3}) of variables*

(ii) *interchanging $(x_1, x_2, x_3) \leftrightarrow (x_4, x_5, x_6)$,*

NF1 *turns out to be the generic situation, while* NF2, …, NF9 *can be regarded as boundary cases.*

Proof. The configuration of the three vertices of the cones K_i, $i = 1, 2, 3$ in V serves to distinguish between the nine cases. The proof of this lemma appears in Appendix 2.

Theorem 2. *The linear span V of quadrics satisfying conditions* C0, C1, *and* C2, *contains a non-degenerate component $\mathscr{C}' \subset \mathscr{C} \subset \mathbb{P}^3$ of rank* 3 *quadrics if and only if the following two conditions are satisfied*:

(i) *V has a basis of the form* NF1 *(besides one other case where V has a basis of the form* NF3, *which is a limiting case of* NF1*) and*

(ii) $\mathscr{C}'=\mathscr{C}$, *where* $\mathscr{C}$ *is either an irreducible rational curve with vertex* $(K_i)\in\mathscr{C}$ *(Case 1) or an elliptic curve (Case 2).*

We now describe in terms of the basis NF1 *the exact conditions leading to Case 1 or Case 2:*

Case 1. $\mathscr{C}=$ *irreducible rational curve with each vertex* $(K_i)\in\mathscr{C}$
$\Leftrightarrow$ *the cones* K_i *are linearly independent and* $K_i\cap K_j=\mathscr{C}\cup \mathit{line}_{ij}$
$\Leftrightarrow$ *the basis* NF1 *of* V *satisfies the conditions*
$\alpha_4=\beta_5=0$, *rank* $A\leqq 2$ *and* $B\leqq 2$
(spelled out in Table 1) with the quantities
$\alpha_5,\beta_4,\alpha_3,\alpha_6,\beta_3,\beta_6$ *and* $\alpha_5^2-\beta_4^2$ *all* $\neq 0$.

Case 2. $\mathscr{C}=$ *elliptic curve*
$\Leftrightarrow$ *the cones* K_i *are linearly dependent*
$\Leftrightarrow$ *the basis* NF1 *of* V *satisfies the condition: rank* $(A,B)\leqq 2$
(spelled out in Table 1) with
$\alpha_5,\beta_4,\ \alpha_3,\ \beta_3\neq 0$
α_6 *or* $\beta_6\neq 0$, α_4 *or* $\beta_5\neq 0$
$\alpha_4^2-\alpha_5^2$ *or* $\beta_4^2-\beta_5^2\neq 0$
$\alpha_4^2-\beta_4^2$ *or* $\alpha_5^2-\beta_5^2\neq 0$
$\alpha_4^2-\beta_5^2$ *or* $\beta_4^2-\alpha_5^2\neq 0$.

Remark. Observe that the normal form NF1 is invariant under the involution

$$(x_1,x_4)\leftrightarrow(x_2,x_5),\ (x_3,x_6)\ \text{ stay},\quad (\alpha_4,\alpha_5)\leftrightarrow(\beta_5,\beta_4),$$
$$(\alpha_3,\alpha_6,\alpha_{36})\leftrightarrow(\beta_3,\beta_6,\beta_{36}).$$

Both the conditions in case 1 and case 2 are invariant as well under this involution, so that every identity appearing in Table 1 is either self-dual or the dual identity holds. Also with regard to Table 1, notice that without loss of generality, we may set $\alpha_{36}=\alpha_3\alpha_6$, $\beta_{36}=\beta_3\beta_6$ by absorbing the sign of α_{36} and β_{36} into α_3 and β_3.

Proof. In view of the special form of Q_1 and Q_2 in (1.3), the set of rank 3 quadrics $\mathscr{C}\subset V$ is given by the following equations

$$\mathscr{C}\equiv K_1\cap K_2\cap K_3$$
$$=\left\{\begin{aligned}&(X,Y,Z,U)\in\mathbb{P}^3,\ H_i\equiv a_ia_{i+3}-a_{i,i+3}^2,\quad i=1,2,3,\\ &\qquad\equiv(X+\alpha_i^2Z+\beta_i^2U)\,(Y+\alpha_{i+3}^2Z+\beta_{i+3}^2U)\\ &\qquad\quad-(\alpha_{i,i+3}Z+\beta_{i,i+3}U)^2\quad.\\ &\qquad\equiv\ell_i(Y,Z,U)X+m_i(Y,Z,U)=0,\\ &\qquad\quad\ell_i(Y,Z,U)\ \text{linear.}\end{aligned}\right\}\tag{1}$$

Then $\mathscr{C}$ contains a nondegenerate irreducible curve $\mathscr{C}'$ (i.e., not in a hyperplane, in particular not in $\ell_1=0$ and not in $U=0$) if and only if

$$\mathscr{C}\cap\{\ell_1=Y+\ldots\neq 0\}=\bigcap_1^3\{\ell_iX+m_i=0\}\cap\{\ell_1\neq 0\}$$
$$=\{(P_1,P_2,P_3)\equiv(\ell_1,\ell_2,\ell_3)\wedge(m_1,m_2,m_3)=0\}\cap\{\ell_1X+m_1=0\}\cap\{\ell_1\neq 0\}$$
$$=\{P_2(Y,Z,U)=0\}\cap\{P_3(Y,Z,U)=0\}\cap\{\ell_1X+m_1=0\}\cap\{\ell_1\neq 0\}$$

Table 1

$$1.\ \begin{cases} \alpha_{36}^2=\alpha_3^2\alpha_6^2, \quad \beta_{36}^2=\beta_3^2\beta_6^2, \qquad \alpha_3, \beta_3, \alpha_5, \beta_4 \neq 0 \\ \alpha_3^2+\beta_3^2=1, \\ (\alpha_{36}+\beta_{36})^2=\alpha_3^2\alpha_5^2+\beta_3^2\beta_4^2, \\ \det(1\ \alpha_i^2\ \beta_i^2)_{4\leq i\leq 6}=0. \end{cases}$$

Case 1

2. $\alpha_4=\beta_5=0,$
3. $\alpha_5^2\alpha_3\beta_6=\beta_4^2\beta_3\alpha_6,$
4. $\alpha_5^2=\alpha_3^{-1}\alpha_6(\alpha_3\alpha_6+\beta_3\beta_6)$ and dual
5. $\beta_3\alpha_6-\beta_6\alpha_3=\alpha_5^{-2}\beta_3\alpha_6(\alpha_5^2-\beta_4^2)$
   with $\alpha_6, \beta_6 \neq 0,$

   $\alpha_5^2-\beta_4^2 \neq 0.$

Case 2

2. $\alpha_6^2-\alpha_3^2\alpha_5^2-\beta_3^2\alpha_4^2=0$ and dual,
3. $(\alpha_3\beta_6-\alpha_6\beta_3)^2=\alpha_3^2\beta_5^2+\beta_3^2\alpha_4^2,$
4. $2\alpha_6\beta_6=-\alpha_3\beta_3 l,$
5. $\alpha_3^2\beta_3^2k^2=4\alpha_4^2\beta_4^2\beta_4^2+4\alpha_5^2\beta_5^2\alpha_3^2,$
6. $\alpha_3^4k^2+\alpha_3^2(4\alpha_5^2\beta_5^2-4\alpha_4^2\beta_4^2-k^2)+4\alpha_4^2\beta_4^2=0,$
7. $(\ell+2\beta_4\varepsilon\alpha_5)^2-4\alpha_4^2\beta_5^2=\left(\dfrac{2}{\alpha_3\beta_3}(\beta_4\beta_3\alpha_6-\varepsilon\alpha_5\alpha_3\beta_6)\right)^2, \quad \varepsilon=\pm 1,$
8. $(\alpha_3\beta_6-\alpha_6\beta_3)(\alpha_6\beta_6-\alpha_3\beta_3(\alpha_5^2-\alpha_4^2))-\beta_5^2\alpha_{36}-\alpha_4^2\beta_{36}=0$ and dual,
9. $\alpha_3\beta_3(\alpha_6^2\beta_5^2+\beta_6^2\alpha_4^2)=(\alpha_3\beta_6-\alpha_6\beta_3)(\alpha_6\beta_6(\alpha_3\beta_6-\alpha_6\beta_3)-\beta_5^2\alpha_{36}+\alpha_4^2\beta_{36}),$
10. $\alpha_3^2(\alpha_6^2\beta_5^2+\beta_6^2\alpha_4^2)=(\alpha_3\beta_6-\alpha_6\beta_3)(\alpha_6^2(\alpha_3\beta_6-\alpha_6\beta_3)+\alpha_4^2(\alpha_3\beta_6+\alpha_6\beta_3)),$
11. $\alpha_6^2\beta_5^2-\beta_3^2(\alpha_6^2\beta_5^2+\beta_6^2\alpha_4^2)=(\alpha_3^4\alpha_5^2\beta_5^2-\beta_3^4\alpha_4^2\beta_4^2),$ and dual.

    Assuming $k\equiv\alpha_4^2+\beta_4^2-\alpha_5^2-\beta_5^2,\ l\equiv\alpha_4^2-\beta_4^2-\alpha_5^2+\beta_5^2,$
    with α_6 or $\beta_6 \neq 0$, α_4 or $\beta_5 \neq 0$, $\alpha_4^2-\alpha_5^2$ or $\beta_4^2-\beta_5^2 \neq 0$
    $\alpha_4^2-\beta_4^2$ or $\alpha_5^2-\beta_5^2 \neq 0$, $\alpha_4^2-\beta_5^2$ or $\beta_4^2-\alpha_5^2 \neq 0.$

(Using the identity $\sum \ell_i P_i = \langle \ell, \ell \wedge m \rangle = 0$ and $\ell_1 \neq 0$) contains a non-degenerate curve $\mathscr{C}'$. Therefore it suffices to investigate under which conditions the varieties $P_2 = 0$ and $P_3 = 0$ have a non-degenerate curve in common.

The polynomials P_2 and P_3 will take on quite different forms according to whether V is spanned by a basis of the form NF1, ..., or NF9. In this proof we shall deal with the basis NF1, while the remaining NF2, ..., NF9 will be investigated in Lemma 2. For the basis NF1, substituting $\alpha_1 = \alpha_{14} = \beta_2 = \beta_{25} = 0$, $\alpha_2 = \beta_1 = 1$, $\alpha_{25} = \alpha_5$, $\beta_{14} = \alpha_4$ into (1) leads to

$$\left\{\begin{array}{c} P_3(Y,Z,U) = Y^2(Z-U) + Y(aZ^2 + bZU + cU^2) + ZU(dZ + eU) \\ \text{with } a = \alpha_4^2, \quad b = \beta_5^2 - \alpha_5^2 + \beta_4^2 - \alpha_4^2, \ c = -\beta_5^2, \\ d = \alpha_4^2(\beta_5^2 - \alpha_5^2), \ e = \beta_5^2(\beta_4^2 - \alpha_4^2) \end{array}\right\}, \tag{2}$$

while $P_2(Y,Z,U)$ is a similar expression. Under what conditions do the polynomials P_3 and P_2 have a common factor which is non-linear? Since the common factor is not allowed to be in the hyperplane $U = 0$, we may set $U = 1$. Now we are facing two possibilities according to whether $P_3(Y,Z,1)$ is reducible or not.

Case 1. $P_3(Y,Z,1)$ is reducible. Then $P_2(Y,Z,1) = 0$ and $P_3(Y,Z,1) = 0$ define a non-degenerate curve $\mathscr{C}$ if and only if they have a common quadratic factor. By careful inspection, $P_3(Y,Z,1)$ can be shown to factor in exactly five different ways, recorded in the first column of Table 2. Each way of factoring implies relations at the level of a, b, c, d, e; they themselves, upon using their expressions (2) in terms of the α, β yield the relations listed in column 3. In the last column it is seen that all cases, but case (iv) a, violates either condition C1 or the non-degeneracy of $\mathscr{C}$. A remark at the end of the proof of this lemma, will sketch the proof of the violation in case (ii) (Table 2) for instance.

Hence from Table 2 it is seen that the only admissible case is (iv)a: α_4 and $\beta_5 = 0$ with α_5 and $\beta_4 \neq 0$. Note from (iv) that also $\alpha_5^2 \neq \beta_4^2$, or else $Y^2(Z-1) = 0$, which would violate the non-degeneracy of $\mathscr{C}$. Equations $\ell_1 X + m_1 = 0$ and $Y(Z-1) + aZ^2 + bZ + c = 0$ provide X and Y in terms of Z, which upon substitution into $H_3(X,Y,Z,1) = 0$, gives rise to a 4th degree polynomial identity $\sum c_i(\alpha, \beta) Z^i \equiv 0$ in Z, if $\mathscr{C}$ is to be a curve. Hence all $c_i(\alpha, \beta)$ vanish, leading to the 5 relations 1. in Table 1 upon using $\alpha_4 = \beta_5 = 0$ and the inequalities β_4, $\alpha_5, \alpha_5^2 - \beta_4^2 \neq 0$. They are easily seen to be equivalent to rank A and rank $B \leqq 2$. The remaining relations in Case 1, Table 2 (which will be useful in later sections) are merely consequences of 1. and 2. With regard to the inequalities, we already have shown $\beta_4, \alpha_5, \alpha_5^2 - \beta_4^2 \neq 0$. As to the remaining inequalities, conditions C1 and C2 imply $\alpha_3, \alpha_6, \beta_3, \beta_6 \neq 0$; indeed using 1. and 3. (Case 1), $\alpha_5, \beta_4 \neq 0$ and condition C1, we see that α_3 and α_6 vanish simultaneously, while $\alpha_3 = \alpha_6 = 0$ would imply $Q_3 = (x_2 + \alpha_5 x_5)^2$ violating condition C2. We proceed similarly for β_3 and β_6.

Case 2. $P_3(Y,Z,1)$ is irreducible. If $P_2 = P_3 = 0$ is to define a curve $\mathscr{C}$, then the polynomials P_2 and P_3 must be proportional and since $(P_3, P_2) = (1, -\alpha_3^2)\, Y^2 Z + \ldots$, we must have in particular $P_2(Y,Z,1) = -\alpha_3^2 P_3(Y,Z,1)$ for all Y, Z. Then also $P_1 = (\alpha_3^2 - 1) P_3$; indeed $\sum_1^3 \ell_i P_i = 0$ yields $\ell_1 P_1 = (\alpha_3^2\, \ell_2 - \ell_3) P_3$, and since the

Table 2

Ways to factor $P_3(Y,Z,1)$		Relations on a,b,c,d,e	Relations on α,β	
(i) Different from (iv) or (v) below:	$[Y(Z-1)+(a+b)Z+c][Y+aZ]$	$d=a(a+b)$, $e=ac$	$\beta_4=\alpha_5=0$	Violates C1
(ii) Different from (iv) or (v) below:	$[Y(Z-1)+(a+b+c)Z][Y+aZ-c]$	$d=a(a+b+c)$, $e=-c(a+b+c)$	$\alpha_4^2=\alpha_5^2$, $\beta_4^2=\beta_5^2$	Violates C1 and non-degeneracy of $\mathscr{C}$
(iii) Different from (iv) or (v) below:	$[Y(Z-1)+(aZ+b+c)Z][Y-C]$	$d=-ac$, $e=-c(b+c)$	$\beta_4=\alpha_5=0$	Violates C1
(iv)	$[Y(Z-1)+aZ^2+bZ+c]Y$	$d=0$, $e=0$	(a) $\alpha_4=\beta_5=0$ or (b) $\alpha_4^2=\beta_4^2$, $\alpha_5^2=\beta_5^2$	Admissible or violates non-degeneracy of $\mathscr{C}$
(v)	$[Y^2+(aZ-c)Y+dZ][Z-1]$	$a+b+c=0$, $d+e=0$	$\alpha_4^2=\beta_5^2$, $\alpha_5^2=\beta_4^2$	Violates non-degeneracy of $\mathscr{C}$

linear expressions ℓ_1 and $\alpha_3^2\ell_2-\ell_3$ do not vanish on the curve $\mathscr{C}$, the polynomials P_1 and P_3 vanish simultaneously; then the irreducibility of P_3 and the fact that $P_1=(\alpha_3^2-1)Y^2Z+\dots$ imply $P_1=(\alpha_3^2-1)P_3$. Consider now the set

$$\begin{aligned}
&\{(\alpha,\beta)\,|\,P_3 \text{ is irreducible and } P_2=-\alpha_3^2P_3 \text{ for all } Y,Z\}\\
&\quad=\{(\alpha,\beta)\,|\,P_3 \text{ is irreducible and } \ell\wedge m\equiv(P_1,P_2,P_3)=(\alpha_3^2-1,\,-\alpha_3^2,1)P_3\}\\
&\quad=\left\{\begin{array}{l}(\alpha,\beta)\,|\,P_3 \text{ is irreducible and there exist functions } \eta_i(\alpha,\beta)\not\equiv 0,\\ \text{such that } \eta=(\eta_1,\eta_2,\eta_3) \text{ satisfies } \langle\eta,\ell\rangle=0,\ \langle\eta,m\rangle=0\end{array}\right\}\\
&\quad=\left\{\begin{array}{l}(\alpha,\beta)\,|\,P_3 \text{ is irreducible and there exist } \eta_i(\alpha,\beta), \text{ not all zero, such that}\\ \langle\eta,\ell X+m\rangle=\sum_1^3\eta_iH_i(X,Y,Z,1)=0 \quad \text{for all} \quad X,Y,Z,\\ \text{i.e., the } H_i \text{ are linearly dependent functions of } X,Y,Z\end{array}\right\}\\
&\quad=\left\{(\alpha,\beta)\,\middle|\,\begin{array}{l}P_3 \text{ is irreducible and the 3 by 8 matrix of coefficients}\\ \text{of the } H_i,\ i=1,2,3, \text{ has rank at most 2, i.e. rank}\\ (A,B)\leqq 2.\end{array}\right\}
\end{aligned}$$

The equalities rank $(A,B)\leqq 2$ are easily seen to be equivalent to all the equalities 1., 2. and 3., of case 2; they in turn imply the remaining equalities of case 2, listed in Table 1. The equalities combined with the conditions C1 and C2 imply a number of inequalities, besides α_5 and $\beta_4\neq 0$ (normal form NF1). Also α_6 or $\beta_6\neq 0$, or else $\alpha_{36}=\beta_{36}=0$, violating condition C1. Condition C2 implies $\alpha_3\neq 0$; indeed $\alpha_3=0$ and Table 2 would imply $\alpha_4^2=\alpha_6^2$ and $Q_3-\alpha_4^2Q_2$ would violate C2, and similarly $\beta_3=0$ is forbidden. Moreover $\alpha_4^2-\beta_4^2$ or $\alpha_5^2-\beta_5^2\neq 0$, because, if not, $\alpha_6^2=\beta_6^2$ and then $2\alpha_6\beta_6\alpha_3\beta_3=0$, and thus, since α_3 and $\beta_3\neq 0$, $\alpha_6=\beta_6=0$, which already has been ruled out. Also $\alpha_4^2-\beta_5^2$ or $\beta_4^2-\alpha_5^2\neq 0$, because otherwise $\alpha_5^2=\beta_4^2$ and the identities in Table 1 yield

$$\begin{aligned}
\pm 2\alpha_3\beta_3\alpha_4\alpha_5&=2\alpha_3\beta_3(\alpha_3\beta_6-\alpha_6\beta_3)(\alpha_3\alpha_6+\beta_3\beta_6)\\
&=2\alpha_3\beta_3\,[(\alpha_3^2-\beta_3^2)\alpha_6\beta_6+(\beta_6^2-\alpha_6^2)\alpha_3\beta_3]\\
&=[\beta_3^2\beta_6^2-\alpha_3^2\alpha_6^2+(\alpha_3^2-\beta_3^2)(\alpha_3\alpha_6+\beta_3\beta_6)^2]\\
&=\beta_3^2\beta_6^2-\alpha_3^2\alpha_6^2+(\alpha_3^2-\beta_3^2)\alpha_5^2\\
&=0
\end{aligned}$$

which contradicts the previous inequalities. Finally, the irreducibility of the polynomial $P_3(Y,Z,1)$ implies $\alpha_4^2-\alpha_5^2$ or $\beta_4^2-\beta_5^2\neq 0$ and also α_4 or $\beta_5\neq 0$, completing the verification of all inequalities, listed in Case 2 of Table 1. This concludes the proof of Theorem 2.

Remark. As an example we now check the non-degeneracy assertion of case (ii) Table 2, as promised. There we found that $\alpha_4^2=\alpha_5^2\neq 0$ and $\beta_5^2=\beta_4^2\neq 0$. Then the curve $\mathscr{C}$ is defined by $\ell_1X+m_1=0$ and the quadratic factor appearing in $P_3(Y,Z,1)$ (see Table 2) [which must appear in $P_2(Y,Z,1)$ as well]. Hence

$$\mathscr{C}:\quad \left\{\begin{array}{c}(\alpha_4^2Z^2-\beta_4^2)X=Z(\beta_4^2-\alpha_4^2Z)\\ (Z-1)Y=Z(\alpha_4^2-\beta_4^2)\end{array}\right\}. \tag{3}$$

By the non-degeneracy of $\mathscr{C}$, we have $\alpha_4^2 \neq \beta_4^2$. Clearly from the above expressions for $\mathscr{C}$, the following five points

$$(X,Y,Z)=(-1,\alpha_4^2-\beta_4^2,\infty),(0,0,0),(-1,\infty,1),(\infty,\beta_4(\mp\alpha_4-\beta_4)^{-1},\pm\beta_4/\alpha_4)),$$

belong to $\mathscr{C}$ and substituting them into $H_3=0$ leads to the respective equalities

$$\alpha_{36}^2=\alpha_3^2\alpha_6^2,\ \beta_{36}^2=\beta_3^2\beta_6^2,\ \alpha_3^2+\beta_3^2=1,\ \alpha_6^2=\alpha_4^2(\alpha_4^2-\beta_4^2)^{-1},\ \beta_6^2=-\beta_4^2(\alpha_4^2-\beta_4^2)^{-1}.$$

Substituting X and Y from (3) into $H_3=0$, and using the above identities, leads to a quartic polynomial identity in Z, whose vanishing contradicts C1 and the non-degeneracy of $\mathscr{C}$.

