Completely integrable systems and symplectic actions[a)]

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We study results on a class of completely integrable systems, for instance, with Hamiltonian $H(x,y)=(1/2)\sum_{i=1}^n y_i^2+\sum_{i<j}(x_i-y_j)^{-2}+\alpha\sum_{i=1}^n x_i^2$, using quotient manifolds induced by symplectic group actions, which enables us to integrate the systems and understand their complete integrability. In addition, we give a natural interpretation for the scattering maps associated with these systems.

1. INTRODUCTION

In this short note we explain the results of "Some finite dimensional integrable systems and their scattering behavior,"[1] by the author, and of Refs. 2–4, in terms of the abstract machinery set up in the paper of Kazhdan, Kostant, and Sternberg, entitled "Hamiltonian group actions and dynamical systems of Calogero type,"[5] which explains systems first discovered by Calogero and Marchioro,[6] and first discussed by Moser,[7].

Briefly, the systems to be discussed have the property that their equations of motion can be expressed as matrix differential equations which can be easily integrated, and moreover, the integration process is seen to occur naturally in a space of much higher dimensionality than the systems in question. The systems to be studied are thus interpreted as quotient systems of the much larger systems, where the quotienting out process is performed by a symplectic action of a group.

The process of quotienting out in mechanics, such as using center of mass coordinates, i.e., ignoring the position of the center mass, is indeed a common practice. We point out that usually quotienting out, or ignoring certain data, is a way of ignoring the symmetries, or integrals of the system, so as to arrive at some basic equations to study. Here the quotienting out does not really involve the integrals, but enables us to pass to the ultimate system to be studied. The integrals are in fact generated in a much more trivial way, for instance through the use of natural Lagrangian submanifolds and simple canonical maps, which of course makes use of the quotient structure. In addition, the so-called scattering maps of these systems have a natural interpretation in this context.

A special case of this symplectic quotienting out process is the coadjoint orbit construction of Kirillov–Kostant (see Ref. 8). This construction is relevant in the n-dimensional Euler spinning top problem of Arnold, as was observed by Dikii in Ref. 9. In addition, the Toda systems and their generalizations, as well as the Korteweg–deVries equation and its generalizations have orbit symplectic structures for their relevant phase spaces. We refer the reader to Kostant,[10] Mumford and Moerbeke,[11] and the author[12] for the former case, and Ref. 12 for the later case.

In the first section we merely summarize the abstract machinery of Ref. 5 of use in the discussion, referring the reader to Ref. 5 and 13, and the paper of Marsden and Weinstein,[8] for a fuller discussion. We then discuss the results of Ref. 1, which entails referring to Ref. 1 frequently. Finally in Sec. 7, we discuss the results of Olshanetsky and Perelomov,[2,3] in the above quotient framework.

2. THE SYMPLECTIC STRUCTURES

We summarize and briefly discuss the necessary abstract machinery needed to discuss Ref. 1. Let (M,ω,G) be a triple, with M an (exact) symplectic manifold with nondegenerate closed 2-form $\omega=d\tau$, and G a Lie group, with elements g, which acts on M with an exact symplectic action. If $\mathcal{L}$ is the Lie algebra of G, with elements denoted by $\mathring{g}$; then the action of G associates with each $\mathring{g}$ the Hamiltonian vector field $\dot{\mathring{g}}$, and the Hamiltonian function $f_{\mathring{g}}(\ )=-\tau(\hat{\mathring{g}})(\ )$, yields a Lie homomorphism, i.e.,

$$\{f_{\mathring{g}_1},f_{\mathring{g}_2}\}=f_{[\mathring{g}_1,\mathring{g}_2]}\quad [\ ,\ ]\text{ the bracket in }\mathcal{L},\tag{2.1}$$

where $\{\ ,\ \}$ is just the usual Poisson bracket, i.e., if

$$X_f\lrcorner\,\omega=df,\quad\text{then } X_{f_2}(f_1)=\omega(X_{f_1},X_{f_2})=\{f_1,f_2\}.\tag{2.2}$$

We define the moment map of Souriau,

$$\Phi:M\to\mathcal{L}^*,\quad\text{by }\Phi(m)(\mathring{g})=f_{\mathring{g}}(m),\tag{2.3}$$

with $\mathcal{L}^*$ the dual of $\mathcal{L}$. The group G acts on itself by conjugation, hence on $\mathcal{L}$ by the linearization of conjugation, Ad, and on $\mathcal{L}^*$ by $(\mathrm{Ad})^*$; and its easy to see that (2.1) is just the infinitesmal, and hence equivalent version of the relation of equivariance,

$$\Phi\circ g=(\mathrm{Ad}\,g^{-1})^*\circ\Phi.\tag{2.4}$$

We then form the orbit of $\alpha\in\mathcal{L}^*$ under $(\mathrm{Ad})^*$, Θ_α, and assume $V=\Phi^{-1}(\Theta_\alpha)$ is a manifold. It is easy to see V is a coisotropic manifold, i.e., $(TV_x)^\perp\subset(TV_x)$, for all $x\in V$, with $\perp$ denoting perpendicularity with respect to ω, and we can thus form $S=V/[$leaves of the foilation induced by $(TV)^\perp]$, taking S to be connected and assuming it to be a manifold. Then as a direct consequence of (2.4), it's not hard to see that S is a covering space of Θ'_α as follows:

$$S\cong\Theta'_\alpha\times\Theta_\alpha,\quad\Theta'_\alpha=\frac{\Phi^{-1}(\alpha)}{G_\alpha},\tag{2.5}$$

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where $/G_\alpha$ means we identify elements $x, y \in M$ if they lie on the same G_α orbit, with G_α the isotropy group of α, i.e., the connected subgroup of G which fixes α by its action on M. Incidentally, this shows S is a manifold precisely if Θ'_α is one. By the coisotropy of V, and the transitivity of G on the fibers Θ_α, ω induces a symplectic structure on $\Theta'_\alpha, \omega_\alpha$ (i.e., we shall tacitly assume ω_α is nondegenerate), where $\{\ ,\ \} \to \{\ ,\ \}_\alpha$ is a homomorphism. The structure $(\Theta'_\alpha, \omega_\alpha)$, G shall be our arena of activity.

We note that by (2.1) and (2.2), functions in M which are G invariant induce Hamiltonian flows on M which *pointwise* fix the image of M under Φ, and hence they induce Hamiltonian flows on Θ'_α. In addition, such functions, if they are in involution with respect to $\{\ ,\ \}$ on M, are, via the homomorphism $\omega \to \omega_\alpha$, automatically in involution in Θ'_α, thought of, by their G invariance, as functions on Θ'_α. This ends our discussion of quotient structures.

In preparation, we discuss some M's which shall come up in the examples.

Let F be the linear manifold of $n \times n$ matrices with complex coefficients, and $T^*F = \Phi$ be the cotangent bundle of F, where we shall identify $T^*F \sim F \times F$ via the bilinear $\langle X, Y\rangle = \mathrm{tr}(XY)$. Then the complex symplectic 2-form ω, naturally associated with T^*F is

$$\omega = \sum dX_{ij} \wedge dY_{ji} = \langle dX, dY\rangle,$$

or alternately, we write Hamilton's equations, with Hamiltonian $H = H(X, Y)$, as

$$\dot{X} = H_Y, \quad \dot{Y} = H_X \tag{2.6}$$

where $[H_X]_{ij} = \partial H/\partial X_{ji}$, etc. If we restrict ω to

$$M_1 = T^*\mathcal{L} = \{(X, Y) \mid X = X^*, Y = Y^*\}, \tag{2.7}$$

where * denotes taking the Hermitian adjoint, i.e., $\mathcal{L}$ is just the self-adjoint matrices [which we shall identify with the Lie algebra of the unitary group $G = \mathrm{U}(n, C)$], ω yields a real symplectic structure, with Hamilton's equations remaining as given in (2.6), where it is understood that H is real.

