Math 221B/Physics 202A
Differential geometry in classical and quantum mechanics

Instructors: Daniel Ruberman and Albion Lawrence

1 Introduction and motivation

We would like to call attention to a new class offered this winter/spring 2014 quarter, being taught jointly by Prof. Daniel Ruberman in Mathematics and Albion Lawrence in Physics. This is being listed jointly as Physics 202a (Quantum Field Theory) and Math 221b (Topics in Topology). It is being team-taught under the auspices of the Brandeis Geometry and Dynamics IGERT program.

This course aims to introduce basic notions of fiber bundles and connections on them, and their application to basic physical examples in classical and quantum mechanics: especially the mechanics of deformable bodies, and Berry's phase. The target audience is mathematics and physics students, and mathematically inclined students in physical chemistry, neuroscience, computer science, and economics. The essential principles here find applications to chemical and neural oscillators and control theory; there have even been suggestions that it is a useful language for describing currency trading.

The mathematics covered here typically appears in advanced courses on quantum and statistical field theory. However, it has much broader applicability, and the instructors felt that studying more elementary physics examples better highlighted the essential mathematics and lead to a broader perspective that would better prepare students to find new and creative uses for the mathematics. Furthermore, they allow us to teach a broader audience, as the essential physics background is straightforward and can be explained without the student needing two years of graduate-level physics courses.

This course is essentially a graduate course, but it is certainly appropriate for senior undergraduates with a solid mathematical background (math and physics majors especially). The modern mathematical language of manifolds and vector bundles will be introduced and used throughout, but with reference to physical and geometric notions. This will provide physics students with an appropriate vocabulary for further study, while mathematics students can try to grasp the intuition behind the formalism. Note that the course satisfies one of the IGERT course requirements; however, we strongly encourage non-IGERT students to entoll.

The course is scheduled to take place Mondays and Wedesdays from 2-3:20 PM.

2 Course Outline

The course outline below is preliminary and aspirational.

1. Introduction: Physics and mathematics of manifolds and connections
   (a) Falling cats.
   (b) Manifolds: surfaces in 3-space; configuration spaces of physical objects.
   (c) Differentiation of functions and vector fields.
   (d) Brief introduction to differential forms.
2. Line bundles and $U(1)$ bundles, and connections on them.
   (a) Curvature of a $U(1)$ connection.
   (b) Chern class of a $U(1)$ connection.
3. Applications of $U(1)$ bundles.
   (a) Electricity and magnetism as a $U(1)$ connection. Example: Dirac monopole.
   (b) Deformable bodies in two dimensions: requires some simple words on conservation of angular momentum.
   (c) (Abelian) Berry's phase. A basic introduction to quantum mechanics: Hilbert space structure, role of Hamiltonian, adiabatic approximation.
4. Vector bundles, tangent, and cotangent bundles.
   (a) Covariant derivatives.
5. Applications

   (a) Nonabelian Berry's phase.

6. Principal bundles
   (a) Frame bundles; $SU(2)$ bundles.
   (b) Configuration space as principal bundle.
7. Applications
   (a) Falling cats
   (b) Gauge theory and 4-manifolds (brief introduction).

3 Prerequisites

A minimum mathematical preparation will include multivariable calculus and linear algebra [Math 15/20 or 22ab], as well as basic notions of analysis such as continuity and differentiability [Math 40a/110a or 34a/104a]. Geometric analysis [Math 110a/140a] would be helpful but is not required. For physics students, a good undergraduate course in quantum mechanics and in advanced classical mechanics (Lagrangian and Hamltonian mechanics) will be helpful.

4 Reading List

The required (and most recommended) reading will consist of online materials placed on the course LATTE page.

4.1 Required reading

Required reading will include:

1. *Lecture Notes on Bundles and Connections*, by Chris Wendl. Online lecture notes from an MIT course on differential geometry.
2. Some parts of the article *Gauge fields in the separation of rotation and internal motions in the n-body problem*, by Robert G. Littlejohn and Matthias Reinsch. *Reviews of Modern Physics* **69**, pages 213-274. This describes the basic kinematics of deformable bodies in some generality, with a very nice two-dimensional example worked out.

3. Further reading on Berry's phase, including Michael Berry's original paper "Quantal phase factors accompanying adiabatic changes", Proceedings of the Royal Society **A392** 45-57.

4.2 Recommended bacgkround reading and reviews

We will update this list as the semester approaches. We welcome your input if you find additional nice readings on the subjects covered here.

1. *Gravitation, Gauge Theories, and Differential Geometry*, by Tohru Eguchi, Peter B. Gilkey, and Andrew J. Hanson, *Physics Reports* **66**, pages 213-393. A very readble introduction to differential geometry, written for physicists, with many instructive examples.

2. *Geometric Phases in Physics*, by Alfred Shapere and Frank Wilczek. World Scientific (1989).

4.3 Original research papers of interest

Also to be updated before and during the semester. Again, we welcome your input here.

1. *A dynamical explanation of the falling cat phenomenon*, T.R. Kane and M.P. Scher, *Int. J. Solids Structures* **5** pp 663-670.

2. *Geometric phase shifts in chemical oscillators*, M.L. Kaplan, T.B. Kepler, and I.R. Epstein, *Nature* **349** p. 506-8.