Lemma 2. *Among the spaces of quadrics V having a basis of the form* NF1, ..., NF9, *and leading to a non-degenerate curve $\mathscr{C}'\subset\mathscr{C}$, we can have a basis of the form* NF1 *(discussed in Theorem 2) or* NF3. *The space V having a basis* NF3 *and having a non-degenerate curve $\mathscr{C}'\subset\mathscr{C}$ is spanned by the following four quadrics*

$$\lim_{\substack{\alpha_5\to\infty\\ \beta_6\to 0}}\left[Q_1,Q_2,\left(\frac{Q_3-\alpha_4^2Q_2}{\alpha_5}\right)\Bigg|_{\alpha_4=\alpha_5+\alpha_5^{-1}},Q_4\right],$$

where Q_1,Q_2,Q_3,Q_4 is of the form NF1, *subject to the conditions of Case 2 (Theorem 2). Hence the only remaining case, besides the cases discussed in Theorem 2 is merely a limiting situation of Case 2. Then the curve $\mathscr{C}$ of rank 3 quadrics is an irreducible rational curve and the cones K_i are linearly dependent.*

Proof. The proof follows that of Theorem 2 and we refer the reader to (1) and there abouts for notation and discussion. We shall discuss one by one the different normal forms NF2, ..., NF9 listed in Appendix 2. Consider

$$NF2:\quad Q_3(x)=(\alpha_1^2x_1^2+\alpha_4^2x_4^2)+(\alpha_2^2x_2^2+\alpha_5^2x_5^2),$$
$$Q_4(x)=(\beta_1^2x_1^2+\beta_4^2x_4^2+2\beta_{14}x_1x_4)+(\beta_5^2x_5^2+2\beta_{25}x_2x_5)+2\beta_{36}x_3x_6,$$

with $\alpha_1,\beta_{14},\beta_{25}\neq 0$, and (without loss of generality) set $\alpha_2=\beta_{36}=1$. From the discussion centered about (1), the locus $\mathscr{C}$ containing a non-degenerate curve leads to two cases:

(a) $P_1(Y,Z,1)$ is irreducible; then if $\mathscr{C}$ is to be a curve, $P_1(Y,Z,1)$ and $P_2(Y,Z,1)$ must be proportional, leading to the following two cases:

(i) $\alpha_1^2=1,\ \beta_1^2=0,\ \beta_{14}^2=\beta_{25}^2,\ \alpha_4^2=\alpha_5^2,\ \beta_4^2=\beta_5^2$, implying
$Q_1+\alpha_5^2Q_2-Q_3=x_3^2+\alpha_5^2x_6^2$, in violation of C2,

or

(ii) $\alpha_4^2,\alpha_5^2,\beta_1^2,\beta_4^2,\beta_5^2=0$ and $(1-\beta_{25}^2)\alpha_1^2=(1-\beta_{14}^2)$,
implying $\mathscr{C}$ is a degenerate curve.

(b) $P_1(Y,Z,1)$ is reducible; check P_1 has the form:

$$P_1=ZY^2+(aZ^2+bZ+c)Y+(aZ+b),\qquad \text{with } c=1-\beta_{25}^2\neq 1,$$

and then one easily checks P_1 can only factor in two ways:

$$P_1 = Y(ZY + c) \quad \text{or} \quad P_1 = Z(Y^2 + aZY + a)$$
$$a, b = 0 \qquad\qquad b, c = 0 \, .$$

Then from $H_3 = XY - 1 = 0$ on $\mathscr{C}$, conclude

$$XP_1 = Y(Z + cX) = 0 \quad \text{or} \quad XP_1 = Z(Y + aZ + aX),$$

implying $\mathscr{C}$ is a degenerate curve; this shows that NF2 can never lead to a curve of rank 3 quadrics.

NF3:
$$Q_3(x) = 2x_2x_5 + 2\alpha_3 x_3 x_6$$
$$Q_4(x) = (\beta_1^2 x_1^2 + \beta_4^2 x_4^2 + 2\beta_{14} x_1 x_4) + \beta_5^2 x_5^2 + \beta_3^2 x_3^2 \text{ [4]}$$

with $\alpha_3, \beta_{14}, \beta_3$ or $\beta_5 \neq 0$.

Observe that Z is determined by X, Y via $H_3 = 0$ $(U = 1)$; also observe that since $H_1 = 0$ (irreducible) and $\alpha_3^2 H_2 - H_3 = 0$ on $\mathscr{C}$ and since both relations are seen to be functions of X, Y only, these two relations must be proportional if $\mathscr{C}$ is to be a curve, leading to the identities and inequalities

$$\beta_1^2 \beta_4^2 = \beta_{14}^2, \; \alpha_3^2 + \beta_3^2 = 1, \; \alpha_3^2 \beta_5^2 + \beta_3^2 \beta_4^2 = 0, \; \alpha_3^2 \neq 1, \; \beta_5^2 \neq 0 \, .$$

That $\mathscr{C}$ is nondegenerate follows from the irreducibility of $H_1 = 0$. That this is the asserted limit of Case 2 follows immediately upon substituting the identities above in NF3 and putting $\beta_1 = 1$; the latter can be done without loss of generality, since $\beta_1^2 \beta_4^2 = \beta_{14}^2 \neq 0$, by setting $Q_4 \curvearrowright \beta_1^{-2} Q_4$.

Remark. The case above may be reparametrized as follows

$$Q_3(x) = 2x_2 x_5 + 2a x_3 x_6$$
$$Q_4(x) = (x_1 + b x_4)^2 + (1 - a^{-2}) b^2 x_5^2 + (1 - a^2) x_3^2, \qquad \text{with} \quad a^2 \neq 1, \, b \neq 0 \, .$$

NF4–8:
$$Q_3(x) = (\alpha_2^2 x_2^2 + \alpha_5^2 x_5^2 + 2\alpha_{25} x_2 x_5) + (x_3^2 + \alpha_6^2 x_6^2 + 2\alpha_{36} x_3 x_6)$$
$$Q_4(x) = 2x_1 x_4 + (\beta_2^2 x_2^2 + \beta_5^2 x_5^2) + \beta_6^2 x_6^2$$

with

$$\alpha_2 \quad \text{or } \beta_2 \neq 0, \quad \beta_4^2 \neq \beta_6^2 \quad \text{if} \quad \beta_2 = 0, \qquad \text{and} \quad \alpha_{25} \text{ and } \alpha_{36} \neq 0 \, .$$

Since

$$P_2(Y, Z, 1) = ZY^2 + (aZ^2 + bZ + 1) Y + (cZ + b), \qquad a - c = -\alpha_{36}^2 \neq 0,$$

$P_2(Y, Z, 1)$ is easily seen to be irreducible, and so if $\mathscr{C}$ is to be a curve, P_2 and P_3 must be proportional; consequently $\alpha_2 \neq 0$, otherwise $\alpha_{25} = 0$, violating C1, but $\alpha_2 \neq 0$ forces $\beta_2 = 0$ and $\beta_5^2 = \beta_6^2$, a contradiction of the above inequalities.

NF9:
$$Q_3(x) = (\alpha_5^2 x_5^2 + 2\alpha_{25} x_2 x_5) + (\alpha_3^2 x_3^2 + \alpha_6^2 x_6^2 + 2\alpha_{36} x_3 x_6),$$
$$Q_4(x) = (\beta_4^2 x_4^2 + 2\beta_{14} x_1 x_4) + (\beta_2^2 x_2^2 + \beta_5^2 x_5^2 + 2\beta_{25} x_2 x_5),$$

[4] The basis given here is easily seen to be equivalent to the basis of NF3 given in Appendix 2, by setting $Q_3 \curvearrowright \alpha_{25} Q_3$ and $Q_4 \curvearrowright Q_4 - \beta_1^2 Q_1 - \beta_4^2 Q_2$, and then relabeling

with $\alpha_{36}\beta_{25} \neq 0$ and $\alpha_6\beta_4 = 0$ and set $\alpha_{25} = \beta_{14} = 1$ without loss of generality. The locus $\mathscr{C}$ containing a non-degenerate curve leads to two cases:

(a) $P_2(Y,Z,1)$ is irreducible; as above $P_2(Y,Z,1)$ and $P_3(Y,Z,1)$ must be proportional, yielding in order α_3, β_2 and $\beta_{25}=0$, a contradiction.

(b) $P_2(Y,Z,1)$ is reducible; then $P_2=0$ takes the form

$$P_2 = fZY^2 + (aZ^2 + bZ + 1)Y + Z(dZ + e) = 0.$$

Observe that $f \neq 0$, because otherwise $\alpha_6\beta_4 = 0$ and P_2 reducible imply $\alpha_4 = \beta_6 = 0$, and hence $P_2 = Y(aZ^2+1) = 0$, which would force $\mathscr{C}$ to be degenerate. Also we show $\beta_4 = 0$, otherwise $\beta_4 \neq 0$ and $\alpha_6 = 0$ forces $d \neq 0$ and $e = 0$, easily leading to $P_2 = 0$ being irreducible, which is a contradiction. To sum up $f \neq 0$, $\beta_4 = 0$; this yields $b, d = 0$, which forces P_2 to factor as follows:

$$P_2 = f(YZ + r)(Y + sZ) = 0, \qquad \text{for some } r \text{ and } s;$$

then, as a consequence of $H_1 = XY - 1 = 0$ on $\mathscr{C}$,

$$XP_2 = f(Z + rX)(Y + sZ) = 0,$$

forcing $\mathscr{C}$ to be degenerate, which is a contradiction. This ends the proof of Lemma 2.

Before considering Theorem 3, the reader is referred to the definition of the space $\mathscr{V}$ of linear spans V of quadrics (with regard to a fixed set of variables $x_1, \ldots, x_6$), as stated in Sect. 1.

Theorem 3. *In parallel with Theorem 2, we distinguish between the two cases*:

Case 1. The stratum $\mathscr{V}_1$ in $\mathscr{V}$ leading to Case 1, with running variables $(\alpha_4^2, \beta_4^2, \alpha_5^2, \beta_5^2, \alpha_3, \alpha_6, \beta_3, \beta_6)$ is a rational variety over

$$\{(\alpha_3, \alpha_6, \beta_3, \beta_6) \in \mathbb{C}^4,\ \alpha_3^2 + \beta_3^2 = 1\}.$$

Case 2. The stratum $\mathscr{V}_2$ in $\mathscr{V}$ leading to Case 2, with running variables $(\alpha_4, \beta_4, \alpha_5, \beta_5, \alpha_3^2, \alpha_6^2, \beta_3^2, \beta_6^2)$ is a hyperelliptic variety: it is a double cover of $\mathbb{C}^4 = \{(\alpha_4, \beta_4, \alpha_5, \beta_5)\}$ ramified over the eight hyperplanes defined by $\alpha_4 \pm \beta_4 \pm \alpha_5 \pm \beta_5 = 0$. On these hyperplanes and only on them, α_3^2 satisfies the relation

$$\alpha_3^4\alpha_5^2\beta_5^2 - \beta_3^4\alpha_4^2\beta_4^2 = 0. \tag{4}$$

Proof. As the description of $\mathscr{V}_1$ follows at once from Table 1, we turn to $\mathscr{V}_2$. The quadratic equation 6. (Table 1, Case 2) in α_3^2

$$\alpha_3^4k^2 + \alpha_3^2(4\alpha_5^2\beta_5^2 - 4\alpha_4^2\beta_4^2 - k^2) + 4\alpha_4^2\beta_4^2 = 0 \tag{5}$$

defines a double cover of $(\alpha_4, \beta_4, \alpha_5, \beta_5) \in \mathbb{C}^4$, ramified over the zero locus of its discriminant

$$\begin{aligned}
&(4\alpha_5^2\beta_5^2 - 4\alpha_4^2\beta_4^2 - k^2)^2 - 16k^2\alpha_4^2\beta_4^2 \\
&\quad = (2\alpha_5\beta_5 + k - 2\alpha_4\beta_4)(2\alpha_5\beta_5 - k + 2\alpha_4\beta_4)(2\alpha_5\beta_5 + k + 2\alpha_4\beta_4)(2\alpha_5\beta_5 - k - 2\alpha_4\beta_4) \\
&\quad = \Pi(\alpha_4 \pm \beta_4 \pm \alpha_5 \pm \beta_5), \quad \text{(this product taken over the 8 possible signs);}
\end{aligned} \tag{6}$$

this defines the eight hyperplanes mentioned in the statement. Moreover, by Table 1, the three remaining quantities β_3^2, α_6^2 and β_6^2 are linear functions of α_3^2 over $(\alpha_4, \beta_4, \alpha_5, \beta_5) \in \mathbb{C}^4$. Hence $\mathscr{V}_1$ can be viewed as a fibering of hyperelliptic curves in the variables (α_3^2, α_4) over the base space $(\beta_4, \alpha_5, \beta_5) \in \mathbb{C}^3$. To prove the last part of the lemma, notice that on any of the eight hyperplanes, the discriminant (6) of (5) vanishes and hence

$$\alpha_3^2 = (2k^2)^{-1}(k^2 - 4(\alpha_5^2\beta_5^2 - \alpha_4^2\beta_4^2)),$$

yielding the identity (4), upon using relation (6). Conversely (4) has for solution (using $\alpha_3^2 + \beta_3^2 = 1$)

$$\alpha_3^2 = \frac{\alpha_4\beta_4}{\alpha_4\beta_4 \pm \alpha_5\beta_5},$$

which put into (5) leads to the equation $k = \pm 2(\alpha_4\beta_4 \pm \alpha_5\beta_5)$ for the hyperplanes. The result holds as well when $\alpha_4\beta_4 \pm \alpha_5\beta_5 = 0$; then (4) implies $\alpha_3^2 = \beta_3^2 = 1/2$, which substituted into (5) yields the eight hyperplanes, concluding the proof of Theorem 3.

4. The Rational Curve of Rank 4 Quadrics and a Canonical Basis for $\bar{V}$

Allowing linear changes of variables, we provide a canonical basis for the space $\bar{V}$, discussed in Case 1 of Theorem 2. This situation relates to one of three distinct strata of left-invariant metrics on $SO(4)$ for which geodesic flow is algebraically completely integrable.

Theorem 4. *Let the space $\bar{V}$ contain an irreducible rational curve $\mathscr{C}$ of rank 4 quadrics containing the vertices of the cones K_i. Then $\bar{V}$ can be spanned by the three rank 3 quadrics $\bar{Q}_i$ $(1 \leqq i \leqq 3)$, corresponding to the three vertices and an appropriate fourth quadric $\bar{Q}_4$, which after a suitable change of variables $x \curvearrowright z$ take on the form*

$$\begin{aligned} \bar{Q}_1 &= z_2^2 - z_3^2 - c_1 z_0^2 \\ \bar{Q}_2 &= z_1^2 - z_6^2 - c_2 z_0^2 \\ \bar{Q}_3 &= z_4^2 - z_5^2 - c_3 z_0^2 \\ \bar{Q}_4 &= -(z_1 - z_4)^2 + 2(z_2 - z_5)^2 + 2(z_3 - z_6)^2 - c_4 z_0^2 . \end{aligned} \tag{1}$$

The curve $\mathscr{C}$ of rank 4 quadrics, with regard to this basis, is described by

$$\begin{aligned} \mathscr{C}:\quad & \frac{\bar{Q}_1}{Z-1} - \frac{\bar{Q}_2}{Z-2} + \frac{\bar{Q}_3}{Z} + \frac{\bar{Q}_4}{2} \\ &= \left(\sqrt{\frac{Z}{2(2-Z)}}\, z_1 + \sqrt{\frac{2-Z}{2Z}}\, z_4\right)^2 + \left(\sqrt{\frac{Z}{Z-1}}\, z_2 - \sqrt{\frac{Z-1}{Z}}\, z_5\right)^2 \\ &\quad + \left(\sqrt{\frac{Z-2}{Z-1}}\, z_3 - \sqrt{\frac{Z-1}{Z-2}}\, z_6\right)^2 - c(Z) z_0^2 , \end{aligned}$$

where

$$c(Z) = \frac{c_1}{Z-1} - \frac{c_2}{Z-2} + \frac{c_3}{Z} + \frac{c_4}{2}.$$

The curve $\mathscr{C}$ in $\bar{V}$ has 6 distinguished points, given by the divisor of poles and zeroes of $c(Z)$. They map into six rank 3 quadrics: the poles correspond to the vertices of the cones K_i, which are the quadrics $\bar{Q}_1, \bar{Q}_2, \bar{Q}_3$ and the zeroes correspond to the three other quadrics which are independent of z_0^2.

The affine intersection $I = \cap\{\bar{Q}_i = 0, z_0 = 1\}$ supports the following two commuting vector fields:

$$\begin{aligned}
X_1:\ \dot{z}_1 &= z_2 z_6 & \qquad X_2:\ \dot{z}_1 &= z_5 z_6 \\
\dot{z}_2 &= \tfrac{1}{2} z_3 (z_1 + z_4) & \dot{z}_2 &= z_3 z_4 \\
\dot{z}_3 &= \tfrac{1}{2} z_2 (z_1 + z_4) & \dot{z}_3 &= z_2 z_4 \\
\dot{z}_4 &= z_3 z_5 & \dot{z}_4 &= z_5 (2z_3 - z_6) \\
\dot{z}_5 &= z_3 z_4 & \dot{z}_5 &= z_4 (2z_3 - z_6) \\
\dot{z}_6 &= z_1 z_2 & \dot{z}_6 &= z_1 z_5 .
\end{aligned}$$

Up to a linear change of variables, the four quadrics Q_i provide the four constants of motion of the completely integrable geodesic flow on the group $SO(4)$ (see Adler and van Moerbeke [4, 5])

$$X_H:\quad \dot{x}' = x' \wedge \frac{\partial H}{\partial x'}, \qquad \dot{x}'' = x'' \wedge \frac{\partial H}{\partial x''} \tag{2}$$

for the left-invariant metric

$$H = \frac{1}{2}\sum_1^6 \lambda_i x_i^2 + \sum \lambda_{i,i+3} x_i x_{i+3},$$

where $(\Lambda_{ij} \equiv \lambda_i - \lambda_j)$

$$\begin{aligned}
&(\lambda_{14}^2, \lambda_{25}^2, \lambda_{36}^2)(\Lambda_{46}\Lambda_{32} - \Lambda_{65}\Lambda_{13})^2 \\
&\quad = \Lambda_{13}\Lambda_{21}\Lambda_{32}\Lambda_{46}\Lambda_{54}\Lambda_{65}\left(\frac{(\Lambda_{32}-\Lambda_{65})^2}{\Lambda_{32}\Lambda_{65}}, \frac{(\Lambda_{13}-\Lambda_{46})^2}{\Lambda_{13}\Lambda_{46}}, \frac{(\Lambda_{21}-\Lambda_{54})^2}{\Lambda_{21}\Lambda_{54}}\right),
\end{aligned}$$

with the sign specification

$$\begin{aligned}
&\lambda_{14}\lambda_{25}\lambda_{36}(\Lambda_{46}\Lambda_{32} - \Lambda_{65}\Lambda_{13})^3 \\
&\quad = \Lambda_{13}\Lambda_{21}\Lambda_{32}\Lambda_{46}\Lambda_{54}\Lambda_{65}(\Lambda_{65} - \Lambda_{32})(\Lambda_{46} - \Lambda_{13})(\Lambda_{54} - \Lambda_{21}).
\end{aligned}$$

Proof. According to Case 1 of Theorem 2, V has a basis of the form NF 1, with the α and β subjected to the conditions of Table 1. Then using the basis Q_i of NF 1, observe that

$$Q_3 = (x_2 + \alpha_5 x_5)^2 + (\alpha_3 x_3 + \alpha_6 x_6)^2, \qquad Q_4 = (x_1 + \beta_4 x_4)^2 + (\beta_3 x_3 + \beta_6 x_6)^2$$

and

$$\begin{aligned}
&\delta((\beta_3\alpha_6 - \alpha_3\beta_6)(\alpha_3\beta_3 Q_1 - \alpha_6\beta_6 Q_2) + \beta_3\beta_6 Q_3 - \alpha_3\alpha_6 Q_4) \\
&\quad = -(\alpha_3\delta x_1 + \alpha_6\beta_4 x_4)^2 + (\beta_3\delta x_2 + \beta_6\alpha_5 x_5)^2, \quad \text{with } \delta = \alpha_3\alpha_6 + \beta_3\beta_6,
\end{aligned}$$

yield three independent rank 2 quadrics. Upon making the change of coordinates

$$\begin{aligned} y_1 &= -\alpha_3\alpha_6(x_1+\beta_4 x_4) & y_4 &= -\alpha_3(\alpha_3\delta x_1+\alpha_6\beta_4 x_4) \\ y_2 &= \beta_3\beta_6(x_2+\alpha_5 x_5) & y_5 &= \beta_3(\beta_3\delta x_2+\beta_6\alpha_5 x_5) \\ y_3 &= \alpha_3\beta_3\beta_6(\alpha_3 x_3+\alpha_6 x_6) & y_6 &= \beta_3\alpha_3\alpha_6(\beta_3 x_3+\beta_6 x_6), \end{aligned}$$

the three rank 2 quadrics, together with Q_1, are seen to be proportional to a new set of quadrics depending only on one parameter $a \equiv 1-\alpha_3^{-2}$, to wit

$$\begin{aligned} Q'_1(y,a) &= y_2^2+(1-a)y_3^2 \\ Q'_2(y,a) &= ay_1^2-(1-a)y_6^2 \\ Q'_3(y,a) &= ay_4^2+y_5^2 \\ Q'_4(y,a) &= (y_1-y_4)^2+(y_2-y_5)^2+(y_3-y_6)^2. \end{aligned} \tag{3}$$

The curve of rank 3 quadrics in V with regard to the basis Q'_i is given by

$$\frac{Q'_1}{s-1}+\frac{Q'_2}{s-a}-\frac{Q'_3}{s}+Q'_4$$
$$=\left(\sqrt{\frac{s}{s-a}}y_1-\sqrt{\frac{s-a}{s}}y_4\right)^2+\left(\sqrt{\frac{s}{s-1}}y_2-\sqrt{\frac{s-1}{s}}y_5\right)^2+\left(\sqrt{\frac{s-a}{s-1}}y_3-\sqrt{\frac{s-1}{s-a}}y_6\right)^2.$$

It is striking to observe that by a rescaling $y \curvearrowright z$ and by picking a new basis, the parameter a in (3) is arbitrary. Indeed

$$\begin{aligned} \frac{s}{s-1}Q'_1(y,a) &= Q'_1(z,b) \\ \frac{s(s-1)}{(s-a)^2}Q'_2(y,a) &= Q'_2(z,b) \\ \frac{s-1}{s}Q'_3(y,a) &= Q'_3(z,b) \\ \left(\frac{Q'_1}{s-1}+\frac{Q'_2}{s-a}-\frac{Q'_3}{s}+Q'_4\right)(y,a) &= Q'_4(z,b), \end{aligned}$$

where a and b are related by a fractional linear map

$$b=\frac{a(s-1)}{s-a}$$

and where the rescaling $y \curvearrowright z$ is obvious from the identities above. Hence we may put $a=2$ in (3), leading to the announced normal form $\bar{Q}_i$ (up to a trivial rescaling) and the curve $\mathscr{C}$. The three vertices of the cones K_i, which belong to $\mathscr{C}$, correspond to the three rank 3 quadrics $\bar{Q}_i$.

Finally one checks that the geodesic flow on $SO(4)$ for the metric mentioned in the statement of Theorem 4, has a set of invariants of the form (3) with $a=\Lambda_{32}/\Lambda_{31}$ upon making the linear transformation $x \curvearrowright y$

$$y_i=e_i(e_i x_i+e_{i+3}x_{i+3}), \quad y_{i+3}=e_{i+3}(e_{i+3}x_i+e_i x_{i+3}), \quad 1 \leqq i \leqq 3,$$

with

$$(e_1^2, \dots, e_6^2) = (A_{46}A_{12}, A_{65}A_{21}, A_{65}A_{31}, A_{45}A_{13}, A_{54}A_{32}, A_{64}A_{32}).$$

Using the same argument as before to scale out a, the set of quadrics (3) can be transformed into the set (1). The Hamiltonian vector fields X_H of (2) defined by $H = \bar{Q}_1$ and $\bar{Q}_3$ in (1) yield the vector fields X_1 and X_2; they commute because clearly the vector field X_1 preserves the quadrics Q_3 and so

$$0 = X_1(Q_3') = X_{Q_1'}(Q_3') = \{Q_1', Q_3'\},$$

where the Poisson bracket is taken with regard to the $SO(4)$ symplectic structure.