It is interesting to map $\Phi \to \Phi$ via

$$\tau: \begin{vmatrix} X \\ Y \end{vmatrix} \to \begin{vmatrix} X \\ XY \end{vmatrix} = \begin{vmatrix} Z_1 \\ Z_2 \end{vmatrix}, \tag{2.8}$$

and one computes

$$H_X = H_{Z_1} + Y H_{Z_2}, \quad H_Y = H_{Z_2} X,$$

$$\dot{Z}_1 = \dot{X} = H_Y, \quad \mathring{Z}_2 = (XY)^\circ = H_Y Y - X H_X.$$

Consequently, we may write Hamilton's equations in (Z_1, Z_2) coordinates as

$$\dot{Z}_1 = H_{Z_2} Z_1, \quad \mathring{Z}_2 = [H_{Z_2}, Z_2] - Z_1 H_{Z_1}. \tag{2.9}$$

Let us now restrict τ to $T^*\mathrm{U}(n)$, i.e., we identify

$$M_2 = T^*\mathrm{U}(n, C) \cong \{(Z_1, Z_2) \mid Z_1 Z_1^* = I, Z_2^* = Z_2\}, \tag{2.10}$$

and restrict

$$\tau \to \tau_{|\tau^{-1}(M_2)} = \overline{\tau}.$$

Note $\overline{\tau}$ is invertible, and of course we may just as well identify $\tau^{-1}(M_2)$ with $T^*\mathrm{U}(n)$. In that case, since by the pairing $\langle\ ,\ \rangle$, X, Y are the usual dual coordinates of $T^*\mathrm{U}(n)$, (2.9) restricted to M_2 precisely yields Hamilton's equations for the natural symplectic structure of $T^*\mathrm{U}(n)$. We must however put in the factor i due to our identification of the Lie algebra of $\mathrm{U}(n)$ with self-adjoint matrices, or equivalently we may think of time as being purely imaginary in (2.9). We omit the necessary, but easy verification that (2.9) restricted to M_2 automatically perserves M_2, which is sufficient to insure the restricted ω is symplectic.

3. EQUATIONS OF MOTION FOR THE SIMPLEST SYSTEM

We now apply the discussion of Sec. 2. We shall let (M, ω, G) of Sec. 2 be $(M_1, (dX, dY), \mathrm{U}(n))$ of Sec. 2. Since $M_1 = T^*\mathcal{L} \sim \mathcal{L} \times \mathcal{L}$, $\mathcal{L}$ the Lie algebra of $\mathrm{U}(n)$, $\mathrm{U}(n)$ acts naturally on $\mathcal{L}$ via Ad, i.e., by conjugation, which extends to a Hamiltonian action on $T^*\mathcal{L}$:

$$U: (X, Y) \to (UXU^{-1}, UYU^{-1}), \quad U \in \mathrm{U}(n),$$

of which the linear version is

$$\mathring{U}: (X, Y) \to ([\mathring{U}, X], \ [\mathring{U}, Y]) \in TM_{1(X,Y)}.$$

Hence, by (2.3), and the above

$$\Phi(X, Y)(\mathring{U}) = f_{\mathring{U}}(X, Y) = \langle [\mathring{U}, X], Y\rangle = \langle [X, Y], \mathring{U}\rangle,$$

and so by the identification of $\mathcal{L}$ with $\mathcal{L}^*$ through $\langle\ ,\ \rangle$

$$\Delta_\alpha \equiv \Phi^{-1}(\alpha) = \{(X, Y) \mid [X, Y] = \alpha\}, \tag{3.1}$$

and we shall once and for all pick α such that

$$[\alpha]_{jk} = i(1 - \delta_{jk}), \quad \text{i.e.,} \quad \alpha = i\{v^* \otimes v\},$$

with $v \in R^n$, $v = (1, 1, 1 \ldots, 1)^T$. Note that the isotropy subgroup $G_\alpha = \{U \mid U \propto U^{-1} = \alpha\}$, and we shall define the effective reduced subgroup

$$G_0 = \{U \mid U(v) = v\}, \text{ with Lie algebra } \mathcal{L}_0 = \{B \mid B(v) = 0\}. \tag{3.2}$$

In Ref. 5 it is shown by a simple linear algebra argument that if $[X, Y] = \alpha$, we can always find a unique $U \in G_0$ such that

$$UXU^{-1} = \mathrm{diag}(x_1, x_2, \ldots, x_n) \equiv \overline{x}, \quad x_i < x_{i+1}, \text{ all } i,$$
$$[UYU^{-1}]_{jk} = \delta_{jk} y_j + i(1 - \delta_{jk})(x_j - x_k)^{-1} \equiv \overline{y} \tag{3.3}$$

and hence $\Theta'_\alpha = \Phi^{-1}(\alpha)/G_\alpha$ is effectively coordinatized by $((x_1, x_2, \ldots, x_n), (y_1, \ldots, y_n)) = (x, y)$; and moreover it is shown in Ref. 5, by a local argument, that

$$\omega \to \omega_\alpha = \sum_{i=1}^{n} dx_i \wedge dy_i, \quad \text{i.e.,} \ (x, y) \tag{3.4}$$

form a set of canonical coordinates. Hence in this case, Θ'_α, and thus S is a manifold (see Sec. 2), and ω_α is nondegenerate, and hence symplectic.

We now wish to find functions on Θ'_α, and by the discussion in Sec. 2, functions of the form

$$H = H(X, Y) = \mathrm{tr}[P(X, Y)], \tag{3.5}$$

with $P(\cdot, \cdot)$ a noncommuting polynomial in its arguments, will certainly do. If we take (3.5) as a Hamiltonian function on M_1, then (2.6) yields for Hamilton's equations,

$$\dot{X}=h_1(X,Y),\quad \mathring{Y}=h_2(X,Y) \tag{3.6}$$

with $h_i(\cdot,\circ)$, $i=1,2$, polynomials in their arguments, uniquely determined by $P(\cdot,\circ)$. As mentioned, $H=\mathrm{tr}[P(X,Y)]$ automatically can be thought of as a function on Θ'_α, in fact via

$$h(x,y)=\mathrm{tr}[P(\bar{x},\bar{y})]; \tag{3.7}$$

and we wish to determine the analog of (3.6) for the system on Θ'_α with Hamiltonian (3.7), or to put it another way, we shall determine how (3.6) transforms in Θ'_α.

So assume we are given initial data for Hamiltonian's equation with Hamiltonian $h(x,y)$ in Θ'_α, which corresponds to $(\bar{x}(0),\bar{y}(0))$, which we may identify, and thus set equal to, $(X(0),Y(0))$ in M_1. Under the Hamiltonian $h(x,y)$, $(\bar{x}(0),\bar{y}(0))\to(\bar{x}(t),\bar{y}(t))$, and correspondingly under the Hamiltonian $H(X,Y)$, $(X(0),Y(0))\to(X(t),Y(t))$. By the previous remarks, we must have

$$X(t)=U\bar{x}(t)U^{-1},\quad Y(t)=U\bar{y}(t)U^{-1},\quad U=U(t), \tag{3.8}$$

with $U(t)\in G_0$ [see (3.2)] uniquely defined, as the $H(X,Y)$ flow in the big manifold M_1 descends to the $h(x,y)$ flow in the little manifold Θ'_α through quotienting out via G_0. Define $B(t)\in\mathcal{L}_0$ [see (3.2)] by $\dot{U}=-UB$, and so by (3.6) and (3.8),

$$\dot{X}=U\delta\bar{x}U^{-1}=h_1(X,Y)=Uh_1(\bar{x},\bar{y})U^{-1},$$

where $\delta\bar{x}=\dot{\bar{x}}-[B,\bar{x}]$; and so we have

$$\delta\bar{x}=h_1(\bar{x},\bar{y}),\quad \delta\bar{y}=h_2(\bar{x},\bar{y}). \tag{3.9}$$

As a consequence of Hamiltonian's equations on Θ'_α, $\dot{x}=(\partial/\partial x)[h(x,y)]$, $\dot{y}=-(\partial/\partial y)[h(x,y)]$, and thus we see how (3.6) is transformed in Θ'_α. Note that from the definitions of δ, (3.9), and $B(v)=0$ [see (3.2)], we can immediately compute the unexpected functional dependence, $B(t)=B(\bar{x}(t),\bar{y}(t))$, since $\bar{x}$ is a diagonal matrix.