5. The Elliptic Curve of Rank 4 Quadrics and Canonical Bases for $\bar{V}$

Consider the situation discussed in Case 2 of Theorem 2, namely where $\bar{V}$ contains an elliptic curve $\mathscr{C}$ of rank 4 quadrics. The purpose of this section is to find a canonical basis for $\bar{V}$ by picking distinguished configurations of points on the curve and by allowing linear changes of variables; these changes depend strongly on the geometry of the curve $\mathscr{C}$. A first canonical basis is obtained by picking a set of 3 quadrics on $\mathscr{C}$, which are simultaneously diagonalizable (Theorem 5). In that form, they tie up with the geodesic flow on the group $SO(4)$ for the Manakov metric. Another canonical basis exhibited in Theorem 6 is constructed by picking three of the four collinear rank 3 quadrics lying on $\mathscr{C}$; this construction is inspired by Kötter [14, 15]. Both canonical bases, which depend on 4 continuous parameters, will be crucial in determining the moduli of the intersection of the quadrics.

In Lemma 3, we show that the elliptic curve is non singular away from the branch locus of the variety $\mathscr{V}_2$ discussed in Theorem 3 and away from 4 additional hyperplanes. Lemma 4 gives the three points on the curve which yield the first canonical basis.

Lemma 3. *In Case 2, described in Theorems 2 and 3, the curve $\mathscr{C} \subseteq \mathbb{P}^3$ is an elliptic curve over the Z-plane ramified at the four points*

$$Z = \frac{-\beta_3(\beta_6 - \delta\beta_3\beta_4)}{\alpha_3(\alpha_6 - \delta\varepsilon\alpha_3\alpha_5)}, \quad \delta = \pm 1, \ \varepsilon = \pm 1;$$

the curve becomes singular when the α_i's and β_i's belong to the eight hyperplanes $\alpha_4 \pm \beta_4 \pm \alpha_5 \pm \beta_5 = 0$, mentioned in Theorem 3, or when they belong to the four hyperplanes $\alpha_4 = 0$, $\alpha_5 = 0$, $\beta_4 = 0$ and $\beta_5 = 0$; the curve is non-singular everywhere else. When $\mathscr{C}$ is non-singular, then the functions $a_i/a_{i,i+3} = a_{i,i+3}/a_{i+3}$ are meromorphic of order 2 on $\mathscr{C}$ and the following inequality $\alpha_3^4\alpha_5^2\beta_5^2 - \beta_3^4\alpha_4^2\beta_4^2 \neq 0$ holds.

Proof. Putting $U = 1$ excludes at most a finite number of points as otherwise $\mathscr{C}$ would be degenerate. Then $P_3(Y, Z, 1) = 0$ defines a double cover of the Z-plane ramified over the zeroes of the quartic discriminant $R(Z)$ of P_3, for which we provide two alternative expressions

$$\begin{aligned} R(Z) &= \prod_{\varepsilon = \pm 1} [\alpha_4^2 Z^2 - (\ell + 2\varepsilon\beta_4\alpha_5)Z + \beta_5^2] \\ &= \prod_{\varepsilon = \pm 1} [\alpha_4^2(Z-1)^2 + (\alpha_4^2 + \alpha_5^2 + \beta_4^2 - \beta_5^2 - 2\varepsilon\beta_4\alpha_5)(Z-1) + (\alpha_5 - \varepsilon\beta_4)^2]; \quad (1) \end{aligned}$$

the roots of $R(Z)$ are given by

$$Z_{\delta,\varepsilon} = (2\alpha_4^2)^{-1}\,[\ell + 2\varepsilon\beta_4\alpha_5 + \delta\sqrt{(\ell+2\varepsilon\beta_4\alpha_5)^2 - 4\alpha_4^2\beta_5^2}\,], \quad \text{with } \delta = \pm 1,$$

$$= -\beta_3\alpha_3^{-1}(\alpha_4^2\beta_3^2)^{-1}(\beta_6 - \delta\beta_3\beta_4)(\alpha_6 + \delta\varepsilon\alpha_3\alpha_5), \text{ using 7. and 4. (Table 1) to replace } \ell \text{ (in that order)}$$

$$= -\frac{\beta_3(\beta_6 - \delta\beta_3\beta_4)}{\alpha_3(\alpha_6 - \delta\varepsilon\alpha_3\alpha_4)} \text{ using } \alpha_4^2\beta_3^2 = (\alpha_6 - \alpha_3\alpha_5)(\alpha_6 + \alpha_3\alpha_5) \text{ (see 1. in Table 1)},$$

proving the first part of Lemma 3.

When does a pair of roots coincide? Notice that $Z_{1,\varepsilon} = Z_{-1,\varepsilon}$ for $\varepsilon = +1$ or -1 if and only if the discriminant

$$(\ell + 2\beta_4\varepsilon\alpha_5)^2 - 4\alpha_4^2\beta_5^2$$
$$= (\alpha_4 + \beta_4 + \beta_5 - \varepsilon\alpha_5)(\alpha_4 + \beta_4 - \beta_5 - \varepsilon\alpha_5)(\alpha_4 - \beta_4 + \beta_5 + \varepsilon\alpha_5)(\alpha_4 - \beta_4 - \beta_5 + \varepsilon\alpha_5);$$

of one of the quadratic factors in $R(Z)$ vanishes; this occurs on the eight hyperplanes in $\mathbb{C}^4$ mentioned in Theorem 3. Also $Z_{\delta,1} = Z_{\delta,-l}$ if and only if $\beta_4\alpha_5\beta_5\alpha_4 = 0$, using the above expressions for $Z_{\delta,\varepsilon}$, combined with the identities. In particular, when $\alpha_5 \to 0$ or $\beta_4 \to 0$, we have that $Z_{\delta,1} = Z_{\delta,-l}$; when $\beta_5 \to 0$, $Z_{-1,1} = Z_{-1,-1} = 0$ and when $\alpha_4 \to 0$, $Z_{1,1} = Z_{1,-1} = \infty$. Thus the curve $\mathscr{C}$ is singular precisely on the twelve hyperplanes given in the lemma.

Finally, the functions $a_i/a_{i,i+3} = a_{i,i+3}/a_{i+3}$ are rational in X, Y, Z, U and can therefore be viewed as meromorphic functions on $\mathscr{C}$. We now check they have order 2, for $i=3$ for instance, the two other cases $i=1,2$ being similar. Solving the quadratic equation $P_3(Y,Z,1)$ with regard to Y shows that the meromorphic function

$$\frac{a_6}{a_{36}} = \frac{Y + \alpha_6^2 Z + \beta_6^2}{\alpha_{36}Z + \beta_{36}}$$

on $\mathscr{C}$ has poles at $Z=1$ and $Z = -\beta_{36}/\alpha_{36}$ only. Then from the second expression (1) for the discriminant $R(Z)$, we have the estimates

$$\frac{a_6}{a_{36}} = \frac{\alpha_5^2 - \beta_4^2}{2(\alpha_{36}+\beta_{36})}\left(\frac{1\pm 1}{Z-1}\right) + O(1) \qquad \text{near } Z=1$$

$$= \frac{\alpha_3^4\alpha_5^2\beta_5^2 - \beta_3^4\alpha_4^2\beta_4^2}{2\alpha_{36}(\alpha_{36}+\beta_{36})}\,\frac{1\pm 1}{(\alpha_{36}Z+\beta_{36})} + O(1) \quad \text{near } Z = -\frac{\beta_{36}}{\alpha_{36}} \tag{2}$$

(where $\pm$ refers to the $\pm$ sheet); it shows a_6/a_{36} has simple poles at $Z=1$ and at $Z = -\beta_{36}/\alpha_{36}$ on the $+$ sheet only, provided the leading terms in (2) differ from 0, or ∞ which is generically so, in view of Theorem 3. Therefore a_6/a_{36} has order two generically and hence always, as long as the parameters α, β stay away from the 8 hyperplanes given in Theorem 3; there we also observed that $\alpha_3^4\alpha_5^2\beta_5^2 - \beta_3^4\alpha_4^2\beta_4^2 = 0$ holds on the 8 hyperplanes only. This concludes the proof of Lemma 3.

Remark. From equation 6. (Table 1) it follows $\alpha_3 \neq 0$, when $\mathscr{C}$ is non-singular and by symmetry also $\beta_3 \neq 0$.

Lemma 4. *Let the elliptic curve $\mathscr{C}$ be non-singular and let $\alpha_6^2\beta_5^2+\beta_6^2\alpha_4^2 \neq 0$; then there exist three points $p_1, p_2, p_3 \in \mathscr{C}$, such that the meromorphic functions a_i, a_{i+3}, $a_{i,i+3}$ $(1 \leqq i \leqq 3)$ on $\mathscr{C}$ satisfy*

$$\frac{a_{14}}{a_1}(p_2)=\frac{a_{14}}{a_1}(p_3), \quad \frac{a_{25}}{a_2}(p_3)=\frac{a_{25}}{a_2}(p_1), \quad \frac{a_{36}}{a_3}(p_1)=\frac{a_{36}}{a_3}(p_2) \tag{3}$$

with $a_{i,i+3} \neq 0$ at $p_1, p_2, p_3 \in \mathscr{C}$.

Proof. Before proceeding to the actual proof, consider the roots $\varrho=\varrho_\pm$ and $\sigma=\sigma_\pm$ of the quadratic polynomials R and S defined by

$$R(\varrho) \equiv \varrho^2+2\alpha_{36}\varrho+\alpha_3^2\beta_3^2(\alpha_6^2\beta_5^2+\beta_6^2\alpha_4^2)\beta_5^{-2}=0$$
$$S(\sigma)=\sigma^2-2\beta_{36}\sigma+\alpha_3^2\beta_3^2(\alpha_6^2\beta_5^2+\beta_6^2\alpha_4^2)\alpha_4^{-2}=0;$$

clearly the roots are related as follows

$$\begin{gathered}\beta_{36}(\varrho_+ + \varrho_-)+\alpha_{36}(\sigma_+ + \sigma_-)=0 \\ \sigma_\mp\varrho_\mp-\sigma_\pm\varrho_\pm=2(\alpha_{36}\sigma_\pm+\beta_{36}\varrho_\mp)=\alpha_{36}(\sigma_\pm-\sigma_\mp)+\beta_{36}(\varrho_\mp-\varrho_\pm).\end{gathered} \tag{4}$$

The independent terms of R and S do not vanish, by assumption and upon using the remark at the end of Lemma 3 and the inequalities of Table 1, the discriminants of R and S have the form

$$4\alpha_3^2\beta_5^2(\alpha_3^4\alpha_5^2\beta_5^2-\beta_3^4\alpha_4^2\beta_4^2) \quad \text{and} \quad -4\beta_3^2\alpha_4^2(\alpha_3^4\alpha_5^2\beta_5^2-\beta_3^4\alpha_4^2\beta_4^2)$$

and do not vanish, since $\mathscr{C}$ is non-singular (by virtue of the last statement in Lemma 3). Hence we have the inequalities

$$\begin{gathered}\varrho_+\varrho_- \neq 0 \qquad \sigma_+\sigma_- \neq 0 \qquad \varrho_+ - \varrho_- \neq 0 \qquad \sigma_+ - \sigma_- \neq 0 \\ 2(\alpha_{36}\sigma_\pm+\beta_{36}\varrho_\pm)=\sigma_\mp\varrho_\mp-\sigma_\pm\varrho_\pm \neq 0;\end{gathered} \tag{5}$$

the latter inequality holds because, in view of (4)

$$\alpha_{36}^2(\sigma_\pm-\sigma_\mp)^2-\beta_{36}^2(\varrho_\pm-\varrho_\mp)^2=\frac{\alpha_3^2\beta_3^2}{\alpha_4^2\beta_5^2}(\alpha_3^4\alpha_5^2\beta_5^2-\beta_3^4\alpha_4^2\beta_4^2)(\alpha_6^2\beta_4^2+\beta_6^2\alpha_4^2) \neq 0,$$

again by the hypothesis and by the fact that $\mathscr{C}$ is non-singular.

Proving the existence of three distinct points $p_1, p_2, p_3 \in \mathscr{C}$ satisfying (3) amounts to finding $p_1, p_2, p_3 \in \mathscr{C}$ (i.e., satisfying $H_i(p_1)=H_i(p_2)=H_i(p_3)=0$, $i=1,2,3$) such that

$$\frac{a_4}{a_{14}}(p_2)=\frac{a_4}{a_{14}}(p_3), \quad \frac{a_5}{a_{25}}(p_3)=\frac{a_5}{a_{25}}(p_1), \quad \frac{a_6}{a_{36}}(p_1)=\frac{a_6}{a_{36}}(p_2). \tag{6}$$

In terms of the coordinates $(X_i, Y_i, Z_i, 1)$ of $p_i \in \mathbb{P}^3$, the relations (6) amount to a linear system of equations in X_i and Y_i, with coefficients depending on Z_i

$$\begin{gathered} D\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \alpha_3\beta_3(\alpha_3\beta_6-\alpha_6\beta_3)(Z_1-Z_2) \\ 0 \\ 0 \end{bmatrix}, \\ D\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} \alpha_6\beta_6(\alpha_3\beta_6-\alpha_6\beta_3)(Z_2-Z_1) \\ \beta_5^2(Z_3-Z_1) \\ \alpha_4^2(Z_2-Z_3) \end{bmatrix}, \end{gathered} \tag{7}$$

where

$$D=\begin{bmatrix} -(\alpha_{36}Z_2+\beta_{36}) & \alpha_{36}Z_1+\beta_{36} & 0 \\ -Z_3 & 0 & Z_1 \\ 0 & -1 & 1 \end{bmatrix}$$

with

$$\Delta=\det(D)=Z_3(\alpha_{36}Z_1+\beta_{36})-Z_1(\alpha_{36}Z_2+\beta_{36}). \tag{8}$$

By formally substituting X_i and Y_i in terms of Z_i obtained from (7), the equation $H_i(p_2)-H_i(p_3)$ is automatically satisfied for $i=1$ and leads to an equation for $i=2$:

$$H_2(p_2)-H_2(p_3)=\Delta^{-1}\alpha_4^2(Z_3-Z_2)^2\,[Z_3(\alpha_{36}Z_1+\beta_{36})+Z_1(\alpha_{36}Z_2+\beta_{36})]=0;$$

upon setting

$$Z_3(\alpha_{36}Z_1+\beta_{36})+Z_1(\alpha_{36}Z_2+\beta_{36})=0, \tag{9}$$

we get the following relations, involving the polynomials R and S introduced in the beginning of the proof

$$H_1(p_3)=\beta_5^2\Delta^{-2}Z_3^2(Z_2-Z_1)^2\,R\left(\frac{2(\alpha_{36}Z_1+\beta_{36})}{Z_2-Z_1}\right)=0 \tag{10}$$

and

$$H_2(p_3)=\alpha_4^2\Delta^{-2}Z_3^2(Z_2-Z_1)^2\,S\left(\frac{2Z_3(\alpha_{36}Z_1+\beta_{36})}{Z_2-Z_1}\right)=0. \tag{11}$$

The remaining relations

$$H_1(p_2)=H_2(p_2)=H_1(p_1)=H_2(p_1)=0$$

are then automatically satisfied, because upon eliminating X_1, Y_1, Z_1, we have

$$H_1(p_1)=H_1(p_2)\quad\text{and}\quad\left(\frac{\alpha_{36}Z_2+\beta_{36}}{\alpha_{36}Z_3+\beta_{36}}\right)^2H_2(p_1)=H_2(p_2).$$

For the points p_i to be all distinct with $a_{i,i+3}\neq 0$, and in order to solve the linear system (7), we must impose the inequalities

$$\Delta\neq 0,\ (Z_1-Z_2)(Z_2-Z_3)(Z_3-Z_1)\neq 0,\ Z_i\neq 0\ \text{ and }\ \alpha_{36}Z_i+\beta_{36}\neq 0\quad i=1,2,3. \tag{12}$$

The problem now reduces to finding three points p_1, p_2, p_3 satisfying the relations (7), (9), (10), (11) and the inequalities (12).

In view of the discussion about the polynomials R and S at the start of this proof, the expressions

$$\frac{2(\alpha_{36}Z_1+\beta_{36})}{Z_2-Z_1}=\varrho_\pm\neq 0\quad\text{and}\quad Z_3\varrho_\pm=\sigma_\pm\neq 0, \tag{13}$$

with the relation

$$Z_3(\alpha_{36}Z_1+\beta_{36})=-Z_1(\alpha_{36}Z_2+\beta_{36}), \tag{14}$$

provide a solution to (9), (10), and (11), and in turn

$$Z_1=\frac{\sigma_\pm}{\varrho_\mp},\quad Z_2=\frac{\sigma_\mp}{\varrho_\pm},\quad Z_3=\frac{\sigma_\pm}{\varrho_\pm} \tag{15}$$

provides the solution to (13) and (14) upon using the relations (4). The inequalities (5) imply at once that the $Z_i (i=1,2,3)$ are non zero and all distinct. Then (13) and (14) imply $\alpha_{36} Z_i + \beta_{36} \neq 0$ for $i=1,2$ and (5) implies that inequality for $i=3$. Therefore also in view of (8) and (14), we have $\Delta \neq 0$. As a result the systems (7) have a unique solution $X_1, X_2, X_3, Y_1, Y_2, Y_3$. Thus the coordinates of the 3 distinct points $p_1, p_2, p_3 \in \mathscr{C}$ have the required properties, ending the proof of Lemma 4.

Theorem 5. *Let the elliptic curve $\mathscr{C}$ be non-singular, then after an appropriate change of coordinates $x \curvearrowright z$, the space $\bar{V}$ can be spanned by a "simultaneously diagonalizable" basis*

$$\begin{aligned}
&\bar{Q}_1 = z_4^2 + z_2^2 - z_3^2 - c_1 z_0^2\\
&\bar{Q}_2 = z_5^2 + z_3^2 - z_1^2 - c_2 z_0^2\\
&\bar{Q}_3 = z_6^2 + z_1^2 - z_2^2 - c_3 z_0^2\\
&\bar{Q}_4 = a z_1 z_4 + b z_2 z_5 + c z_3 z_6 - c_4 z_0^2, \qquad \text{with} \quad a^2+b^2+c^2=0, \quad abc=1,
\end{aligned} \tag{16}$$

depending on four parameters a, c_1, c_2 and c_3; c_4 can be made 1. The curve $\mathscr{C}$ of rank 4 quadrics has, with regard to this basis, the following form

$$\begin{aligned}
\mathscr{C}:\ & -a^2 Z\bar{Q}_1 + (b^2 Z + c^2)\bar{Q}_2 + Z(b^2 Z + c^2)\bar{Q}_3 - 2W\bar{Q}_4\\
&= -\left(\frac{W}{\sqrt{Z}} z_1 + a\sqrt{Z} z_4\right)^2 + \left(b\sqrt{Z(1-Z)}\, z_2 - \frac{W}{\sqrt{Z(1-Z)}} z_5\right)^2\\
&\quad + \left(c\sqrt{1-Z}\, z_3 - \frac{W}{\sqrt{1-Z}} z_6\right)^2 - c(Z,W) z_0^2,
\end{aligned} \tag{17}$$

where

$$W^2 = Z(1-Z)(b^2 Z + c^2)$$

and where

$$c(Z,W) \equiv -a^2 Z c_1 + (b^2 Z + c^2) c_2 + Z(b^2 Z + c^2) c_3 - 2W c_4$$

is a meromorphic function on $\mathscr{C}$. Moreover $\mathscr{C}$ contains four collinear points $p_\alpha (1 \leqq \alpha \leqq 4)$ corresponding to four rank 3 quadrics $\bar{Q}(p_\alpha)$ (i.e., with z_0^2 missing) having the property

$$\sum_1^4 \bar{Q}(p_\alpha) \equiv 0.$$

The affine intersection $I = \cap \{\bar{Q}_i = 0,\ z_0 = 1\}$ supports two commuting vector fields

$$\begin{aligned}
X_1:\ &\dot{z}_1 = a z_5 z_6 & X_2:\ &\dot{z}_1 = a z_2 z_3\\
&\dot{z}_2 = b z_6 z_4 & &\dot{z}_2 = c z_4 z_6 + a z_1 z_3\\
&\dot{z}_3 = c z_4 z_5 & &\dot{z}_3 = -b z_4 z_5 + a z_1 z_2\\
&\dot{z}_4 = -b z_2 z_6 + c z_3 z_5 & &\dot{z}_4 = -c z_2 z_6 - b z_3 z_5\\
&\dot{z}_5 = -c z_3 z_4 + a z_1 z_6 & &\dot{z}_5 = b z_3 z_4\\
&\dot{z}_6 = -a z_1 z_5 + b z_2 z_4 & &\dot{z}_6 = c z_2 z_4
\end{aligned}$$

The quadrics $\bar{Q}_i$ in (16) *provide the set of invariants for the geodesic flow X_H, given by* (4.2)*; in the left-invariant metric H the λ_i and $\lambda_{i,i+3}$ satisfy (see* [4, 5]*)*

$$\Lambda_{14}=\Lambda_{25}=\Lambda_{36} \quad \textit{and} \quad \lambda_{14}^2\Lambda_{32}+\lambda_{25}^2\Lambda_{13}+\lambda_{36}^2\Lambda_{21}+\Lambda_{32}\Lambda_{13}\Lambda_{21}=0. \quad \textit{(Manakov metric)}. \tag{18}$$

Proof. We first assume $\alpha_6^2\beta_5^2+\beta_6^2\alpha_4^2 \neq 0$; then, at any point on $\mathscr{C}$ and in particular at the points $p_1, p_2, p_3 \in \mathscr{C}$ found in Lemma 4, the relations $a_i a_{i+3}=a_{i,i+3}^2$ hold and hence the corresponding three quadrics $Q(p_1)$, $Q(p_2)$ and $Q(p_3)$ have the form

$$Q=\sum_{i=1}^{3} a_i\left(x_i+\frac{a_{i,i+3}}{a_i}x_{i+3}\right)^2$$

with a_i and a_{i+3} evaluated at p_1, p_2 and p_3. Observe that, from Lemma 4,

$$\frac{a_{14}}{a_1}(p_2)=\frac{a_{14}}{a_1}(p_3)\neq\frac{a_{14}}{a_1}(p_1), \quad \frac{a_{25}}{a_2}(p_3)=\frac{a_{25}}{a_2}(p_1)\neq\frac{a_{25}}{a_2}(p_2),$$

$$\frac{a_{36}}{a_3}(p_1)=\frac{a_{36}}{a_3}(p_2)\neq\frac{a_{36}}{a_3}(p_3),$$

the inequalities holding because otherwise the meromorphic functions $a_{i,i+3}/a_i$ would have at least order 3 which has been ruled out by Lemma 3. Also the a_i are nonzero at p_1, p_2, p_3, since there $a_i a_{i+3}=a_{i,i+3}^2 \neq 0$. Therefore it follows, at once, that the linear map $y=Lx$

$$L=\begin{bmatrix} a_1(p_3)^{1/2} & a_{14}(p_3)a_1(p_3)^{-1/2} & 0 & 0 & 0 & 0 \\ a_1(p_1)^{1/2} & a_{14}(p_1)a_1(p_1)^{-1/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2(p_1)^{1/2} & a_{25}(p_1)a_2(p_1)^{-1/2} & 0 & 0 \\ 0 & 0 & a_2(p_2)^{1/2} & a_{25}(p_2)a_2(p_2)^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_3(p_2)^{1/2} & a_{36}(p_2)a_3(p_2)^{-1/2} \\ 0 & 0 & 0 & 0 & a_3(p_3)^{1/2} & a_{36}(p_3)a_3(p_3)^{-1/2} \end{bmatrix},$$

maps $Q(p_1)$, $Q(p_2)$, $Q(p_3)$ into

$$\begin{aligned} Q(p_1)&=y_4^2+y_2^2+a_3(p_1)a_3(p_2)^{-1}y_3^2\equiv y_4^2+y_2^2+a_3'y_3^2 \\ Q(p_2)&=y_5^2+y_3^2+a_1(p_2)a_1(p_3)^{-1}y_1^2=y_5^2+y_3^2+a_1'y_1^2 \\ Q(p_3)&=y_6^2+y_1^2+a_2(p_3)a_2(p_1)^{-1}y_2^2=y_6^2+y_1^2+a_2'y_2^2, \qquad \text{with} \quad a_i'\neq 0. \end{aligned} \tag{19}$$

Carrying out the program of Lemma 4 and implementing the linear map L, yields the basis of (16) in terms of the basis NF1: $Q_i(x)-c_i'x_0^2$, $i=1,\dots,4$. Indeed the change of coordinates $x \curvearrowright z$ and the values of $a, b, c, c_1, c_2, c_3, c_4$ are specified below:

$$z_i=\sqrt{a_i^+}\,(x_i+b_i^+x_{i+3}), \quad z_{i+3}=\sqrt{a_i^-}\,(x_i+b_i^-x_{i+3}), \quad i=1,2,3,$$

with

$$\begin{array}{lll}
a_1^+ = -\alpha_3^2(\alpha_{36}+\beta_{36})+U, & a_2^+ = -\beta_3^2(\alpha_{36}+\beta_{36})+V, & a_3^+ = -\beta_3^2 U-\alpha_3^2 V,\\
a_1^- = -a_1^+(-U), & a_2^- = a_2^+(-V), & a_3^- = a_3^+(-U,V)\\
b_1^+ a_1^+ = \beta_4(-\alpha_{36}+U), & b_2^+ a_2^+ = \alpha_5(-\beta_{36}+V), & b_3^+ a_3^+ = -\alpha_{36}V-\beta_{36}U,\\
b_1^- = b_1^+(-U), & b_2^- = b_2^+(-V), & b_3^- = b_3^+(-U/V),\\
U = i\dfrac{\alpha_3}{\beta_5}W, & V = \dfrac{\beta_3}{\alpha_4}W, & W^2 = -\alpha_5^2\beta_5^2\alpha_3^4+\alpha_4^2\beta_4^2\beta_3^4
\end{array}$$

and

$$\begin{array}{lll}
a = i\alpha_4\beta_4\beta_3^2, & b = -\alpha_5\beta_5\alpha_3^2, & c = W,\\
c_1 = c(-U,V), & c_2 = c(U,-V), & c_3 = -c(U,V)
\end{array}$$

with

$$\begin{aligned}
c(U,V) = {} & (\beta_3^2\alpha_{36}-\alpha_3^2\beta_{36})c_1' + (-\beta_5^2 U+\alpha_4^2 V+\beta_6^2\alpha_{36}-\alpha_6^2\beta_{36})c_2'\\
& + (\beta_{36}-V)c_3' - (\alpha_{36}-U)c_4',
\end{aligned}$$

and

$$c_4 = \frac{\alpha_3^2\beta_3^2\alpha_4\beta_5}{(\alpha_3^2\beta_{36}-\beta_3^2\alpha_{36})}\left[\begin{array}{l}\frac{1}{2}(\alpha_4^2+\alpha_5^2-\beta_4^2-\beta_5^2)(\alpha_3^2\beta_3^2 c_1'-\alpha_{36}\beta_{36}c_2')\\ +(\alpha_4^2\beta_{36}+\beta_5^2\alpha_{36})\left(\dfrac{\alpha_{36}}{\alpha_4^2}c_3'-\dfrac{\beta_{36}}{\beta_5^2}c_4'\right)\end{array}\right].$$

Therefore assuming $\alpha_6^2\beta_5^2+\beta_6^2\alpha_4^2 \neq 0$, we have shown the existence of three independent quadrics of the general form (16).

Next assume $\alpha_6^2\beta_5^2+\beta_6^2\alpha_4^2=0$; this relation combined with Eq. (2) of Table 1 enables one to express α_3^2 and β_3^2 and hence α_6^2 and β_6^2 as rational functions of $\alpha_4^2, \alpha_5^2, \beta_4^2, \beta_5^2$; it also implies that the right hand side of 10. (Table 1) vanishes, leading to two cases:

(i) $\alpha_3\beta_6-\alpha_6\beta_3=0$; squaring this expression and using 3. of Table 1 and the expressions for α_3^2 and β_3^2 (alluded to above) imply $\beta_4^2+\beta_5^2-\alpha_4^2-\alpha_5^2=0$. Upon substituting these expressions into the basis NF1 of V, we find

$$Q_3 = \alpha_4^2 x_4^2 + (x_2+\alpha_5 x_5)^2 + \frac{\alpha_4^2}{\alpha_4^2-\beta_5^2}\left(x_3+\sqrt{\alpha_5^2-\beta_5^2}\,x_6\right)^2;$$

upon dualizing Q_3 and upon using $\beta_4^2+\beta_5^2-\alpha_4^2-\alpha_5^2=0$, we get

$$Q_4 = (x_1+\beta_4 x_4)^2 + \beta_5^2 x_5^2 - \frac{\beta_5^2}{\alpha_4^2-\beta_5^2}\left(x_3+\sqrt{\alpha_5^2-\beta_5^2}\,x_6\right)^2;$$

consider also

$$\alpha_4^2\beta_5^2 Q_2 - \beta_5^2 Q_3 - \alpha_4^2 Q_4 = -\alpha_4^2(x_1+\beta_4 x_4)^2 - \beta_5^2(x_2+\alpha_5 x_5)^2 + \alpha_4^2\beta_5^2 x_6^2.$$

The two first quadrics have the x_3, x_6 piece in common, the first and third the x_2, x_5 piece and the second and third the x_1, x_4 piece. Thus, after some minor rescaling they have the form of the three quadrics (19).

(ii) $\alpha_6^2(\alpha_3\beta_6-\alpha_6\beta_3)+\alpha_4^2(\alpha_3\beta_6+\alpha_6\beta_3)=0$; this equation and identities 1., 3. and 4. of Table 1 imply $(\alpha_4^2+\alpha_5^2)(\beta_4^2-\beta_5^2)^2-(\beta_4^2+\beta_5^2)(\alpha_4^2-\alpha_5^2)^2=0$. Upon renaming $x_i'=x_{i+3}$ and $x_{i+3}'=x_i$ and upon taking into account the inequalities $\alpha_4^2-\alpha_5^2\neq 0$ and $\beta_4^2-\beta_5^2\neq 0$, one checks that

$$\begin{aligned}\hat{Q}_3 &\equiv (\alpha_5^2-\alpha_4^2)^{-1}\left[Q_3+\alpha_4^2(\alpha_5^2-\alpha_4^2)^{-1}Q_1-\alpha_4^2Q_2\right]\\ &=\alpha_4'^2x_4'^2+(x_2'+\alpha_5'x_5')^2+(\alpha_3'x_3'+\alpha_6'x_6')^2;\end{aligned}$$

and by dualizing,

$$\hat{Q}_4=(x_1'+\beta_4'x_4')^2+\beta_5'x_5'^2+(\beta_3'x_3'+\beta_6'x_6')^2.$$

In the expressions above,

$$\begin{aligned}&\alpha_4'^2=\alpha_4^2(\alpha_5^2-\alpha_4^2)^{-2}, &&\alpha_3'^2=\alpha_3^2,\\ &\alpha_5'^2=\alpha_5^2(\alpha_5^2-\alpha_4^2)^{-2}, &&\alpha_6'^2=\alpha_6^2(\alpha_5^2-\alpha_4^2)^{-2},\end{aligned}$$

with dual expressions for the β_i' in terms of the β_i. Observe now that the α_i' and β_i' satisfy the relations of Table 1 (Case 2), besides $\alpha_3'\beta_6'-\alpha_6'\beta_3'=0$ and $\beta_4'^2+\beta_5'^2-\alpha_4'^2-\alpha_5'^2=0$. This is to say that case (ii) brings us back to case (i).

Having shown that, in all circumstances, the elliptic curve $\mathscr{C}$ possesses three points corresponding to three independent quadrics of the form (19), there exists a fourth quadric Q (all together spanning V) of the following form, after knocking off y_4^2, y_5^2 and y_6^2 with $Q(p_1)$, $Q(p_2)$ and $Q(p_3)$:

$$Q\equiv d_1y_1^2+d_2y_2^2+d_3y_3^2+2(b_1y_1y_4+b_2y_2y_5+b_3y_3y_6).$$

Of course, the span $V=V(Q(p_1),Q(p_2),Q(p_3),Q)$ must still contain the same non-degenerate elliptic curve $\mathscr{C}$. Then expressing the linear dependence of the three cones $H_i=0$ (with regard to the coordinates of the basis just mentioned), yields

$$d_1=d_2=d_3=0,\ a_1'a_2'a_3'+1=0\quad\text{and}\quad b_1^2a_3'-b_2^2a_1'a_3'-b_3^2=0$$

with all b_i, $a_i'\neq 0$. Using the rescaled variables z_i defined by

$$(y_1,y_4,y_2,y_5,y_3,y_6)=(z_1,\ \sqrt{a_1'a_3'}\,z_4,\ \sqrt{a_1'a_3'}\,z_2,\ \sqrt{-a_1'}\,z_5,\ \sqrt{-a_1'}\,z_3,z_6)$$

leads at once to the four quadrics (16). With this notation, one easily checks that the curve of rank 4 quadrics has the form (17). It contains exactly 4 points corresponding to rank 3 quadrics; this happens whenever $c(Z,W)=0$, i.e., at the four intersection points $p_\alpha=(X_\alpha,\dots,U_\alpha)$, $1\leqq\alpha\leqq 4$ of the hyperplane

$$\langle c,p\rangle\equiv c_1X+c_2Y+c_3Z+c_4U=0$$

with the elliptic curve $\mathscr{C}$. Thus $\langle c,p_\alpha\rangle=0$ for $\alpha=1,\dots,4$ and as a result the four points p_α lie in a hyperplane, i.e., $\sum\lambda_\alpha p_\alpha=0$ for some $\lambda_\alpha\in\mathbb{C}$. Then we have that

$$\begin{aligned}\sum_1^4\lambda_\alpha Q(p_\alpha)&=\sum_1^4\lambda_\alpha(X_\alpha\bar{Q}_1+Y_\alpha\bar{Q}_2+Z_\alpha\bar{Q}_3+U_\alpha\bar{Q}_4)\\ &=\left(\sum_1^4\lambda_\alpha X_\alpha\right)\bar{Q}_1+\dots+\left(\sum_1^4\lambda_\alpha U_\alpha\right)\bar{Q}_4\\ &=0.\end{aligned}$$

Absorbing λ_α into $Q(p_\alpha)$ leads to the announced result.

Consider the geodesic flow mentioned in the statement of this theorem. Then perform the coordinate change $x \curvearrowright (X_{ij})_{1 \leq i,j \leq 4} \in so(4)$ defined by

$$X_{ij} = \tfrac{1}{2}(x_k + x_{k+3}), \qquad X_{i4} = \tfrac{1}{2}(x_i - x_4),$$

where the (i, j, k) are the cyclic permutations of $(1, 2, 3)$. In these new variables, the geodesic flow becomes

$$X_H: \quad \dot{X} = \left[X, \frac{\partial H}{\partial X}\right], \qquad X \in so(4)$$

for the metric

$$H(X, \alpha, \beta) = \sum_{1 \leq i < j \leq 4} \left(\frac{\beta_i - \beta_j}{\alpha_i - \alpha_j}\right) X_{ij}^2,$$

this form being a consequence of the relations (18) on the metric. Observe that whatever be the values of the β_i, the quadrics $H(X, \alpha, \beta)$ are invariants of the same flow. In particular, taking limits for $\beta_i \uparrow \infty$, we find that the invariants of the flow above are given by the quadrics

$$Q_i = \lim_{\beta_i \uparrow \infty} H(X, \alpha, \beta)\,(1 \leq i \leq 3) \quad \text{and} \quad Q_4 = \sqrt{\text{determinant}\,(X)},$$

which upon an obvious rescaling leads to the quadrics (16). The vector fields X_1 and X_2 are Hamiltonian flows X_H above for $H = Q_1 + Q_2 + Q_3$ and $H = Q_1$ respectively; thus they commute. This ends the proof of Theorem 5.

Theorem 6. *Consider the elliptic curve $\mathscr{C}$ of rank 4 quadrics, as before. Then after an appropriate change of coordinates $x \curvearrowright y$, an alternative basis for V has the following form*

$$\bar{Q}'_1 = \sum_1^6 y_i^2$$

$$\bar{Q}'_2 = 2 \sum_1^3 y_i y_{i+3}$$

$$\bar{Q}'_3 = \sum_{i=1}^3 (b_i y_i^2 + b_i^{-1} y_{i+3}^2)$$

$$\bar{Q}'_4 = \sum_{i=1}^3 \left(L^2 \frac{y_i^2}{b_4^{-1} - b_i^{-1}} + 2KLb_4 \frac{y_i y_{i+3}}{b_i - b_4} + K^2 \frac{y_{i+3}^2}{b_i - b_4}\right) - K^2 y_0^2,$$

with parameters $b_1, b_2, b_3, b_4 = b_1 b_2 b_3$ and K/L; notice $y_0 = z_0$ does not appear in $\bar{Q}'_i\,(1 \leq i \leq 3)$, but only in $\bar{Q}'_4$. The b_i are all distinct and different from 0 and 1. With regard to this basis, the elliptic curve $\mathscr{C}$ is a double cover of the X-plane ramified at the 4 points linearly equivalent to

$$\frac{b_j K}{b_4 L} + \frac{b_4 L}{b_j K}, \qquad j = 1, 2, 3, 4;$$

and the 4 collinear rank 3 quadrics (alluded to in Theorem 5) on $\mathscr{C}$ are given by

$$\bar{Q}'_1 \pm \bar{Q}'_2 \quad \textit{and} \quad \bar{Q}'_3 \pm \bar{Q}'_2.$$

The affine variety $I = \cap\{\bar{Q}_i' = 0, y_0 = 1\}$ supports two commuting vector fields X_1 and X_2 given by

$$\begin{bmatrix} \dot y_1 \\ \dot y_2 \\ \dot y_3 \\ \dot y_4 \\ \dot y_5 \\ \dot y_6 \end{bmatrix} = K \begin{bmatrix} \lambda_{65} y_6 y_5 \\ \lambda_{46} y_4 y_6 \\ \lambda_{54} y_5 y_4 \\ \lambda_6 y_2 y_6 - \lambda_5 y_3 y_5 \\ \lambda_4 y_3 y_4 - \lambda_6 y_1 y_6 \\ \lambda_5 y_1 y_5 - \lambda_4 y_2 y_4 \end{bmatrix} - \frac{L}{b_4} \begin{bmatrix} \lambda_3 y_3 y_5 - \lambda_2 y_2 y_6 \\ \lambda_1 y_1 y_6 - \lambda_3 y_3 y_4 \\ \lambda_2 y_2 y_4 - \lambda_1 y_1 y_5 \\ \lambda_{32} y_3 y_2 \\ \lambda_{13} y_1 y_3 \\ \lambda_{21} y_2 y_1 \end{bmatrix}$$

and

$$\begin{bmatrix} \dot y_1 \\ \dot y_2 \\ \dot y_3 \\ \dot y_4 \\ \dot y_5 \\ \dot y_6 \end{bmatrix} = K \begin{bmatrix} \lambda'_{65} y_3 y_2 \\ \lambda'_5 y_3 y_1 - \lambda'_4 y_4 y_6 \\ \lambda'_4 y_4 y_5 - \lambda'_6 y_2 y_1 \\ \lambda'_6 y_2 y_6 - \lambda'_5 y_3 y_5 \\ \lambda'_{54} y_4 y_3 \\ \lambda'_{46} y_2 y_4 \end{bmatrix} - \frac{L}{b'_4} \begin{bmatrix} \lambda'_3 y_3 y_5 - \lambda'_2 y_2 y_6 \\ \lambda'_{21} y_1 y_6 \\ \lambda'_{13} y_5 y_1 \\ \lambda'_{32} y_6 y_5 \\ \lambda'_2 y_4 y_6 - \lambda'_1 y_3 y_1 \\ \lambda'_1 y_2 y_1 - \lambda'_3 y_4 y_5 \end{bmatrix},$$

where $\lambda_{ij} \equiv \lambda_i - \lambda_j$ and $\lambda'_{ij} \equiv \lambda'_i - \lambda'_j$, with

$$\lambda_i = \frac{1}{b_4^{-1} - b_i^{-1}}, \quad \lambda'_i = \frac{1}{b_4'^{-1} - b_i'^{-1}}, \quad \lambda_{i+3} = \frac{1}{b_4 - b_i}, \quad \lambda'_{i+3} = \frac{1}{b'_4 - b'_i}$$

and $(b'_1, b'_2, b'_3, b'_4) = (b_4, b_2, b_3, b_1)$.

Proof. From Theorem 5, there are four collinear points p_α on $\mathscr{C}$ such that

$$\bar{Q}(p_\alpha) = \sum_{k=1}^{3} (a_i(p_\alpha) z_k + a_{i+3}(p_\alpha) z_{k+3})^2$$

with $\sum_\alpha \bar{Q}(p_\alpha) \equiv 0$. For each k, define y_k, y_{k+3}, b_k, b'_k, d_k, and d'_k such that

$$\begin{aligned} y_k - y_{k+3} &= a_k(p_1) z_k + a_{k+3}(p_1) z_{k+3} \\ i(y_k + y_{k+3}) &= a_k(p_2) z_k + a_{k+3}(p_2) z_{k+3} \\ b_k y_k + b'_k y_{k+3} &= a_k(p_3) z_k + a_{k+3}(p_3) z_{k+3} \\ i(d_k y_k + d'_k y_{k+3}) &= a_k(p_4) z_k + a_{k+3}(p_4) z_{k+3}, \end{aligned}$$

which can always be done for a generic choice of constants c_i, appearing in (16). Expressing the fact that $\sum Q(p_\alpha) = 0$ yields, for every $1 \leqq k \leqq 3$,

$$(y_k - y_{k+3})^2 - (y_k + y_{k+3})^2 + (b_k y_k + b'_k y_{k+3})^2 - (d_k y_k + d'_k y_{k+3})^2 \equiv 0$$

leading to

$$b_k^2 = d_k^2, \quad b_k'^2 = d_k'^2,$$
$$b_k b'_k - d_k d'_k = 2,$$

implying $b'_k = b_k^{-1}$, $d_k = b_k$, $d'_k = -b_k^{-1}$. Hence the 4 collinear quadrics take on the form

$$\bar{Q}(p_1) = \sum_{k=1}^{3} (y_k - y_{k+3})^2 \qquad \bar{Q}(p_2) = -\sum_{k=1}^{3} (y_k + y_{k+3})^2$$

$$\bar{Q}(p_3) = \sum_{k=1}^{3} (b_k y_k + b_k^{-1} y_{k+3})^2 \qquad \bar{Q}(p_4) = -\sum_{k=1}^{3} (b_k y_k - b_k^{-1} y_{k+3})^2 .$$

For a generic choice of c_i, we have that all $b_k \neq 0, \pm 1$, distinct and different from $b_4 \equiv b_1 b_2 b_3$. Taking appropriate linear combinations of the above quadrics leads to the first three quadrics $\bar{Q}'_i$ $(i = 1, 2, 3)$ of Theorem 6.

In order to complete the basis of $\bar{V}$ of Theorem 6, we need to find a fourth quadric $\bar{Q}'_4$, which (without loss of generality) has the form

$$\bar{Q}_4 = \sum_{2}^{3} (d_i y_i^2 + 2d_{i,i+3} y_i y_{i,i+3} + d_{i+3}^2 y_{i+3}^2) - c y_0^2 .$$

Expressing the fact that $\bar{V}$ contains an (elliptic) curve $\mathscr{C}$ of rank 4 quadrics, i.e., that the polynomials $H_i(X, Y, Z, U)$ are linearly dependent, we find after some row operations that

$$\operatorname{rank} \begin{pmatrix} b_2 + b_2^{-1} - b_1 - b_1^{-1} & d_2 + d_5 & d_5 b_2 + d_2 b_2^{-1} & d_{25} & d_2 d_5 - d_{25}^2 \\ b_3 + b_3^{-1} - b_1 - b_1^{-1} & d_3 + d_6 & d_6 b_3 + d_3 b_3^{-1} & d_{36} & d_3 d_6 - d_{36}^2 \end{pmatrix} = 1 .$$

Comparing the columns with the first one, leads to expressions for d_2, d_5, d_{25}^2, and d_{36}^2 in terms of d_3 and d_6. Letting

$$K^2 = b_1 b_2 b_3 (d_6 b_3 (b_2 - b_1) + d_3 (b_1 b_2 - 1)) \quad \text{and} \quad L^2 = d_6 b_3 (b_1 b_2 - 1) + d_3 (b_2 - b_1),$$

solving these expressions for d_3 and d_6 in terms of K^2, L^2, and b_i, and putting them into the formulas for d_2, d_5, d_{25}^2, and d_{36}^2 lead to

$$\begin{aligned} \bar{Q}'_4 = & \left(\frac{L^2}{b_4^{-1} - b_2^{-1}} + \frac{K^2}{b_3 - b_1} \right) y_2^2 + \left(\frac{L^2}{b_1^{-1} - b_3^{-1}} - \frac{K^2}{b_4 - b_2} \right) y_5^2 \\ & + 2KL \frac{b_4 (b_1 - b_1^{-1})}{(b_3 - b_1)(b_4 - b_2)} y_2 y_5 \\ & + \left(\frac{L^2}{b_4^{-1} - b_3^{-1}} + \frac{K^2}{b_2 - b_1} \right) y_3^2 + \left(\frac{L^2}{b_1^{-1} - b_2^{-1}} - \frac{K^2}{b_4 - b_3} \right) y_6^2 \\ & + 2KL \frac{b_4 (b_1 - b_1^{-1})}{(b_2 - b_1)(b_4 - b_3)} y_3 y_6 - y_0^2 . \end{aligned}$$

Since the quadrics $\bar{Q}'_1$, $\bar{Q}'_2$, and $\bar{Q}'_3$ are invariant under cyclic permutation $y_1 \curvearrowright y_2 \curvearrowright y_3 \curvearrowright y_1, y_4 \curvearrowright y_5 \curvearrowright y_6 \curvearrowright y_4$ and $b_1 \curvearrowright b_2 \curvearrowright b_3 \curvearrowright b_1$, the space $\bar{V}$ spanned by the $\bar{Q}'_i$ contains besides $\bar{Q}'_4$ two other quadrics, obtained by cyclically permuting the indices. Then summing up these three quadrics leads to the quadric Q'_4 announced in Theorem 6, while the rest of the statement follows from a straightforward but lengthy computation, ending the proof of Theorem 6.

6. Intersection of Quadrics, Abelian Surfaces, their Moduli and Geodesic Flow on $SO(4)$ for Left-Invariant Metrics

The punch line of this paper is Theorem 7, which we state and prove in this section.