We specialize to the case $H=H_f=\mathrm{tr}[f(Y)]$, for which we compute [see (3.6)] $h_1(X,Y)=f'(Y)$, $h_2=0$, and thus conclude from (3.6) and (3.9),

$$\dot{X}=f'(Y),\quad Y'=0, \tag{3.10}$$

$$\delta\bar{x}=f'(\bar{y}),\quad \delta\bar{y}=0. \tag{3.11}$$

Since (3.10) is immediately solvable, we have in fact solved (3.11) by the use of (3.8).

We also note that since the H_f's clearly are in involution on M_1, being function only of Y, that by the homomorphism $\omega\to\omega_\alpha$, the h_f's, $h_f=\mathrm{tr}[f(\bar{y})]$, are in involution on Θ'_α, and in fact are generated by n independent functions $h^{(j)}=\mathrm{tr}[(\bar{y})^j]$, $j=1,2,\ldots,n$. Thus $h=\mathrm{tr}(\frac{1}{2}y^2)$ gives rise to a completely integrable Hamiltonian system.

4. SCATTERING MAPS

Upon inspection, one observes that the map

$$\eta:(X,Y)\to(Y,X),\quad \eta|_{\Delta_\alpha}:\Delta_\alpha\to\Delta_{-\alpha},\quad [\text{see (3.1)}], \tag{4.1}$$

is a canonical map with multiplier -1 in M_1, and hence so is its projection η_p in Θ'_α, since $\omega\to\omega_\alpha$ is a homomorphism. This map, η_p, is precisely the scattering map for system (3.11), with $f(s)=\frac{1}{2}s^2$, which is discussed in Ref. 1, Theorem 6, as was observed by J. Moser (in a personal communication). More precisely, in the above case, (3.11) with $f(s)=\frac{1}{2}s^2$, one shows the time evolution of the system is given by

$$(x,y)=(qt+p+O(t^{-1}),p+O(t^{-2})),\quad t\to+\infty,$$

with

$$\eta_p:(x(0),y(0))\to(q,p).$$

Similarly one defines the $\pi/4$ rotation map

$$\hat{\eta}:(X,Y)\to 2^{-1/2}(X+Y,X-Y),\quad \hat{\eta}|_{\Delta_\alpha}:\Delta_\alpha\to\Delta_{-\alpha}, \tag{4.2}$$

which is easily seen to be canonical with multiplier -1 on M_1 (here one uses that X,Y are Hermitian), and the corresponding canonical projection, $\hat{\eta}_p$ on Θ'_α. If $h(x,y)=\frac{1}{2}\mathrm{tr}(\bar{y}^2-\bar{x}^2)$ in (3.7), then the time evolution of the system (3.9) is given by

$$(x(t),y(t))=2^{-1/2}\times(q^{\pm}e^{\pm t}+p^{\pm}e^{\pm t},q^{\pm}e^{\pm t}-pe^{\pm t})+O(e^{-2|t|}),$$

$$t\to\pm\infty$$

with

$$\hat{\eta}_p(x(0),y(0))=(q,p).$$

This is shown in Ref. 1, Theorem 4. Moreover, as one easily checks

$$\eta=\hat{\eta}\circ\rho\circ\hat{\eta}^{-1},\quad \rho:(X,Y)=(X,-Y),$$

hence

$$\hat{\eta}_p=\hat{\eta}_p\circ\rho_p\circ\hat{\eta}_p^{-1},$$

and by (3.3), $\rho_p(x,y)=(x,-y)$.

Note that we have shown, by the time reversibility of system (3.9), that $\eta_p:(q^-,p^-)\to(q^+,p^+)$. These latter statements are shown in Ref. 1, Theorems 5 and 6.

5. TWO EQUIVALENT SYSTEMS

We now investigate the systems on Θ'_α with Hamiltonians respectively

$$h_1(x,y)=\mathrm{tr}[f(\bar{x}\cdot\bar{y})], \tag{5.1a}$$

$$h_2(x,y)=\mathrm{tr}\{f(\tfrac{1}{2}[\bar{x}+\bar{y}]\cdot[\bar{x}-\bar{y}])\}. \tag{5.1b}$$

By the canonical map $\hat{\eta}_p$ of Sec. 4, it is only necessary to investigate case (a), and then via $\hat{\eta}_p$, transpose the results to case (b). Also, via the transformation formalism (3.8), and (3.6)$\mapsto$(3.9), it is only necessary to study the equations on the full manifold M_1, rather than the quotient manifold Θ'_α. The formalism says substitute δ for d/dt, and $(\bar{x},\bar{y})$ for $X,Y)$, which enables one to compute the generator B and solve the equation on the big manifold, M_1, and via (3.8), to pass tothe solution on the quotient manifold Θ'_α.

We thus need only study the system on M_1 with Hamiltonian

$$H_1=H_f(X,Y)=\mathrm{tr}[f(XY)], \tag{5.2}$$

and we first show the H_f's are in involution. Specifically, assume

$$H^{(1)}=\mathrm{tr}[f_1(XY)],\quad H^{(2)}=\mathrm{tr}[f_2(XY)],$$

then, taking increments,

$$\delta H^{(1)}=\mathrm{tr}\{f_1'(XY)\cdot[\delta X\cdot Y+X\cdot\delta Y]\}+O^{(2)},$$

where $O^{(2)}$ means terms of at least second order; and we have

$$H_X^{(1)} = Y \cdot f_1'(XY), \quad H_Y^{(1)} = f_1'(XY) \cdot X, \tag{5.3}$$

etc., for $H^{(2)}$. By (2.6) and (2.2), the Poisson bracket $\{\ ,\ \}$ of $H^{(1)}, H^{(2)}$ is given by [where $\langle A, B\rangle = \mathrm{tr}(AB)$] $\{H^{(1)}, H^{(2)}\} = \langle H_X^{(1)}, H_Y^{(2)}\rangle - \langle H_Y^{(1)}, H_X^{(2)}\rangle$, and thus substituting in (5.3), we find

$$\{H^{(1)}, H^{(2)}\} = \langle Yf_1', f_2'X\rangle - \langle f_1'X, Yf_2'\rangle$$
$$= \mathrm{tr}(f_1'f_2'XY) - \mathrm{tr}(f_1'f_2'XY) = 0,$$

where we have used $[f_2'(XY), XY] = 0$, and so $\{H^{(1)}, H^{(2)}\} = 0$.