Theorem 7. *The moduli for the intersection of four quadrics having the form*

$$\sum (\gamma_i^2 x_i^2 + \gamma_{i+3}^2 x_{i+3}^2 + 2\gamma_{i,i+3} x_i x_{i+3}) - c x_0^2 = 0,$$

satisfying conditions C0, C1, *and* C2 *and having a non-degenerate curve of rank* 4 *quadrics in their linear span breaks up into two pieces: the moduli of abelian surfaces of principal polarization and polarization* (1,2). *The affine intersection I of the quadrics can be completed into an abelian surface A by adjoining to I a divisor $\tilde{D}$, which can be viewed as an* 8*-fold cover of $\mathscr{C}$ in two different ways, as indicated in Fig.* 3. *The representation of $\mathscr{C}$ as a curve of rank* 4 *quadrics is obtained by substituting the quadrics $\bar{Q}_i$ of Theorems 4 and 5 for c_i in the expressions $c(Z, W)$. The moduli for each of the cases are given in terms of the quadrics $\bar{Q}_i$ in Theorems 4 and 6, as follows:*

Case 1	*Case 2*
hyperelliptic Jacobians $A = \mathrm{Jac}(D)$	*Prym variety* $A = \mathrm{Prym}(D/\mathscr{C})$ *(polarization* (2,4)*)*
moduli: c_1/c_4, c_2/c_4, c_3/c_4	*moduli* b_1, b_2, b_3
D: $U^2 = c(Z)$	D: $\{\mathscr{C}, U^2 = c(Z, W)\}$
	$[\tilde{D}] \ni 8$*-fold unramified*[5] *cover of* $\mathscr{H}$
	$\mathscr{H}$: $V^2 = Z(Z-b_1)(Z-b_2)(Z-b_3)(Z-b_1 b_2 b_3)$
	$\mathrm{Jac}(\mathscr{H})$ *is a double unramified cover of* A.

Proof. According to Theorem 1, the curve $\mathscr{E} = \mathscr{E}'$ is a four-fold cover of $\mathscr{C}$; it is unramified, because z_1^2/z_3^2 and z_2^2/z_3^2 have a divisor on $\mathscr{C}$ divisible by 2; the latter follows from expressing (2.1) in terms of the curves of rank 4 quadrics appearing in Theorems 4 and 5. Also from Theorem 1, $\bar{I}$ has a normal crossing along $\mathscr{E}$ which upon normalization turns $\mathscr{E}$ into the curve $\tilde{D} = \{\mathscr{E}, U^2 = c(Z, W)\}$.

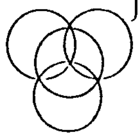
In *case 1*, $\bar{I} \cap \{z_0 = 0\} = \mathscr{E} + \mathscr{E}^c$, $\mathscr{E} = 4$ lines C_1 and $\mathscr{E}^c = 4$ lines C_2, where $\mathscr{E}$ and $\mathscr{E}^c$ intersect according to Fig. 4. The surface $\bar{I}$ has a normal crossing along $\mathscr{E}$ and is smooth along $\mathscr{E}^c$. Blowing up $\bar{I}$ along $\mathscr{E}$ and blowing it down along $\mathscr{E}^c$ lead to the desired abelian surface with a divisor $\tilde{D}$, consisting of 4 isomorphic hyperelliptic curves. In *Case 2*, $\bar{I}$ must be blown up along $\mathscr{E} = \bar{I} \cap \{x_0 = 0\}$ to yield an abelian surface carrying a smooth curve of genus 16.

The proof of these statements is based on arguments in [5], which we now summarize. The set of quadrics in Theorems 4 and 5 supports the two commuting vector fields X_1 and X_2. Then X_1 admits a family of Laurent solutions $z(t) = t^{-1}(z^{(0)} + z^{(1)} t + \ldots)$ with simple pole, parametrized by $\tilde{D}$. After reduction by the invariants and upon substituting the Laurent solutions, the space

[5] $[\tilde{D}]$ denotes the linear system of $\tilde{D}$

Case 1

$$\begin{tikzcd}[column sep=small]
 & \mathbb{P}^{15} \supset A = \mathrm{Jac}(D) \supset \tilde{D} = \left\{\begin{array}{l}\text{4 hyperelliptic curves}\\ \text{intersecting in 4 ordinary}\\ \text{triple points}\end{array}\right\},\quad g(\tilde{D}) = 17, \arrow[dl, "\substack{\text{ramified}\\ 2-1}"'] \arrow[dr, "\substack{4-1\\ \text{unramified}}"] & \\
\mathscr{E} = \{4 \text{ lines}\} \arrow[dr, "\substack{\text{unramified}\\ 4-1}"'] & & D = \left\{\begin{array}{c}\mathscr{C}\\ U^2 = c(Z) = \dfrac{c_1}{Z-1} - \dfrac{c_2}{Z-2} + \dfrac{c_1}{Z} + \dfrac{c_4}{2}\end{array}\right\} = \begin{array}{l}\text{hyperelliptic}\\ \text{curve}\end{array} \arrow[dl, "\substack{2-1\\ \text{ramified at the 6 points}\\ \text{where } U=0 \text{ and } \infty}"] \\
 & \mathscr{C} = \{\text{line: } W = Z\} &
\end{tikzcd}$$



Case 2

$$\begin{tikzcd}[column sep=small]
 & \mathbb{P}^{7} \supset A = \mathrm{Prym}(D/\mathscr{C}) \supset \tilde{D} = \left\{\begin{array}{c}D\\ X^2 = 1-Z,\ Y^2 = Z\end{array}\right\},\quad \text{smooth},\quad g(\tilde{D}) = 9 \arrow[dl, "\substack{\text{ramified}\\ \text{over 16}\\ \text{points}\\ 2-1}"'] \arrow[dr, "\substack{4-1\\ \text{unramified}}"] & \\
\mathscr{E} = \left\{\begin{array}{c}\mathscr{C}\\ X^2 = 1-Z,\ Y^2 = Z\end{array}\right\} \arrow[dr, "\substack{\text{unramified}\\ 4-1}"'] & & D = \left\{\begin{array}{c}\mathscr{C}\\ U^2 = c(Z, W) = -a^2 Z c_1 + (b^2 Z + c^2) c_2 + \\ + Z(b^2 Z + c^2) c_3 - 2W c_4\end{array}\right\} \arrow[dl, "\substack{2-1\\ \text{ramified at}\\ \text{the 4 points } U=0}"] \\
 & \mathscr{C}: W^2 = Z(1-Z)(b^2 Z + c^2) &
\end{tikzcd}$$

Fig. 3

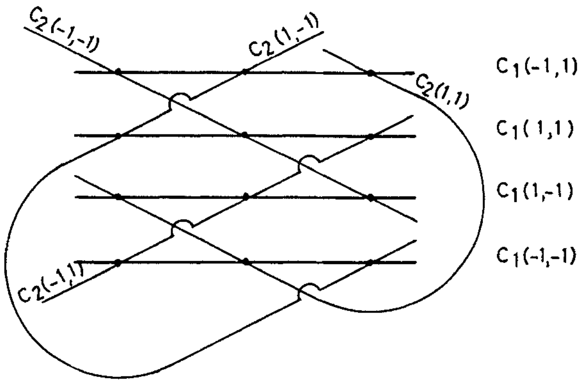


Fig. 4

of polynomials in $z_1, \ldots, z_6$ having a simple pole is spanned by 16 functions 1, $z_1, \ldots, z_6, \ldots, z_{15}$ in Case 1 and the 8 functions $1, z_1, \ldots, z_6$ and $z_7 \equiv bz_1z_4 - az_2z_5$ in Case 2. These functions and their residues at $t=0$ map the affine surface I and the divisor $\tilde{D}$ smoothly into $\mathbb{P}^N$, where $N=15$ and 7, respectively. One then shows, with effort, that the trajectories of the vector field X_1 issuing from $\tilde{D}$ form a smooth surface strip around $\tilde{D}$; this procedure is used to glue the curve $\tilde{D}$ onto the affine surface I, yielding a smooth surface A embedded into $\mathbb{P}^N$. The complex Arnold-Liouville theorem and the existence of two commuting vector fields imply that A is a complex torus carrying the divisor $\tilde{D}$. The functions 1, $z_1, \ldots, z_N$ defined above generate $L(\tilde{D})$, in addition to embedding A into $\mathbb{P}^N$. Hence by Chow's theorem, A is an Abelian surface with very ample divisor $\tilde{D}$. Its period matrix can always be given by

$$\begin{pmatrix} \delta_1 & 0 & a & c \\ 0 & \delta_2 & c & b \end{pmatrix}, \quad \operatorname{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0, \quad \delta_i \in \mathbb{Z}, \quad \delta_1 \mid \delta_2$$

with $\delta_1\delta_2 = g(\tilde{D}) - 1 = N+1$, leaving only a few possibilities for (δ_1, δ_2). Then $(\delta_1, \delta_2) = (4,4)$ in Case 1, and $=(2,4)$ in Case 2, by counting the even sections (θ-functions) of the line bundle associated with $\tilde{D}$.

Since, in case 1, A contains a divisor $\tilde{D}$, consisting of 4 isomorphic hyperelliptic curves D, we have that $A = \operatorname{Jac}(D)$. To see that in Case 2, $A = \operatorname{Prym}(D/\mathscr{C})$, we use the flows X_1 and X_2 of Theorem 5, and the differentials dt_1 and dt_2 defined on A by $dt_i(X_j) = \delta_{ij}$; dt_1 and dt_2 restricted to the divisor $\tilde{D}$ turn out to be odd holomorphic differentials ω_1 and ω_2 on the curve D. To actually obtain these differentials, we pick two coordinates z_α and z_β, viewed on A as functions of t_1 and t_2. Differentiating $1/z_\alpha$ and z_β/z_α with regard to the vector fields X_1 and X_2, yields

$$\begin{bmatrix} d\left(\dfrac{1}{z_\alpha}\right) \\ d\left(\dfrac{z_\beta}{z_\alpha}\right) \end{bmatrix} = \begin{bmatrix} X_1\left(\dfrac{1}{z_\alpha}\right) & X_2\left(\dfrac{1}{z_\alpha}\right) \\ X_1\left(\dfrac{z_\beta}{z_\alpha}\right) & X_2\left(\dfrac{z_\beta}{z_\alpha}\right) \end{bmatrix} \begin{bmatrix} dt_1 \\ dt_2 \end{bmatrix}.$$

Then substituting the Laurent solutions in the above and evaluating the result at $t=0$, we obtain the two holomorphic differentials ω_1 and ω_2. The oddness of ω_i and the decomposition

$$\operatorname{Jac}(D)=\operatorname{Prym}(D/\mathscr{C})\oplus\mathscr{C}$$

imply that A is isogenous to Prym $(D/\mathscr{C})$. A more careful analysis involving cycles of D shows $A=\operatorname{Prym}(D/\mathscr{C})$; see Haine [10].

With regard to Case 2, we now prove that the linear system $[\tilde{D}]$ contains an 8-fold unramified cover of the hyperelliptic curve $\mathscr{H}$ given in the statement of this theorem. To do this, consider the quadrics $\bar{Q}_i'$ of Theorem 6 expressed in the coordinates y. Inspired by Kötter [14, 15], we consider the following function

$$f\equiv\sum_1^3\frac{y_{i+3}^2}{b_i-b_4}$$

in $L(2\tilde{D})$; notice this expression appears explicitly in the quadric $\bar{Q}_4'$. We now show

$$(f)=-2\tilde{D}+2\ (\text{8-fold unramified cover of }\mathscr{H}).$$

Consider the curve

$$D_0\equiv\bigcap_1^4\{\bar{Q}_i'(y)=0\}\cap\{f=0\}. \tag{1}$$

The following expressions

$$y_{i+3}=Z\sqrt{X-b_i}\sqrt{\frac{b_4-b_i}{\varphi'(b_i)}}\qquad y_i=y_{i+3}\frac{V}{(X-b_i)(X-b_4)\sqrt{X}} \tag{2}$$

provide a solution of the equations defining D_0, in which

$$\begin{aligned}&Z^2=c^{-1}\left(\frac{K}{L}\right)^2\frac{X-b_4}{X},\qquad (c\text{ rational function of }b_i)\\&\varphi(u)=\prod_1^3(u-b_i)\quad\text{and}\quad V^2=X\prod_1^4(X-b_i).\end{aligned} \tag{3}$$

The main tool here is Jacobi's wonderful device, which consists of evaluating the contour integral

$$\int\frac{R(u)}{\varphi(u)}du,\qquad R\text{ rational}$$

around a small circle about $u=\infty$. In this way, one verifies that

$$\sum_4^6 y_i^2=Z^2,\quad\sum_1^3 y_i^2=-Z^2,\quad\sum_1^3 y_iy_{i+3}=0,\quad\sum_1^3\frac{y_iy_{i+3}}{b_i-b_4}=0,$$

$$\sum_1^3\frac{y_{i+3}^2}{b_i-b_4}=0,\quad\sum_1^3\frac{y_i^2}{b_4^{-1}-b_i^{-1}}=c\frac{XZ^2}{X-b_4},\quad\sum_1^3(b_iy_i^2+b_i^{-1}y_{i+3}^2)=0,$$

showing that (2) is a solution of (1) and thus parametrizes a connected component D_0' of D_0.

Using (2), we show that D_0' is equivalent to the curve D_0'':

$$D_0''=\left\{\begin{array}{l} y_4y_5y_6=\varepsilon\sqrt{X\prod_1^4(X-b_i)}:\mathscr{H} \\ \dfrac{1}{y_4y_5y_6}(y_1y_4,y_2y_5,y_3y_6)=\delta\sqrt{X} \\ \text{and} \\ y_4y_5=\kappa\sqrt{(X-b_1)(X-b_2)} \\ y_5y_6=\lambda\sqrt{(X-b_2)(X-b_3)} \end{array}\right\},$$

where all equalities are valid up to multiplication by rational functions of X and where ε, δ, κ, and λ take on ± 1 independently of each other. Indeed, a point on D_0'', given by $(X,\varepsilon,\delta,\kappa,\lambda)$ specifies a unique point $(y_1,\dots,y_6)$ in D_0' as is seen from taking appropriate ratios of the formulas defining D_0'' and conversely given $(y_1,\dots,y_6)$ on the curve D_0', we recover uniquely Z^2, X, ε, δ, κ, and λ from $\sum_4^6 y_i^2=Z^2$, (3) and the formulas in D_0''. Hence $D_0'=D_0''$ and D_0 contains a curve D_0' which can be viewed as an 8-fold unramified cover of the genus 2 hyperelliptic curve $\mathscr{H}$; thus the curve D_0' has genus 9. The various sign flips associated with ε, δ, κ, and λ correspond to involutions on the curve D_0' as summarized in the figure below.

Genus		*Involutions*
9	D_0'	
	unram. $\downarrow$	$\lambda\curvearrowright-\lambda:(y_1,y_4,y_2,y_5,y_3,y_6)\curvearrowright(-y_1,-y_4,-y_2,-y_5,y_3,y_6)$
5	$\mathscr{H}_2$	
	unram. $\downarrow$	$\kappa\curvearrowright-\kappa:(y_1,y_4,y_2,y_5,y_3,y_6)\curvearrowright(y_1,y_4,-y_2,-y_5,-y_3,-y_6)$
3	$\mathscr{H}_1$	
	unram. $\downarrow$	$\delta\curvearrowright-\delta:(y_1,y_2,y_3,y_4,y_5,y_6)\curvearrowright(-y_1,-y_2,-y_3,y_4,y_5,y_6)$
2	$\mathscr{H}$	
	ram. $\downarrow$	$\varepsilon\curvearrowright-\varepsilon:(y_1,y_2,y_3,y_4,y_5,y_6)\curvearrowright(y_1,y_2,y_3,-y_4,-y_5,-y_6)$
	$\mathbb{C}$	

Fig. 5

To establish $D_0=D_0'$ and $(f)=-2D+2D_0$, we use the flows X_1 and X_2 of Theorem 6 and we first show that $\partial f/\partial t_i=0$ for $i=1,2$; indeed, one checks that

$$\begin{aligned}\left.\frac{\partial f}{\partial t_1}\right|_{D_0'}&=-\frac{L}{b_4}y_1y_2y_3\left.\left[\frac{y_4}{y_1}\lambda_4\lambda_{32}+\frac{y_5}{y_2}\lambda_5\lambda_{13}+\frac{y_6}{y_3}\lambda_6\lambda_{21}\right]\right|_{D_0'}\\ &=-\frac{L}{b_4}y_1y_2y_3\Big|_{D_0'}\frac{(X-b_4)\sqrt{X}}{V}\\ &\quad\cdot[(X-b_1)\lambda_4\lambda_{32}+(X-b_2)\lambda_5\lambda_{13}+(X-b_3)\lambda_6\lambda_{21}],\ \text{using (2)}\\ &\equiv 0,\end{aligned}$$

while

$$\left.\frac{\partial f}{\partial t_2}\right|_{D_0'} = -\left.\frac{\partial f}{\partial t_1}\frac{dt_1}{dt_2}\right|_{D_0'} = 0,$$

showing that D_0' is a double zero of f. Thus $(f) \geqq 2D_0' - 2D$, with $g(D) = g(D_0') = 9$, and hence f must have a double pole along D, establishing $(f) = 2D_0 - 2D$; thus taking the square root of f shows that D_0 is in the linear system of D.

Finally we show that A is isogeneous to $\mathrm{Jac}(\mathscr{H})$. It suffices to show, in view of Theorem 4 of [5], that the holomorphic differentials $\omega_i = dt_i|_{D_0}$ depend on $\mathscr{H}$ only and thus descend to the hyperelliptic differentials. Taking the differentials of two independent functions F and G defined on A, we find

$$\begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix} = \frac{1}{\left(\dfrac{\partial F}{\partial t_1}\dfrac{\partial G}{\partial t_2} - \dfrac{\partial F}{\partial t_2}\dfrac{\partial G}{\partial t_1}\right)} \begin{bmatrix} \dfrac{\partial G}{\partial t_2} & -\dfrac{\partial F}{\partial t_2} \\ -\dfrac{\partial G}{\partial t_1} & \dfrac{\partial F}{\partial t_1} \end{bmatrix} \begin{bmatrix} dF \\ dG \end{bmatrix}.$$

Setting $F = y_4 y_5 y_6$ and $G = y_1 y_2 y_3$, observe that $\partial F/\partial t_i$ and $\partial G/\partial t_i$ are functions of $y_j y_{j+3}$ and y_j^2 only, and therefore F, G, $\partial F/\partial t_i$ and $\partial G/\partial t_i$ are invariant under the λ and κ involution of figure 5. Consequently the same holds for $\omega_i = dt_i|_{D_0} (i = 1, 2)$. Therefore not only are ω_1 and ω_2 holomorphic differentials on D_0, but also holomorphic on $\mathscr{H}_1$ (see Fig. 5). Then by Theorem 4 of [5], we have

$$A \subset \mathrm{Jac}(\mathscr{H}_1) = \mathrm{Jac}(\mathscr{H}) \oplus \text{Elliptic curve, modulo isogenies}$$

and since A is irreducible, $A = \mathrm{Jac}(\mathscr{H})$, moduls isogenies. Moreover $\mathrm{Jac}(\mathscr{H})$ is a double cover of the abelian surface A, as follows from the arguments in Horozov-van Moerbeke [25] and Adler-van Moerbeke [23]. Hence b_1, b_2, and b_3 form a set of moduli for A, concluding the proof of Theorem 7.

Remark. An exact computation shows that

$$\omega_1 = dt_1|_{D_0} = \frac{(X - b_1)\,dx}{V}, \qquad \omega_2 = dt_2|_{D_0} = \frac{(X - b_4)\,dX}{V}.$$

This shows that the flows X_1 and X_2 are both doubly tangent to D_0 at the 8 points covering $X = b_4$ and $X = b_1$ respectively.

Corollary. *The Abelian surfaces A of polarization* $(2, 4)$ *are completely defined by the intersection of the 6 quadrics in* $\mathbb{P}^7$

$$\bar{Q}_1 = z_4^2 + z_2^2 - z_3^2 - c_1 z_0^2$$
$$\bar{Q}_2 = z_5^2 + z_3^2 - z_1^2 - c_2 z_0^2$$
$$\bar{Q}_3 = z_6^2 + z_1^2 - z_2^2 - c_3 z_0^2$$
$$\bar{Q}_4 = \frac{a}{c_4} z_1 z_4 + \frac{b}{c_4} z_2 z_5 + \frac{c}{c_4} z_3 z_6 - z_0^2$$
$$\bar{Q}_5 = -b z_1 z_4 + a z_2 z_5 + z_0 z_7$$
$$\bar{Q}_6 = -c^2(c_1 z_1^2 + c_2 z_2^2 + c_3 z_3^2) - c_4(a z_1 z_4 + b z_2 z_5 - c z_3 z_6) - z_7^2$$

or equivalently by the following 6 quadrics, the summation running from 1 to 3:

$$\bar{Q}'_1 = \sum (y_i + y_{i+3})^2$$
$$\bar{Q}'_2 = \sum (y_i - y_{i+3})^2$$
$$\bar{Q}_3^{\pm\,\prime} = \sum (b_i y_i \pm b_i^{-1} y_{i+3})^2$$
$$\bar{Q}'_4 = \sum \frac{y_i^2}{b_4^{-1} - b_i^{-1}} - y_7^2$$
$$\bar{Q}'_5 = \sum \frac{y_i y_{i+3}}{b_i - b_4} - \frac{y_7 y_8}{b_4}$$
$$\bar{Q}'_6 = \sum \frac{y_{i+3}^2}{b_i - b_4} - y_8^2, \quad b_4 = b_1 b_2 b_3,$$

where b_1, b_2, $b_3 \in \mathbb{C}^*$ form a set of moduli for A. Moreover

$$\operatorname{Jac}\{y^2 = x(x-b_1)(x-b_2)(x-b_3)(x-b_4)\}$$

is a double unramified cover of A.

Proof. In the proof of Theorem 7, it was observed that $L(\tilde{D}) = \{z_0 = 1, z_1, \ldots, z_6, z_7 = bz_1 z_4 - az_2 z_5\}$ smoothly embeds A into $\mathbb{P}^7$, and it is easy to check (as first observed by Haine [10]) that this smooth embedding is given by augmenting the relations of the embedding in $\mathbb{P}^6$ (5.16) by the definition of z_7, to wit $\bar{Q}_5 = 0$, and one relation involving z_7^2, namely $\bar{Q}_6 = 0$; this leads to the first basis. Moreover, the locus of rank 4 quadrics in the span $X\bar{Q}_1 + \ldots + W\bar{Q}_6$ is given by the intersection of the four quadratic cones

$$K_i = \{p \,|\, a_i a_{i+3} - a_{i,i+3}^2 = 0\}, \qquad K_4 = \{p \,|\, a_0 a_7 - a_{0,7}^2 = 0\}, \quad i = 1, 2, 3,$$

having the explicit form:

$$K_1 = \left\{4(Z - Y - c^2 c_1 W)X - \left(\frac{a}{c_4} U - bV - ac_4 W\right)^2 = 0\right\}$$
$$K_2 = \left\{4(X - Z - c^2 c_2 W)Y - \left(\frac{b}{c_4} U + aV - bc_4 W\right)^2 = 0\right\}$$
$$K_3 = \left\{4(Y - X - c^2 c_3 W)Z - \left(\frac{c}{c_4} U + cc_4 W\right)^2 = 0\right\}$$
$$K_4 = \{4(c_1 X + c_2 Y + c_3 Z + U)W - V^2 = 0\}.$$

By inspection (using $a^2 + b^2 + c^2 = 0$) we have $K_1 + K_2 + K_3 + c^2 K_4 = 0$, and therefore $K = \bigcap_1^4 K_i$ defines a surface. The hyperplane section with K

$$K \cap \{\lambda^2 (c_1 X + c_2 Y + c_3 Z + U) - \kappa^2 W\}$$
$$= K_1 \cap K_2 \cap \{c_1 X + c_2 Y + c_3 Z + U = \kappa^2,\ V = 2\kappa\lambda,\ W = \lambda^2\}$$

is seen, after eliminating first U, V, W and then Z, to be an elliptic curve; the latter can be viewed as a curve of rank 3 quadrics in the affine linear span

$$Q_1 - c_1 Q_4,\ Q_2 - c_2 Q_4,\ Q_3 - c_3 Q_4,\ Q = \kappa^2 Q_4 + 2\kappa\lambda Q_5 + \lambda^2 Q_6 .$$

According to Theorem 6, the above basis may be replaced, after a block preserving change of variables, by a basis of the form:

$$Q_1', Q_2', Q_3^{+\prime},\ Q_{KL} = L^2 Q_4' + 2KL b_4 Q_5' + L^2 Q_6',$$

for an appropriate choice of K and L. To show that the linear spans of the two sets of 6 affine quadrics given in the corollary match, we observe that the quadrics Q_1', Q_2', $Q_3^{+\prime}$ are in the span of $Q_1 - c_1 Q_4$, $Q_2 - c_2 Q_4$, $Q_3 - c_3 Q_4$, and the rest of the argument proceeds by picking three distinct values of K/L. Thus the bases $Q_1, \ldots, Q_6$ and $Q_1', \ldots, Q_6'$ have the same span, and we conclude by the preceding arguments that the basis $\bar{Q}_1, \ldots, \bar{Q}_6$ can be replaced, after a block preserving change of coordinates, by the basis $\bar{Q}_1', \bar{Q}_2', \bar{Q}_3', Q_4' - q_4(z_0, z_7)$, $Q_5' - q_5(z_0, z_7)$, $Q_6' - q_6(z_0, z_7)$, where $q_i(z_0, z_7) \equiv a_i z_0^2 + b_i z_7^2 + 2c_i z_0 z_7$. It was shown in Theorem 7 that the divisor (Q_6') has the structure $(Q_6') = 2D_0 - 2D$, from which it follows that $q_6(z_0, z_7)$ is a perfect square $(ez_0 + fz_7)^2 \equiv y_7^2$, and by symmetry, so is $q_4(z_0, z_7) = y_8^2$. Since the span $X\bar{Q}_1' + \ldots + W\bar{Q}_6'$ must support a surface of rank 4 quadrics defined by 4 linearly dependent quadratic cones, we immediately find that $q_5(z_0, z_7) = \pm y_7 y_8 / b_4$, concluding the proof of the corollary.

Remark: Since the four quadrics $\bar{Q}_1', \bar{Q}_2', \bar{Q}_3^+, \bar{Q}_3^-$ satisfy $\bar{Q}_1' - \bar{Q}_2' = \bar{Q}_3^+ - \bar{Q}_3^-$, we may take any pair of them to be the first two quadrics, and the other pair to be the latter two quadrics. Recomputing the new basis of 6 quadrics, involves a linear change of coordinates and a fractional linear change of the $\sqrt{b_i}$'s. There are clearly 3 essential choices of such bases, resulting all together in 6 sections corresponding to $y_7 = 0$ or $y_8 = 0$; they define 8–1 unramified covers of 6 hyperelliptic genus 2 curves, coming in 3 distinct pairs. All the above is computable in a linear fashion. This has been observed by L. Haine.

7. The Degenerate Curve of Rank 4 Quadrics and $K3$ Surfaces

In this section, we deal with the possibility excluded earlier, where the curve of rank 4 quadrics $\mathscr{C} \subset \bar{V} \simeq \mathbb{P}^3$ contains a degenerate component.

Theorem 8. *The situation where $\mathscr{C} \subset \bar{V}$ contains a degenerate component, which is not a line, breaks up generically into two cases for which we provide a canonical basis in some appropriate coordinates; namely $\bar{V}$ has a basis containing, besides a generic quadric $\bar{Q}_4$ of the usual block form, the following three quadrics*:

case (A): $\bar{Q}_1 = x_1^2 + x_2^2 + x_3^2,\ \bar{Q}_2 = x_4^2 + x_5^2 + x_6^2$ *and* $\bar{Q}_3 = x_1 x_4 + x_2 x_5 + x_3 x_6 - x_0^2$

case (B): $\bar{Q}_1 = x_1^2 + x_2^2 + x_3^2,\ \bar{Q}_2 = x_4^2 + x_5^2 + x_6^2$ *and* $\bar{Q}_3 = x_1 x_4 + x_2 x_5 + a_3 x_3^2 - x_0^2$,

where $\bar{Q}_1$ and $\bar{Q}_2$ do not contain any x_0^2 term.

Then in case (A) (case (B) being analogous), the affine intersection I of the quadrics can be completed into a K3 surface A by adjoining to I a divisor $\tilde{D}$, which is an 8-fold cover of $\mathscr{C}$ in two different ways, as indicated in Fig. 6. An appropriate rescaling leads to

$$\bar{Q}_4 = (x_2^2 + x_5^2 + 2\alpha_{25} x_2 x_5) + (\alpha_3^2 x_3^2 + \alpha_6^2 x_6^2 + 2\alpha_{36} x_3 x_6) - c_4 x_0^2$$

and in this basis $\mathscr{C}$ is a rational curve:

$$\mathscr{C}:\ Z^2 \bar{Q}_1 + \bar{Q}_2 - 2Z\bar{Q}_3 = \sum_1^3 (Zx_i - x_{i+3})^2 - 2Zx_0^2 .$$

The surface $\bar{I}$ experiences a 2-fold normal crossing along $\mathscr{E}'(=\mathscr{E})$ with eight pinch points occuring at $Z=0$ or ∞; moreover $\mathscr{E}$ can be given by

$$\mathscr{E} = \left\{ \begin{array}{c} \mathscr{C}:\ W = Z \\ \dfrac{x_1^2}{P_1(Z)} = \dfrac{x_2^2}{P_2(Z)} = \dfrac{x_3^2}{P_3(Z)} \end{array} \right\}$$

with

$$P_1 \equiv -P_2 - P_3, \qquad P_2(Z) \equiv \alpha_3^2 + \alpha_6^2 Z^2 + 2\alpha_{36} Z, \qquad P_3(Z) \equiv -1 - Z^2 - 2\alpha_{25} Z .$$

Upon normalizing $\bar{I}$ along $\mathscr{E}$, the curve $\mathscr{E}$ turns into the curve $\tilde{D} = \{\mathscr{E}, U^2 = Z\}$ of genus 9 (see Fig. 6).

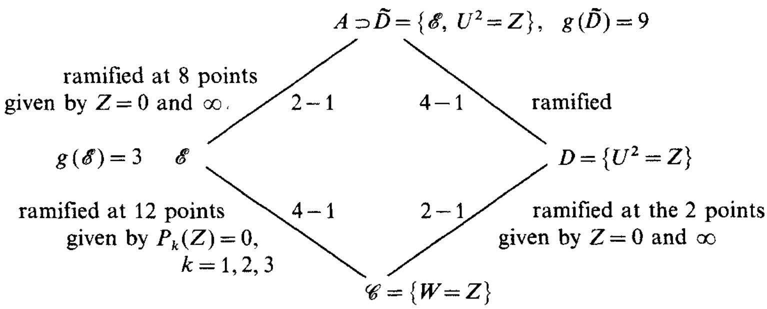


Fig. 6

Proof. According to the Lemma in Appendix 1, V contains the announced canonical basis $\bar{Q}_1, \bar{Q}_2, \bar{Q}_3$ and one other (generic) quadric $\bar{Q}_4$; after appropriately subtracting a linear combination of Q_1, Q_2, and Q_3 and after rescaling x_4, x_5, and x_6 simultaneously, Q_4 can be taken to have the form given above.

One then computes $\mathscr{E}'$ by applying Theorem 1, with Q_4 replaced by Q_2 and one checks by direct calculation that $\mathscr{E}' = \mathscr{E}$. The statements concerning the normal crossing, pinch points and the formula for $\tilde{D}$ follows from Theorem 1, with $c(p) = 2Z$. All this taken together and the Riemann-Hurwitz formula yield Fig. 6. It is interesting to point out that the roots of $P_k(Z) = 0$, $k = 1, 2, 3$, discussed in Fig. 6, coincide with the points where $K_i \cap K_j$ intersect $\mathscr{C}$.

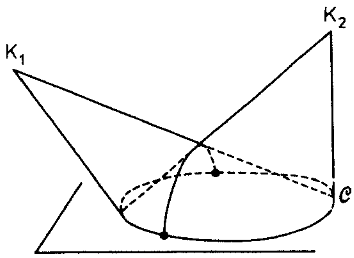


Fig. 7

It will now be shown that the smooth surface $A = \tilde{I}$ obtained by separating the two sheets of $\bar{I}$ at $\mathscr{E}$ is a $K3$ surface. Let $\tilde{\mathscr{E}} = \tilde{D}$ be the curve on $\tilde{I}$ obtained by blowing up $\mathscr{E}$. From the Enriques classification of surfaces it suffices to prove that: (i) the canonical divisor is trivial and (ii) there are no holomorphic 1-forms. Let π: $\tilde{I} \to \bar{I}$ be the natural projection. Then the canonical divisor

$$\begin{aligned} K_{\tilde{I}} &= \pi^*(K_{\bar{I}}) - \tilde{\mathscr{E}} \\ &= \pi^*((\deg Q_1 + \deg Q_2 + \deg Q_3 + \deg Q_4) \cdot H + K_{\mathbb{P}^6})|_{I} - \tilde{\mathscr{E}} \\ &= \pi^*(8H - 7H) - (\tilde{I} \cap \{x_0 = 0\}) \\ &= 0, \end{aligned}$$

i.e., the canonical divisor is trivial. Statement (ii) amounts to

$$\dim H^0(\Omega^1_{\tilde{I}}) = \dim H^{0,1}_{\bar{\partial}}(\tilde{I}) + \dim H^1(\mathcal{O}_{\tilde{I}}) = 0,$$

or what is the same

$$\chi(\mathcal{O}_{\tilde{I}}) = \dim H^0(\mathcal{O}_{\tilde{I}}) - \dim H^1(\mathcal{O}_{\tilde{I}}) + \dim H^2(\mathcal{O}_{\tilde{I}}) = 1 - 0 + 1 = 2.$$

In view of the exact sequences

$$\begin{gathered} 0 \to \mathcal{O}_{\bar{I}} \to \pi_* \mathcal{O}_{\tilde{I}} \to \pi_* \mathcal{O}_{\tilde{I}} / \mathcal{O}_{\bar{I}} \to 0, \\ 0 \to \mathcal{O}_{\mathscr{E}} \to \pi_* \mathcal{O}_{\tilde{\mathscr{E}}} \to \pi_* \mathcal{O}_{\tilde{\mathscr{E}}} / \mathcal{O}_{\mathscr{E}} \to 0, \end{gathered}$$

we have

$$\begin{aligned} \chi(\mathcal{O}_{\tilde{I}}) &= \chi(\pi_* \mathcal{O}_{\tilde{I}}) \\ &= \chi(\mathcal{O}_{\bar{I}}) + \chi(\pi_* \mathcal{O}_{\tilde{I}} / \mathcal{O}_{\bar{I}}) \\ &= \chi(\mathcal{O}_{\bar{I}}) + \chi(\pi_* \mathcal{O}_{\tilde{\mathscr{E}}} / \mathcal{O}_{\mathscr{E}}) \\ &= \chi(\mathcal{O}_{\bar{I}}) + \chi(\mathcal{O}_{\tilde{\mathscr{E}}}) = \chi(\mathcal{O}_{\mathscr{E}}), \end{aligned}$$

with $\chi(\mathcal{O}_{\mathscr{E}}) = 1 - \text{genus}(\mathscr{E}) = 1 - 3 = -2$; also $\chi(\mathcal{O}_{\tilde{\mathscr{E}}}) = -8$. By an argument due to Mumford [19], $\chi(\mathcal{O}_{\bar{I}}) = 8$, and so $\chi(\mathcal{O}_{\tilde{I}}) = 8 - 8 - (-2) = 2$, proving (ii) and thus the theorem.

Remark. It is reasonable to *conjecture* that the $K3$ surfaces obtained here are *Kummer* surfaces.

8. A Curve of Rank 4 Quartics and Abelian Surfaces of Polarization (1,6)

In Theorem 7, it was pointed out that the abstract curve $\mathscr{C} \subset \mathbb{P}^3$ can be represented as a family of rank 4 quadrics in $\bar{V}$ by substituting $\bar{Q}_i$ for c_i into $c(Z, W)$; see Fig. 3. This procedure can now be generalized to a situation of quartics, where much of the geometry carries over. In the classification of the left-invariant metrics on $SO(4)$ for which geodesic flow

$$\dot{x}' = x' \wedge \frac{\partial H}{\partial x'}, \qquad \dot{x}'' = x'' \wedge \frac{\partial H}{\partial x''}$$

is algebraically completely integrable, we found a one-parameter family of metrics (see Adler and van Moerbeke [2–4]) given by

$$H = x_1^2 + a^2 x_4^2 + \frac{(1-a)(1+3a)}{2} x_1 x_4 - \frac{(1+a)(1-3a)}{2} x_2 x_5$$
$$- \frac{(1+a)(1-3a)^3}{16a} x_3^2 - \frac{(1+a)^3(1-3a)}{16a} x_6^2 - \frac{(1-a^2)(1-9a^2)}{8a} x_3 x_6$$

with $a \in \mathbb{C}$, $a \neq \pm 1, \pm 1/3, 0$. The associated flow has besides the orbit invariants $T_1 = \|x'\|^2$, $T_2 = \|x''\|^2$ and the energy H above, a *quartic invariant* Q_4 to be exhibited below. To this end, perform a linear change of coordinates $x \curvearrowright y$, given in Appendix 3; there also we define the quadratic expressions $G_1, \ldots, G_8$ in y; expressed in terms of the G_i, the system admits the following invariants:

$$\bar{Q}_1 \equiv aG_2 + \frac{1-a}{1+3a} G_7 - c_1 y_0^2$$

$$\bar{Q}_2 = -aG_1 + \frac{1+a}{1-3a} G_8 - c_2 y_0^2$$

$$\bar{Q}_3 = -\frac{2G_6}{(1-3a)(1+3a)} - \frac{G_1}{1+3a} - \frac{G_2}{1-3a} - c_3 y_0^2 \tag{1}$$

$$\bar{Q}_4 = \frac{1}{(3a-1)(3a+1)} ((1-a)(1-3a)(G_1^2 + G_4^2) + (1+a)(1+3a)(G_2^2 + G_5^2)$$
$$+ 3(1-a^2)(2G_1G_2 - G_3^2) + 4(1+a)G_2(G_6 + G_8) + 4(1-a)G_1(G_6 + G_7) - c_4 y_0^4.$$

It is shown in [3, 5] that the affine intersection $I = \bigcap_1^4 \{\bar{Q}_i = 0, y_0 = 1\}$ completes into an abelian surface A with polarization (2, 12). Associated with this polarization, there is a very ample divisor D on A of geometric genus 25, with 8 normal crossings; thus the smooth version D of $\tilde{D}$ has genus 17. The space of sections $L(\tilde{D})$ is 24-dimensional and contains 10 odd and 14 even sections, showing the period matrix must have the form

$$\begin{pmatrix} 2 & 0 & a & c \\ 0 & 12 & c & b \end{pmatrix}, \qquad \operatorname{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0.$$

This curve D can be viewed as a 16-fold cover of a rational curve $\mathscr{C}$ in two different ways as illustrated in Fig. 8. *On the one hand*, D is a $4-1$ ramified cover of the component $\mathscr{E}' \subseteq \bar{I} \cap \{y_0 = 0\} \subset \mathbb{P}^6$, which is "reached" by the trajectories of the flow. The curve $\mathscr{E}'$ is rational and along there $\bar{I}$ experiences a 4-fold normal crossing with various pinchings. In turn, $\mathscr{E}'$ is a $4-1$ ramified cover of a rational curve $\mathscr{C}$, which will play the same role as the curve of rank 4 quadrics for the polarizations $(1,1)$ and $(1,2)$.

On the other hand, in order to identify $\mathscr{C}$ as a curve of rank 4 quartics, it is useful to consider D as a cover of $\mathscr{C}$ in a different way. Namely D is a $4-1$ unramified cover of a genus 5 curve D_0, itself a $4-1$ ramified cover of $\mathscr{C}$, an intermediate curve being the hyperelliptic curve

$$\mathscr{H}:\quad W^2(\alpha Z+\beta)^2 - 2WP_3(Z) + P_4(Z) = 0,$$

where

$$\begin{aligned} P_4(Z) &= (c_0(1-Z)Z + c_1 Z + c_2(1-Z))^2 - Z(1-Z)((\gamma Z+\delta)c_4 + 4c_1c_2 - \delta\bar{\delta}c_3^2) \\ P_3(Z) &= (\alpha Z+\beta)P_2(Z) - 2Z(1-Z)T_1 \end{aligned} \tag{2}$$

with

$$\begin{aligned} P_2(Z) &= -c_0(1-Z)Z + c_1 Z + c_2(1-Z) \\ c_0 &= c_3(3a^2+1) - c_1 - c_2 \\ T_1 &= \|x'\|^2 = \beta c_1 + \bar{\beta}c_2 - c_3(a^2-1)\delta\bar{\delta} \end{aligned}$$

$$\alpha = 16a^3 \qquad \beta = (a-1)^3(3a+1) \qquad \gamma = 4a \qquad \delta = (a-1)(3a+1);$$

in the formulas above $\bar{\ }$ denotes the involution $a \curvearrowright -a$.

To summarize, D is a $1-4$ unramified cover of the curve D_0 of genus 5, while D_0 is a double cover of the hyperelliptic curve $\mathscr{H}$, ramified at 4 points, where $P_4(Z) = 0$. We are now in a position to make the following statement:

$$\begin{tikzcd}[column sep=small]
\mathrm{Prym}(D_0/\mathscr{H}) \supset A \supset \tilde{D}(g=25) & D(g=17) \arrow[l] \arrow[dl, "\text{ramified}"', "4-1"] \arrow[dr, "4-1\ \text{unramified}"] & \\
\mathscr{E}' = \left\{\begin{matrix} \mathscr{C} \\ X^2 = Z \\ Y^2 = 1-Z \end{matrix}\right\} \arrow[ddr, "\text{ramified}"', "4-1"] & & D_0 = \left\{\begin{matrix} \mathscr{H} \\ W = U^2(1-Z)Z \end{matrix}\right\} \ (g=5) \arrow[d, "2-1"', "\text{ramified at 4 points where } P_4(Z)=0"] \\
 & & \mathscr{H} = \{W^2(\alpha Z+\beta)^2 - 2WP_3(Z) + P_4(Z) = 0\} \ (g=2) \arrow[dl, "2-1"', "\text{ramified at } \infty \text{ and the 5 roots of } P_3^2(Z) - (\alpha Z+\beta)^2 P_4(Z) \equiv Z(1-Z)R_3(Z)"] \\
 & \mathscr{C} = \{W = Z\} &
\end{tikzcd}$$

Fig. 8

Theorem 9. *Given the divisor D and the space of quartics*

$$\bar{V} = V(Q_i Q_j - c_i c_j y_0^4,\ 1 \leqq i \leqq j \leqq 3,\ \bar{Q}_4),$$

the curve $\mathscr{C}$ can be realized as a curve of "quartics of rank 4"

$$\mathscr{C}:\ \sum_1^3 (\alpha_i(Z) y_i + \alpha_{i+3}(Z) y_{i+3})^2 F_i(y) - c(Z) y_0^4, \quad F_i \text{ quadrics} \tag{3}$$

in $\bar{V}$, by substituting Q_i for c_i (given by (1)) in $P_4(Z)$. Conversely, given this curve $\mathscr{C}$ of "quartics of rank 4", there is a $1-4$ map from $\mathscr{C}$ to a component $\mathscr{E}'$ of $\cap\{\bar{Q}_i = 0,\ y_0 = 0\}$, along which the surface $\bar{I} \subset \mathbb{P}^6$ has a four-fold normal crossing with a number of pinch points. Blowing up $\bar{I}$ along $\mathscr{E}'$ turns $\mathscr{E}'$ into the curve D defined by

$$(U^2(1-Z)Z)^2(\alpha Z + \beta)^2 - 2U^2(1-Z) Z P_3(Z) + P_4(Z) = 0, \tag{4}$$
$$X^2 = Z, \qquad Y^2 = 1 - Z$$

where $P_3(Z)$ and $P_4(Z)$ are the polynomials defined in (2).

Proof. Substituting $\bar{Q}_i$ for c_i in $P_4(Z)$ realizes $\mathscr{C}$ as a curve of rank 4 quartics in $\bar{V}$

$$\mathscr{C}:\ \bar{Q}_Z = z_1^2 F_1 + z_2^2 F_2 + z_3^2 F_3 - P_4(Z) z_0^4, \tag{5}$$

where the z_i denote the linear expressions of y appearing in (3) and the F_i quadrics of the usual block form; it is convenient to express the quadrics F_i in terms of the z_i (as done in Appendix 3), although they must be thought of as functions of y_i. The Z-dependent linear map $y \curvearrowright z$ and the quadrics F_i are given in Appendix 3.

In analogy with Theorem 1 and based on the form (5) of $\bar{Q}_Z \in \mathscr{C}$, we define a natural map from $\mathscr{C}$ to $\mathscr{E}' = \cap\{\bar{Q}_i = 0,\ z_0 = 0\}$. The intersection

$$\{\bar{Q}_Z = 0,\ z_0 = 0\} \cap \{\partial \bar{Q}_Z / \partial Z = 0,\ z_0 = 0\}$$

contains the plane[6] $z_0 = z_1 = z_2 = z_3 = 0$. Thus the intersection of the plane with two generic quadrics Q' and Q'' in the span $V(Q_1, Q_2, Q_3)$ will consist of four points dependent on Z. This procedure can easily be carried out by choosing for Q' and Q'' the following two quadrics

$$G_Z = P_3(Z)|_{c_1 = Q_1,\, c_2 = Q_2,\, c_3 = Q_3} \quad \text{with} \quad \bar{G}_Z = G_Z - P_3(Z) z_0^2$$
$$H_Z = x_4^2 + x_5^2 + x_6^2 = (\delta y_1 + \bar{\delta} y_4)^2 + (\bar{\delta} y_2 + \delta y_5)^2 - (\delta y_3 + \bar{\delta} y_6)^2,$$

by performing the change of variables $y \curvearrowright z$ and by observing that

$$G_Z|_{z_1 = z_2 = z_3 = 0} = \left(\frac{2a(a-1)}{3a+1} z_4^2 + \frac{2a(a+1)}{3a-1} z_5^2 + \frac{a^2-1}{2(9a^2-1)} z_6^2\right) Z(Z-1)$$
$$H_Z|_{z_1 = z_2 = z_3 = 0} = 4\gamma^2 \delta^2 (Z-1) z_4^2 + 4\gamma^2 \bar{\delta}^2 Z z_5^2 - \delta^2 \bar{\delta}^2 z_6^2.$$

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[6] Moving with Z

One then verifies that

$$\begin{aligned}\mathscr{E}' &= \{G_Z=0\} \cap \{H_Z=0\} \cap \{z_0=z_1=z_2=z_3=0\} \\ &= \{(z_0^2, z_1^2, \dots, z_6^2) = (0,0,0,0,-\bar{\delta}^2, \delta^2, 4\gamma^2)\} \\ &= \left\{(y_0, y_1, \dots, y_6) = \left(0, \frac{\sqrt{1-Z}}{Z}, \frac{\sqrt{Z}}{1-Z}, -\frac{1}{Z}, \frac{1}{\sqrt{1-Z}}, \frac{1}{\sqrt{Z}}, -\frac{1}{1-Z}\right)\right\}. \end{aligned} \tag{6}$$

This is a $4-1$ cover of $\mathscr{C}$ (Z plane), ramified at $Z=0$ and 1. In exactly the same way as in Theorem 1 one shows that $\bar{I}$ is singular along $\mathscr{E}'$.