We have thus, through $\hat{\eta}$, shown the H_2's, $H_2 = H_f = \mathrm{tr}\{f(\frac{1}{2}[X+Y]\cdot[X-Y]\}$, are in involution, and thus by the homomorphism $\omega \to \omega_\alpha$, so are the $\{\mathrm{tr}[f(\overline{xy})]\}$'s $\{\mathrm{tr}[f(\frac{1}{2}[\bar{x}+\bar{y}]\cdot[\bar{x}-\bar{y}])]\}$'s, respectively; and so $\mathrm{tr}(\frac{1}{2}\bar{x}y^-)$, $\mathrm{tr}(\frac{1}{2}[\bar{x}+\bar{y}]\cdot[\bar{x}-\bar{y}])$, respectively give rise to completely integrable Hamiltonian systems on Θ_α', as was observed proven in Ref. 1. This is, thus, the second and more pleasant proof of that fact.

Now by (2.6) and (5.3), Hamiltonian's equations of the system of (5.2) are

$$\dot{X} = f'(XY)\cdot X, \quad \dot{Y} = -Yf'(XY) \tag{5.4}$$

and since $(XY)^{\cdot} = 0$ as a consequence of $[f'(XY), XY] = 0$, we immediately integrate (5.4) to obtain

$$X = \exp(f'(X_0Y_0)t)\cdot X_0, \quad Y = Y_0\cdot\exp(-f'(X_0Y_0)t), \tag{5.5}$$

where the subscript $_0$ denotes evaluation at $t = 0$. Thus by (3.9), the corresponding equations on Θ_α' for the systems of (5.1a) are

$$\delta\bar{x} = f'(\bar{x}\bar{y})\cdot\bar{x}, \quad \delta\bar{y} = -\bar{y}f'(\bar{x}\bar{y}), \tag{5.6}$$

while (3.8) implies the time evolution of (5.6) is given by

$$\mathrm{diag}(x_1, x_2, \ldots, x_n)(t) = U^{-1}\exp(f'(X_0Y_0)t)\cdot X_0U, \ \text{etc.} \tag{5.7}$$

The canonical map $\hat{\eta}$ (5.4), implies the corresponding equations and time evolution for the system with Hamiltonian $H_2 = \mathrm{tr}[f(\frac{1}{2}[X+Y]\cdot[X-Y])]$ are

$$(X+Y)^{\cdot} = -f'(\tfrac{1}{2}[X+Y]\cdot[X-Y])\cdot[X+Y],$$
$$(X-Y)^{\cdot} = [X-Y]\cdot f'(\tfrac{1}{2}[X+Y]\cdot[X-Y]),$$

with

$$\{\tfrac{1}{2}[X+Y]\cdot[X-Y]\}^{\cdot} = 0, \tag{5.8a}$$

and

$$\begin{aligned}(X+Y) &= [\exp - \{tf'(\tfrac{1}{2}[X_0+Y_0]\cdot[X_0-Y_0])\}]\cdot[X_0+Y_0],\\ (X-Y) &= [X_0-Y_0]\cdot[\exp\{\tfrac{1}{2}tf'[X_0+Y_0]\cdot[X_0-Y_0]\}],\end{aligned} \tag{5.8b}$$

where the changing of signs $t \to -t$, comes about because $\hat{\eta}$ is the canonical multiplier -1. We now use (3.8) and (3.9) to transpose (5.8) to Θ_α', i.e., system (5.1b), in particular concluding

$$2\delta\bar{x} = [\bar{x}, f'] - [f', \bar{y}]_+, \quad 2\delta\bar{y} = [\bar{y}, f'] - [f', \bar{x}]_+, \tag{5.9}$$

where $[A, B]_+ = AB + BA$, $f' = f'(\frac{1}{2}[\bar{x}+\bar{y}]\cdot[\bar{x}-\bar{y}])$. Note the simplicity of (5.8) and (5.9), when $f(s) = s$. We could equally well study the systems gotten by "stretching" $X \to \mu\cdot X$, in (5.1) and (5.2), which tend to have compact behavior and thus give rise to periodic solutions for μ purely imaginary, see Ref. 1, in fact for $f(s) = s$, $\mu = \sqrt{-1}$, all solutions are periodic with one and the same nonprimitive period.

Note that our studying the case $H = \mathrm{tr}[f(XY)]$, $h = \mathrm{tr}[f(\bar{x}\bar{y})]$, makes it unnecessary to study the (Sutherland) case where our manifold is

$$T^*U(n, C) \cong \{(U, R)\,|\,UU^* = I, \ R = R^*\},$$

and our Hamiltonian is $H(U, R) = H_f = \mathrm{tr}[f(R)]$, with Hamilton's equations given by (2.9), $(U, R) = (Z_1, Z_2)$; for after the change $t \to it$, we would get the same formal results as (5.4)—(5.7), including the involution statement, via the map τ, (2.8), where we identify $(X, XY) = (Z_1, Z_2)$ with (U, R). We note that the condition $(X, Y) \in \Delta_\alpha$, namely $[X, Y] = \alpha$ is transformed into $[U, U^{-1}R] = \alpha$, i.e., $R - U^{-1}RU = \alpha$.

6. YET ANOTHER SYSTEM ON θ_α'

We now discuss the system of Sec. 6 of Ref. 1.

One may either regard the Hamiltonian of this system on Θ_α' as

$$h_1 = \mathrm{tr}[\tfrac{1}{2}(\bar{x}\bar{y})^2 + \bar{x}], \tag{6.1}$$

or

$$h_2 = \mathrm{tr}(\tfrac{1}{2}\bar{x}\,\bar{y}^2\bar{x} + \bar{x}), \tag{6.2}$$

as h_1, h_2 differ by a constant. Of course in the full manifold the corresponding Hamiltonians,

$$H_1 = h[\tfrac{1}{2}(XY)^2 + X], \tag{6.3}$$

$$H_2 = \mathrm{tr}[\tfrac{1}{2}(XY^2X) + X], \tag{6.4}$$

are far from identical. Although it is shown in Ref. 1 that (6.1) and (6.2) are completely integrable systems, we shall not show that (6.3) and (6.4) are completely integrable systems, in fact we have not been able to do this.

We shall study both (6.3) and (6.4) and then relate them in case the associated differential equations on Θ_α' have the same initial data. Since the calculations are so similar to those of Sec. 5, we just give the results. For simplicity we set $XY = Z$. Then with the Hamiltonian of (6.3) we calculate, from (2.6),

$$\dot{X} = ZX, \quad \dot{Z} = -X,$$

and since $[X, Y] = \alpha$, $(YX)^{\cdot} = -X$, we have $\frac{1}{2}(XY^2X) + X = e_1$, e_1 a constant, and thus we arrive at, again using $[X, Y] = \alpha$,

$$\tfrac{1}{2}Z^2 - \tfrac{1}{2}Z\alpha - \dot{Z} = e_1. \tag{6.5}$$

Letting $Z = -2a_1^{-1}\dot{a}_1$, we find $\ddot{a}_1 = \frac{1}{2}a_1e_1 - \frac{1}{2}\dot{a}_1\alpha$, hence we have

$$\begin{pmatrix} a_1 \\ \dot{a}_1 \end{pmatrix} = \begin{pmatrix} a_1(0) \\ \dot{a}_1(0) \end{pmatrix}\cdot\exp C_1t, \tag{6.6}$$

$$C_1 = D_1(X, Y) = \begin{bmatrix} 0, & I \\ \frac{1}{4}(XY^2X) + \frac{1}{2}X, & -\alpha/2 \end{bmatrix} \text{ at } t = 0. \tag{6.7}$$

For the Hamiltonian of (6.4), we find

$$\dot{Z} - [\tfrac{1}{2}YX, Z] = -X, \quad \dot{X} - [\tfrac{1}{2}YX, X] = \tfrac{1}{2}(XZ + ZX),$$

which motivates us to define the derivation δ,

$$\delta(\cdot)=\frac{d}{dt}(\cdot)-[\tfrac{1}{2}YX,\cdot],$$

and thus we have from the above,

$$\delta Z=-X,\quad \delta X=\tfrac{1}{2}(XZ+ZX). \tag{6.8}$$

Clearly, if $U(0)=I$, $\dot{U}=-U(\frac{1}{2}YX)$, we have the following rule of transformations for matrices $A=A(t)$: If $\hat{A}\equiv UAU^{-1}$, then $(d/dt)^j\hat{A}=U\delta^jAU^{-1}$. From (6.8) we conclude, using that δ is a derivation,

$$\delta(\tfrac{1}{2}Z^2+X)=0,\quad \delta(\tfrac{1}{2}Z^2-\delta Z)=0; \tag{6.9}$$

hence by our rule of transformation,

$$\tfrac{1}{2}\hat{Z}^2-\dot{\hat{Z}}=e_2,\quad e_2 \text{ a constant;}$$

and thus letting $\hat{Z}=-2a_2^{-1}\dot{a}_2$, we find

$$\ddot{a}_2=\tfrac{1}{2}a_2e_2,$$

i.e.,

$$\begin{pmatrix}a_2\\ \dot{a}_2\end{pmatrix}=\begin{pmatrix}a_2(0)\\ \dot{a}_2(0)\end{pmatrix}\cdot\exp C_2t,$$

$$C_2=D_2(X,Y)=\begin{bmatrix}0, & I\\ \frac{1}{4}(XY)^2+\frac{1}{2}X, & 0\end{bmatrix}\text{ at } t=0. \tag{6.10}$$

We now consider the case where the $X(0)$ of both systems are the same in Θ'_α, and moreover $x_1(0)>0$. Then since both Eqs. (6.3) and (6.4) are the same as seen in Θ'_α, they both must have the same long term behavior of X as is projected down into Θ'_α. In Ref. 1, Sec. 6, it is shown that system (6.3) and (6.4) have the following long term "scattering" behavior,

$$(\log x, xy)=(\pm\lambda t+\beta^\pm+O(t^{-1}), \pm\lambda+O(t^{-2})),\quad t\to\pm\infty,$$

where $\log x=(\log x_1,\ldots,\log x_n)$, $xy=(x_1y_1,\ldots,x_ny_n)$. By arguments in that same section, it's clear that the spectrum of C_1, or alternately C_2, precisely carry the data λ, and hence we must have

$$D_1(X,Y)\sim D_2(X,Y),$$

where $\sim$ denotes spectral equivalence. By (6.7), (6.10), and (3.8), this implies

$$D_1(\bar{x}\,\bar{y})\sim D_2(\bar{x},\bar{y}). \tag{6.11}$$

Moreover it follows from (6.5) and (6.9) that $D_1(X,Y)$, $D_2(X,Y)$ are isospectral matrices of the differential equations (6.3 and (6.4) respectively, and so in particular $D_1(\bar{x},\bar{y})$, $D_2(\bar{x},\bar{y})$ are for the (6.1) and (6.2) flow; and thus we arrive at

$$D_1(\bar{x}(0),\bar{y}(0))\sim\lim_{t\to\pm\infty}D_1(\bar{x}(t),\bar{y}(t))\sim\lim_{t\to\pm\infty}D_2(\bar{x}(t),\bar{y}(t)). \tag{6.12}$$

This yields the scattering behavior of system (6.1) and (6.2) as discussed in Corollaries 11.2, 11.3, of Ref. 1, which essentially maintains that the system scatters as if it is completely decoupled, and is just constrained to maintain a fixed order on the line.

7. THE GENERALIZATIONS OF OLSHANETSKY AND PERELOMOV

We now apply the considerations of Sec. 2, and Ref. 5, to generalizations of the Calogero—Moser systems in the context of complex semisimple Lie algebras as considered by Olshanetsky and Perelomov.[2,3] We shall recover their results along the way. They actually work in the setting of real Lie algebras, but the natural setting for a full analysis is the complex Lie algebras and their real conpact decompositions. We remark that we shall make seemingly stronger hypotheses than they do. As this section is in effect an abstraction of the work of the previous sections, we shall tend to sketch arguments in order to avoid unnecessary redundancy. We first need to enumerate some well-known facts about Lie algebras which we mention without proof (see Ref. 14).

Let $\tilde{\mathcal{L}}$ be a semisimple Lie algebra over C, the complex numbers, with connected Lie group $\tilde{G}$ and with (nondegenerate) Killing form $\langle X,Y\rangle=\mathrm{tr}(\mathrm{ad}X\,\mathrm{ad}Y)$. Note $\langle\cdot,\cdot\rangle$ has the following important properties (where $[\cdot,\cdot]$ denotes the Lie bracket):

$$\langle X,[Y,Z]\rangle=\langle Y[Z,X]\rangle=\langle Z,[X,Y]\rangle, \tag{7.1}$$
$$\langle \mathrm{Ad}_gX,\mathrm{Ad}_gY\rangle=\langle X,Y\rangle,\quad g\in\tilde{G}.$$

Let h be a Cartan subalgebra of $\tilde{\mathcal{L}}$, with $\tilde{\Delta}$ the corresponding set of nonzero roots, and Δ the subset of roots which are positive with respect to some ordering. Pick root vectors $E_\beta\subset\tilde{\mathcal{L}}^\beta$, $\tilde{\mathcal{L}}^\beta$ a root subspace corresponding to the root $\beta\in h^*$, having the following properties:

$$\begin{aligned}&[E_\beta,E_{-\beta}]=H^{(\beta)}\in h,\quad \langle H^{(\beta)},H^{(\beta)}\rangle=1,\\ &[H,E_\beta]=(\beta\langle H\rangle)\cdot E_\beta\equiv H_\beta\cdot E_\beta,\quad\text{for all } H\in h.\end{aligned} \tag{7.2}$$

Let $\mathcal{L}\subset\tilde{\mathcal{L}}$ denote the compact *R-linear* subspace

$$\mathcal{L}=\sum_{\beta\in\Delta}R\cdot(iH^{(\beta)})+\sum_{\beta\in\Delta}R\cdot(E_\beta-E_{-\beta})+\sum_{\beta\in\Delta}R\cdot(i(E_\beta+E_{-\beta})); \tag{7.3}$$

and thus we have the *R-linear* direct sum

$$\tilde{\mathcal{L}}=\mathcal{L}+(i\mathcal{L}),\text{ and moreover }[\mathcal{L},\mathcal{L}]\subset\mathcal{L}. \tag{7.4}$$

Let G denote the connected Lie group with Lie algebra $\mathcal{L}$.

From (7.2), we have $\langle\cdot,\cdot\rangle_{|\mathcal{L}\times\mathcal{L}}\to R$ and by the nondegeneracy of $\langle\cdot,\cdot\rangle,\langle\cdot,\cdot\rangle$ restricted to $\mathcal{L}\times\mathcal{L}$, $(i\mathcal{L})\times(i\mathcal{L})$, respectively, is nondegenerate (and real). This allows us to identify $T^*(i\mathcal{L})\cong(i\mathcal{L})\times(i\mathcal{L})\equiv M_3$. From (7.4), G acts on $(i\mathcal{L})$ via the adjoint action, which one extends symplectically to $T^*(i\mathcal{L})$. Hence for the triple (M,ω,G), of Sec. 2 we take $((i\mathcal{L})\times(i\mathcal{L}), d\langle X,dY\rangle, G)$, where we take (X,Y) as running coordinates on $M_3=(i\mathcal{L})\times(i\mathcal{L})$. As in Sec. 3 we compute the associated moment map, using (7.1), $\Phi(X,Y)(\dot{g})=\langle[X,Y],\dot{g}\rangle$; hence by the nondegeneracy of $\langle\cdot,\cdot\rangle_{|\mathcal{L}\times\mathcal{L}}$ $\Phi^{-1}(\alpha)=\{(X,Y)\,|\,[X,Y]=\alpha\}$, $\alpha\in\mathcal{L}$.