In order to establish the converse, we first notice the following relation between the quadrics F_i and G_Z

$$F_i = \mu_i G_Z(0,0,0,z_4,z_5,z_6) + \sum_{j=1}^{3} z_j \ell_{ij}(z_j, z_{j+3}) \tag{7}$$

with μ_i depending on Z and the ℓ_{ij} linear functions of z; this shows in particular that $F_i = 0$ along $\mathscr{E}'$. It is convenient to replace the quartic $\bar{Q}_Z$ by a new quartic $\hat{Q}_Z$, which vanishes as well on $\bar{I}$ and which is defined by

$$\begin{aligned}\hat{Q}_Z &\equiv \bar{Q}_Z - \left(\sum \mu_i z_i^2\right) \bar{G}_Z \\ &= \sum_1^3 z_i^2 \left(\sum_1^3 z_j m_{ij}(z)\right) + \left(\sum_1^3 \mu_i z_i^2\right) P_3(Z) z_0^2 - P_4(Z) z_0^4, \quad \text{using (7)}\end{aligned}$$

with linear functions $m_{ij}(z)$.

Consider now the chart on $\bar{I}$ around most of $\mathscr{E}'$, which is defined by the non-homogeneous equation $y_3 y_6 + y_3 + y_6 = 0$; this is legitimate since that relation vanishes automatically along $\mathscr{E}'$, as seen from (6). As was done in the second half of Theorem 1, we must extend the variable Z to a neighborhood of $\mathscr{E}'$ in $\bar{I}$, by introducing a relation between Z and y, which is satisfied on $\mathscr{E}'$. In view of (6), this is achieved by setting

$$Z = -\frac{1}{y_3} = 1 + \frac{1}{y_6}$$

in the chart $y_3 y_6 + y_3 + y_6 = 0$.

In this chart, $z_3 = 0$, $z_6 = 2\gamma$ and hence $z_3 z_6 = 0$; we now perform a first change of coordinates

$$(y_0, y_1, \dots, y_6) \curvearrowright (y_0, Z, u_1, \dots, u_5)$$

defined by

$$z_i z_{i+3} = u_i y_0 \, (i=1,2), \quad z_4^2 + \bar{\delta}^2 = u_4 y_0, \quad z_5^2 - \delta^2 = u_5 y_0, \quad Z = -\frac{1}{y_3} = 1 + \frac{1}{y_6}.$$

Here, the z_i must be viewed as functions of y_i and Z. Using this change of coordinates in the quadric $\hat{Q}_Z = 0$, adequately prepared [see (1) in Appendix 3], leads to

$$((u_1 - u_2)^2 + O(y_0)) y_0^3 = 0 .$$

Upon dividing by y_0^3 and upon taking the limit $y_0 = 0$, we find $(u_1 - u_2)|_{y_0=0} = 0$.

This conclusion suggests, in a second step, an amended change of coordinates

$$(y_0, y_1, \dots, y_6) \curvearrowright (y_0, Z, u_1, u_2, u_4, u_5)$$

defined in the same affine chart by

$$z_1 z_4 = u_1 y_0 \qquad z_4^2 + \delta^2 - c u_1 y_0 = u_4 y_0^2$$
$$z_2 z_5 - z_1 z_4 = u_2 y_0^2 \qquad z_5^2 - \delta^2 - \bar{c} u_1 y_0 = u_5 y_0^2 \tag{8}$$
$$Z = -\frac{1}{y_3} = 1 + \frac{1}{y_6}.$$

Expressing the quadrics $\bar{Q}_i$ [see (1)] in terms of the z_i and noticing that $\bar{Q}_i$ vanishes along $\mathscr{E}'$, we have that

$$\begin{aligned}\bar{Q}_i &= \bar{Q}_i - \bar{Q}_i|_{(z_0^2,\dots,z_6^2)=(0,\dots,0,-\delta^2,\delta^2,4\gamma^2)}\\ &= \alpha_i(z_4^2-\delta^2)+\beta_i(z_5^2-\delta^2)+\gamma_i(z_6^2-4\gamma^2)+\delta_i z_1 z_4\\ &\quad + \text{linear function }(z_2z_5-z_1z_4,\ z_3z_6,\ z_1^2,z_2^2,z_3^2)-c_i y_0^2=0.\end{aligned} \tag{9}$$

with α_i, β_i, and γ_i rational in Z. Performing the change of coordinates (8), the second line in (9) is divisible by y_0^2, whereas the first line is divisible by y_0; however by choosing c appropriately in (8), the first line has order y_0^2 as well. Dividing by y_0^2, one easily finds

$$M(Z,u_1)(u_2,u_4,u_5)^T = f(u_1,Z) + y_0\, g(u,Z,y_0), \tag{10}$$

where the 3 by 3 matrix $M(Z,u_1)$ has determinant $u_1^3 h(Z)$; so M is invertible for most values of u_1 and Z. Finally, we use the coordinate transformation (8) in $\hat{Q}_Z$

$$\begin{aligned}\hat{Q}_Z|_{z_3=0,z_6=2\gamma} &= R_Z + S_Z\\ &= ((\alpha Z+\beta)^2\tilde{u}_1^2 - 2P_3(Z)\tilde{u}_1 + P_4(Z))y_0^4 + p(u,Z,y_0)y_0^5,\end{aligned} \tag{11}$$

where R_Z and S_Z lead to terms of order y_0^4 and y_0^5 respectively and where

$$\beta\bar{\beta}\,\tilde{u}_1 = \left(\frac{u_1}{\gamma Z+\delta}\right)^2 (1-Z)Z(a^2-1).$$

Dividing (11) by y_0^4 and letting $y_4 \to 0$ in (10) and (11) lead to

$$(\alpha Z+\beta)^2\tilde{u}_1^2 - 2P_3(Z)\tilde{u}_1 + P_4(Z) = 0 \quad \text{and} \quad M(Z,u_1)(u_2,u_4,u_5)^T = f(u_1,Z),$$

showing that u_1, u_2, u_4, and u_5 are all rational functions on the curve announced in (4). At a few places along $\mathscr{E}'$, like $Z=0$ and $Z=1$, the coordinate change breaks down and hence a separate argument must be made, ending the proof of Theorem 9.

Remark. The divisor $\tilde{D}$ on the abelian surface is the same as the curve D, but with 8 normal crossings. The latter cover the point $Z = -\delta/\gamma$, which is also the locus where the quartic $\bar{Q}_Z$, $z_0 = 0$ is a product of two quadrics; the two quadrics happen to be the orbit invariants T_1 and T_2 introduced in the beginning of this section.

Appendix 1

Lemma. (i) *The curve of rank 3 quadrics $\mathscr{C} \subset V \simeq \mathbb{P}^3$ contains a line component $\mathscr{C}'$ if and only if, in appropriate coordinates, V contains the following two quadrics*

$$Q_1 = x_1^2 + x_2^2, \qquad Q_2 = x_2^2 + x_3^2.$$

Then, with regard to Q_1 *and* Q_2,

$$\mathscr{C}' = \{Q_1 + tQ_2 = x_1^2 + (1+t)x_2^2 + tx_3^2, \quad t \in \mathbb{C}\}.$$

(ii) $\mathscr{C}$ *contains a degenerate component* $\mathscr{C}'$, *which is not a line, if and only if in appropriate coordinates,* V *contains the following three quadrics*

either case (A):

$$Q_1 = x_1^2 + x_2^2 + x_3^2,\ Q_2 = x_4^2 + x_5^2 + x_6^2 \quad \textit{and} \quad Q_3 = x_1x_4 + x_2x_5 + x_3x_6$$

or case (B):

$$Q_1 = x_1^2 + x_2^2 + x_3^2,\ Q_2 = x_4^2 + x_5^2 + x_3^2 \quad \textit{and} \quad Q_3 = x_1x_4 + x_2x_5 + a_3x_3^2.$$

Then, with regard to Q_1, Q_2, Q_3

$$\mathscr{C}' = \left\{t^2Q_1 + Q_2 + 2tQ_3 = \sum_1^3 (tx_i + x_{i+3})^2,\ t \in \mathbb{C}\right\} \subset \mathscr{C} \qquad \textit{in case } A$$

$$= \left\{t^2Q_1 + Q_2 + 2tQ_3 = \sum_1^2 (tx_i + x_{i+3})^2 + (t^2 + a_3t + 1)x_3^2, t \in \mathbb{C}\right\} \subset \mathscr{C} \qquad \textit{in case } B.$$

Proof. (i) Let $\mathscr{C}'$ be a line component of $\mathscr{C}$. Since V satisfies condition C2, any point of $\mathscr{C}'$ has the form $x_1^2 + x_2^2 + \gamma_3^2x_3^2$ (with γ_3 possibly 0), maybe after a linear change of coordinates. Any other point on $\mathscr{C}'$ must have the form $\sum_1^3 (\alpha_ix_i + \alpha_{i+3}x_{i+3})^2$; so

$$\mathscr{C}' = \{s(x_1^2 + x_2^2 + \gamma_3^2x_3^2) + \sum_1^3 (\alpha_ix_i + \alpha_{i+3}x_{i+3})^2 = \text{sum of 3 squares, for all } s \in \mathbb{C}\}.$$

For this quadric to be a sum of 3 squares, $\alpha_{i+3} = 0$ $(i = 1, 2, 3)$ must hold. Then, using condition C2 once more, one checks, upon some rescaling, that

$$\mathscr{C}' = \text{linear span of the quadrics } x_1^2 + x_2^2 \text{ and } x_2^2 + x_3^2,$$

concluding the proof of (i).

(ii) Next consider the case of $\mathscr{C}$ containing a degenerate component $\mathscr{C}'$, being different from a line. So $\mathscr{C}' \subset$ two dimensional plane $\subset V$. All quadrics in $\mathscr{C}$ have rank 2 or 3, but never 1 by condition C2. This rank must be generically 3, for otherwise the curve $\mathscr{C}'$ would reduce to the vertex of one of the cones K_i, which is impossible, as $\mathscr{C}'$ is a curve. Therefore we may pick one quadric

$$Q_1 = x_1^2 + x_2^2 + x_3^2.$$

Besides, not all remaining quadrics [modulo permutations of (1, 4), (2, 5), (3, 6)] on $\mathscr{C}'$ have the form

$$(\alpha_1x_1 + \alpha_4x_4)^2 + \alpha_2^2x_2^2 + \alpha_3^2x_3^2; \tag{1}$$

indeed, if all quadrics in $\mathscr{C}'$ would have that form, then since $\mathscr{C}'$ lies in a plane and since $\mathscr{C}'$ is not a line, there would be three independent quadrics of the form (1); therefore their linear span would contain a quadric $\beta_1^2x_1^2 + \beta_4^2x_4^2 + 2\beta_{14}x_1x_4$, violating condition C2. Therefore, the remaining quadrics in $\mathscr{C}'$ must generically have the form

$$\text{either case A: } Q_2 = x_4^2 + x_5^2 + x_6^2 \quad \text{or case B: } Q_2 = x_4^2 + x_5^2 + x_3^2.$$

Hence let us pick Q_2 as in case A. Then the plane containing $\mathscr{C}'$ must be spanned by the following three quadrics

$$Q_1 = x_1^2 + x_2^2 + x_3^2$$
$$Q_2 = x_4^2 + x_5^2 + x_6^2$$
$$Q_3 = 2x_1x_4 + \sum_2^3 (\alpha_i^2 x_i^2 + \alpha_{i+3}^2 x_{i+3}^2 + 2\alpha_{i,i+3} x_i x_{i+3}),$$

and expressing that the linear span $XQ_1 + YQ_2 + ZQ_3$ contains a curve of rank 3 quadrics amounts to the relations

$$XY = Z^2$$
$$(X + \alpha_i^2 Z)(Y + \alpha_{i+3}^2 Z) = \alpha_{i,i+3}^2 Z^2, \quad i = 2, 3;$$

for this to define a curve, you must have $\alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = 0$, $\alpha_{25}^2 = \alpha_{36}^2 = 1$, and hence, perhaps after flipping some signs, Q_3 takes on the form announced in (ii). Finally, picking Q_2 as in case B leads to the other normal form in (ii), thus ending the proof of the Lemma.

The corollary below refers to the space $\bar{V}$ of quadrics of the form $\bar{Q}_i \equiv Q_i - c_i x_0^2$.

Corollary. *Whenever $\mathscr{C}$ contains a degenerate component $\mathscr{C}'$ which is not a line, then for generic c_i, $\bar{V}$ contains the following three quadrics*

case (A)

$$\bar{Q}_i = x_1^2 + x_2^2 + x_3^2, \bar{Q}_2 = x_4^2 + x_5^2 + x_6^2 \quad \textit{and} \quad \bar{Q}_3 = x_1x_4 + x_2x_5 + x_3x_6 - x_0^2$$

case (B)

$$\bar{Q}_1 = x_1^2 + x_2^2 + x_3^2, \bar{Q}_2 = x_4^2 + x_5^2 + x_3^2 \quad \textit{and} \quad \bar{Q}_3 = x_1x_4 + x_2x_5 + a_3x_3^2 - x_0^2$$

where $\bar{Q}_1$ and $\bar{Q}_2$ do not contain any x_0^2 term.

Proof. As both cases are similar, we prove the theorem for case (A) only. From the Lemma above, $\bar{Q}_i$ can be picked as

$$\bar{Q}_1 = x_1^2 + x_2^2 + x_3^2 - c_1x_0^2,$$
$$\bar{Q}_2 = x_4^2 + x_5^2 + x_6^2 - c_2x_0^2$$

and

$$\bar{Q}_3 = x_1x_4 + x_2x_5 + x_3x_6 - c_3x_0^2.$$

Then the curve $\mathscr{C}'$ in the hyperplane $V = V(\bar{Q}_1, \bar{Q}_2, \bar{Q}_3) \subseteq \bar{V}$ has the form

$$\mathscr{C}' = \left\{ t^2\bar{Q}_1 + \bar{Q}_2 + 2t\bar{Q}_3 = \sum_1^3 (tx_i + x_{i+3})^2 - (t^2c_1 + 2tc_3 + c_2)x_0^2, \quad t \in \mathbb{C} \right\};$$

for generic c_i, the coefficient of x_0^2 vanishes for distinct t_i. Picking the corresponding points in $\mathscr{C}'$ leads to $\bar{Q}_1$ and $\bar{Q}_2$ as promised and then repeating the argument in the above lemma and rescaling x_0 leads to $\bar{Q}_3$ as announced in the corollary.

Appendix 2

Normal Forms for a Basis V

The purpose of this section is to prove Lemma 1 of Sect. 3. We now list the nine normal forms, which are classified according to the position in $\mathbb{P}^3$ of the vertices $(\alpha_i^2\beta_{i,i+3}-\beta_i^2\alpha_{i,i+3}, \alpha_{i+3}^2\beta_{i,i+3}-\beta_{i+3}^2 a_{i,i+3}, -\beta_{i,i+3}, \alpha_{i,i+3})$ of the cones K_i.

The proof of Lemma 1 depends crucially on the following:

Lemma. *Let the linear span V with basis Q_i, $i=1,\dots,4$ of Sect. 1 have the property that every pair of columns in the matrix*

$$\begin{pmatrix} \alpha_{14} & \alpha_{25} & \alpha_2^2-\alpha_1^2 \\ \beta_{14} & \beta_{25} & \beta_2^2-\beta_1^2 \end{pmatrix} \tag{1}$$

is independent. Then and only then after a linear change of the basis, V has a basis of the form NF1.

Proof of Lemma. By condition C1 of Sect. 1, we may, after a linear change of basis, assume $\alpha_{14}, \beta_{14}, \alpha_{25}$, and $\beta_{25} \neq 0$. Then in order to find Q_3 and Q_4 of NF1, we find (X, Y, Z, U) and $(X', Y', Z', U') \in K_1 \cap K_2$ with $Z \neq 0$ and $U' \neq 0$ and in addition satisfying

$$\begin{aligned} &\text{(a)} \quad (X+\alpha_1^2Z+\beta_1^2U)(Y+\alpha_4^2Z+\beta_4^2U)=(Z\alpha_{14}+U\beta_{14})^2=0 \\ &\qquad \text{with} \quad X+\alpha_1^2Z+\beta_1^2U=0, \\ &\text{(b)} \quad (X+\alpha_2^2Z+\beta_2^2U)(Y+\alpha_5^2Z+\beta_5^2U)=(Z\alpha_{25}+U\beta_{25})^2 \neq 0, \end{aligned} \tag{2}$$

and

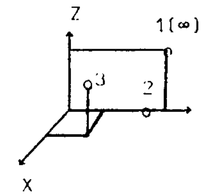
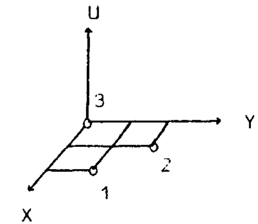
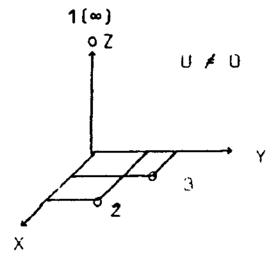
$$\begin{aligned} &\text{(a)} \quad (X'+\alpha_1^2Z'+\beta_1^2U')(Y'+\alpha_4^2Z'+\beta_4^2U')=(Z'\alpha_{14}+U'\beta_{14})^2 \neq 0, \\ &\text{(b)} \quad (X'+\alpha_2^2Z'+\beta_2^2U')(Y'+\alpha_5^2Z'+\beta_5^2U')=(Z'\alpha_{25}+U'\beta_{25})^2=0 \\ &\qquad \text{with} \quad X'+\alpha_2^2Z'+\beta_2^2U'=0. \end{aligned} \tag{3}$$

Indeed, system (2) has a solution with $Z \neq 0$: at first solve (a) for U and X in terms of Z and then (b) is solvable in Y by the independence of the first and last columns in (1). Similarly solve (3) by using the independence of the last two columns. The independence of the first two columns of (1) assures the inequality in (2) and (3). Then the new quadrics $XQ_1+YQ_2+ZQ_3+UQ_4$ and $X'Q_1+Y'Q_2+Z'Q_3+U'Q_4$ have respectively the form

$$a_4^2x_4^2+(a_2x_2+a_5x_5)^2+\text{ term containing } x_3, x_6, \quad a_2, a_5 \neq 0,$$

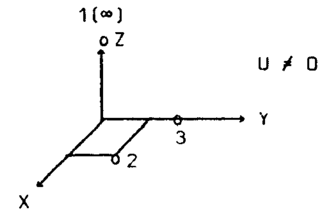
$$(b_1x_1+b_4x_4)^2+b_5^2x_5^2+\text{ term containing } x_3, x_6, \quad b_1, b_4 \neq 0.$$

These quadrics, together with Q_1 and Q_2 span V. The proof of the Lemma is finished by noticing that if V admits the basis NF1, then any other basis with Q_1 and Q_2 as before satisfies the condition placed on matrix (1).

	$Q_1 = x_1^2 + x_2^2 + x_3^2, \quad Q_2 = x_4^2 + x_5^2 + x_6^2$	Inequalities all $\alpha_{i,i+3}, \beta_{i,i+3} \neq 0$	Position of the Vertices 1,2,3 of the cones K_1, K_2, K_3
NF1 :	$Q_3 = \alpha_4^2 x_4^2 + (x_2 + \alpha_5 x_5)^2$ $+ (\alpha_3^2 x_3^2 + \alpha_6^2 x_6^2 + 2\alpha_{36} x_3 x_6)$ $Q_4 = (x_1 + \beta_4 x_4)^2 + \beta_5^2 x_5^2$ $+ (\beta_3^2 x_3^2 + \beta_6^2 x_6^2 + 2\beta_{36} x_3 x_6)$	$\alpha_5, \beta_4 \neq 0$	 $U \neq 0$
NF2 :	$Q_3 = (\alpha_1^2 x_1^2 + \alpha_4^2 x_4^2) + (\alpha_2^2 x_2^2 + \alpha_5^2 x_5^2)$ $Q_4 = (\beta_1^2 x_1^2 + \beta_4^2 x_4^2 + 2\beta_{14} x_1 x_4)$ $+ (\beta_5^2 x_5^2 + 2\beta_{25} x_2 x_5) + 2\beta_{36} x_3 x_6$	$\alpha_1 \neq 0,\ \alpha_2 \neq 0$ (a) $\alpha_1^2 \neq \alpha_2^2,\ \alpha_4^2 \neq \alpha_5^2$, all $\alpha_i \neq 0$, or (b) $\alpha_1^2 \neq \alpha_2^2,\ \alpha_5 = 0$, α_1, α_2 and $\alpha_4 \neq 0$, or (c) $\alpha_1^2 \neq \alpha_2^2,\ \alpha_4 = \alpha_5 = 0$, α_1 and $\alpha_2 \neq 0$,	$Z \neq 0$ 
NF3 :	$Q_3 = 2\alpha_{25} x_2 x_5 + 2\alpha_{36} x_3 x_6$ $Q_4 = 2\beta_{14} x_1 x_4 + (\beta_2^2 x_2^2 + \beta_5^2 x_5^2)$ $+ (\beta_3^2 x_3^2 + \beta_6^2 x_6^2)$	$\beta_2^2 \neq \beta_3^2$ or $\beta_5^2 \neq \beta_6^2$	 $U \neq 0$

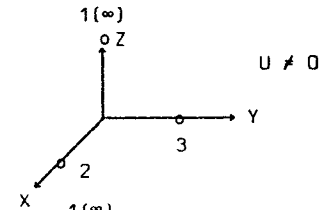
NF4 : $Q_3 = 2\alpha_{25}x_2x_5 + (\alpha_3^2x_3^2 + 2\alpha_{36}x_3x_6)$ $\qquad \alpha_3 \neq 0$

$Q_4 = 2\beta_{14}x_1x_4 + (\beta_2^2x_2^2 + \beta_5^2x_5^2) + \beta_6^2x_6^2$ $\qquad \beta_2 \neq 0$ or $\beta_5^2 \neq \beta_6^2$



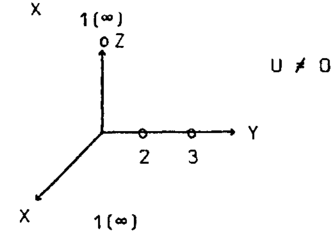
NF5 : $Q_3 = (2\alpha_{25}x_2x_5 + \alpha_5^2x_5^2) + (\alpha_3^2x_3^2 + 2\alpha_{36}x_3x_6)$ $\qquad \alpha_3, \alpha_5 \neq 0$

$Q_4 = 2\beta_{14}x_1x_4 + \beta_2^2x_2^2 + \beta_6^2x_6^2$ $\qquad \beta_2 \neq 0$ or $\beta_6^2 \neq 0$



NF6 : $Q_3 = \alpha_2^2x_2^2 + 2\alpha_{25}x_2x_5 + \alpha_3^2x_3^2 + 2\alpha_{36}x_3x_6$ $\qquad$ with $\alpha_2, \alpha_3 \neq 0$

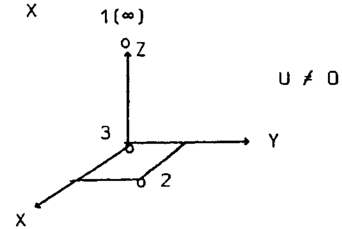
$Q_4 = 2\beta_{14}x_1x_4 + \beta_5^2x_5^2 + \beta_6^2x_6^2$ $\qquad$ with $\beta_5^2 \neq \beta_6^2$



NF7 : $Q_3 = 2\alpha_{25}x_2x_5 + \alpha_3^2x_3^2 + 2\alpha_{36}x_3x_6 + \alpha_6^2x_6^2$ $\qquad$ with $\alpha_3, \alpha_6 \neq 0$

$Q_4 = 2\beta_{14}x_1x_4 + \beta_2^2x_2^2 + \beta_5^2x_5^2$ $\qquad$ with $\beta_2 \neq 0$

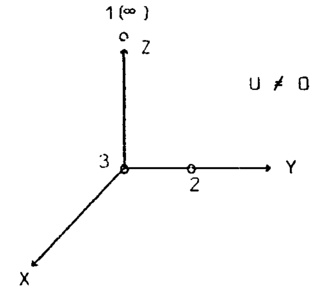
or $\beta_5 \neq 0$



NF8 : $Q_3 = \alpha_2^2 x_2^2 + 2\alpha_{25} x_2 x_5 + \alpha_3^2 x_3^2 + 2\alpha_{36} x_3 x_6 + \alpha_6^2 x_6^2$

$Q_4 = 2\alpha_{14} x_1 x_4 + \beta_5^2 x_5^2$

with $\alpha_2, \alpha_3, \alpha_6 \neq 0$



NF9 : $Q_3 = \alpha_5^2 x_5^2 + 2\alpha_{25} x_2 x_5 + \alpha_3^2 x_3^2 + \alpha_6^2 x_6^2 + 2\alpha_{36} x_3 x_6$

$Q_4 = \beta_4^2 x_4^2 + 2\beta_{14} x_1 x_4 + \beta_2^2 x_2^2 + \beta_5^2 x_5^2 + 2\beta_{25} x_2 x_5$

with

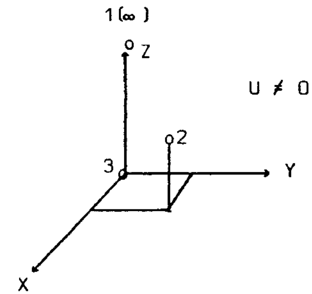
a) α_5^2 and $\beta_4^2 = 0$

or

b) α_5^2 and $\beta_4^2 \neq 0$

and

α_6^2 and $\beta_5^2 = 0$



Proof of Lemma 1. If the condition of the Lemma or some permutation thereof is satisfied, then V admits a basis NF1 permuted accordingly. Therefore an obstruction will occur whenever every one of the six matrices $(1 \leqq i < j \leqq 3)$

$$M_{ij} = \begin{pmatrix} \alpha_{i,i+3} & \alpha_{j,j+3} & \alpha_j^2 - \alpha_i^2 \\ \beta_{i,i+3} & \beta_{j,j+3} & \beta_j^2 - \beta_i^2 \end{pmatrix}, \quad M'_{ij} = \begin{pmatrix} \alpha_{i,i+3} & \alpha_{j,j+3} & \alpha_{j+3}^2 - \alpha_{i+3}^2 \\ \beta_{i,i+3} & \beta_{j,j+3} & \beta_{j+3}^2 - \beta_{i+3}^2 \end{pmatrix}$$

admits a pair of dependent columns. All the different cases of this fact will now be investigated, leading to the exceptional forms NF2, ..., NF9.

Case 1. Assume the first two columns are dependent in at least three of the six matrices; then simple inspection shows ($\sim$ means proportional)

$$(\alpha_{14}, \alpha_{25}, \alpha_{36}) \sim (\beta_{14}, \beta_{25}, \beta_{36}),$$

in which case every matrix M_{ij} has already two proportional columns. The proportionality relation shows that Q_3 can be replaced by an appropriate linear combination of Q_3 and Q_4 with no cross-terms, while x_3^2 and x_6^2 can be removed by subtracting an appropriate linear combination of Q_1 and Q_2, leading to the quadric Q_3 appearing in NF2.

This case breaks up further into three cases.

(a) $Q_3 = \alpha_1^2 x_1^2 + \alpha_4^2 x_4^2 + \alpha_2^2 x_2^2 + \alpha_5^2 x_5^2$ with all $\alpha_i \neq 0, \alpha_1^2 \neq \alpha_2^2, \alpha_4^2 \neq \alpha_5^2$,

(b) $Q_3 = \alpha_1^2 x_1^2 + \alpha_4^2 x_4^2 + \alpha_2^2 x_2^2$ with all $\alpha_i \neq 0, \alpha_1^2 \neq \alpha_2^2$,

(c) $Q_3 = \alpha_1^2 x_1^2 + \alpha_2^2 x_2^2$ with all $\alpha_i \neq 0, \alpha_1^2 \neq \alpha_2^2$,

and Q_4 as before. Indeed in (c), not satisfying the inequalities would violate condition C2; in (b), $\alpha_2 = 0$ would violate C2 and $\alpha_4 = 0$ would imply (c); $\alpha_1 \neq 0$ can always be made to hold, because if $\alpha_1 = 0$, $Q_3 - \alpha_4^2 Q_2$ would have the coefficient of x_5^2 non-zero, which could be renamed into x_1^2 via the relabeling $(1,4,2,5,3,6) \curvearrowright (6,3,4,1,5,2)$ which puts us back into case (b); and $\alpha_1^2 = \alpha_2^2$ leads back to the case $\alpha_1 = 0$, just considered, by relabeling. Finally, violating any of the inequalities (a) leads back to (b) or (c). Moreover, in Q_4 the terms x_3^2 and x_6^2 can be removed by forming $Q_4 - \beta_3^2 Q_1 - \beta_6^2 Q_2$. The three cases above can be summarized by the conditions going with NF2.

Case 2. The first columns are dependent in precisely two of the six matrices, while in the other four, the dependence involves the last column. The only case to be considered is the one where M_{ij} and M'_{ij} both have proportional first columns, for some choice of i, j, since any other case would imply (Case 1) $(\alpha_{14}, \alpha_{25}, \alpha_{36}) \sim (\beta_{14}, \beta_{25}, \beta_{36})$. In view of the relabelings it can be assumed without loss of generality that $(\alpha_{25}, \alpha_{36})$ is proportional to (β_{25}, β_{36}) in the matrices M_{23} and M'_{23}. Then in each of the remaining four matrices, the last column is a multiple of the column $(\alpha_{14}, \beta_{14})^T$ or $(\alpha_{25}, \beta_{25})^T$; in short, the four columns whose lead entries are $(\alpha_2^2 - \alpha_1^2, \alpha_3^2 - \alpha_1^2, \alpha_5^2 - \alpha_4^2, \alpha_6^2 - \alpha_4^2)$ are proportional to columns with lead entries α_{14} or α_{25}; we list all alternatives for the lead entries of the columns (up to relabeling):

$$(\alpha_{14}, \alpha_{14}, \alpha_{14}, \alpha_{14}), \quad (\alpha_{14}, \alpha_{14}, \alpha_{14}, \alpha_{25}), \quad (\alpha_{14}, \alpha_{14}, \alpha_{25}, \alpha_{25}),$$

$$(\alpha_{14}, \alpha_{25}, \alpha_{14}, \alpha_{25}), \quad (\alpha_{14}, \alpha_{25}, \alpha_{25}, \alpha_{14}), \quad (\alpha_{25}, \alpha_{25}, \alpha_{25}, \alpha_{14}).$$

To get the above list, rule out the case $(\alpha_{25}, \alpha_{25}, \alpha_{25}, \alpha_{25})$ by condition C2 and keep in mind $(\alpha_{25}, \alpha_{36}) \sim (\beta_{25}, \beta_{36})$. After taking appropriate linear combinations of the basis of V, the following choices for the pair Q_3 and Q_4 remain, corresponding to the above ordering:

$$\text{(i)} \qquad \begin{cases} Q_3 = 2\alpha_{25}x_2x_5 + 2\alpha_{36}x_3x_6 \\ Q_4 = 2\beta_{14}x_1x_4 + \beta_2^2x_2^2 + \beta_5^2x_5^2 + \beta_3^2x_3^2 + \beta_6^2x_6^2, \end{cases}$$

$$\text{(ii)} \qquad \begin{cases} Q_3 = 2\alpha_{25}x_2x_5 + 2\alpha_{36}x_3x_6 + \alpha_6^2x_6^2, \\ Q_4 = 2\beta_{14}x_1x_4 + \beta_2^2x_2^2 + \beta_5^2x_5^2 + \beta_3^2x_3^2, \end{cases}$$

$$\text{(iii)} \qquad \begin{cases} Q_3 = 2\alpha_{25}x_2x_5 + \alpha_5^2x_5^2 + 2\alpha_{36}x_3x_6 + \alpha_6^2x_6^2, \\ Q_4 = 2\beta_{14}x_1x_4 + \beta_2^2x_2^2 + \beta_3^2x_3^2, \end{cases}$$

$$\text{(iv)} \qquad \begin{cases} Q_3 = 2\alpha_{25}x_2x_5 + \alpha_3^2x_3^2 + 2\alpha_{36}x_3x_6 + \alpha_6^2x_6^2, \\ Q_4 = 2\beta_{14}x_1x_4 + \beta_2^2x_2^2 + \beta_5^2x_5^2, \end{cases}$$

$$\text{(v)} \qquad \begin{cases} Q_3 = 2\alpha_{25}x_2x_5 + \alpha_5^2x_5 + \alpha_3^2x_3 + 2\alpha_{36}x_3x_6 \\ Q_4 = 2\beta_{14}x_1x_4 + \beta_2^2x_2^2 + \beta_6^2x_6^2, \end{cases}$$

$$\text{(vi)} \qquad \begin{cases} Q_3 = \alpha_2^2x_2^2 + \alpha_5^2x_5^2 + 2\alpha_{25}x_2x_5 + \alpha_3^2x_3^2 + 2\alpha_{36}x_3x_6, \\ Q_4 = 2\beta_{14}x_1x_4 + \beta_6^2x_6^2, \end{cases}$$

with all α_{ij} and β_{ij} nonzero by condition C1. We indicate the argument in the first case only, where $(k = \alpha_{14}/\beta_{14})$

$$(\alpha_2^2 - \alpha_1^2, \alpha_3^2 - \alpha_1^2, \alpha_5^2 - \alpha_4^2, \alpha_6^2 - \alpha_4^2, \alpha_{14}^2) = k(\beta_2^2 - \beta_1^2, \beta_3^2 - \beta_1^2, \beta_5^2 - \beta_4^2, \beta_6^2 - \beta_4^2, \beta_{14}^2)$$

and

$$(\alpha_{25}, \alpha_{36}) = k'(\beta_{25}, \beta_{36}),$$

with $k, k' \neq 0$, and $k \neq k'$, to insure independence of Q_3 and Q_4. Then by subtracting appropriate combinations of Q_1 and Q_2 from Q_3 and Q_4, we may assume α_1, α_4, β_1 and $\beta_4 = 0$; then replace

$$\begin{aligned} Q_3 \curvearrowright Q_3 - kQ_4 &= 2(\alpha_{25} - k\beta_{25})x_2x_5 + 2(\alpha_{36} - k\beta_{36})x_3x_6, \\ Q_4 \curvearrowright Q_3 - k'Q_4 &= 2(\alpha_{14} - k'\beta_{14})x_1x_4 + (\alpha_2^2 - k'\beta_2^2)x_2^2 + (\alpha_5^2 - k'\beta_5^2)x_5^2 \\ &\quad + (\alpha_3^2 - k'\beta_3^2)x_3^2 + (\alpha_6^2 - k'\beta_6^2)x_6^2, \end{aligned}$$

and relabel. Since $k \neq k'$, these Q_3 and Q_4 along with Q_1 and Q_2 form a basis of V and by C1, the new α_{ij} and β_{ij} do not vanish; this is case (i) with $\beta_2^2 \neq \beta_3^2$ or $\beta_5^2 \neq \beta_6^2$, for otherwise $Q_4 - \beta_2^2 Q_1 - \beta_5^2 Q_2$ would violate condition C2. This is precisely form NF3. Similar arguments lead to (ii), (iii), (iv), (v) and (vi).

Case (ii) can be renamed into NF4 with $\alpha_3 \neq 0$, otherwise it reduces to a special case of NF3. Case (iii) can be renamed into NF6 with $\alpha_2 \neq 0$, otherwise it reduces to a special case of NF4; also, if $\alpha_3 = 0$ would hold, the permutation $2 \leftrightarrow 3$, $5 \leftrightarrow 6$ would turn it into a special case of NF4. Case (iv) is precisely NF7, where $\alpha_3 \neq 0$, for otherwise the permutation $1 \leftrightarrow 4$, $2 \leftrightarrow 5$, $3 \leftrightarrow 6$ would turn it into NF4; also $\alpha_6 \neq 0$ so this is not a special case of NF4. Case (v) is precisely NF5, where $\alpha_5 \neq 0$ since otherwise it is a special case of NF4 and also $\alpha_3 \neq 0$, for otherwise the permutation $3 \leftrightarrow 5$, $2 \leftrightarrow 6$, $1 \leftrightarrow 4$ would turn it into NF4. Finally, the permutation $5 \leftrightarrow 6$, $2 \leftrightarrow 3$ turns (vi) into NF8, with $\alpha_2 \neq 0$, since otherwise it is a special case of NF7; also

$\alpha_6 \neq 0$ or else this is a special case of NF6 and finally $\alpha_3 \neq 0$ or otherwise this is a special case of NF5 under the $2 \leftrightarrow 5$, $3 \leftrightarrow 6$ permutation. The inequalities on the β's in NF3 to NF8 are present to satisfy condition C2.

Case 3. In each matrix M_{ij}, the last column is proportional to one of the remaining columns. Most of the cases violate condition C2 or the independence of the quadrics or they can be reduced to another one by renaming, so that only two cases remain, namely:

$$\text{(a)}\qquad \begin{aligned}
&(\alpha_{14}, \alpha_2^2-\alpha_1^2, \alpha_5^2-\alpha_4^2) \sim (\beta_{14}, \beta_2^2-\beta_1^2, \beta_5^2-\beta_4^2),\\
&(\alpha_{36}, \alpha_3^2-\alpha_1^2, \alpha_6^2-\alpha_4^2) \sim (\beta_{36}, \beta_3^2-\beta_1^2, \beta_6^2-\beta_4^2),\\
&(\alpha_{25}, \alpha_3^2-\alpha_2^2, \alpha_6^2-\alpha_5^2) \sim (\beta_{25}, \beta_3^2-\beta_2^2, \beta_6^2-\beta_5^2),
\end{aligned}$$

$$\text{or (b)}\qquad \begin{aligned}
&(\alpha_{14}, \alpha_2^2-\alpha_1^2, \alpha_6^2-\alpha_4^2) \sim (\beta_{14}, \beta_2^2-\beta_1^2, \beta_6^2-\beta_4^2),\\
&(\alpha_{25}, \alpha_3^2-\alpha_2^2, \alpha_5^2-\alpha_4^2) \sim (\beta_{25}, \beta_3^2-\beta_2^2, \beta_5^2-\beta_4^2),\\
&(\alpha_{36}, \alpha_3^2-\alpha_1^2, \alpha_6^2-\alpha_5^2) \sim (\beta_{36}, \beta_3^2-\beta_1^2, \beta_6^2-\beta_5^2).
\end{aligned}$$

These cases amount to normal forms NF9, a) and b), concluding the proof of the Lemma.

Appendix 3

The linear change of coordinates $x \curvearrowright y$, defined by

$$\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = i \begin{pmatrix} a-1 & -1 \\ 3a+1 & 1 \end{pmatrix} \begin{pmatrix} (a-1)y_1 \\ (3a-1)(a+1)y_4 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ x_5 \end{pmatrix} = -i \begin{pmatrix} a+1 & -1 \\ 3a-1 & 1 \end{pmatrix} \begin{pmatrix} (a+1)y_2 \\ (3a+1)(a-1)y_5 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ x_6 \end{pmatrix} = - \begin{pmatrix} a-1 & a+1 \\ 3a+1 & 3a-1 \end{pmatrix} \begin{pmatrix} (a-1)y_3 \\ (a+1)y_6 \end{pmatrix},$$

transforms the geodesic flow of Sect 8 into a new system, whose constants of motion are given by (1); the quadratic expressions $G_1, \ldots, G_8$ are given by

$$G_1 = y_4^2 - y_2 y_5 \qquad G_4 = \frac{-2}{1-3a}(y_2 y_3 - y_5 y_6)$$

$$G_3 = y_1 y_2 - y_4 y_5 \qquad G_6 = y_1 y_4 + y_2 y_5 - y_3 y_6$$

$$G_2 = y_5^2 - y_1 y_4 \qquad G_5 = \frac{-2}{1+3a}(y_1 y_6 - y_4 y_3)$$

$$G_7 = y_1^2 - y_3^2 + y_1 y_4$$

$$G_8 = y_2^2 - y_6^2 + y_2 y_5 .$$

Consider now the linear functions $z_1,\dots,z_6$ of $y_1,\dots,y_6$, dependent on the parameter Z

$$\begin{aligned}
&\tilde{z}_1 = Zy_1+(Z-1)y_4 \quad && \tilde{z}_4=(a-1)\tilde{z}_1+\bar{\delta}y_4\\
&\tilde{z}_2 = (Z-1)y_2+Zy_5 \quad && \tilde{z}_5=(a+1)\tilde{z}_2+\delta y_5\\
&\tilde{z}_3 = Zy_3+(Z-1)y_6 \quad && \tilde{z}_6=\delta\tilde{z}_3+8a(Z-1)y_6
\end{aligned}$$

$$(z_1,z_2,z_3)=(\gamma Z+\delta)((Z-1)^{-1/2}\tilde{z}_1,\, Z^{-1/2}\tilde{z}_2,\tilde{z}_3)$$

$$(z_4,z_5,z_6)=((Z-1)^{1/2}\tilde{z}_4,\, Z^{1/2}\tilde{z}_5,\tilde{z}_6)\,.$$

Here and in the sequel, we use α, β, γ, δ defined in (2) of Sect. 8 and $\varepsilon\equiv(a-1)(3a-1)$; the involution $\bar{\ }$ denotes the sign flip $a \curvearrowright -a$. Then referring to Sect. 8, compute

$$G_Z(0,0,0,z_4,z_5,z_6)=Z(Z-1)\left(2a\varepsilon z_4^2+2a\bar{\varepsilon}z_5^2+\frac{a^2-1}{2}z_6^2\right)$$

and

$$\begin{aligned}
&G_Z(z)-G_Z(0,0,0,z_4,z_5,z_6)\\
&\quad = z_1g_1(z)+z_2g_2(z)+z_3g_3(z)\\
&\quad \equiv Z(1-Z)((a+1)\bar{\delta}z_1z_4+(a-1)\delta z_2z_5+4a^2z_3z_6)\\
&\qquad +\frac{Z(1-Z)(1-9a^2)}{2(\gamma Z+\delta)^2}(4a\bar{\delta}(Z-1)z_1^2+4a\delta Z^2z_2^2+\delta\bar{\delta}z_3^2)\,.
\end{aligned}$$

Moreover

$$\begin{aligned}
F_1 &\equiv \frac{2Z(Z-1)}{(\gamma Z+\delta)}\bar{\delta}G_Z\bigg|_{z_1=z_2=z_3=0}\\
&\quad -\frac{Z^2(1-Z)^2(3a-1)}{(\gamma Z+\delta)}(8a^2\bar{\varepsilon}z_2z_5+2(a-1)^2(a+1)z_3z_6-(a-1)^3z_3^2)\\
F_2 &\equiv \frac{2Z(Z-1)}{(\gamma Z+\delta)}\delta G_Z\bigg|_{z_1=z_2=z_3=0}\\
&\quad -\frac{Z^2(1-Z)^2(3a+1)}{(\gamma Z+\delta)}(8a\varepsilon z_1z_4+2(a+1)^2(a-1)z_3z_6-4a^3(3a-1)z_1^2)\\
F_3 &\equiv \frac{8Z(Z-1)}{(\gamma Z+\delta)}aG_Z\bigg|_{z_1=z_2=z_3=0}\\
&\quad -\frac{Z^2(1-Z)^2}{(\gamma Z+\delta)}(8a(a-1)\varepsilon z_1z_4+8(a+1)\bar{\varepsilon}z_2z_5-(a+1)^2\bar{\varepsilon}z_2^2)\,.
\end{aligned}$$

Making occasional use of the substitutions

$$\bar{\delta}^2=(z_4^2+\bar{\delta}^2)-z_4^2 \quad\text{and}\quad \delta^2=(z_5^2+\delta^2)-z_5^2\,,$$

the quartic $\hat{Q}_Z$ evaluated at $z_3=0$ and $z_6=2\gamma$ turns into

$$\hat{Q}_Z|_{z_3=0,\,z_6=2\gamma}\equiv R+S \tag{1}$$

with

$$\begin{aligned} z_4^4 z_5^4 R \equiv\, & 2z_4^2 z_5^4 (z_1 z_4)^2 (z_4^2+\bar{\delta}^2)((a+1)z_1 z_4-(3a+1)z_2 z_5) \\ & + 2z_4^4 z_5^2 (z_2 z_5)^2 (z_5^2-\delta^2)((3a-1)z_1 z_4-(a+1)z_2 z_5) + \frac{4a(1-9a^2)}{(\gamma Z+\delta)^2} \\ & \cdot \left(\begin{matrix} (\bar{\delta}(z_1 z_4)^2 z_5^2+\delta(z_2 z_5)^2 z_4^2)(\delta(1-Z)^2 (z_1 z_4)^2 z_5^2+\bar{\delta} Z^2 (z_2 z_5)^2 z_4^2) \\ -a^2(\gamma Z+\delta)^2 z_4^2 z_5^2 (z_1 z_4 z_2 z_5)^2 \end{matrix} \right) \\ & - 2z_4^2 z_5^2 (\bar{\delta}(z_1 z_4)^2 z_5^2+\delta(z_2 z_5)^2 z_4^2)\frac{P_3(Z)}{Z(1-Z)} z_0^2 - z_4^4 z_5^4 (\gamma Z+\delta)\frac{P_4(Z)}{Z^2(1-Z)^2} z_0^4 \end{aligned}$$

and

$$S \equiv -((1+a)z_1 z_4+(1-a)z_2 z_5)(z_1 z_4-z_2 z_5)^2 .$$

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