We now pick a very special α, $\alpha=\sum_{\beta\in\Delta}c_\beta(E_\beta-E_{-\beta})$, $c_\beta\neq 0$, real, $\beta\in\Delta$, satisfying:

Property A: Let G_α be the connected isotropy group of α with Lie algebra $\mathcal{L}_\alpha$, which we assume is specified by a relation of the form

$$\mathcal{L}_\alpha=\{\dot{g}\,|\,P_h\dot{g}+L(I-P_h)\dot{g}=0,\ \dot{g}\in\mathcal{L}_\alpha\},$$

where $P_h:\tilde{\mathcal{L}}\to\tilde{\mathcal{L}}$ is specified by $P_h(\Sigma f_\beta H_\beta+\Sigma l_\beta E_\beta)=\Sigma f_\beta H_\beta$ [see (7.3) and (7.4)] I is the identity operator in $\tilde{\mathcal{L}}$, and L is some linear map; moreover, if $[X,Y]=\alpha$, then there exists $g \in G_0$ such that $\mathrm{Ad}_g X \equiv \bar{x} \in h$ (which shall imply $\bar{x}$ is regular). The only freedom in the element g is that we may specify which Weyl chamber $\bar{x}$ is contained in, and we shall always take it to be the positive Weyl chamber with respect to the ordering, h^+; and then only a finite number of choices remain for g.

Using Property A of α, we can easily compute $\Theta'_\alpha = \Phi^{-1}(\alpha)/G_\alpha$. For if $[X, Y] = \alpha$, we pick $g \in G_0$ such that $\mathrm{Ad}_g X = \bar{x}$, hence $\mathrm{Ad}_g Y = \tilde{y} + \Sigma_{\beta\in\Delta} c_\beta \bar{x}_\beta^{-1}(E_\beta + E_{-\beta}) \equiv \bar{y}$, $\tilde{y} \in h$. Note automatically $\beta\langle\bar{x}\rangle = \bar{x}_\beta \neq 0$, $\beta \in \Delta$, hence $\bar{x}$ is regular. We shall specify g uniquely by requiring $\bar{x}_\beta > 0$, $\beta \in \Delta$. We now pick an orthonormal basis of h, with respect to $\langle\cdot,\cdot\rangle$, $\{H_1, H_2, \ldots H_n\}$, and define $\bar{x} = \Sigma_{i=1}^n x_i H_i$, $\tilde{y} = \Sigma_{i=1}^n y_i H_i$. From the previous remarks it follows that we may take $(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x, y)$ as coordinates on Θ'_α, noting that they are constrained by the open relations $\bar{x}_\beta > 0$, $\beta \in \Delta$, to lie in the positive Weyl chamber.

We claim that the symplectic structure induced on Θ'_α is $\omega \to \omega_\alpha = \Sigma_{i=1}^n dx_i \wedge dy_i$. For if $P_h X = \Sigma X_i H_i$, $P_h Y = \Sigma Y_i H_i$, $\omega = \langle dX, dY\rangle = \Sigma dX_i \wedge dY_i + \{(I - T \in X, \ (I - P_h)\,Y$ terms$\}$; hence if we show for $X_\epsilon = \bar{x} + \epsilon T$, $T \in (1 - P_h)(i\mathcal{L})$, that $[X_\epsilon, Y_\epsilon] = \alpha$ implies $\mathrm{Ad}_{g_\epsilon} X_\epsilon = \bar{x} + \Theta(\epsilon^2) \in h^+$, $g_0 = e$, $g \in G_0$, we will have proven our (local) assertion up to "second order," which is sufficient (see the argument given in Ref. 5). First by the regularity of $\bar{x}$ (and implicitly the uniqueness of g_ϵ), it is easy to see—in fact by sharpening the argument to be given—that g_ϵ is smooth in ϵ. Let $g_\epsilon = e + \epsilon V + \Theta(\epsilon^2)$, with the notation having the obvious meaning. As we must have $\mathrm{Ad}_{g_\epsilon}(\bar{x} + \epsilon T) = \bar{x} + \epsilon\bar{x}_1 + \Theta(\epsilon^2) \in h$, for all ϵ small, it follows, upon expanding the left side of the equation in ϵ, that $[V, \bar{x}] + T = \bar{x}_1 \in h$; but $\bar{x} \in h$ implies $[V, \bar{x}] \in (1 - P_h)\mathcal{L}$, hence $P_h([V, \bar{x}] + T) = 0$. Thus $\bar{x}_1 = 0$, as was to be shown.

For Hamiltonians on M_3 which induce Hamiltonians on Θ'_α, we pick the G invariant functions $H = H(X, Y) = \mathrm{tr}[P(\mathrm{ad}X, \mathrm{ad}Y)]$, with P a noncommuting polynomial of its arguments. As is usual, we define the gradient of $H \in C^1(M_3)$ as follows: $\delta H \equiv \langle\delta X, H_X\rangle + \langle H_Y, \delta Y\rangle$, H_X, $H_Y \in (i\mathcal{L})$. This uniquely specifies H_X, H_Y. Note the adjoint invariance of H, hence of δH [and the adjoint invariance of $\langle\cdot,\cdot\rangle$ in (7.1)], implies

$$\mathrm{Ad}_g H_X(X, Y) = H_X(\mathrm{Ad}_g X, \mathrm{Ad}_g Y), \tag{7.5}$$

similarly for H_Y. In particular if $H = (a/2)\langle X, X\rangle + (b/2)\times\langle Y, Y\rangle$, $H_X = aX$, $H_Y = bY$. By the form of $\omega = \langle dX, dY\rangle$, Hamilton's equation on M_3 for the Hamiltonian H are

$$\dot{X} = H_Y(X, Y), \quad \dot{Y} = -H_X(X, Y); \tag{7.6}$$

which upon using $\mathrm{Ad}_g\bar{x} = X$, $\mathrm{Ad}_g\bar{y} = Y$ and (7.5), are easily seen, as in Sec. 3, to transform on Θ'_α to the equations

$$\delta\bar{x} = H_Y(\bar{x}, \bar{y}), \quad \delta\bar{y} = -H_X(\bar{x}, \bar{y}), \tag{7.7}$$

with

$$\delta(\cdot) = \frac{d\cdot}{dt} - [B, \cdot], \ \dot{g} \equiv -(L_g)_* B(t), \ g(0) = e,$$

where $L_g : g_1 \mapsto g g_1$. Of course we have as usual on Θ'_α,

$$\frac{dx_i}{dt} = \frac{\partial h(x, y)}{dy_i}, \ \frac{dy_i}{dt} = -\frac{\partial h}{\partial x_i}, \quad i = 1, \ldots n,$$

$$h = h(x, y) = H(\bar{x}, \bar{y}).$$

Since $\bar{x} \in h$ is regular, (7.7) implies $[B, \bar{x}] = (1 - P_h) H_Y(\bar{x}, \bar{y})$; and thus determines $(1 - P_h)B$, and hence B, since $B \in \mathcal{L}_\alpha$, as a function of t through $B = B(x, y)$.

Since automatically the $H^{(j)} = (1/2j)\mathrm{tr}(\mathrm{ad}\bar{y})^{2j}$, $j = 1, \ldots, n$, are in involution on M_3, so are the $h^{(j)} = (1/2j)\mathrm{tr}(\mathrm{ad}\bar{y})^{2j}$, $j = 1, \ldots, n$, on Θ'_α; and hence they give rise to a completely integral system on Θ'_α. The corresponding Hamiltonian differential equations are (where the subscript 0 shall denote initial conditions), $X = H_Y^{(j)}(Y)$, $\dot{Y} = 0$, hence $X(t) = H_Y^{(j)}(Y_0)\cdot t + X_0$, $Y(t) = Y_0$ on M_3. Therefore, on Θ'_α we have $\delta\bar{x} = H_Y^j(\bar{y})$, $\delta\bar{y} = 0$, hence $\bar{x}(t) = \mathrm{Ad}_{g^{-1}(t)}([H_Y^{(j)}(\bar{y}_0)]t + \bar{x}_0)$, $\bar{y}(t) = \mathrm{Ad}_{g^{-1}}\bar{y}_0$, $\delta\cdot = (d\cdot/dt) - [B_j, \cdot]$, for an appropriate $g(t)$. In particular for $j = 1$, $H_Y^{(1)}(Y) = Y$, and thus the above implies

$$\delta\bar{x} = \bar{y}, \quad \delta\bar{y} = 0, \quad \text{hence } \delta(\bar{x} - t\bar{y}) = 0; \tag{7.8}$$

which in particular implies $(1 - P_h)B_1 = \Sigma_{\beta\in\Delta}\bar{x}_\beta^{-2}c_\beta \times (E_\beta - E_{-\beta})$. It is easy to show (see Ref. 1), that

$$(x(t), y(t)) \mapsto (qt + p + \Theta(t^{-1}), p + \Theta(t^{-2})), \quad t \to \infty; \tag{7.9}$$

and that as a consequence of (7.8) and (7.9) the canonical map $(x, y) \mapsto (q, p)$ is just the projection of the canonical map $\eta : (X, Y) \mapsto (Y, X)$ onto Θ'_α (se Sec. 4 of Ref. 1). Note that from (7.2) $h^{(1)} = \frac{1}{2}\langle\bar{y}, \bar{y}\rangle = \frac{1}{2}\Sigma_{i=1}^n y_i^2 + \Sigma_{\beta\in\Delta} c_\beta^2 \bar{x}_\beta^{-2}$, which is just the Calogero–Moser potential of Olshanetsky and Perelomov. We remark that the proof of *complete* integrability given here is the only algebraic proof we know of, i.e., the only one that doesn't depend on the scattering map.

If we let $H = \frac{1}{2}(\langle Y, Y\rangle - a^2\langle X, X\rangle)$, then Hamilton's equations on M_3 are $\dot{X} = Y$, $\dot{Y} = a^2 X$; which transform to the equations $\delta\bar{x} = \bar{y}$, $\delta\bar{y} = a^2\bar{x}$, on Θ'_α, with the appropriate $\delta(\cdot) = (d\cdot/dt) - [B, \cdot]$. Since $\delta\bar{x} = \bar{y}$, $B = B(x, y) = B_1(x, y)$, the B_1 going with the Hamiltonian $h^{(1)}$. Note the Hamiltonian function on Θ'_α is $h^{(1)} - (a^2/2)\Sigma_{i=1}^n x_i^2$. Since on M_3, $2X(t) = (X_0 + a^{-1}Y_0)e^{ta} + (X_0 - a^{-1}Y_0)e^{-ta}$, $2\bar{x}(t) = \mathrm{Ad}_{g^{-1}}[(\bar{x}_0 + a^{-1}\bar{y}_0)e^{ta} + (\bar{x}_0 - a^{-1}\bar{y}_0)e^{-ta}]$, $\dot{g} = -(L_g)_* B$, etc., for $\bar{y}(t)$.

The scattering maps for this system are discussed precisely as in Sec. 4 using the maps η and $\hat{\eta}$, the crucial tool being $\delta[(\bar{y} \pm a\bar{x})e^{\mp a t}] = 0$, (see Ref. 1). The preceding equation also implies $\mathrm{tr}[\mathrm{ad}(\bar{y} + a\bar{x}) \cdot \mathrm{ad}(\bar{y} - a\bar{x})]^v$, $v = 1, \ldots, n$, are integrals for the above system. If we let $a = ib$, b real, in the above equations, we get a periodic system with period $2\pi/b$.

We now consider the Sutherland type systems. Namely, let us identify, with the usual symplectic structure, $T^*G \cong G \times (i\mathcal{L}) \equiv M_4$, via the (nondegenerate) inner product $-i\langle\cdot,\cdot\rangle_{|(i\mathcal{L})\times(\mathcal{L})}$, where M_4 has the running coordinates (Q, R). The group G acts on itself by conjugation which extends naturally to a symplectic action on $T^*G = M_4$, via $g : (Q, R) \mapsto (gQg^{-1}, \mathrm{ad}_{g^{-1}}R)$; and thus we have for the triple of Sec. 2, $(M, \omega, G) = (G \times (i\mathcal{L}), d(i\langle R, Q^{-1}dQ\rangle), G)$ (see Ref. 2). Note that the linearized version of this group action about (e, R) is just the previous group action. The discussion we give here is towards a different purpose than the one in Sec. 2, and of necessity proceeds differently, although the formulas are of course related. Since $i\omega_{|(Q,R)} = -\langle dR, Q^{-1}dQ\rangle + \langle R, Q^{-1}dQ \wedge Q^{-1}dQ\rangle$, if (η_1, β_1), (η_2, β_2) are elements of $T(T^*G)_{|(Q,R)} \cong \mathcal{L} + (i\mathcal{L}) = \mathcal{L}^c$, we have

$$i\omega((\eta_1, \beta_1), (\eta_2, \beta_2))_{|(Q,R)} = \langle R, [\eta_1, \eta_2]\rangle + (\langle\beta_2, \eta_1\rangle - \langle\beta_1, \eta_2\rangle). \tag{7.10}$$

Given $H = H(Q,R) \in C^1(M_4)$, we define the gradient of H in the ususal way (which is slightly different from the way of Sec. 2),

$$i\delta H = \langle \delta R, H_R\rangle + \langle H_Q, Q^{-1}\delta Q\rangle, \quad H_R \in \mathcal{L}, \quad H_Q \in (i\mathcal{L}); \qquad (7.11)$$

and from the definition $\omega(X_H, Y) \equiv Y(H)$, one finds

$$X_H = (L_Q)_* H_R \partial_Q + ([R, H_R] - H_Q)\partial_R; \qquad (7.12)$$

and thus we have the Poisson bracket $\{\cdot,\cdot\}$,

$$\{F^{(1)}, F^{(2)}\} = \omega(X_{F^{(1)}}, X_{F^{(1)}}) = i\langle R, [F_R^{(2)}, F_R^{(1)}]\rangle + i(\langle F_Q^{(1)}, F_R^{(2)}\rangle - \langle F_Q^{(2)}, F_R^{(1)}\rangle). \qquad (7.13)$$

Note that if $H = H(R)$ is an adjoint invariant function, then if $g(t) = e + \dot{g}t + \Theta(t^2)$, $0 = (d/dt)(iH(\mathrm{Ad}_g R)) = \langle [\dot g, R], H_R\rangle = \langle R, [H_R, \dot g]\rangle$, for all $\dot g \in \mathcal{L}$, hence by (7.13) all adjoint invariant functions of R are in involution.

As is easily computed (see Ref. 5), the moment map for our action is given by $\Phi(Q,R) = \mathrm{Ad}_Q R - R$ [note $\Phi(Q,R) \in (i\mathcal{L})$. We now pick an $\alpha = \sum_{\beta\in\Delta} i c_\beta (E_\beta - E_{-\beta})$, $c_\beta \neq 0$, real, $\beta \in \Delta$, satisfying:

Property B: Let $\mathcal{L}_\alpha$ be specified by a relation of the form given in Property A, such that if $\mathrm{Ad}_Q R - R = \alpha$, then there exists a $g \in G_0$ such that $gQg^{-1} \equiv \exp(2i\bar q)$, $\bar q \in h$ (which shall imply $\bar q$ is regular). The only freedom in g is that we may specify which Weyl chamber $\bar q$ is contained in, and we shall always take it to be in the positive Weyl chamber, h^+, with respect to the ordering, and then only a finite number of choices remain for g.

As usual we compute $\Theta'_\alpha = \Phi^{-1}(\alpha)/G_\alpha$. If $\mathrm{Ad}_Q R - R = \alpha$, by Property B there is a $g \in G_0$, such that $gQg^{-1} \equiv \exp \equiv \exp 2i\bar q$, $\bar q \in h$. If we apply Ad_g to the equation $\mathrm{Ad}_Q R - R = \alpha$, we have $\mathrm{Ad}_{(gQg^{-1})}(\mathrm{Ad}_g R) - (\mathrm{Ad}_g R) = \alpha$; and so upon making use of the identity $\mathrm{Ad}_{\exp H} E_\alpha = e^{H\alpha} \cdot E_\alpha$, $H \in h$, which is a consequence of (7.2), we conclude,

$$\mathrm{Ad}_q R = \bar p + \sum_{\beta\in\Delta} i c_\beta \frac{(e^{-i\bar q_\beta} E_\beta + e^{i\bar q_\beta} E_{-\beta})}{(e^{i\bar q_\beta} - e^{-i\bar q_\beta})} \equiv \bar r, \quad \bar p \in h.$$

Note that automatically $\bar q_\beta \neq n\pi$, $n \in Z$, $\beta \in \Delta$, and thus in particular $\bar q$ is regular; and by property B, we may also assume $\bar q_\beta > 0$, $\beta \in \Delta$. If now $\{H_1, \ldots, H_n\}$ is an orthonormal basis of h, let $\bar q = \sum_{i=1}^n q_i H_i$, $p = \sum_{i=1}^n \bar p_i H_i$, then from the previous remarks we may take $(q_1, \ldots, q_n, p_1, \ldots, p_n) = (q,p)$ as coordinates on Θ'_α, subject to the open relations $\bar q_\beta > 0$, $\bar q_\beta \neq n\pi$, $n \in Z$, $\beta \in \Delta$.

The symplectic structure $\omega \to \omega_\alpha = 2\sum_i^n dq_i \wedge dp_i$. To establish this (local) fact, it is sufficient to show that at the point $(Q_\epsilon, R_\epsilon) = (\exp(2i\bar q + \epsilon T), R_\epsilon)$, $\bar q \in h$, $P_h T = 0$, $\Phi(Q_\epsilon, R_\epsilon) = \alpha$, the $\langle R, [\eta_1, \eta_2]\rangle$ piece in (7.10) is really "second order" in ϵ. This reduces to showing, as in the previous case, that for the (unique) $g_\epsilon = e + \epsilon V + \Theta(\epsilon^2)$ discussed in Property B, $g_\epsilon \exp(2i\bar q + \epsilon T) g_\epsilon^{-1} = \exp[2i\bar q + \Theta(\epsilon^2)]$, which upon linearizing about $\epsilon = 0$ is just the familiar fact that $P_h([V, 2i\bar q] + T) = 0$.

We now consider the adjoint invariant Hamiltonians on M_4, $H^{(j)} = (1/2j)\,\mathrm{tr}(\mathrm{ad}R)^{2j}$, $j = 1, 2, \ldots, n$, which as discussed before, are in involution. Since the $H^{(j)}$'s are G invariant functions, they project down to Hamiltonians, $h^{(j)}(q,p) = (1/2j)\mathrm{tr}[(\mathrm{ad}\bar r)^{2j}]$, $j = 1, \ldots, n$, on Θ'_α; which moreover are in involution, thus giving rise to a completely integrable system. From (7.12), the $H^{(j)}$'s lead to the Hamiltonian differential equations on M_4, $\dot R = 0$, $\dot Q = (L_Q)_* H_R$ (where we have used the fact $[R, H_R^{(j)}] = 0$); which has the solution $R(t) = R_0$, $Q(t) = (L_{Q_0})_* \exp((H_R(R_0)t)$. To study the equations projected to Θ'_α, set $g^{-1}Qg(t) = \exp[2i\bar q(t)]$, $g(0) = e$, $\dot g \equiv -(Lg)_* B_j$. The previous equations imply, in a now familiar manner, $\bar r = \mathrm{Ad}_{g^{-1}} \bar r_0$, $q = -(i/2)\exp^{-1}(g^{-1}(L_{Q_0}\exp[H_R(r_0)t])g)$, $Q_0 = \exp(2i\bar q_0)$. The associated differential equtions for $\bar r$, $\bar q$, are easily computed using $g^{-1}Qg = \exp(2i\bar q)$, $\mathrm{Ad}_{g^{-1}} R = \bar r$, and the adjoint invariance of $H = H^{(j)}(R)$, i.e., $\mathrm{Ad}_g H_R^{(j)}(R) = H_R^{(j)}(\mathrm{Ad}_g R)$. They are found to be

$$\delta\bar r = 0, \quad 2i\dot{\bar q} = \mathrm{ad}_Q B_j - B_j + H_R(\bar r_0), \quad Q = \exp(-2i\bar q). \qquad (7.14)$$

In the case of $j = 1$, $H^{(1)}(R) = iR$, and (7.14) implies $(1 - P_h)B_1 = -\frac{1}{4}\sum_{\beta\in\Delta} c_\beta (E_\beta - E_{-\beta})\sin^{-2}\bar q_\beta$, which thus determines B_1. We also note that $\dot\alpha = 0$, $\delta\alpha = 0$, hence $\delta(r + \frac{1}{2}\alpha) = 0$ for all the above differential equations; and so $r^+ = r + \frac{1}{2}\alpha = p + \frac{1}{2}\sum_{\beta\in\Delta} c_\beta \cot q_\beta (E_\beta + E_{-\beta})$ satisfies the "Lax equation" $\delta r^+ = 0$ for all the above flows. Finally note that $h^{(1)} = \frac{1}{2}\langle \bar r, \bar r\rangle = \frac{1}{2}\sum p_i^2 + \frac{1}{4}\sum_{\beta\in\Delta} c_\beta^2 \cot^2 \bar q_\beta$, the "Sutherland" potential. We may of course scale these algebraic equations, $q \to aq$, and take a purely imaginary to get the noncompact case, or let $a \to 0$ to get the x^{-2} potential, which corresponds to linearizing the group action about (e, R).

We remark that although Properties A and B imply the computational "Lax" criteria of Olshanetsky and Perelomov, (see Refs. 2, 3), and hence seem stronger, we suspect that in fact they are equivalent. In practice, from the remarks in Ref. 5, it is clear that in order to find α satisfying Properties A and B, one should look for α's whose orbits have dimension $2\{\mathrm{rank}_C \mathcal{L}\}$ under the adjoint action of G. Thus one would expect that the α's in Properties A and B are the same up to the factor $\sqrt{-1}$. It is of course an interesting question to investigate Θ'_α's for symplectic actions other than the ones discussed here, and hopefully find more integrable systems. Finally, we mention that given a concrete matrix representation of $\tilde G$, we can of course represent the preceding equations as matrix equations; and hence recover results of preceding sections.

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